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Verjetnostna Zeta Funkcija na Končni Mreži

(The Probabilistic Zeta Function of a Finite Lattice)

Ime in priimek: Besfort Shala
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Mentor: izr. prof. Russ Woodroofe

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Izvleček: V tem delu obravnavamo naravno alternativno definicijo verjetnostne funkcije zeta za končne mreže in dobimo razvoj njenega inverza v splošno Dirichletovo vrsto. Funkcijo zeta izračunamo na številnih primerih končnih mrež, nato pa rezulate primerjamo z dobro znanimi identitetami. Na koncu preučimo še splošne lastnosti verjetnostne funkcije zeta, s poudarkom na primerih, kjer je splošna Dirichletova vrsta kar enaka običajni Dirichletovi vrsti.

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Abstract: In this paper, we consider a natural alternative definition of the probabilistic zeta function for finite lattices and obtain a general Dirichlet series expression for its reciprocal. We compute the zeta function on a number of examples of finite lattices, pointing out connections to well-known identities. Finally, we study general properties of the probabilistic zeta function, mainly focusing on when the general Dirichlet series turns out to be an ordinary one.

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i.e.	that is
e.g.	for example
poset	finite partially ordered set
sq.f.	square-free

1 INTRODUCTION

1.1 Motivation & Structure of the Paper

Let G be a finite group and s a non-negative integer. Define $P(G, s)$ to be the probability that a randomly chosen s-tuple from G^s generates G. In [5], Hall finds a finite ordinary Dirichlet series expression for $P(G, s)$, which may be used to extend the definition of $P(G, s)$ to the complex plane. We present the derivation in the next section for completeness. The **probabilistic zeta function** of G is then defined to be the reciprocal of the complex function $P(G, s)$. The name is motivated by the probabilistic interpretation of $1/\zeta(2)$, where ζ is the Riemann Zeta function (see Theorem 332 in [6] for more details). In [2], Brown defined an analogous probability function $P(L, s)$ for finite lattices in order to show that $P(G, s)$ depends only on the coset lattice of G. The aim of this project was to study this definition in its own right, that is, in a lattice-theoretic context.

We begin this paper with a section on preliminaries, including the derivation of the finite Dirichlet series expression for $P(G, s)$. In the second chapter, we propose a natural alternative definition of the probability function $P(L, s)$ for finite lattices, which may be better-suited for non-atomistic lattices. Furthermore, we compute $P(L, s)$ for a number of examples of finite lattices and point out some connections with well-known identities. In the third chapter, we define and study some general properties of $P(L, s)$, partly motivated by certain group theoretic concepts presented in [2]. Finally, in the fourth chapter, we provide summary of the paper and present possibile directions for further work on this topic. The fifth chapter contains a translation of the fourth chapter to Slovene.

1.2 Preliminaries

In this section, we list a number of well-known definitions, propositions and theorems that we will use throughout the paper.

1.2.1 The Probabilistic Zeta Function of a Finite Group & Möbius Inversion

Recall that for a finite group G and a non-negative integer s, we defined $P(G,s)$ to be the probability that a randomly chosen s-tuple in G^s generates G. Clearly, $P(G,s)\cdot|G|^s$ is the number of s-tuples which generate G. Since any s-tuple in G^s generates some subgroup $H \leq G$, we have

$$|G|^s = \sum_{H\leq G} P(H,s)\cdot|H|^s. \tag{1.1}$$

It is tempting to use a more general version of Möbius inversion to extract $P(G,s)$ from (1.1). We may do this via Möbius inversion for an arbitrary finite partially ordered set (from here on, **poset**) $\mathscr{P}$ with partial order $\leq$, a technique first introduced by Hall in [5]. We digress to present a suitable version of it here. Let $f,g:\mathscr{P}\to\mathbb{R}$ be two real valued functions such that

$$f(x) = \sum_{y\leq x} g(y). \tag{1.2}$$

We would like to find a function $\mu:\mathscr{P}\times\mathscr{P}\to\mathbb{R}$ which allows us to express $g(x)$ in terms of $f(y)$ as $g(x)=\sum_{y\leq x}\mu(y,x)f(y)$. Let us rewrite the latter expression as follows by using (1.2).

$$\begin{aligned} g(x) = \sum_{y\leq x}\mu(y,x)f(y) &= \sum_{y\leq x}\mu(y,x)\sum_{z\leq y} g(z)\\ &= \sum_{z;\ z\leq x} g(z) \sum_{y;\ z\leq y\leq x}\mu(y,x)\\ &= g(x)\mu(x,x) + \sum_{z;\ z<x} g(z)\sum_{y;\ z\leq y\leq x}\mu(y,x). \end{aligned}$$

Since the right hand side must be $g(x)$, it is but natural to recursively define μ on $\mathscr{P}\times\mathscr{P}$ by setting $\mu(x,x)=1$ for all $x\in\mathscr{P}$ and $\sum_{y;\ z\leq y\leq x}\mu(y,x)=0$ so that $\mu(z,x)=-\sum_{y;\ z<y\leq x}\mu(y,x)$ for each $z<x$. Define μ to be 0 elsewhere (for our purposes, we only need the value of $\mu(y,x)$ when $y\leq x$).

Returning to (1.1) and using Möbius inversion on the poset of subgroups of G, as well as noticing that (1.1) holds for any subgroup H of G since it holds for an arbitrary group, we obtain

$$P(G,s) = \sum_{H\leq G}\frac{\mu(H,G)}{[G:H]^s} = \sum_{n=1}^{\infty}\frac{\sum_{H\leq G;\ [G:H]=n}\mu(H,G)}{n^s}. \tag{1.3}$$

The Dirichlet series on the right hand side of (1.3) is finite, has integer coefficients and may be used to extend the domain of $P(G,s)$ to the entire complex plane $\mathbb{C}$.

1.2.2 Lattices

Because we will only be interested in finite lattices, we drop writing the word "finite" and require a lattice to be a finite one by definition.

Definition 1.1. Let L be a finite non-empty set with two commutative, associative and idempotent binary operations $\vee : L \times L \to L$ (the **join**) and $\wedge : L \times L \to L$ (the **meet**) such that for all $x, y \in L$, the meet and the join are related in the following way:

$$x \vee (x \wedge y) = x \text{ and } x \wedge (x \vee y) = x.$$

Then the algebraic structure $(L, \vee, \wedge)$ is said to be a **lattice**.

The above definition is equivalent to the following one from order theory (see [3, Chap. I, Sect. 1] for details) and we shall use them interchangeably as deemed appropriate.

Proposition 1.2. *A lattice L is a poset with order $x \leq y$ if and only if $x \vee y = y$ with the property that any two elements of L have a unique supremum and a unique infimum, which coincide with the join and the meet of the two elements, respectively.*

It can be easily shown that a lattice has unique identities for both of its binary operations. We will write $\widehat{0}$ (and call it the **bottom** element) and $\widehat{1}$ (the **top** element) for the identities of L with respect to $\vee$ and $\wedge$, respectively. We say an element $x \in L \setminus \{\widehat{0}\}$ is **join irreducible** if it cannot be written as a non-trivial join of elements of L. Equivalently (due to finiteness), an element x is join irreducible if $x = a \vee b$ implies that $a = x$ or $b = x$. Furthermore, given an element $x \in L \setminus \{\widehat{0}\}$, it is either join irreducible or else may be expressed as a non-trivial join, say $a \vee b$. Repeating the same argument for a and b and continuing in this manner, due to finiteness, we obtain a factorization of x as a join of join irreducible elements of L. In this sense, join irreducible elements serve as building blocks for lattices. The following easy observation shall be important for us.

Proposition 1.3. *Let L be a lattice with distinct identities (so that L does not consist of a single element). Then, for any $x \in L \setminus \{\widehat{0}\}$, the join of all join irreducible elements which are at most x is x. In particular, the join of all join irreducible elements in L is $\widehat{1}$.*

Proof. If the specified join was not x, this would imply that x is join irreducible, meaning it would have been present in the join initially, contradicting that the join is not x. ∎

1.2.3 Results From Number Theory

The theory that follows has intriguing relations to number theory. We shall need the following theorems.

Theorem 1.4 (The Prime Number Theorem [1]). *For a real number x, let $\pi(x)$ be the number of prime numbers less than or equal to x. Then,*

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1.$$

It easily follows that for a given $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{\pi((1+\varepsilon)x)}{\frac{x}{\log x}} = (1+\varepsilon) \lim_{x\to\infty} \frac{\pi((1+\varepsilon)x)}{\frac{x(1+\varepsilon)}{\log((1+\varepsilon)x)}} = 1+\varepsilon,$$

so that

$$\lim_{n\to\infty} \frac{\pi((1+\varepsilon)n) - \pi(n)}{\frac{n}{\log n}} = \varepsilon.$$

Since $\lim_{n\to\infty} \frac{n}{\log n} = \infty$, we have that for any $\varepsilon > 0$, $\lim_{n\to\infty} \pi((1+\varepsilon)n) - \pi(n) = \infty$ also. This gives that for any $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that for all $n \geq N$, $\pi((1+\varepsilon)n) - \pi(n) \geq 1$, i.e. that there is a prime number between n and $(1+\varepsilon)n$ for $n \geq N$. While the prime number theorem provides a solid foundation for asymptotic results, Nagura was able to prove the following much more explicit result in [8].

Theorem 1.5 (Nagura's Theorem [8]). *There exists a prime between n and $6n/5$ for any $n \geq 25$.*

In the context of our remark following the statement of the prime number theorem, Nagura's result states that $N(\varepsilon = 1/5) = 25$. Lastly, we recall Legendre's formula and a particular case of it, which for a given integer n, gives a formula for largest exponent of a power of a prime p which divides $n!$, denoted by $v_p(n!)$.

Proposition 1.6 (Legendre's formula). *Let n be a non-negative integer and let p be a prime number. Then,*

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

In particular, if p is such that $p^2 > n$, then $v_p(n!) = \lfloor n/p \rfloor$.

2 THE PROBABILISTIC ZETA FUNCTION OF A FINITE LATTICE

Let L be a lattice with distinct identities $\widehat{0}$ and $\widehat{1}$. Furthermore, let $J = J(L)$ be the set of join irreducible elements in L (recall that we have excluded $\widehat{0}$ being join irreducible by definition). We say an s-tuple $(x_1, \ldots, x_s)$ in J^s **generates up to** $x \in L \setminus \{\widehat{0}\}$ if $\bigvee_{i=1}^{s} x_i = x$ and in this case, $x_i \leq x$ for each i, therefore all components of such s-tuples come from $J_x = \{j \in J : j \leq x\}$. If $(x_1, \ldots, x_s)$ generates up to $\widehat{1}$, we say the s-tuple **generates** L. Let $P(L, x, s)$ be the probability that a randomly chosen s-tuple from J_x^s generates up to x and define $P(L, s) = P(L, \widehat{1}, s)$. Similarly as before, $P(L, x, s)|J_x|^s$ is the number of s-tuples in J_x^s which generate up to x. Since every s-tuple in J_x^s generates up to some $\widehat{0} < y \leq x$, we have

$$|J_x|^s = \sum_{\widehat{0} < y \leq x} P(L, y, s)|J_y|^s.$$

Applying Möbius inversion on $L \setminus \{\widehat{0}\}$ as the underlying poset and plugging in $x = \widehat{1}$, we obtain

$$P(L, s) = \sum_{x \in L \setminus \{\widehat{0}\}} \frac{\mu(x, \widehat{1})}{[J : J_x]^s}, \tag{2.1}$$

where $[J : J_x] = |J|/|J_x|$ (notice that $J_{\widehat{1}} = J$). As before, the expression on the right hand side could be used to extend the domain of $P(L, s)$ to the entire complex plane $\mathbb{C}$. We refer to $P(L, s)$ as the **probability function** of L, and to $1/P(L, s)$ as the **probabilistic zeta function** of L.

Some important remarks are due here. The definition of $P(L, s)$ found in the last section of [2] is different from the one we have just given, since Brown sets J to be the set of minimal elements of $L \setminus \{\widehat{0}\}$, namely the **atoms** of L. However, we may well be in a situation where the join of all atoms is not $\widehat{1}$, and consequently where none of the s-tuples would generate the whole lattice. Any chain of length at least two is an example of this situation; a more interesting example is the divisibility lattice of any non-square free integer. As shown by Proposition 1.3, we avoid this degeneracy via our definition. Indeed, the two definitions are equivalent for **atomistic** lattices, namely lattices where the join irreducible elements are precisely the atoms.

Naturally, one might wonder what is the connection between the probability function of a finite group and that of a lattice. Brown provided an answer of this in [2] (in fact, this is the primary reason why Brown introduced the concept of the probability function for lattices). For a finite group G, let $\mathscr{C}(G)$ be the set of all cosets of all subgroups of G, together with the empty set. Then, $\mathscr{C}(G)$ is a lattice with meet given by set intersection, and join given by $x_1H_1 \vee x_2H_2 = x_1H = x_2H$ for $H = \langle x_1^{-1}x_2, H_1, H_2\rangle$. We shall refer to $\mathscr{C}(G)$ as the **coset lattice** of G. Indeed, an element of $\mathscr{C}(G)$ is join irreducible if and only if it is a coset of the identity group, so $J(\mathscr{C}(G))$ corresponds to G. Brown proved that

$$P(\mathscr{C}(G), s+1) = P(G, s) \tag{2.2}$$

by noticing that an $(s+1)$-tuple $(x_0, x_1, \ldots, x_s)$ generates $\mathscr{C}(G)$ if and only if the s-tuple $(x_0^{-1}x_1, x_0^{-1}x_2, \ldots, x_0^{-1}x_s)$ generates G. Of course, all join irreducible elements in $\mathscr{C}(G)$ are atoms, hence we are justified in claiming that (2.2) holds also for our definition of the probability function.

2.1 Examples

In this section, we compute the probability function on a number of examples of lattices, establishing connections with well-known identities.

2.1.1 Divisibility Lattice

Let $\mathcal{O}_n = \{d \in \mathbb{N} : d \mid n\}$ be the set of positive divisors of a positive integer $n > 1$ with canonical factorization $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_i \geq 1$ for all $i = 1, 2, \ldots, r$. Indeed, $\mathcal{O}_n$ is a lattice with join lcm and meet gcd. The join irreducible elements of $\mathcal{O}_n$ are precisely the prime powers $p_i, p_i^2, \ldots, p_i^{\alpha_i}$ for each $i = 1, 2, \ldots, r$, that is, there are $\sum_1^r \alpha_i$ join irreducible elements in total. For a given $d \in \mathcal{O}_n$, we may write $d = p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$, where $0 \leq \beta_i \leq \alpha_i$ for all $i = 1, 2, \ldots, r$; thus the number of join irreducible elements less than d is $\beta_d = \sum_1^r \beta_i$ (in particular, $\beta_n = \sum_1^r \alpha_i$). The Möbius numbers of the divisibility lattice (as a poset) are closely related to the usual number-theoretic Möbius function. This is the content of the following lemma, the proof of which can be found in [9, Chap. 3, Sect. 8].

Proposition 2.1. *The Möbius numbers $\mu(d, n)$ of the divisibility lattice $\mathcal{O}_n$ are given by $\mu(d, n) = \mu(n/d)$, where the μ on the right hand side is the usual number-theoretic Möbius function.*

As an immediate corollary, $\mu(d,n)$ is non-zero if and only if n/d is square-free (sq.f.) and in this case, $\mu(d,n) = \mu(n/d) = (-1)^{\beta_n-\beta_d}$ and $\beta_d \geq \sum_{i=1}^{r}(\alpha_1 - 1) = \beta_n - r$. Putting everything together, we obtain

$$\begin{aligned} P(\mathcal{O}_n, s) &= \sum_{1<d|n} \frac{\mu(d,n)}{[J:J_d]^s} = \sum_{\substack{1<d|n;\\ n/d \text{ sq.f.}}} \frac{(-1)^{\beta_n-\beta_d}}{(\beta_n/\beta_d)^s} \\ &= \frac{1}{\beta_n^s} \sum_{k=\beta_n-r}^{\beta_n} (-1)^{\beta_n-k} \sum_{\substack{1<d|n;\\ n/d \text{ sq.f.};\\ \beta_d=k}} k^s \\ &= \frac{(-1)^{\beta_n}}{\beta_n^s} \sum_{k=\beta_n-r}^{\beta_n} (-1)^k \binom{r}{\beta_n-k} k^s, \end{aligned} \tag{2.3}$$

where the last equality follows as there are precisely $\binom{r}{\beta_n-k}$ divisors d of n such that n/d is square free and $\beta_d = k$ (from the r primes, we may choose any $\beta_n - k$ of them to set their power equal to $\alpha_i - 1$).

Let us now consider the special case when $\beta_n = r$, i.e. when n is square free. The expression (2.3) simplifies to

$$P(\mathcal{O}_{p_1\cdots p_r}, s) = \frac{(-1)^r}{r^s} \sum_{k=1}^{r} (-1)^k \binom{r}{k} k^s. \tag{2.4}$$

Firstly, since $\mathcal{O}_{p_1\cdots p_r}$ and the Boolean lattice $\mathcal{B}_r$ of all subsets of $\{1, 2, \ldots, r\}$ ordered by inclusion are isomorphic, (2.4) is also the probability function of $\mathcal{B}_r$.

Secondly, notice that for positive integer values of s, (2.4) may be rewritten as $r!S(s,r)/r^s$ where $S(s,r)$ is the Stirling number of the second kind (see [9, 1.94a]), namely the number of ways of partitioning s elements into r non-empty parts. Of course, an s-tuple generates $\mathcal{O}_{p_1\cdots p_r}$ if and only if it contains all of the primes $p_1, \ldots, p_r$ and for r given primes, there are precisely $r!S(s,r)$ ways of distributing the positions $1, 2, \ldots, s$ of the s-tuple to the r ordered parts determined by the primes. In other words, $r!S(s,r)$ is the number of s-tuples in $\{p_1, \ldots, p_r\}^s$ which contain all of $p_1, \ldots, p_r$.

Returning to (2.3) and noticing that an s-tuple generates $\mathcal{O}_n$ if and only if it contains all of the r maximal prime powers $p_1^{\alpha_1}, \ldots, p_r^{\alpha_r}$, we get that (2.3) is a possible generalization of the Stirling numbers of the second kind, in the sense that $\beta_n^s P(\mathcal{O}_n, s)$ is the number of s-tuples over a set with β_n elements which contain all elements of a fixed subset with r elements. Generalizations of the Stirling numbers of the second kind have been studied in [4, 7].

2.1.2 Subspace Lattice of a Finite Dimensional Vector Space Over a Finite Field

Let q be a prime power and consider the set $\mathcal{S}(\mathbb{F}_q^n)$ of all subspaces of the vector space $\mathbb{F}_q^n$ over the field with q elements $\mathbb{F}_q$. Indeed, $\mathcal{S}(\mathbb{F}_q^n)$ is a lattice with join $+$ (i.e. addition of vector subspaces) and meet $\cap$. The join irreducible elements of $\mathcal{S}(\mathbb{F}_q^n)$ are precisely its atoms, that is, the 1-dimensional subspaces of $\mathbb{F}_q^n$. Thus, the number of join irreducible elements in $\mathcal{S}(\mathbb{F}_q^n)$ is $(q^n-1)/(q-1)$, as each of the q^n-1 non-trivial elements of the vector space generates a 1-dimensional vector subspace with $q-1$ non-trivial elements.

Now, let $V \leq \mathbb{F}_q^n$ be a vector subspace. Notice that $|J_V|$, the number of join irreducible elements less than V, that is to say, the number of 1-dimensional vector subspaces of V, is precisely $(q^{\dim V}-1)/(q-1)$, for essentially the same reason as above. The relevant Möbius numbers were found by Hall; the proof may be found in [5].

Proposition 2.2. *The Möbius numbers $\mu(V,\mathbb{F}_q^n)$ of the subspace lattice $\mathcal{S}(\mathbb{F}_q^n)$ are given by $\mu(V,\mathbb{F}_q^n) = (-1)^{n-\dim V} q^{\binom{n-\dim V}{2}}$.*

Thus, we immediately obtain

$$\begin{aligned} P(\mathcal{S}(\mathbb{F}_q^n),s) &= \frac{1}{(q^n-1)^s} \sum_{0\neq V\leq \mathbb{F}_q^n} (-1)^{n-\dim V} q^{\binom{n-\dim V}{2}} (q^{\dim V}-1)^s \\ &= \frac{1}{(q^n-1)^s} \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (q^k-1)^s, \end{aligned}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$. We recall the explicit expression (see [9, Chap 1, Sect. 7] for a proof)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)},$$

which is a "q-analog" of the binomial coefficient (see [11]).

The subspace lattice $\mathcal{S}(\mathbb{F}_q^n)$ is a natural extension of the Boolean lattice $\mathcal{B}_n$, in the sense that many results about $\mathcal{S}(\mathbb{F}_q^n)$ degenerate to results about the Boolean lattice $\mathcal{B}_n$ when $q \to 1$. Thus, in many ways $\mathcal{B}_n$ plays the role of the subspace lattice of a n-dimensional vector space over a field with one lement (if one were to exist). The following proposition concretizes this relationship and lifts it to a "q-analog" relation between the probability functions.

Proposition 2.3. *If we regard $P(\mathcal{S}(\mathbb{F}_q^n), s)$ as a continuous function of q with a removable singularity at $q = 1$, then $\lim_{q\to 1} P(\mathcal{S}(\mathbb{F}_q^n), s) = P(\mathcal{B}_n, s)$.*

Proof. This immediately follows by the well-known limit $\lim_{q\to 1} \frac{q^i-1}{q^j-1} = \frac{i}{j}$ which gives $\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$. Now,

$$\lim_{q\to 1} \frac{1}{(q^n-1)^s} \sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (q^k-1)^s = \frac{(-1)^n}{n^s} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^s = P(\mathcal{B}_n, s),$$

as desired. ∎

2.1.3 Partition Lattice

Let Π_n be the set of all partitions of $\{1, 2, \ldots, n\}$, ordered by refinement. Then, Π_n is a lattice (see [9, Chap. 3, Sect. 10]). The join irreducible elements of Π_n are precisely its atoms, i.e. partitions with only one non-trivial part of size 2. To see this, notice that any non-trivial partition $P \in \Pi_n$ with parts $P_1, P_2, \ldots, P_k$ may be written as a join of atoms determined by elements in $\binom{P_i}{2}$ (i.e. the set of all 2-element subsets of P_i) for $i = 1, 2 \ldots, k$. As such, we have $|J| = \binom{n}{2}$, for atoms bijectively correspond to 2-element subsets of $\{1, 2, \ldots, n\}$.

Now, let P be as before. By similar reasoning as above, the join irreducible elements less than P are precisely the partitions corresponding to all possible pairs of elements in the parts of P, i.e. $|J_P| = |\bigsqcup_{i=1}^{k} \binom{P_i}{2}|$, meaning that $|J_P| = \sum_{i=1}^{k} \binom{|P_i|}{2}$. The Möbius number of a partition P with respect to the top partition of Π_n is given in the following proposition; the proof may be found in [9, Chap. 3, Sect. 10].

Proposition 2.4. *For a partition $P \in \Pi_n$, the Möbius number $\mu(P, \widehat{1}_{\Pi_n})$ is given by $(-1)^{|P|-1}(|P|-1)!$, where $|P|$ is the number of parts of P.*

So, we may write $P(\Pi_n, s)$ in the following form.

$$P(\Pi_n, s) = \frac{1}{\binom{n}{2}^s} \sum_{k=1}^{n} \sum_{P\in\Pi_n;\ |P|=k} (-1)^{k-1}(k-1)! \left(\sum_{i=1}^{k} \binom{|P_i(P)|}{2} \right)^s,$$

where $P_1(P), P_2(P), \ldots, P_k(P)$ are the parts of the partition P, given in some ordering.

3 PROPERTIES

3.1 Coset-Like Behavior

Notice that in (2.1), the ratio $[J : J_x]$ need not be an integer. Consequently, $P(L, s)$ need not be an ordinary Dirichlet series. In this regard, the obtained expression is a finite general Dirichlet series over the integers. Since for a coset xH in the coset lattice $\mathscr{C}(G)$ of a finite group G, the index $[J : J_{xH}] = |G : H|$ is always an integer (by Lagrange's theorem), we call lattices with this property coset-like. More precisely:

Definition 3.1. We say a lattice L is **strong coset-like** if $|J_x|$ divides $|J|$ for every $x \in L \setminus \{\widehat{0}\}$.

In principle, this divisibility condition need not be necessary (although it is sufficient) for $P(L, s)$ to be a finite ordinary Dirichlet series, for the Möbius numbers may happen to cancel out in just the right way to eliminate problematic non-integer ratios. Thus:

Definition 3.2. We say a lattice L is **weak coset-like** if $P(L, s)$ is a finite ordinary Dirichlet series.

The examples computed in the previous chapter show that the divisibility lattice $\mathcal{O}_n$ (hence also the Boolean lattice $\mathcal{B}_r$) and the subspace lattice $\mathcal{S}(\mathbb{F}_q^n)$ are typically not weak coset-like, hence also not strong coset-like. This is not entirely apparent for the partition lattice Π_n. As a first step, it is easy to prove that Π_n is typically not strong coset-like.

Proposition 3.3. *The partition lattice Π_n is strong coset-like if and only if $n \leq 4$.*

Proof. It is trivial that Π_2, Π_3 are strong coset-like, as there are no non-trivial elements which are not join irreducible, and $|J_P| = 1$ for all of them. For Π_4, it suffices to check that $\binom{3}{2} = 3$ and $\binom{2}{2} + \binom{2}{2} = 2$ divide $\binom{4}{2} = 6$.

Suppose now that $n \geq 5$. Consider the partition $P_0 = \{\{1\}, \{2, 3, 4, \ldots, n\}\}$. Then, $|J_{P_0}| = \binom{n-1}{2}$ does not divide $\binom{n}{2} = \binom{n-1}{2} + n - 1$, for this is equivalent to $\binom{n-1}{2} \mid n - 1$, yet $\binom{n-1}{2} > n - 1$ for $n \geq 5$. ∎

The lattices Π_2 and Π_3 are both isomorphic to coset lattices, namely the coset lattices of the trivial group and $\mathbb{Z}_3$, respectively. However, Π_4 is a more interesting example of a strong coset-like lattice, as it is not isomorphic to the coset lattice of any group. If $\Pi_4 \cong \mathscr{C}(G)$ for some group G, the group would need to have order 6 because $\mathscr{C}(G)$ has precisely $|G|$ join irreducible elements and Π_4 has 6 join irreducible elements, meaning that $G \cong \mathbb{Z}_6$ or $G \cong S_3$. But $\mathscr{C}(\mathbb{Z}_6)$ consists of 13 elements: the group itself, 2 cosets yielded by the unique subgroup of order 3, 3 cosets yielded by the unique subgroup of order 2, 6 cosets yielded by the trivial subgroup, together with the empty set; while $\mathscr{C}(S_3)$ consists of 19 elements: the group itself, 2 cosets yielded by the alternating group, $3 \cdot 3 = 9$ cosets yielded by each of the three subgroups of order 2, 6 cosets yielded by the trivial subgroup, together with the empty set. In contrast Π_4 has 15 elements, so it is not isomorphic to either.

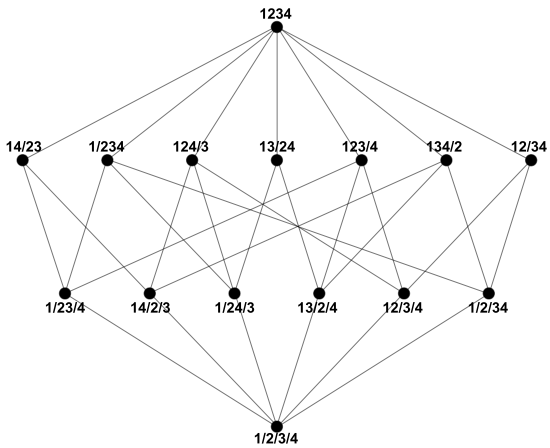

Figure 1: Hasse diagram of Π_4

The following lemma gives a sufficient condition for when we may extend a “not strong coset-like" result to a “not weak coset-like" one.

Lemma 3.4. *Let L be a lattice with distinct identities $\widehat{0}, \widehat{1}$. If there exists $x \in L \setminus \{\widehat{0}, \widehat{1}\}$ such that*

i. $|J_x| \geq |J_y|$ for all $y \in L \setminus \{\widehat{0}, \widehat{1}\}$, where the inequality is strict if y is not maximal in L,

ii. $|J_x|$ does not divide $|J|$ (i.e. the strong coset-like condition for L fails at x),

then L is not weak coset-like.

Proof. Firstly, we show that the conditions of the lemma imply that x is a maximal element. If x was not maximal, then there would exist $y \in L$ with $\widehat{1} > y > x$. Then, $|J_y| \geq |J_x| \geq |J_y|$ implies that $|J_x| = |J_y|$ which together with $J_x \subseteq J_y$ gives $J_x = J_y$. Proposition 1.3 then gives $x = \bigvee_{j \in J_x} j = \bigvee_{j \in J_y} j = y > x$, a contradiction.

Since x is maximal, $\mu(x,\widehat{1}) = -1$ and because if $y \in L$ has $|J_x| = |J_y|$, then y is maximal, thus also $\mu(y,\widehat{1}) = -1$, it follows that the term $1/[J : J_x]^s$ appears in $P(L,s)$ with a non-zero coefficient, as desired to show that $P(L,s)$ is not an ordinary Dirichlet series. ■

We obtain the full answer regarding coset-like properties of Π_n as a corollary.

Proposition 3.5. *The partition lattice Π_n is weak coset-like if and only if $n \leq 4$.*

Proof. By Proposition 3.3, it suffices to show that $P_0 = \{\{1\},\{2,3,\dots,n\}\} \in \Pi_n$ satisfies the conditions of Lemma 3.4 for $n \geq 5$. Let $P = \{P_1, P_2 \dots, P_k\}$ be any non-trivial ($1 < k < n$) partition in Π_n, set $l = \lfloor k/2 \rfloor$ and consider $P' = \{P'_1, P'_2\}$, where $P'_1 = \sqcup_{i=1}^{l} P_i$ and $P'_2 = \sqcup_{i=l+1}^{k} P_i$. Thus, P' is maximal and by reverse subadditivity of the function $x \mapsto \binom{x}{2}$ (i.e. $\binom{x}{2} + \binom{y}{2} < \binom{x+y}{2}$ and induction), we get

$$|J_P| = \sum_{i=1}^{l} \binom{|P_i|}{2} + \sum_{i=l+1}^{k} \binom{|P_i|}{2} \leq \binom{|P'_1|}{2} + \binom{|P'_2|}{2} \leq \binom{n-1}{2} = |J_{P_0}|.$$

The last inequality follows by the fact that the function $x \mapsto \binom{x}{2} + \binom{n-x}{2}$ is symmetric with respect to $x = n/2$, strictly decreasing on $[1, n/2]$ and strictly increasing on $[n/2, n-1]$, so that its maximum on $[1, n-1]$ is attained at the boundary point $x = 1$. Furthermore, the inequality $|J_P| \leq \binom{n-1}{2}$ is strict when P is not maximal, since reverse subadditivity is strict unless $k = 2$. ■

3.1.1 d-Divisible Partition Lattice

Now, we turn our attention to the d-divisible partition lattice Π_{dn}^d, namely the set of all partitions of $\{1, 2, \dots, dn\}$ with the property that each part is of cardinality divisible by d, together with the empty set, ordered by refinement. There was initially more hope for positive results regarding coset-like behavior of Π_{dn}^d, motivated by EL-labeling considerations in [12]. Although we unexpectedly obtained negative results, they are certainly more interesting than the previous results for Π_n.

The join irreducible elements of Π_{dn}^d are precisely its atoms, i.e. partitions where all parts are of cardinality exactly equal to d. To see this, notice that a part of any partition $P \in \Pi_{dn}^d$ may be partitioned further into parts with cardinalities precisely equal to d, thus the same is true for P. In order to find the number of all join irreducible elements, notice that the dn elements in $\{1, 2, \dots, dn\}$ may first be permuted in $(dn)!$ ways and for a given permutation, we create a partition with n parts, each of size d, by taking the first part to consist of the first d elements, the second part of the second d elements, and so on. Of course, the order of the elements in each part does not matter and nor does the order of the parts, hence we are counting the same partition precisely $(d!)^n n!$ times, giving that $|J| = (dn)!/((d!)^n n!)$.

Now, let P be as before and set $p_i = |P_i|/d$ for $i = 1, 2, \ldots, k$. By similar reasoning as above, the join irreducible elements less than P are all the possible partitions with n parts, each of size d, obtained by partitioning parts of P. Notice that a part P_i may be partitioned further into p_i parts of size d in precisely as many ways as there are join irreducible elements in $\Pi^d_{dp_i}$. So, the number of ways P could be partitioned further into n parts of size d, is the product $|J_P| = \prod_{i=1}^k (dp_i)!/((d!)^{p_i} p_i!)$, where $\sum_{i=1}^k p_i = n$. So, whether Π^d_{dn} is a strong coset-like lattice reduces to the following number theoretic question:

Let n be a positive integer and let $k \in \{1, 2, \ldots, n\}$. Is it true that for any positive integers $p_1, p_2 \ldots, p_k$ such that $\sum_{i=1}^k p_i = n$, the divisibility relation

$$\prod_{i=1}^{k} (dp_i)!/p_i! \mid (dn)!/n!$$

holds?

Notice that the multinomial $(\sum x_i)!/\prod x_i!$ being an integer implies both that the numerator of the product divides $(dn)!$ as well as that the denominator of the product divides $n!$, and this is inconvenient for us. Indeed, taking $d = 2$ and $n = 4$ (so that we consider the 2-divisible partition lattice over 8 elements Π^2_8; we will later show that this is in fact the minimal counterexample for 2-divisible partition lattices), then the partition $P = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$ has $|J|/|J_P| = 8!/4!/(4!/2!)^2 = 35/3$, which is not an integer.

In order to show that $\Pi^2_{2\cdot 2m}$ is not strong coset-like for any $m \geq 2$, it suffices to follow the same strategy as for Π^2_8 and show that the partition

$$P = \{\{1, 2 \ldots, 2m\}, \{2m + 1, 2m + 2, \ldots, 4m\}\}$$

is always problematic. That is, we would like to show that $((2m)!/m!)^2$ does not divide $(4m)!/(2m)!$ and this is equivalent to the condition that $\binom{2m}{m}$ does not divide $\binom{4m}{2m}$ for any $m \geq 2$. Although it was certainly expected, we were not able to find the proof of this anywhere in the literature.

Lemma 3.6. *Let $m \geq 2$ be an integer. Then, $\binom{2m}{m}$ does not divide $\binom{4m}{2m}$.*

Proof. Notice that

$$\frac{\binom{4m}{2m}}{\binom{2m}{m}} = \frac{(4m-1)(4m-3)\cdots(2m+1)}{(2m-1)(2m-3)\cdots 1} = \frac{(4m-1)!!}{(2m-1)!!^2}.$$

The identity $(2n-1)!! = (2n-1)!/(2^{n-1}(n-1)!)$ allows us to calculate $v_p((2n-1)!!)$ for $p > 2$ as $v_p((2n-1)!) - v_p((n-1)!)$. For $m \geq 16$, Theorem 1.5 (taking $6n/5 = 2m-1$) guarantees the existence of a prime p_m on the interval $[\frac{5}{6}(2m-1), 2m-1]$. Then, $p_m^2 > 4m-1$ and so the particular case of Legendre's formula (Proposition 1.6) gives

$$\begin{aligned} v_{p_m}\left(\frac{(4m-1)!!}{(2m-1)!!^2}\right) &= \left\lfloor \frac{4m-1}{p_m} \right\rfloor - 3\left\lfloor \frac{2m-1}{p_m} \right\rfloor \\ &\leq \left\lfloor \frac{4m-1}{\frac{5}{6}(2m-1)} \right\rfloor - 3\left\lfloor \frac{2m-1}{2m-1} \right\rfloor = 2 - 3 \cdot 1 = -1, \end{aligned}$$

as desired to show that $\binom{4m}{2m}/\binom{2m}{m}$ is not an integer for $m \geq 16$. The remaining cases $(2 \leq m \leq 15)$ follow by a computer check on `https://www.wolframalpha.com/`. ■

Proposition 3.7. *Π_{2n}^2 is strong coset-like if and only if $n < 4$ or $n = 5$.*

Proof. It is straightforward to check by hand that Π_{2n}^2 is strong coset-like if $n < 4$ or $n = 5$.

If $n \geq 4$ and n is even, the previous lemma gives that Π_{2n}^2 is not strong coset-like. Assume now that $n > 5$ is odd and write it as $n = 2m+1$ for some $m \geq 3$. Consider the partition $P = \{\{1, 2, \ldots, 2m\}, \{2m+1, 2m+2, \ldots, 4m+2\}\} \in \Pi_{2\cdot(2m+1)}^2$. Then, $|J_P| \mid |J|$ reads $(2m)!/m! \cdot (2m+2)!/(m+1)! \mid (4m+2)!/(2m+1)!$ or equivalently,

$$(2m+1)\binom{2m}{m} \mid (4m+1)\binom{4m}{2m}. \tag{3.1}$$

As in the proof of the previous lemma, we consider the ratio of the two terms in (3.1) and for $m \geq 16$, we fix a prime $p_m \in [\frac{5}{6}(2m-1), 2m-1]$. Then,

$$v_{p_m}\left(\frac{(4m+1)\binom{4m}{2m}}{(2m+1)\binom{2m}{m}}\right) = v_{p_m}\left(\frac{\binom{4m}{2m}}{\binom{2m}{m}}\right) + v_{p_m}(4m+1) - v_{p_m}(2m+1) < 0,$$

where the inequality follows by $v_{p_m}(2m+1) \geq 0$ and the previous lemma. Note that the choice of p_m also guarantees that $v_{p_m}(4m+1) = 0$, since p_m lies strictly between $(4m+1)/3$ and $(4m+1)/2$. The remaining cases $(3 \leq m \leq 15)$ follow by a computer check on `https://www.wolframalpha.com/`. ■

For $d > 2$, we focus on obtaining an asymptotic result, for which we shall utilize the prime number theorem.

Theorem 3.8. *For any even value of $d \geq 2$, there exists $N = N(d)$ such that Π_{dn}^d is not strong coset-like for any $n \geq N$.*

Proof. For $d = 2$, we may take $N = 4$ (this is the content of the previous proposition). Now, assume that $d > 2$ and firstly, let $n = 2m$ for some m. Consider the partition

$$P = \{\{1, 2, \ldots, dm\}, \{dm+1, dm+2, \ldots, 2dm\}\} \in \Pi_{2dm}^d.$$

The divisibility condition $|J_P| \mid |J|$ is equivalent to

$$\left(\frac{(dm)!}{m!}\right)^2 \mid \frac{(2dm)!}{(2m)!}.$$

We consider the ratio and choose N_1 large enough so that primes $p > m$ satisfy $p^2 > 2dm$. That is to say, for $2m = n \geq N_1$, the particular case of Legendre's formula gives

$$v_p\left(\frac{(2dm)!m!^2}{(2m)!(dm)!^2}\right) = \left\lfloor \frac{2dm}{p} \right\rfloor - \left\lfloor \frac{2m}{p} \right\rfloor - 2\left\lfloor \frac{dm}{p} \right\rfloor.$$

Fix $\varepsilon = \frac{1}{d+2}$ and let $N = \max\{N(\varepsilon), N_1\}$, where $N(\varepsilon)$ is obtained by the remark following the statement of the prime number theorem. Then, for all $n = 2m \geq N$, there exists a prime p_m in the interval $[(1-\varepsilon)2m, 2m)$. So, for even values of d, we obtain

$$\begin{aligned} v_{p_m}\left(\frac{(2dm)!m!^2}{(2m)!(dm)!^2}\right) &\leq \left\lfloor \frac{2dm}{(1-\varepsilon)2m} \right\rfloor - \left\lfloor \frac{2m}{2m} \right\rfloor - 2\left\lfloor \frac{dm}{2m} \right\rfloor \\ &= \left\lfloor \frac{d}{1-\frac{1}{d+2}} \right\rfloor - 1 - 2\left\lfloor \frac{d}{2} \right\rfloor \\ &= \left\lfloor \frac{d(d+1)+d}{d+1} \right\rfloor - 1 - d = -1, \end{aligned}$$

as desired to show that $|J_P| \mid |J|$ fails.

Next, suppose that $n = 2m+1$. Consider the partition

$$P = \{\{1, 2, \ldots, dm\}, \{dm+1, dm+2, \ldots, 2dm+d\}\} \in \Pi^d_{2dm+d}.$$

The divisibility relation $|J_P| \mid |J|$ is equivalent to

$$(dm+d-1)\cdots(dm+1)\cdot\frac{(dm)!^2}{m!^2} \mid (2dm+d-1)\cdots(2dm+1)\cdot\frac{(2dm)!}{(2m)!}.$$

By the first part of the proof, it suffices to show that a prime $p_m \in [(1-\varepsilon)2m, 2m)$ does not divide any $s \in \{2dm+1, \ldots, 2dm+d-1\}$. If this was the case, we would have

$$d < \frac{2dm+1}{2m} \leq \frac{s}{p_m} \leq \frac{(2dm+d-1)(d+2)}{2(d+1)m},$$

but

$$\lim_{m\to\infty} \frac{(2dm+d-1)(d+2)}{2(d+1)m} = \frac{d(d+2)}{d+1} < d+1,$$

giving that $s/p_m \in (d, d+1)$ for large enough values of m, an absurdity. ■

In the second part of the above proof, we have shown that also for odd values of d, proving that Π^d_{dn} is not strong coset-like for even n suffices to also show the same for odd n. However, the argument in the first part of the proof fails for odd d, for $\lfloor d/2 \rfloor$ is too large. We believe that the same result holds also for odd values of d, but we do not prove it in this paper.

3.2 Products

We recall that two groups G and H are **coprime** if no proper subgroup of $G \times H$ surjects onto both factors. In [2], Brown proved that for coprime finite groups G and H, the relation $P(G \times H, s) = P(G, s)P(H, s)$ holds. On the level of lattices, the identity reads $P(\mathscr{C}(G \times H), s) = P(\mathscr{C}(G), s)P(\mathscr{C}(H), s)$. Thus, we are naturally led to the problem of finding an appropriate product $\star$ on lattices for which we would be able to prove $P(L \star K, s) = P(L, s)P(K, s)$.

The first place to look at is, of course, the Cartesian product. For two lattices L, K, the Cartesian product $L \times K$ with both the meet and the join defined component-wise, is a lattice (see [3, Chap. I, Sect. 3]). The following multiplicativity statement about the Möbius numbers is easy to prove (see [9, Chap. 3, Sect. 8]).

Proposition 3.9. *Let P and Q be posets and let μ_P, μ_Q and $\mu_{P\times Q}$ be the Möbius functions of P, Q and $P \times Q$, respectively. Then, for any $(x, y), (z, w) \in P \times Q$,*

$$\mu_{P\times Q}((x, y), (z, w)) = \mu_P(x, z)\mu_Q(y, w).$$

In particular, if P, Q are lattices, then

$$\mu_{P\times Q}((x, y), (\widehat{1}, \widehat{1})) = \mu_P(x, \widehat{1})\mu_Q(y, \widehat{1}).$$

It is therefore natural to ask that the same happens for the number of join irreducible elements less than a given element (x, y) in $L \times K$, i.e. that $|J_{(x,y)}| = |J_x| \cdot |J_y|$. However, it is immediate that $P(L, s)$ does not behave well with Cartesian products, for the probability function of a chain of length 1 is identically equal to 1, but this is not true for the Cartesian product of two chains of length 1. The reason for this is that in general, for two lattices L and K, the element (x, y) need not be join irreducible if x, y are, for $(x, y) = (x, \widehat{0}) \vee (\widehat{0}, y)$.

3.2.1 Lower Reduced Products

One possible remedy is to consider the lower reduced product $\star$ of lattices, defined as $L \star K = (L \setminus \{\widehat{0}\}) \times (K \setminus \{\widehat{0}\}) \cup \{\widehat{0}\}$, with component-wise meet and join. This is a natural product to consider, because for groups G, H for which all subgroups of $G \times H$ may be factored as $S \times T$ for some $S \leq G$ and some $T \leq H$, the coset lattices satisfy $\mathscr{C}(G \times H) = \mathscr{C}(G) \star \mathscr{C}(H)$. This happens, for example, when G and H have coprime orders (see [10, Chap. 2, Sect. 4]). It is worth noting that in [2], Brown made a more general related observation involving homotopy equivalence.

Nonetheless, further assumptions must be made. If x is join irreducible in L and y is join irreducible in K, then for

$$(x, y) = (a, b) \vee (c, d) = (a \vee c, b \vee d),$$

neither (a, b) nor (c, d) need be (x, y) if x, y are not atoms. This is because we may write

$$(x, y) = (x, y') \vee (x', y)$$

if x', y' are such that $\widehat{0} < x' < x$ and $\widehat{0} < y' < y$. On the other hand, if x is an atom, then by join irreducibility of x, without loss of generality, we may take $a = x$. If $c = \widehat{0}$, this forces $d = \widehat{0}$ due to the definition of $\star$, so $b = y$, i.e. $(a, b) = (x, y)$. So, assume that $c = x$ also. If $b \neq y$, join irreducibility of y implies that $d = y$, so that $(c, d) = (x, y)$, as desired to show that (x, y) is join irreducible. Thus, if we assume that L is atomistic, then $P(L \star K, s)$ factors through the lower reduced product.

Proposition 3.10. *Let L and K be lattices, where L is an atomistic lattice. Then,*

$$P(L \star K, s) = P(L, s)P(K, s).$$

Proof. By Proposition 3.9, the Möbius function remains multiplicative over the lower reduced product. That is, $\mu((x, y), (\widehat{1}, \widehat{1})) = \mu(x, \widehat{1})\mu(y, \widehat{1})$, as in $P(L \star K, s)$, the Möbius numbers are taken over the poset $(L \star K) \setminus \{\widehat{0}\} = (L \setminus \{\widehat{0}\}) \times (K \setminus \{\widehat{0}\})$, a Cartesian product of posets.

Furthermore, if $(x, y) \in L \star K$ is join irreducible, then for $x = a \vee c$ and $y = b \vee d$, we have $(x, y) = (a, b) \vee (c, d)$, meaning that $(a, b) = (x, y)$ or $(c, d) = (x, y)$, as desired to show that both x, y are join irreducible.

By the remark preceding the statement of this proposition, we conclude that (x, y) is join irreducible in $L \star K$ if and only if x and y are join irreducible in L and K, respectively, i.e. $|J_{(x,y)}| = |J_x| \cdot |J_y|$. We finish the proof with a simple computation.

$$\begin{aligned}
P(L \star K, s) &= \sum_{\widehat{0} < (x,y) \in L \star K} \frac{\mu((x, y), (\widehat{1}, \widehat{1}))}{[J(L \star K) : J_{(x,y)}]^s} \\
&= \sum_{\widehat{0} < x \in L} \sum_{\widehat{0} < y \in K} \frac{\mu(x, \widehat{1}) \cdot \mu(y, \widehat{1})}{[J(L) : J_x]^s \cdot [J(K) : J_y]^s} \\
&= \left(\sum_{\widehat{0} < x \in L} \frac{\mu(x, \widehat{1})}{[J(L) : J_x]^s} \right) \left(\sum_{\widehat{0} < y \in K} \frac{\mu(y, \widehat{1})}{[J(K) : J_y]^s} \right) \\
&= P(L, s)P(K, s),
\end{aligned}$$

as we desired to show. ∎

4 CONCLUSION

In this final project paper, we considered the probabilistic zeta function of a finite lattice defined by Brown in [2], and proposed a natural extension to non-atomistic lattices. In an entirely similar fashion as in [2], we obtained a finite general Dirichlet series expression for the probability function (the reciprocal of the probabilistic zeta function), by using the technique of Möbius inversion on partially ordered sets, firstly introduced by Hall in [5]. We then computed the probability function on a number of standard examples of finite lattices and obtained the following results.

i. *Divisibility Lattice:* For the divisibility lattice of a square-free integer (namely the Boolean lattice), we obtained a probability function closely related to the Stirling numbers of the second kind. Then, the probabilistic interpretation of the probability function for the general divisibility lattice provides a possible generalization of the Stirling numbers of the second kind in a counting s-tuples setting (which is perhaps unusual for the Stirling numbers).

ii. *Subspace Lattice of a Finite Dimensional Vector Space Over a Finite Field:* This lattice is a natural extension of the Boolean lattice, for the latter would be the subspace lattice of a finite dimensional vector space over a one-element field, if one were to exist. We concretized this relationship by showing that the probabilistic zeta function of the Boolean lattice is a limit of that of the subspace lattice.

iii. *Partition Lattice:* Using previously known results on the Möbius numbers of the partition lattice as well as elementary counting arguments, we wrote down the probabilistic zeta function of the partition lattice. However, the expression is not as explicit as for the previous examples, thus requiring further analysis.

For the coset lattice of a group, the probability function always turns out to be an ordinary Dirichlet series. This motivated us to define coset-like lattices, namely strong coset-like and weak coset-like lattices, where as semantically suggested, strong implies weak. Because the expression for the probability function of the partition lattice is not so explicit, we investigated coset-like properties of this lattice in more detail, ultimately showing that it typically fails to be weak coset-like. To do this, we first showed that the partition lattice fails to be strong coset-like and then formulated, proved and used a lemma to translate strong to weak. Interestingly, however, a small partition lattice provided us with an example of a strong coset-like lattice which is not the coset lattice of any group. We were not able to find any examples of weak coset-like lattices which are not strong coset-like. It might be true that the two definitions are in fact equivalent and this would certainly be interesting to prove.

After the partition lattice, we turned our attention to the d-divisible partition lattice, where there was initially more hope for positive coset-like behavior. However, we again obtained negative results. On the bright side, the problems that arose were quite a bit more interesting compared to the partition lattice. During the process, we showed that the divisibility relation $\binom{2m}{m} \mid \binom{4m}{2m}$ of central binomial coefficients fails for $m \geq 2$. It was surprising that we were not able to find this result explicitly stated anywhere in the literature. For the general case when d is even, we obtained an asymptotic result by using the prime number theorem. We believe that the same asymptotic result holds for odd values of d, although we were not able to show this as part of this project paper.

Lastly, we considered products of lattices, showing that Cartesian products do not behave well with respect to the probabilistic zeta function. The right product to consider turned out to be the lower reduced product, even though an additional assumption of having at least one atomistic lattice in the product was unavoidable.

To conclude, we list a few possible directions for further research in this topic, some of which we unfortunately did not pursue due to time restrictions.

i. Extend the definition of the probabilistic zeta function to infinite lattices in a sensible way;

ii. Investigate possibilities of factoring the probabilistic zeta function by considering a relative version involving quotient lattices (and/or join homomorphisms);

iii. Write down $P(\Pi_{dn}^{d}, s)$ in a sensible way and investigate weak coset-like behavior;

iv. Prove or disprove that the definitions of weak coset-like and strong coset-like are equivalent;

v. Obtain a purely lattice theoretic characterization of strong and/or weak coset-like lattices.

5 POVZETEK NALOGE V SLOVENSKEM JEZIKU

Pri temu zaključnemu projektnemu delu smo preučevali verjetnostno funkcijo zeta na končnih mrežah, kar je uvedel Brown v svojem članku [2]. Brown je nadalje predlagal naravno alternativno definicijo, ki je morda primernejša za neatomarne mreže. Na povsem podoben način kot v [2] smo z uporabo Möbiusove inverzije na delno urejenih množicah, ki jo je prvi predstavil Hall v članku [5], dobili končen razvoj v splošno Dirichletovo vrsto za verjetnostno funkcijo (inverz k verjetnostni funkciji zeta). Nato smo izračunali verjetnostno funkcijo na številnih standardnih primerih končnih mrež in dobili naslednje rezultate.

i. *Mreža deljivosti:* Za mrežo deljivosti celega števila brez kvadratov (to je Boolovo mrežo) smo dobili verjetnostno funkcijo, ki je tesno povezana s Stirlingovimi števili druge vrste. Verjetnostna interpretacija verjetnostne funkcije za mrežo deljivosti v splošnem primeru omogoča možno posplošitev Stirlingovih števil druge vrste s pomočjo štetja s-teric (kar je za Stirlingova števila morda nenavadno).

ii. *Mreža podprostorov končno-dimenzionalnega vektorskega prostora nad končnim poljem:* Ta mreža predstavlja naravno razširitev Boolove mreže, saj bi bila slednja pravzaprav mreža podprostorov končno-dimenzionalnega vektorskega prostora nad poljem z enim elementom, če bi le-to obstajalo. To razmerje smo konkretizirali tako, da smo pokazali, da je verjetnostna funkcija zeta Boolove mreže limita funkcij na mreži podprostorov.

iii. *Mreža particij:* Z uporabo znanih rezultatov o Möbius številih za mreže particij in tudi z osnovnimi tehnikami preštevanja smo uspeli zapisati verjetnostno funkcijo zeta za mrežo particij. Ker ta ni izražena tako eksplicitno, kot v prejšnjih primerih, je potrebna nadaljnja analiza.

Za mrežo odsekov grupe se izkaže, da je verjetnostno funkcijo vedno mogoče razviti v običajno Dirichletovo vrsto. To nas je spodbudilo, da smo definirali odsekovne mreže in sicer krepko in šibko odsekovno mrežo, kjer, kot sugerira že samo ime, krepka implicira šibko. Ker izraz za verjetnostno funkcijo mreže particij ni tako ekspliciten, smo podrobneje raziskali odsekovne lastnosti te mreže in pokazali, da običajno ni šibka odsekovna mreža (zanimivo pa je, da smo našli primer majhne mreže particij, ki nam je dala primer krepke odsekovne mreže, ki pa ni mreža odsekov nobene grupe). To smo naredili tako, da smo najprej pokazali, da mreža particij ni krepka odsekovna mreža. Nato smo formulirali, dokazali in uporabili lemo, ki prevede krepko v šibko. Nismo uspeli najti noben primer šibke odsekovne mreže, ki ne bi bila tudi krepka odsekovna mreža. Vsi poskusi konstruiranja primerov so spodleteli. Morda celo drži, da sta obe definiciji ekvivalentni. Vsekakor bi to bilo zanimivo dokazati.

Za mrežami particij smo se posvetili študiju d-deljivostne mreže particij, kjer je bilo sprva več upanja, da bi se mreža obnašala tako kot odsekovna mreža. Vendar smo spet dobili negativne rezultate. Kljub temu so bili problemi, ki so se pojavili, precej zanimivejši v primerjavi s tistimi pri mrežah particij. Obenem smo pokazali tudi, da razmerje deljivosti centralnih binomskih koeficientov $\binom{2m}{m} \mid \binom{4m}{2m}$ ne drži za $m \geq 2$ (presenetljivo je, da tega rezultata nismo zasledili v literaturi). V splošnem primeru, ko je d sodo število, smo z uporabo izreka o praštevilih dobili asimptotični rezultat. Verjamemo, da enak asimptotični rezultat velja za lihe vrednosti d, čeprav tega nismo uspeli dokazati.

Nazadnje smo obravnavali še produkte mrež. Pokazali smo, da se kartezični produkti ne obnašajo lepo za verjetnostno funkcijo zeta. Izkazalo se je, da je najbolje uporabiti spodnji reducirani produkt, čeprav se dodatni predpostavki, da imamo v produktu vsaj eno atomarno mrežo, ni bilo mogoče izogniti.

Za konec naštejemo nekaj možnih smeri za nadaljnje raziskave na tem področju, ki jih žal nismo uspeli podrobneje raziskati, predvsem zaradi časovnih omejitev.

i. Razširitev definicije verjetnostne funkcije zeta na neskončne mreže na smiselen način;

ii. Preučiti možnosti faktorizacije verjetnostne funkcije zeta z upoštevanjem relativne različice, ki vključuje kvocientne mrež (in/ali pridružitvene homomorfizme);

iii. Zapisati $P(\Pi^d_{dn}, s)$ na smiselen način in raziskati šibko odsekovno vedenje;

iv. Dokazati ali ovreči, da sta definiciji šibke odsekovne mreže in krepke odsekovne mreže ekvivalentni;

v. Določiti karakterizacijo krepkih in šibkih odsekovnih mrež v jeziku teoorije mrež.

BIBLIOGRAPHY

[1] T.M. Apostol. *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics. Springer New York, 1998. ISBN: 9780387901633. URL: https://books.google.com/books?id=Il64dZELHEIC.

[2] Kenneth S. Brown. “The Coset Poset and Probabilistic Zeta Function of a Finite Group”. In: *Journal of Algebra* 225.2 (2000), pp. 989–1012. ISSN: 0021-8693. DOI: https://doi.org/10.1006/jabr.1999.8221. URL: https://www.sciencedirect.com/science/article/pii/S0021869399982212.

[3] George Grätzer. *General Lattice Theory, second edition*. Jan. 1998. ISBN: 0-12-295750-4; 3-7643-5239-6; 3-7643-6996-5. DOI: 10.1007/978-3-0348-7633-9.

[4] Martin Griffiths and István Mezo. “A generalization of Stirling numbers of the second kind via a special multiset”. English. In: *Journal of Integer Sequences* 13.2 (2010), pp. 1–23. ISSN: 1530-7638.

[5] P. Hall. “The Eulerian Functions of a Group”. In: *The Quarterly Journal of Mathematics* os-7.1 (Jan. 1936), pp. 134–151. ISSN: 0033-5606. DOI: 10.1093/qmath/os-7.1.134. eprint: https://academic.oup.com/qjmath/article-pdf/os-7/1/134/4504936/os-7-1-134.pdf. URL: https://doi.org/10.1093/qmath/os-7.1.134.

[6] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Fourth. Oxford, 1975.

[7] Mihoubi Miloud, Asmaa Rahim, and Said Taharbouchet. “The multiset partitions and the generalized Stirling numbers”. In: *Afrika Matematika* 31 (Feb. 2020). DOI: 10.1007/s13370-020-00763-8.

[8] Jitsuro Nagura. “On the interval containing at least one prime number”. In: *Proceedings of the Japan Academy* 28.4 (1952), pp. 177–181. DOI: 10.3792/pja/1195570997. URL: https://doi.org/10.3792/pja/1195570997.

[9] Richard P. Stanley. *Enumerative Combinatorics: Volume 1*. 2nd. USA: Cambridge University Press, 2011. ISBN: 1107602629.

[10] Michio Suzuki. *Group theory. I.* Vol. 247. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Translated from Japanese by the author. Springer-Verlag, Berlin-New York, 1982, pp. xiv+434. ISBN: 3-540-10915-3.

[11] Eric W. Weisstein. "q-Analog." In: *From MathWorld–A Wolfram Web Resource.* `https://mathworld.wolfram.com/q-Analog.html` (). URL: `https://mathworld.wolfram.com/q-Analog.html`.

[12] Russ Woodroofe. "Cubical Convex Ear Decompositions". In: *The Electronic Journal of Combinatorics* 16.2 (June 2009). ISSN: 1077-8926. DOI: `10.37236/83`. URL: `http://dx.doi.org/10.37236/83`.