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Master's thesis
(Magistrsko delo)

# Locally 2-arc transitive graphs and quasiprimitive groups: the twisted wreath product case

(Lokalno 2-ločno tranzitivni grafi in kvaziprimitivne grupe: zasukani venčni produkt)

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Abstract: In this thesis we give an overview of the Giudici-Li-Praeger program of global analysis of locally $s$-arc-transitive graphs and study their properties. We then focus on locally 2-arc transitive graphs with a group of automorphisms that acts quasiprimitively on only one orbit of twisted wreath type. Furthermore, we construct examples of such graphs which we believe are part of an infinite family, and we verify their existence for two cases. We conclude the thesis with a discussion of the possible full automorphism groups of the constructed graphs.

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# 1 INTRODUCTION

## 1.1 MOTIVATION AND RELATED WORK

The motivation for this thesis came from studies of $s$-arc transitive graphs, first investigated by Tutte [28,29]. An *automorphism* of a graph is a permutation of vertices which preserves adjacency. All combinations via composition of automorphisms of a graph form a group, called the *automorphism group* of the graph. It is typically difficult to work out the whole automorphism group of a graph, but we usually get away with understanding a subgroup of it.

The analysis of transitivity on paths of length $s$ emanating from a single vertex is performed via searching for subgroups of certain *quasiprimitive type*. A transitive permutation group $G$ on a set $\Omega$ is said to be *quasiprimitive* if every nontrivial normal subgroup of $G$ acts transitively on $\Omega$. A graph is *locally* $(G, s)$*-arc transitive* if the stabilizer in $G$ of a vertex $v$ is transitive on the $s$-arcs emanating from $v$. Tutte showed that locally $(G, s)$-arc transitive graphs of valency three that are also vertex transitive satisfy $s \leq 5$. Later Weiss [31] used the classification of finite simple groups to show that if the graph is vertex transitive and has valency at least three then $s \leq 7$. A recent remarkable theorem proved by van Bon and Stellmacher [30] showed that in the vertex intransitive case, $s \leq 9$. These results arose by analysing the local structure and possible stabilizers of two adjacent vertices.

In this thesis we give an overview of the Giudici-Li-Praeger [13] program of global analysis of locally $s$-arc-transitive graphs and study their properties. This framework deals with the case when $s \geq 2$ and $G$ acts intransitively on vertices. Such graphs are bipartite and the two parts of the bipartition are $G$-orbits. It is shown that if $G$ has a nontrivial normal subgroup intransitive on both $G$-orbits, then the graph arises as a "cover" of a smaller locally $s$-arc transitive graph. This reduces the problem to finding all examples where $G$ acts quasiprimitively on at least one of the two orbits.

The O'Nan-Scott Theorem for quasiprimitive groups [20] is used to study the graphs for which there are no suitable normal quotients, which are referred to as "basic graphs". We dedicate a section to this theorem, and describe the classification in detail. This theorem categorizes every finite quasiprimitive group as a subgroup of the holomorph of an abelian, simple or compound group, a twisted product group, an almost simple group, a simple or compound diagonal group, and a product action group.

We shorten the notation to HA, HS, HC, TW, AS, SD, CD, and PA, respectively. Interestingly, the original versions of O'Nan-Scott Theorem [25] for finite primitive groups incorrectly omitted the twisted wreath product case, which was only pointed out afterwards by Aschbacher [1]. There are eight types of quasiprimitive groups and Praeger's classification provides examples for each of the possible cases. Since the automorphism group of a graph is not necessarily transitive on the vertices of the graph, the action of $G$ may be different on each orbit, so we say that $G$ acts of *type* {X, Y} if $G$ acts quasiprimitively of type $X$ on one orbit and of type $Y$ on the other one. One of the main outcomes of the Giudici-Li-Praeger [13] analysis shows that if $G$ acts faithfully and quasiprimitively on both orbits, then usually $G$ acts quasiprimitively of the same type HA, TW, AS or PA on both orbits and the only other possibility is that $G$ is of quasiprimitive type {SD, PA}. If $G$ acts faithfully on both its orbits but quasiprimitively on only one of them, then the quasiprimitive action is of type HA, HS, AS, PA or TW.

In the Giuidici-Li-Praeger [13] global analysis we encounter examples of locally $(G, s)$-arc transitive graphs with $G$ quasiprimitive of type HA, TW, AS, PA and {SD, PA} on both orbits. In [21], it was shown that there exist nonbipartite $(G, 2)$-arc transitive graphs with $G$ quasiprimitive of type HA, TW, AS and PA on both orbits, as the only possible types. Hence in the global analysis we can use standard double covers of those graphs to get locally $(G, 2)$-arc transitive graphs with $G$ quasiprimitive of the aforementioned types. For the {SD, PA} case, a family of locally 3-arc transitive graphs of valencies $q$ and $q + 1$ is constructed. A general construction of locally $(G, 2)$-arc transitive graphs of {SD, PA} type is given in [15]. In the global analysis, the case with $G$ acting quasiprimitively on only one orbit is separated into the HA, PA, HS, AS or TW types. The following five examples are presented:

1. HA stars: a family of locally 3-arc transitive graphs of valencies $q$ and $\frac{q^d-1}{q-1}$.
2. PA stars: a family of locally 3-arc transitive graphs of valencies $k$ and $n$.
3. HS stars: a family of locally 3-arc transitive graphs of valencies $q$ and $q + 1$.
4. AS stars: a family of locally 3-arc transitive graphs of valencies 3 and 8.
5. TW star: a locally 3-arc transitive graph of valencies 6 and 16.

Also, they give $G$-locally primitive graphs that are not locally $(G, 2)$-arc transitive graphs. The possible types are:

1. {CD, PA}, and the graph has valencies $n^2$ and $|A_{n-1}|^2$.
2. HS on both orbits, and the graph has valency $|T : C_T(\sigma)|$.

3. SD on both orbits, and the graph has valency $|T : C_T(\sigma)|$.

4. HC on both orbits, and the graph has valency $|T : C_T(\sigma)^2|$.

5. CD on both orbits, and the graph has valency $|T : C_T(\sigma)^2|$.

The case where $G$ acts quasiprimitively on only one orbit has been further investigated in [14]. In this paper, the case where $G$ is of type HA has been completely determined with a construction. The case where $G$ is of type HS has been completely determined by using coset graph constructions. Minor adjustments to the five infinite families of HS type also led to the construction of five infinite families of locally $s$-arc transitive graphs of {SD, PA} type. Finally they gave characterizations for the case where $G$ is of type AS and for the case where $G$ is of type PA and preserves a product structure $\Delta^k$ on $\Delta_1$, which denotes one part of the bipartition. There are no known examples of type PA where $G$ does not preserve a product structure on $\Delta_1$. The case where $G$ is of type TW has not been investigated and the one example known so far is given in the global analysis, hence the motivation for finding new examples, which are described at the end of this thesis.

## 1.2 STRUCTURE OF THE THESIS

In Chapter 2, we introduce the preliminary theory consisting of necessary definitions regarding groups which are mentioned in this work and we prove a collection of useful results concerning group actions and products of groups.

As mentioned before, the O'Nan-Scott Theorem for quasiprimitive groups is of great importance to the study of locally arc transitive graphs and so in Chapter 3 we shall give an overview and a few examples related to the first seven quasiprimitive types.

In Chapter 4 we consider twisted wreath products and their properties as the last quasiprimitive type and illustrate a few examples. Our aim is to deliver an in-depth overview of these groups because they are complex and thus less investigated than the other quasiprimitive types. We then summarize Chapter 3 and 4 in Table 1.

In Chapter 5 we review notation and definitions concerning graphs and groups acting on graphs. This allows us to present various constructions of edge transitive graphs in terms of double covers, coset graphs and normal quotients.

In Chapter 6 we prove Theorem 6.1.1 and Lemma 6.1.2, which give a characterization of locally 2-arc transitive graphs that admit a group of automorphisms that acts quasiprimitively of twisted wreath type on only one orbit. Moreover, we utilize Theorem 6.1.1 and Lemma 6.1.2 to prove that assuming the two results hold, locally 2-arc transitive graphs with the aforementioned condition exist and their quotients amount to stars $K_{1,n}$.

In Chapter 7 we work out a construction of locally 2-arc transitive graphs that admit $\mathrm{PSL}(2,p) \operatorname{twr}_\phi \mathrm{ASL}(2,p)$ as a group of automorphisms. Furthermore, we state a conjecture for the existence of an infinite family of locally 2-arc transitive graphs with the aforementioned property and verify it for two cases using GAP [12]. Additionally, in Lemma 7.1.2 we prove that these graphs are not locally 3-arc transitive with respect to $\mathrm{PSL}(2,p) \operatorname{twr}_\phi \mathrm{ASL}(2,p)$.

We conclude the last chapter with a discussion about full automorphism groups of these graphs and exclude some cases based on their properties and already known results about arc transitive graphs. We manage to show that the stabilizer of the bipartition in the full automorphism group is not quasiprimitive on only one orbit of type HA, HS, AS, or PA and neither quasiprimitive on both orbits of type HA, TW or {SD, PA}. The remaining cases are left open and thus could be interesting for future research.

# 2 PRELIMINARIES

Groups are a fundamental structure in abstract algebra, which are used to describe *symmetries* of an object. We can think of a symmetry as a type of transformation applied to the object that preserves its structure. A group allows us to model and study all symmetries of an object, using a few axioms that then lead to many mathematical applications. A *group action*, which is an operation of a group on a set, provides a way to think of any abstract group as a group of symmetries. For the preliminary theory regarding groups in this chapter, we refer to [4, 11, 23, 24].

## 2.1 GROUPS AND GROUP ACTIONS

A *group* is a non-empty set $G$ with a binary operation $G \times G \to G$ such that $(g,h) \to gh$ satisfies the following laws:

1. (Closure law): $g,h \in G$ then $gh \in G$.
2. (Associative law): $g(hk) = (gh)k$ for all $g,h,k \in G$.
3. (Identity law): There exists an element $1 \in G$ such that $g1 = 1g = g$ for all $g \in G$.
4. (Inverse law): For each $g \in G$, there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.

A group $G$ is called *abelian* if the binary operation is commutative, i.e. $gh = hg$ for all $g,h \in G$. The *order* of a group $G$ is the number of elements in $G$, denoted by $|G|$. The *order* of an element $g \in G$ is the least positive integer $n$ such that $g^n = 1$. If no such $n$ exists, then $g$ has infinite order. A *subgroup* $H$ of a group $G$ is a non-empty subset of $G$ that forms a group under the same binary operation as $G$. A subgroup $N$ of $G$ is said to be normal in $G$ if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$. The $gng^{-1}$ operation is called *conjugation*, sometimes denoted by $n^g$. A *field* is a set $F$ with two binary operations $+$ and $\cdot$, called *addition* and *multiplication*, respectively, such that $F$ is a group for both operations, multiplication is distributive and both operations are commutative.

**Example 2.1.1.** The set of $n \times n$ invertible matrices with entries from a field $F$, forms a group with respect to matrix multiplication. It is called the *General Linear Group*, which is denoted by $\mathrm{GL}_n(F)$ or $\mathrm{GL}(n, F)$. If $F$ is finite, we sometimes replace $F$ with its order. Since $\det(AB) = \det(A)\det(B) \neq 0$ for any two matrices $A, B \in \mathrm{GL}(n, F)$, the set is closed under multiplication. Matrix multiplication is associative, which we take as a known fact from linear algebra. The identity matrix, denoted $I_n$, has 1s in the diagonal, and 0s elsewhere. Since we have invertible matrices, each matrix has an inverse. Matrix multiplication is not commutative for $n \geq 2$, so this group is not abelian. An interesting subgroup of $\mathrm{GL}(n, F)$ is the set of $n \times n$ invertible matrices with determinant equal to 1, which is known as the *Special Linear Group*, denoted by $\mathrm{SL}(n, F)$.

A *permutation* of a set $\Omega$ is a bijective function $\pi : \Omega \to \Omega$. The composition of permutations $\pi_1$ and $\pi_2$ applies $\pi_1$ first and then $\pi_2$. The set of all permutations of $\Omega$ with the operation of composition is a group, called the *symmetric group* on $\Omega$. We denote it by $\mathrm{Sym}(\Omega)$, $S_\Omega$, or $S_n$, if $\Omega = \{1, 2, \ldots, n\}$.

Given $g \in G$ and $H$ a subgroup of $G$, a *left coset* of $H$ in $G$ is a set $gH := \{gh : h \in H\}$ for $g \in G$. Right cosets are defined similarly. We write $[G : H]$ to denote the *coset space* of $H$, which consists of all left cosets for $H$ in $G$. The number of all cosets of $H$ in $G$ is the *index* of $H$ in $G$, denoted by $|G : H|$. Lagrange's Theorem states that $|G : H| = \frac{|G|}{|H|}$, if $G$ is a finite group. A left (right) *transversal* for a subgroup $H$ in $G$ is a set of left (right) coset representatives for the cosets of $H$.

The *core* of a subgroup $H$ in $G$, denoted $\mathrm{core}_G(H)$, is the largest normal subgroup of $G$ that is contained in $H$ (or equivalently, the intersection of the conjugates of $H$ in $G$, denoted by $\mathrm{core}_G(H) = \bigcap_{g \in G} H^g$). A subgroup $H$ is said to be *core-free* if its core is the trivial subgroup. Note that the core of a normal subgroup is the subgroup itself, because if $N \trianglelefteq G$ then $N^g = N$ for all $g \in G$, so $\bigcap_{g \in G} N^g = N$.

Let $G$ be a group. The *centralizer* of $g \in G$ is the set of elements $C_G(g)$ such that every $x$ in $C_G(g)$ commutes with $g$. The *normalizer* of $g \in G$ is the set of elements $N_G(g)$ which fix $g$ under conjugation. These definitions lift up to subgroups, so for each $H \leq G$ we have

$$C_G(H) = \{g \in G : gh = hg \text{ for all } h \in H\},$$

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

Let $G$ and $H$ be groups. A map $\phi : G \to H$ is a *homomorphism* if $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in G$. If the map is also bijective, then it is called an *isomorphism* and we say that $G$ and $H$ are *isomorphic*. In this case, we consider $G$ and $H$ to be essentially the same. An *automorphism* is an isomorphism which maps from $G$ to itself. We now prove a few useful lemmas.

**Lemma 2.1.2** (Dedekind's rule)**.** *Let $H, K, L$ be subgroups of $G$ such that $K \leq H$. Then $H \cap (KL) = K(H \cap L)$.*

*Proof.* Let $h \in H \cap (KL)$. Then $h = kl$ for some $k \in K$ and $l \in L$. We can rewrite the equality and get $k^{-1}h = l$ which belongs to both $H$ and $L$, since $K \leq H$. Thus $h \in K(H \cap L)$. For the other inclusion, note that $K(H \cap L) \subseteq H \cap (KL)$ since $K \leq H$. □

**Lemma 2.1.3.** *Let $G$ be a group and let $H, K \subseteq G$ such that $G = HK$. Then $G = KH$.*

*Proof.* Let $h \in H$ and $k \in K$. Note that $kh \in KH$ but also $kh = (h^{-1}k^{-1})^{-1}$ is in $HK$ since $HK$ is a group containing $h^{-1}k^{-1}$. So $KH \subseteq HK$. Since $HK$ is a group, $hk$ is the inverse of some element $h'k'$. Then $hk = (h'k')^{-1} = k'^{-1}h'^{-1} \in KH$, so $HK \subseteq KH$. □

**Lemma 2.1.4.** *Let $H$ and $K$ be subgroups of $G$ with* $\gcd(|H|, |K|) = 1$. *Then $H \cap K = \{1\}$.*

*Proof.* The subgroup $H \cap K$ is subgroup of $H$ so by Lagrange's Theorem, $|H \cap K|$ divides $|H|$. Similarly, $|H \cap K|$ divides $|K|$. Now $\gcd(|H|, |K|) = 1$, so $|H \cap K| = 1$. Thus $H \cap K = \{1\}$. □

A *generating set* of a group $G$ is a subset $S$ of $G$ such that every element in the group can be expressed as the combination of finitely many elements in $S$ and their inverses. By $\langle S \rangle$ we denote the smallest subgroup generated by $S$ where $S \subseteq G$. If $G = \langle S \rangle$, then we say that $S$ *generates* $G$. A group is *cyclic* if it is generated by a single element, called a *generator* of $G$.

An *action* of a group $G$ on a set $\Omega$ is a function $\mu : G \times \Omega \to \Omega$ with the following properties:

1. $\mu(g, \mu(h, \omega)) = \mu(gh, \omega)$ for all $g, h \in G$ and $\omega \in \Omega$.

2. $\mu(1, \omega) = \omega$ for all $\omega \in \Omega$, where 1 is the identity of $G$.

We usually write $\omega^g$ to denote $\mu(g, \omega)$. We say that $G$ *acts* on $\Omega$. The action is *faithful* if for every $g \neq 1$ in $G$ there exists $\omega \in \Omega$ such that $\omega^g \neq \omega$, or simply put, different elements of $G$ induce different permutations of $\Omega$. The cardinality $|\Omega|$ is called the *degree* of the action.

The *orbit* of an element $\omega$ in $\Omega$ is the set $\omega^G := \{\omega^g : g \in G\}$ and the *stabilizer* of $\omega$ is the set $G_\omega := \{g \in G : \omega^g = \omega\}$. We say that the action is *transitive* if there is just one orbit, and *intransitive* otherwise. An action is said to be *semiregular* if the stabilizer of every element is trivial. If an action is transitive and semiregular, then it is called *regular*. A well-known result about group actions is the Orbit-stabilizer theorem, which states that given an action of a finite group $G$ on $\Omega$ and $\omega \in \Omega$, we have $|G| = |\omega^G| \cdot |G_\omega|$.

**Lemma 2.1.5.** *Let $G$ be a group acting transitively on a set $\Omega$ and let $H \leq G$. Then $H$ acts transitively on $\Omega$ if and only if $G = HG_\omega$ for some $\omega \in \Omega$.*

*Proof.* First, let $H \leq G$ act transitively on $\Omega$. Then there exists $h \in H$ such that $\omega^g = \omega^h$ for $g \in G$ and $\omega \in \Omega$. Then $\omega = \omega^{hg^{-1}}$ so $hg^{-1} = k$ for some $k \in G_\omega$. Thus $g = k^{-1}h \in G_\omega H$ and by Lemma 2.1.3 $g \in HG_\omega$.

For the other direction let $G = G_\omega H$. By transitivity of $G$, for any $\alpha \in \Omega$ there exists $g \in G$ such that $\alpha = \omega^g$. Since $G = HG_\omega$ we can write $g = xh$ for $h \in H$ and $x \in G_\omega$. Then $\alpha = \omega^g = \omega^{xh} = \omega^h$ so $H$ acts transitively on $\Omega$. □

## 2.2 PRIMITIVE GROUPS

Let $G \leq S_\Omega$ be transitive.

**Definition 2.2.1.** A *block* of $G$ is a non-empty subset $\Delta \subseteq \Omega$ such that for all $g \in G$, either $\Delta^g = \Delta$ or $\Delta \cap \Delta^g = \emptyset$.

If $\Delta = \{\alpha\}$ for some $\alpha \in \Omega$ or $\Delta = \Omega$, then $\Delta$ is a *trivial block*. Any other block is *nontrivial*. Note that if $\Delta$ is a block, then $\Delta^g$ is also a block for every $g \in G$, and is called a *conjugate block of* $\Delta$. The set of all blocks conjugate to $\Delta$ given by $\{\Delta^g : g \in G\}$ is a partition of $\Omega$ and is called a *block system*.

**Definition 2.2.2.** A group $G \leq S_\Omega$ is *primitive* if it admits no nontrivial blocks. Otherwise, $G$ is *imprimitive*.

**Example 2.2.3.** Here are a few examples of primitive groups.

1. $S_n$ and $A_n$ are primitive.

2. Let $G = \langle (1\,2), (1\,3), (4\,5), (4\,6), (1\,4)(2\,5)(3\,6) \rangle \cong (S_3 \times S_3) \rtimes C_2$. Then $G$ is imprimitive with blocks $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

3. Let $D_{2n}$ be the dihedral group of degree $n$ and order $2n$. Suppose that $k$ divides $n$ and let $m = \frac{n}{k}$. Then $\{1, 1+k, 1+2k, \ldots, 1+(m-1)k\}$ is a block for $D_{2n}$, since rotation by $k$ steps fixes this set. In fact, $D_{2n}$ is primitive if and only if $n$ is a prime.

**Lemma 2.2.4.** *Let $G \leq S_\Omega$, let $\alpha \in \Omega$ and let $G_\alpha \leq H \leq G$. Then $\Delta = \alpha^H$ is a block and $|\Delta| = |H : G_\alpha|$.*

*Proof.* Suppose that there exists $g \in G$ such that $\Delta \cap \Delta^g \neq \emptyset$. Then there exist $h, k \in H$ such that $\alpha^h = \alpha^{kg}$. Then $\alpha = \alpha^{kgh^{-1}}$, so $kgh^{-1} \in G_\alpha < H$. As $k, h^{-1} \in H$, we have that $g \in H$ as well. Then $\Delta^g = (\alpha^H)^g = \alpha^H$ i.e. $\Delta^g = \Delta$ and $\Delta$ is a block. Further, by the Orbit-stabilizer theorem $|\Delta| = |H : H_\alpha|$ and $H_\alpha = G_\alpha$ so $|\Delta| = |H : H \cap G_\alpha| = |H : G_\alpha|$. □

**Theorem 2.2.5.** *Let $G \leq S_\Omega$ be transitive and let $\alpha \in \Omega$. Then $G$ is primitive if and only if $G_\alpha$ is a maximal subgroup.*

*Proof.* Suppose that $G_\alpha$ is not maximal, i.e. that there exists a proper subgroup $H$ in $G$ such that $G_\alpha < H < G$. Let $\Delta = \alpha^H$. By Lemma 2.2.4, $\Delta = \alpha^H$ is a block. Now we show that $\Delta$ is nontrivial. By Lemma 2.2.4, $|\Delta| = |H : G_\alpha| > 1$. If $\Delta = \Omega$ then for any $g \in G$ there exists an element $h \in H$ such that $\alpha^g = \alpha^h$ so $gh^{-1} \in G_\alpha < H$ so $g \in H$ and thus $H = G$, which is not possible. Therefore $\Delta$ is a nontrivial block and $G$ is not primitive.

For the other direction, suppose for a contradiction that $G$ admits a nontrivial block system and let $\Delta$ be the block containing $\alpha$. Let $H = G_\Delta$. We claim that $H$ acts transitively on $\Delta$. Let $\beta, \gamma \in \Delta$. By transitivity of $G$, there exists $g \in G$ such that $\beta = \gamma^g$. So $\beta \in \Delta \cap \Delta^g$, which implies $\Delta = \Delta^g$ so $g \in H$. Let $g \in G_\alpha$. Then $\alpha = \alpha^g$ so $\alpha \in \Delta \cap \Delta^g$, implying that $\Delta = \Delta^g$ and $g \in H$. By Lemma 2.2.4, $|\Delta| = |H : G_\alpha| \neq 1$, we have $G_\alpha < H$ and as $|\Delta| \neq |\Omega|$ we have $H < G$, so $G_\alpha$ is not maximal, which is a contradiction. □

Let $G$ act on a set $\Omega$. If $G$ acts transitively on $\{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2\}$, which is the set of distinct ordered pairs of $\Omega$, then $G$ is 2-*transitive.*

**Theorem 2.2.6.** *If $G$ is 2-transitive, then $G$ is primitive.*

*Proof.* Let $\Delta \subseteq \Omega$ be a nontrivial block. Then there exist $\alpha, \beta \in \Delta$ and there exists $\gamma \in \Omega \setminus \Delta$. By 2-transitivity, there exists $g \in G$ such that $(\alpha, \beta)^g = (\alpha, \gamma)$. Thus $\alpha = \alpha^g$, so $\alpha \in \Delta \cap \Delta^g$ which implies $\Delta = \Delta^g$. However, this gives $\beta^g = \gamma \in \Delta$, a contradiction. □

**Theorem 2.2.7.** *If $N$ is a normal subgroup of a primitive group $G$ then either $N$ is trivial or $N$ is transitive.*

*Proof.* Let $\alpha \in \Omega$ and let $\Delta = \alpha^N$. We claim that $\Delta$ is a block. Let $g \in G$ and $n \in N$. Then $(\alpha^n)^g = (\alpha^g)^{g^{-1}ng} \in (\alpha^g)^N$. Then $\Delta^g$ is also an $N$-orbit so either $\Delta = \Delta^g$ or $\Delta \cap \Delta^g = \emptyset$, i.e. $\Delta$ is a block. As $G$ is primitive either $|\Delta| = 1$ and $\alpha^N = \{\alpha\}$ for all $\alpha \in \Omega$, so that $N$ is trivial, or $\Delta = \Omega$ and $N$ is transitive. □

We now present semidirect products, as a generalization of direct products. Then we write sets of functions as a direct product, by using pointwise multiplication as the operation. This gives a direct decomposition of sets of functions with natural projections. Abstract wreath products arise as semidirect products of sets of functions and other groups, and are characterized by their primitive or imprimitive actions. In the last section we define minimal normal subgroups and their product called the socle, which plays a central role in the description of groups.

## 2.3 SEMIDIRECT PRODUCTS

Let us first define direct products. Let $G_1, G_2, \ldots, G_n$ be groups. The Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ can be turned into a group via coordinate-wise multiplication

$$(g_1, g_2, \ldots, g_n) \cdot (g_1', g_2', \ldots, g_n') = (g_1 g_1', g_2 g_2', \ldots, g_n g_n'),$$

for any $(g_1, g_2, \ldots, g_n), (g_1', g_2', \ldots, g_n') \in G_1 \times G_2 \times \cdots \times G_n$. This is called the *external direct product* of $G_1 \times G_2 \times \cdots \times G_n$. For each $i$, there exists a *natural projection* $\pi_i : G \to G_i$ defined by $\pi(g_1, \ldots, g_n) = g_i$.

The *internal* notion arises if we have a given group $G$ which can be written as a direct product of its certain subgroups. First note that for any two normal subgroups $H, K$ in $G$ the product $HK$ is a subgroup of $G$. More generally, if $G$ has normal subgroups $H_1, H_2, \ldots, H_n$ then $H_1 H_2 \cdots H_n$ is also a subgroup of $G$. We have the following theorem that defines $G$ as the internal product of the $H_i$.

**Theorem 2.3.1.** *Let $H_1, H_2 \ldots, H_n \trianglelefteq G$ such that $G = H_1 H_2 \cdots H_n$ and in addition $H_i \cap (H_1 \cdots H_{i-1} H_{i+1} \cdots H_n)$ is trivial for all $i = 1, \ldots n$. Then $G \cong H_1 \times H_2 \times \cdots \times H_n$.*

*Proof.* As $H_i, H_j \trianglelefteq G$ we have that the commutator $h_i h_j h_i^{-1} h_j^{-1} \in H_i \cap H_j$. However the second condition on the $H_i$ implies that $H_i \cap H_j$ is trivial, so $h_i h_j = h_j h_i$ and $H_i, H_j$ commute for all $i \neq j$. Finally, define $\phi : H_1 \times H_2 \times \cdots \times H_n \to G$ by $\phi(h_1, h_2, \ldots, h_n) = h_1 h_2 \cdots h_n$. Since $h_i h_j = h_j h_i$, it follows that $\phi$ is an isomorphism. □

Next we describe the concept of a *semidirect* product as a generalization of a direct product. The direct product is defined via component-wise multiplication which is intuitive but this is not the only way to combine elements of a Cartesian product. Let $H$ and $K$ be groups and suppose that we have an action of $H$ on $K$ that preserves the group structure of $K$. Let $\phi : H \to \mathrm{Aut}(K)$ be a homomorphism. Let $G := \{(k, h) : k \in K, h \in H\}$ and define a product on $G$ by

$$(k_1, h_1)(k_2, h_2) := (k_1 k_2^{\phi(h_1^{-1})}, h_1 h_2), \tag{2.1}$$

for all $(k_1, h_1), (k_2, h_2) \in G$.

**Proposition 2.3.2.** *The product defined in Equation 2.1 defines a group structure on $G$.*

*Proof.* Let $(k_1, h_1), (k_2, h_2) \in G$. Then the product $(k_1, h_1)(k_2, h_2)$ is in $G$ since conjugation by elements of $H$ preserves the group structure of $K$ and $H, K$ are closed under multiplication. The element $(1, 1)$ is the identity since conjugating by 1 fixes elements of $K$ and $\phi(1)$ is the identity homomorphism. Finally if we let $(k_3, h_3) \in G$ we have

$$\begin{aligned} ((k_1, h_1)(k_2, h_2))(k_3, h_3) &= (k_1 k_2^{\phi(h_1^{-1})}, h_1 h_2)(k_3, h_3) \\ &= (k_1 k_2^{\phi(h_1^{-1})} k_3^{\phi(h_2^{-1} h_1^{-1})}, h_1 h_2 h_3), \end{aligned}$$

and

$$\begin{aligned}(k_1,h_1)((k_2,h_2)(k_3,h_3)) &= (k_1,h_1)(k_2k_3^{\phi(h_2^{-1})},h_2h_3)\\ &= (k_1(k_2k_3^{\phi(h_2^{-1})})^{\phi(h_1^{-1})},h_1h_2h_3),\end{aligned}$$

so we have equality by properties of homomorphisms and the product is associative. □

It is easy to see that $G$ contains subgroups $H^* = \{(1,h) : h \in H\}$ and $K^* = \{(k,1) : k \in K\}$ which are isomorphic to $H$ and $K$ respectively, and $G = K^*H^*$ with $K^* \cap H^* = 1$. Moreover $K^* \trianglelefteq G$ and the action of $H^*$ on $K^*$ reflects the action of $H$ on $K$ as

$$(k,1)^{(1,h)} = (k^{\phi(h)},1) \text{ for all } h \in H, k \in K.$$

We call $G$ the *semidirect product* of $K$ by $H$, denoted by $K \rtimes H$.

Suppose now that $G$ is group with subgroups $H, K$ such that $K \trianglelefteq G$, $G = KH$ and $K \cap H = 1$. Then $G \cong K \rtimes H$ where the action of $H$ on $K$ is the conjugation in $G$. We call $G$ a *split extension* of $K$ by $H$. We sometimes write $K.H$ to denote the product of $K$ by $H$, where the extension is not necessarily a split extension.

## 2.4 SETS OF FUNCTIONS AS A DIRECT PRODUCT

This section follows Praeger and Schneider's book [22] combined with Dixon and Mortimer's book [5].

Let $\Gamma$ be a finite nonempty set and let $K$ be a finite group. We define $\mathrm{Fun}(\Gamma, K)$ to be the set of all functions from $\Gamma$ to $K$. We can define pointwise multiplication on $\mathrm{Fun}(\Gamma, K)$ as follows

$$(fg)(\gamma) := f(\gamma)g(\gamma) \in K \text{ for all } f, g \in \mathrm{Fun}(\Gamma, K) \text{ and } \gamma \in \Gamma,$$

so $\mathrm{Fun}(\Gamma, K)$ acquires a group structure.
For $\gamma \in \Gamma$ let

$$K_\gamma := \{f \in \mathrm{Fun}(\Gamma, K) : f(\gamma') = 1 \text{ for all } \gamma' \neq \gamma\}$$

and define the map $\sigma_\gamma : \mathrm{Fun}(\Gamma, K) \to K_\gamma$ by

$$\sigma_\gamma : f \to f_\gamma \text{ where } f_\gamma(\gamma') = \begin{cases} f(\gamma) & \text{if } \gamma' = \gamma \\ 1 & \text{if } \gamma' \neq \gamma \end{cases}. \tag{2.2}$$

**Proposition 2.4.1.** *The set $K_\gamma$ is a subgroup of $\mathrm{Fun}(\Gamma, K)$ and $K_\gamma \cong K$. Moreover, the set $\{K_\gamma : \gamma \in \Gamma\}$ is a direct decomposition of $\mathrm{Fun}(\Gamma, K)$ and the $\sigma_\gamma$ are the natural projections.*

*Proof.* Consider the set $K_\gamma$. If $f, g \in \mathrm{Fun}(\Gamma, K)$ and $\gamma \in \Gamma$ such that $f(\gamma) = g(\gamma) = 1$ then by definition $f(\gamma)g(\gamma) = (fg)(\gamma) = 1$. So $K_\gamma$ is closed under multiplication and hence a subgroup of $\mathrm{Fun}(\Gamma, K)$. Let $\psi : K \to K_\gamma$ be such that $\psi(k) = f_k$ where $f_k$ is the function in $\mathrm{Fun}(\Gamma, K)$ such that $f_k(\gamma) = k$ and $f_k(\gamma') = 1$ for all $\gamma' \neq \gamma$. For $k, h \in K$, if $\psi(k) = \psi(h)$ then $f_k(\gamma) = f_h(\gamma)$ so $k = h$ and $\psi$ is injective. Let $f \in K_\gamma$. Then $f(\gamma') = 1$ for all $\gamma' \neq \gamma$ so $f(\gamma) = x$ for some $x \in K$. Hence $\psi$ is surjective.

For $k, h \in K$ we have $\psi(kh) = f_{kh}$ and $\psi(k)\psi(h) = f_k f_h$. As we have pointwise multiplication $f_k f_h(\gamma) = f_k(\gamma) f_h(\gamma) = kh$ and $f_k f_h(\gamma') = f_k(\gamma') f_h(\gamma') = 1$ for $\gamma' \neq \gamma$. Also $f_{kh}(\gamma) = kh$ and $f_{kh}(\gamma') = 1$ for $\gamma' \neq \gamma$ so $\psi$ is a homomorphism. Thus $K \cong K_\gamma$ for all $\gamma \in \Gamma$.

Now let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Define $\phi : \mathrm{Fun}(\Gamma, K) \to K_{\gamma_1} \times \cdots K_{\gamma_n}$ such that $\phi(f) = (f_{\gamma_1}, \ldots, f_{\gamma_n})$ where $f_{\gamma_i} \in K_{\gamma_i}$ as in (2.2). If $f \in \mathrm{Fun}(\Gamma, K)$ such that $\phi(f) = 1$, then $f_\gamma = 1$ for all $\gamma \in \Gamma$. Then $f(\gamma) = 1$ for all $\gamma \in \Gamma$ so $f = 1$ and $\phi$ is injective. As $K$ and $\Gamma$ are both finite $|\mathrm{Fun}(\Gamma, K)| = |K_{\gamma_1}| \cdots |K_{\gamma_n}| = |K|^n$ so $\phi$ must be surjective. Finally let $f, g \in \mathrm{Fun}(\Gamma, K)$ and $\gamma \in \Gamma$. The function $(fg)_\gamma$ maps $\gamma$ to $fg(\gamma)$ which equals $f(\gamma)g(\gamma)$ and maps $\gamma' \neq \gamma$ to 1. Thus $(fg)_\gamma = f_\gamma g_\gamma$ and so

$$\phi(fg) = ((fg)_{\gamma_1}, \ldots, (fg)_{\gamma_n}) = (f_{\gamma_1} g_{\gamma_1}, \ldots, f_{\gamma_n} g_{\gamma_n}) = \phi(f)\phi(g)$$

which implies that $\phi$ is a homomorphism, and hence an isomorphism. We conclude that $\mathrm{Fun}(\Gamma, K) \cong K_{\gamma_1} \times \cdots K_{\gamma_n}$ and the maps $\sigma_\gamma$ are the natural projections. □

## 2.5 WREATH PRODUCTS

We may now describe abstract wreath products. Let $K$ and $H$ be groups and suppose $H$ acts on a nonempty set $\Gamma$. Then the *wreath product* of $K$ by $H$ with respect to this action is defined to be the semidirect product $\mathrm{Fun}(\Gamma, K) \rtimes H$ where $H$ acts on $\mathrm{Fun}(\Gamma, K)$ via

$$f^x(\gamma) := f(\gamma^{x^{-1}}) \text{ for all } f \in \mathrm{Fun}(\Gamma, K), \gamma \in \Gamma \text{ and } x \in H.$$

We denote this group by $K \wr_\Gamma H$, and call the subgroup

$$B := \{(f, 1) : f \in \mathrm{Fun}(\Gamma, K)\} \cong \mathrm{Fun}(\Gamma, K)$$

the *base group* of the wreath product. To check that $f^x$ gives an action of $H$ on $\mathrm{Fun}(\Gamma, K)$ we check that $f^1(\gamma) = f(\gamma)$ and $f^{xy}(\gamma) := (f^x)^y(\gamma)$ for all $f \in \mathrm{Fun}(\Gamma, K), \gamma \in \Gamma$ and $x, y \in H$. The first equality holds as the inverse of the identity is the identity itself. For the second equation, note that $f^{xy}(\gamma) = f(\gamma^{(xy)^{-1}}) = f(\gamma^{y^{-1}x^{-1}})$ and $(f^x)^y(\gamma) = f^x(\gamma^{y^{-1}}) = f((\gamma^{y^{-1}})^{x^{-1}}) = f(\gamma^{y^{-1}x^{-1}})$ so we have equality. Thus it was necessary to introduce $x^{-1}$ instead of $x$ into the definition since the group is not necessarily abelian. If $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ then we can identify the base group $B$ with $K^m$

as shown in the previous section. The action of $H$ on $B$ corresponds to permuting the components:

$$(u_1,\ldots,u_m)^{x^{-1}} = (u_{1'},\ldots,u_{m'}) \text{ where } x = \begin{pmatrix} 1 & 2 & \ldots & m \\ 1' & 2' & \ldots & m' \end{pmatrix},$$

for all $(u_1,\ldots,u_m)^{x^{-1}} \in B$ and $x \in H$. Clearly, $|K \wr_\Gamma H| = |K|^m|H|$.

### 2.5.1 IMPRIMITIVE WREATH PRODUCT

Let $G = K \wr_\Gamma H$. If $K$ acts on a set $\Delta$, then there is an action of $G$ on $\Delta \times \Gamma$ given by

$$(\delta,\gamma)^{(f,u)} := (\delta^{f(\gamma)},\gamma^u), \text{ for all } (\delta,\gamma) \in \Delta \times \Gamma,$$

where $(f,u) \in G = \mathrm{Fun}(\Gamma,K) \rtimes H$. This is called the *imprimitive* action since $\{(\delta,1) : \delta \in \Delta\}$ is a block.

### 2.5.2 PRIMITIVE WREATH PRODUCT

Let $H$ and $K$ be groups acting on sets $\Gamma$ and $\Delta$, respectively. Then $\mathrm{Fun}(\Gamma,K)$ is isomorphic to the direct product of $|\Gamma|$ copies of $K$ and as such acts in a natural way on the Cartesian product $\Omega$ of $|\Gamma|$ copies of $\Delta$. We also have $H$ acting on $\Omega$ in a natural way by permuting the components. We combine these actions to give a wreath product. Let $\Omega := \mathrm{Fun}(\Gamma,\Delta)$ and let $W := K \wr_\Gamma H = \mathrm{Fun}(\Gamma,K) \rtimes H$. We want to define the action of $W$ on $\Omega$. Let $\phi \in \Omega$ and let $(f,x) \in W$. Define $\phi^{(f,x)}$ by

$$\phi^{(f,x)}(\gamma) := \phi(\gamma^{x^{-1}})^{f(\gamma^{x^{-1}})} \text{ for all } \gamma \in \Gamma.$$

Then $\phi^{(1,1)}(\gamma) = \phi(\gamma^{1^{-1}})^1$ for all $\gamma \in \Gamma$ so $\phi^{(1,1)} = \phi$. We have $(f,x)(g,y) = (fg^{x^{-1}},xy)$ in $W$ so to prove we have an action we need to show that $\phi^{(f,x)(g,y)} = \phi^{(fg^{x^{-1}},xy)}$ for all $\phi \in \Omega$ and all $(f,x)(g,y) \in W$. We have

$$\phi^{(f,x)(g,y)}(\gamma^{xy}) = \phi^{(f,x)}(\gamma^x)^{g(\gamma^x)} = \phi(\gamma)^{f(\gamma)g(\gamma^x)},$$

$$\phi^{(fg^{x^{-1}},xy)}(\gamma^{xy}) = \phi(\gamma)^{f(\gamma)g^{x^{-1}}(\gamma^x)} = \phi(\gamma)^{f(\gamma)g(\gamma^x)},$$

so replacing $\gamma$ with $\gamma^{(xy)^{-1}}$ gives the required identity. This action of $K \wr_\Gamma H$ on $\Omega$ is called the *product action* of the wreath product. The product action of $W$ is faithful when the given actions of $H$ and $K$ are both faithful. The degree $|\Omega|$ of $W$ equals $|\Delta|^{|\Gamma|}$.

## 2.6 MINIMAL NORMAL SUBGROUPS

In this section we define the *socle* of a group $G$, which is generated by the smallest possible normal subgroups in $G$. Its importance is highlighted by the O'Nan-Scott type theorems, where the classification of primitive and quasiprimitive groups is based on their socles.

A subgroup $H$ of a group $G$ is called a *characteristic* subgroup, denoted by $H$ char $G$, if for every automorphism $\phi \in \mathrm{Aut}(G)$, $H^\phi = H$ holds. Every characteristic group is normal, though the converse is not guaranteed. A non-trivial group $K$ is *characteristically simple* if it has no nontrivial proper characteristic subgroups. A simple group is characteristically simple, though the converse is not guaranteed.

**Example 2.6.1.** Let $G$ be a group. The *commutator* subgroup (or *derived* subgroup) $G' = \langle [g,h] : g,h \in G \rangle$, where $[g,h] = g^{-1}h^{-1}gh$, is a characteristic subgroup of $G$. The *center* of the group $Z(G) = \{z \in G : gz = zg \text{ for all } g \in G\}$ is also a characteristic subgroup.

**Definition 2.6.2.** A nontrivial subgroup $K$ of a group $G$ is called a *minimal normal subgroup of* $G$ if it is normal, and for any normal subgroup $H$ of $G$ such that $H \leq K$, either $H = K$ or $H$ is trivial. The *socle* of a group $G$, denoted *soc*$(G)$, is the subgroup generated by the minimal normal subgroups of $G$.

**Example 2.6.3.** Let $G = A_5 \times A_6$. Then the normal subgroups of $G$ are the trivial subgroup, $A_5$, $A_6$ and $G$ since $A_n$ is simple for $n \geq 5$. Then the minimal normal subgroups of $G$ are $A_5$ and $A_6$.

**Lemma 2.6.4.** *Let $K$ be a minimal normal subgroup in $G$. Then $K$ is characteristically simple.*

*Proof.* Suppose that $H$ is a characteristic subgroup of $K$. For $g \in G$, conjugation by $g$ induces an automorphism of $K$ so $H^g = H$ for all $g \in G$. So $H \trianglelefteq G$ and either $H = \{1\}$ or $H = K$ by minimality of $K$. Hence $K$ is characteristically simple. □

**Lemma 2.6.5.** *If $K$ is characteristically simple, then it is the direct product of isomorphic simple groups.*

*Proof.* Let $T$ be a minimal normal subgroup of $K$. If $\alpha \in \mathrm{Aut}(K)$ then $T^\alpha$ is also a minimal normal subgroup. As $T, T^\alpha$ are minimal normal subgroups $T \cap T^\alpha \trianglelefteq K$, either $T \cap T^\alpha = \{1\}$ or $T = T^\alpha$. If $T \cap T^\alpha = \{1\}$ then $[T, T^\alpha] \leq T \cap T^\alpha = \{1\}$ then $T$ and $T^\alpha$ commute and $TT^\alpha \cong T \times T^\alpha$. Consider the set

$$\mathcal{D} = \{N \trianglelefteq K : N = T_1 \times T_2 \times \cdots \times T_k \text{ with each } T_i \cong T\}.$$

We have an internal direct product such that $T_i \cap T_1 \cdots T_{i-1}T_{i+1} \cdots T_k = \{1\}$ for each $i$ and $N = T_1 \cdots T_k$. Since $T_i \trianglelefteq K$, we have $T_i \trianglelefteq N$. Note that $\mathcal{D}$ contains $T$ so it is nonempty. Choose $N \in \mathcal{D}$ to be the subgroup of largest possible order. We want to show that $N = K$. Suppose $N \neq K$. As $K$ is characteristically simple $N$ is not characteristic in $K$ so there exists $\alpha \in \mathrm{Aut}(K)$ that does not fix $N$. Let $N = T_1 \times T_2 \times \cdots \times T_k$. Then there exists $i$ such that $T_i^\alpha \not\leq N$. Now $N \cap T_i^\alpha$ is a normal subgroup of $K$ and is properly contained in $T_i^\alpha$. As $T_i^\alpha$ is a minimal normal subgroup, $N \cap T_i^\alpha$ must be trivial. Then $[N, T_i^\alpha] \leq N \cap T_i^\alpha = \{1\}$ so $[N, T_i^\alpha]$ is trivial. Then $NT_i^\alpha \cong N \times T_i^\alpha \cong T_1 \times T_2 \times \cdots \times T_k \times T_i^\alpha \trianglelefteq K$. Thus $NT_i^\alpha \in \mathcal{D}$, which contradicts the maximality of $N$. Thus $K = N = T_1 \times T_2 \times \cdots \times T_k$, where each $T_i$ is a minimal normal subgroup isomorphic to $T$. We finally check that $T$ is simple. Suppose $H \trianglelefteq T_1$. Then $H \trianglelefteq T_1 \times T_2 \times \cdots \times T_k = K$. As $T_1$ is a minimal normal subgroup of $K$, either $H$ is trivial or $H = T_1$ and hence $T_1$ is simple so we are done. $\square$

# 3 QUASIPRIMITIVE GROUPS

In this chapter, we describe quasiprimitivity, which is a weaker condition than primitivity. We also give an overview of the O'Nan-Scott Theorem for quasiprimitive groups and describe their structure. The groups are classified into eight types according to the structure of their minimal normal subgroups.

## 3.1 DEFINITIONS AND EXAMPLES

**Definition 3.1.1.** A transitive permutation group $G$ on a set $\Omega$ is said to be *quasiprimitive* if every nontrivial normal subgroup of $G$ acts transitively on $\Omega$.

**Example 3.1.2.** Let $G = S_n$ be the symmetric group on the elements of $\Delta = \{1, \dots, n\}$ and let $\Omega = \{(i,j) : i,j \in \Delta, i \neq j\}$. Consider the action of $S_n$ on $\Omega$, given by $(i,j)^\sigma = (i^\sigma, j^\sigma)$ for all $\sigma \in S_n$. Let $G_1$ be the stabilizer of 1 in $G$, so that $G_1 \cong S_{n-1} < G$. Now the stabilizer of $(1,2)$ equals $G_1 \cap G_2$, so $G_{(1,2)} < G_1 < G$. This shows that the stabilizer of a point in $\Omega$ is not maximal in $G$, so the action is not primitive. The only normal subgroup of the symmetric group $S_n$ acting on $n$ points for $n > 4$ is $A_n$, which acts transitively on $\Omega$, so we have a quasiprimitive action for $n > 4$. For $n = 4$, we have the Klein four-group as a normal subgroup in $S_4$ that is not transitive on $\Omega$. If $n = 3$, then $G_{(1,2)}$ is trivial and $A_3$ is not transitive so the action is neither primitive nor quasiprimitive.

**Example 3.1.3.** Consider the action of $S_6$ on the 3-element subsets of $\{1, \dots, 6\}$. Then $\{\{1,2,3\},\{4,5,6\}\}$ is a block for $S_6$, so the action is not primitive. The only non-trivial normal subgroup of $S_6$ is $A_6$, which is transitive on the subsets, so the action is quasiprimitive. If $n$ is even, this holds for $S_n$ acting on $\frac{n}{2}$-element subsets of $\{1, \dots \dots, n\}$, where the set $\{\{1, \dots, \frac{n}{2}\}, \{\frac{n}{2}+1, \dots, n\}\}$ is a block of imprimitivity.

## 3.2 THE O'NAN-SCOTT THEOREM FOR QUASIPRIMITIVE GROUPS

The O'Nan-Scott Theorem is a famous theorem which classifies finite primitive permutation groups. The different cases are typically distinguished by their group theoretical structure, the nature of the action or the nature of the socle of the group. Praeger [20] gave an analogue to the O'Nan-Scott Theorem for quasiprimitive groups and showed that a finite quasiprimitive group is a subgroup of one of eight types:

1. HA (holomorph of an abelian group).
2. HS (holomorph of a simple group).
3. HC (holomorph of a compound group).
4. TW (a twisted wreath product).
5. AS (an almost simple group).
6. SD (a simple diagonal group).
7. CD (a compound diagonal group).
8. PA (a product action group).

The theory involving the twisted wreath product quasiprimitive type is described in detail in Chapter 4 since such groups are of great interest for this thesis. Throughout, let $T$ be a finite nonabelian simple group. The first three types of quasiprimitive groups are subgroups of the *holomorph* $\mathrm{Hol}(N)$ of a certain group $N$. This is defined as the semidirect product $\mathrm{Hol}(N) = N.\mathrm{Aut}(N)$ acting on $\Omega = N$ where for all $n \in N$ and $\sigma \in \mathrm{Aut}(N)$:

$$n\sigma : x \to x^{\sigma} n^{\sigma} \text{ for all } x \in \Omega.$$

Then $N$ is normal in $\mathrm{Hol}(N)$ and acts regularly by right multiplication.

## 3.3 HOLOMORPH OF AN ABELIAN GROUP

The first quasiprimitive type is HA, which are subgroups of the affine general linear group $\mathrm{AGL}(d,p)$, for some prime $p$ and positive integer $d$, acting on the points of the affine space $\mathrm{AG}(d,p)$. These are in fact primitive groups and they are characterized by their unique elementary abelian minimal normal subgroup $N \cong C_p^d$, which consists of all translations in $G$ and $G = N : G_0$ where the stabilizer of the zero vector $G_0$ must be an irreducible subgroup of $\mathrm{GL}(d,p)$. Note that $\mathrm{AGL}(d,p) = \mathrm{Hol}(N)$.

## 3.4 HOLOMORPH OF A SIMPLE GROUP

The second quasiprimitive type is HS, which are subgroups of the holomorph $\mathrm{Hol}(T) = T.\mathrm{Aut}(T)$, containing $T.\mathrm{Inn}(T)$ and acting on $\Omega = T$. Then this group has two minimal normal subgroups, each isomorphic to $T$ and each acts regularly on $\Omega$, one by right multiplication and one by left multiplication. Then the socle, *soc*($G$), of $G$ is $T \times T = T.\mathrm{Inn}(T)$ which acts on $\Omega$ by

$$(t_1, t_2) : s \to t_2^{-1} s t_1 \text{ for all } s \in T.$$

All such groups are primitive.

## 3.5 HOLOMORPH OF A COMPOUND GROUP

The third quasiprimitive type is HC, where we take $G \leq \mathrm{Hol}(T^k)$ for $k \geq 2$ acting on $\Omega = T^k$. This is again the holomorph action and $G$ has two minimal normal subgroups, both regular and isomorphic to $T^k$. Now $\mathrm{Inn}(T^k) \leq G_1 \leq \mathrm{Aut}(T^k)$ and we require $G_1$ to act transitively on the $k$ simple direct factors of $T^k$.

*Remark* 3.5.1. The groups of types HS and HC are the only quasiprimitive groups with two minimal normal subgroups, as the rest have only one.

## 3.6 ALMOST SIMPLE GROUPS

The next type of quasiprimitive group is AS of the form $T \leq G \leq \mathrm{Aut}(T)$ which are the almost simple groups with transitive socle $T$. We do not have much information about the action in this case, so $T$ could act regularly on $\Omega$. We have a primitive action of $G$ if and only if a point stabilizer is a maximal subgroup of $G$ not containing $T$.

**Example 3.6.1.** Let $T = A_n$ for $n \geq 5$. Since $T$ is a nonabelian simple subgroup of $S_n$ and $\mathrm{Aut}(T) = S_n$ (except when $n \neq 6$), then $S_n$ is an almost simple group as $T \leq S_n \leq \mathrm{Aut}(T)$. The action of $T$ on $\Omega = \{1, \ldots, n\}$ is transitive and primitive, but not regular since the stabilizer of 1 is not trivial because it contains elements such as $(2\,3\,4)$, for example.

## 3.7 SIMPLE DIAGONAL GROUPS

The next quasiprimitive type is SD. Let $N = T^k$ for $k \geq 2$. A *full diagonal subgroup* of $N$ is a subgroup isomorphic to $T$ whose projection onto every coordinate is also isomorphic to $T$. Let $N$ act on the full diagonal subgroup $N_\alpha = \{(t, t, \ldots, t) : t \in$

$T\} \leq K$ and consider the right coset space $\Omega = [N : N_\alpha]$ of $N_\alpha$ in $N$. Then $\Omega$ can be identified with $T^{k-1}$. Observe that

$$N(t_1, \dots, t_k) = N(t_k^{-1}t, t_k^{-1}t_2, \dots, t_k^{-1}t_{k-1}, 1),$$

so we can denote an element of $\Omega$ by $[t_1, \dots, t_{k-1}, 1]$. Then $\phi \in \mathrm{Aut}(T)$ acts on $\Omega$ via

$$[t_1, \dots, t_{k-1}, 1]^\phi = [t_1^\phi, \dots, t_{k-1}^\phi, 1].$$

Next, $N$ acts on $\Omega$ via

$$\begin{aligned}[t_1, \dots, t_{k-1}, 1]^{(s_1,\dots,s_k)} &= [t_1s_1, \dots, t_{k-1}s_{k-1}, s_k] \\ &= [s_k^{-1}t_1s_1, s_k^{-1}t_2s_2, \dots, s_k^{-1}t_{k-1}s_{k-1}, 1].\end{aligned}$$

Then the action of $N_\alpha \leq N$ is the same as those induced by the inner automorphisms of $T$. Let $W$ be the normalizer of $N$ in $S_\Omega$. Then for each $\phi \in \mathrm{Aut}(T)$, the permutation $[t_1, \dots, t_{k-1}, 1]^\phi = [t_1^\phi, \dots, t_{k-1}^\phi, 1]$ induced by $\phi$ is in $W$. Finally, $S_k$ acts on $\Omega$ by an induced action on the copies of $T$, so for each $\sigma \in S_k$, if we let $t_k = 1$ then

$$[t_1, \dots, t_{k-1}, 1] \to [t_{1\sigma^{-1}}, \dots, t_{k\sigma^{-1}}] = [t_{k\sigma^{-1}}^{-1}t_{1\sigma^{-1}}, \dots, t_{k\sigma^{-1}}^{-1}t_{(k-1)\sigma^{-1}}, 1]$$

is in $W$. Then $W = \langle N, \mathrm{Aut}(T), S_k\rangle \cong T^k.(\mathrm{Out}(T) \times S_k)$, where the extension is not necessarily a split extension. A quasiprimitive group of type SD is a group $G$ such that $N \trianglelefteq G \leq W$ and $G$ acts transitively by conjugation on the set of simple direct factors of $N$, that is, $N$ is the unique minimal normal subgroup of $G$. The action is primitive if and only if $G$ acts primitively on the $k$ simple direct factors of $N$.

**Example 3.7.1.** Consider $T = A_5$ and $k = 2$ so that $N = T^2$. Now $N_\alpha = \{(t,t) \mid t \in A_5\}$. Let $N$ act on the set of cosets $\Omega = [N : N_\alpha] \cong A_5$ by $[t,1]^{(s_1,s_2)} = [s_2^{-1}ts_1, 1]$. This is indeed an action, since $[t,1]^{(1,1)} = [t,1]$ and

$$\left([t,1]^{(s_1,s_2)}\right)^{(r_1,r_2)} = [s_2^{-1}ts_1, 1]^{(r_1,r_2)} = [r_2s_2^{-1}ts_1r_1, 1] = [t,1]^{(s_1r_1, s_2r_2)}.$$

The group $\mathrm{Out}(A_5)$ is isomorphic to $S_5/A_5 = C_2$, where the outer automorphism is conjugation by an odd permutation. We have $W \cong (A_5 \times A_5) \rtimes (C_2 \times C_2)$. The permutation $\sigma = (1,1,1,x)$ where $x \in C_2$ acts on $\Omega$ by its induced action on the copies of $T$

$$(t,1,1,1)^\sigma = (1,t,1,1)$$

so $W$ acts transitively by conjugating on the simple direct factors of $N$ and hence it is a quasiprimitive group of type SD. To see that this is actually a primitive group, consider the stabilizer of $[1,1]$. We have

$$[1,1]^{(r_1,r_1)} = [1^{r_1}, 1] = [1,1],$$

$$[1,1]^{\sigma} = [1^{\sigma},1] = [1,1],$$
$$[1,1]^{(1\,2)} = [1^{(1\,2)},1] = [1,1],$$

so $W_{[1,1]} = A_5 \rtimes (C_2 \times C_2) \cong S_5 \times C_2$ which is maximal in $W$, and hence the action is primitive.

## 3.8 COMPOUND DIAGONAL GROUPS

These groups are built from the SD type. Let $H$ be a quasiprimitive group of type SD on $\Delta$ with a unique minimal normal subgroup $T^l$. Let $k$ be a positive integer divisible by $l$. If $G$ satisfies $N = T^k \leq G \leq H \wr S_{k/l}$, then $G$ acts on $\Delta^{k/l}$ with the product action of the wreath product by

$$(\delta_1, \ldots, \delta_{k/l})^h = (\delta_1^{h_1}, \ldots, \delta_{k/l}^{h_{k/l}}),$$
$$(\delta_1, \ldots, \delta_{k/l})^{\sigma} = (\delta_{1\sigma^{-1}}, \ldots, \delta_{(k/l)\sigma^{-1}})$$

for $(\delta_1, \ldots, \delta_{k/l}) \in \Omega, h = (h_1, \ldots, h_{k/l})$ and $\sigma \in S_{k/l}$. This action is quasiprimitive if and only if $G$ acts transitively by conjugation on the set of simple direct factors of $N$, and it is primitive if $H$ is primitive. $N$ is the unique minimal normal subgroup of $G$.

## 3.9 PRODUCT ACTION GROUPS

The next quasiprimitive type is PA. Here $G$ preserves some partition $\mathcal{P}$ (possibly with parts of size 1) of $\Omega$ upon which $G$ acts faithfully preserving a product structure on $\Delta^k$. Further $N = T^k \leq G \leq H \wr S_k$, where $H$ acts quasiprimitively on $\Delta$ of type AS with nonregular socle $T$ and $G$ acts transitively by conjugation on the set of simple direct factors of $N$. The action of $G$ is primitive if and only if $\mathcal{P}$ is trivial and the action of $H$ on $\Delta$ is primitive.

**Example 3.9.1.** Let $G = (A_5 \times A_5) \rtimes C_2 = A_5 \wr C_2$. Then $G$ acts on the set of cosets of $G_\omega = (A_4 \times A_4) \rtimes C_2$. We would like to show that this action is primitive by showing that $G_\omega$ is maximal in $G$. Suppose that $G_\omega < K$ for some $K \leq G$. We would like to show that $K = G$. Let $g \in K \setminus G_\omega$. Then $gh \notin G_\omega$ for all $h \in G_\omega$, as otherwise we multiply by $h^{-1}$ to get $g \in G_\omega$. We may write $g$ as as $g = (g_1, g_2, g_3)$ where $g_1$ is in the first copy of $A_4$, $g_2$ is in the second copy of $A_4$ and $g_3$ is in $C_2$. Note that $h = (1, 1, g_3^{-1}) \in G_\omega$ since $g_3 \in C_2$. Then $gh = (g_1, g_2, 1)$. Clearly, $(g_1, g_2, 1) \in A_5 \times A_5$ and $(g_1, g_2, 1) \notin A_4 \times A_4$, since otherwise $g \in G_\omega$. This means either $g_1$ is not in the first copy of $A_4$ or $g_2$ is not in the second copy of $A_4$, so without loss of generality, suppose $g_1$ is not in the first copy of $A_4$. Then $\langle (g_1, g_2, 1), A_4 \times A_4 \rangle \subseteq K$. Note that $g_1$ and the first copy of $A_4$ generate $A_5$ as $A_4$ is maximal in $A_5$. If we take $(g_1, g_2, 1)^h = (g_2, g_1, 1)$, we get another copy of $A_5$ in $K$. Then $\langle A_5 \times A_5, h\rangle \subseteq K$ and $\langle A_5 \times A_5, h\rangle$ generates $G$, so $K = G$. Therefore $G_\omega$ is maximal in $G$ and the action of $G$ on cosets of $G_\omega$ is primitive.

We now construct a quasiprimitive action of $G$. Let $G_\sigma = (A_4 \times A_4) \rtimes C_2$ and let $G_\omega = (A_3 \times A_3) \rtimes C_2 \leq G_\sigma$. Let $\Omega = [G : G_\omega]$ and note that the action of $G$ on $\Omega$ is not primitive since $G_\omega < G_\sigma < G$. Let $N \trianglelefteq G$. Then $N \cap (A_5 \times A_5)$ is a normal subgroup of $A_5 \times A_5$ and $G$. Note that $A_5 \times A_5$ is a direct product of the simple group $A_5$, so a normal subgroup in $A_5 \times A_5$ is either trivial, all of $A_5 \times A_5$ or one of the $A_5$ factors. If $N \cap (A_5 \times A_5)$ is trivial, then $N$ is a conjugate of $C_2$, which is not possible. If $N \cap (A_5 \times A_5)$ is one of the $A_5$ factors then $C_2$ does not normalize it which contradicts its normality in $G$. So we are left with $N \cap (A_5 \times A_5) = A_5 \times A_5$. Note that $(A_5 \times A_5)G_\omega = G$ so we have a quasiprimitive action on $\Omega$ by Lemma 2.1.5.

# 4 TWISTED WREATH PRODUCTS

In this chapter, we provide the theory to describe twisted wreath product groups as the last quasiprimitive type, and give examples which are of quasiprimitive type. They were first constructed by B.H. Neumann [19] in 1963. In 1982, Suzuki [26] gave a more elegant description. We closely follow Praeger's [20] and Baddeley's [2] notation.

## 4.1 DEFINITION AND A FEW FACTS

Let $T$ be a finite nonabelian simple group, let $P$ be an arbitrary group and let $Q \leq P$ together with a specified homomorphism $\phi : Q \to \mathrm{Aut}(T)$. Let $\mathcal{T}$ be a left transversal for $Q$ in $P$. We can define an action of $P$ on $\mathrm{Fun}(P,T)$ preserving the group pointwise multiplication via

$$f^p(x) := f(px) \text{ for } f \in \mathrm{Fun}(P,T) \text{ and } p, x \in P.$$

This is indeed an action since $f^1(x) = f(1x) = f(x)$ for all $x \in P$ and for $f \in \mathrm{Fun}(P,T)$ and $p_1, p_2, x \in P$ we have

$$(f^{p_1})^{p_2}(x) = f^{p_1}(p_2 x) = f(p_1(p_2 x)) = f((p_1 p_2)x) = f^{p_1 p_2}(x).$$

Consider the semidirect product $\mathrm{Fun}(P,T) \rtimes P$ with respect to this action. Now define

$$B_\phi = \{f \in \mathrm{Fun}(P,T) \mid f(pq) = f(p)^{\phi(q)} \text{ for all } p \in P, q \in Q\}.$$

This is called the *$\phi$-base* group.

**Lemma 4.1.1.** *The set $B_\phi$ is a subgroup of* $\mathrm{Fun}(P,T)$ *which is invariant under the action of $P$. Further, the restriction mapping $f \to f|_R$ is an isomorphism of $B_\phi$ onto* $\mathrm{Fun}(P,T)$.

*Proof.* Let $f, g \in B_\phi$. Then

$$\begin{aligned} fg^{-1}(pq) &= f(pq)g^{-1}(pq) = f(p)^{\phi(q)} g^{-1}(p)^{\phi(q)} \\ &= (f(p)g^{-1}(p))^{\phi(q)} = (fg^{-1}(p))^{\phi(q)} \end{aligned}$$

is in $B_\phi$ as $f, g \in B_\phi$ and the condition holds for all $p \in P$ and $q \in Q$. Thus $B_\phi \leq \mathrm{Fun}(P,T)$. The action of $P$ on $B_\phi$ is given by

$$f^p(x) = f(px) \text{ for } f, f^p \in B_\phi \text{ and } p, x \in P.$$

Let $p_1, p_2, q \in P$ and $f \in B_\phi$. Then

$$\begin{aligned} f^{p_1}(q)f^{p_2}(q) &= f(p_1q)f(p_2q) = f(p_1)^{\phi(q)}f(p_2)^{\phi(q)} = (f(p_1)f(p_2))^{\phi(q)} \\ &= f(p_1p_2)^{\phi(q)} = (f^{p_1}(p_2))^{\phi(q)} = f^{p_1}(p_2q) = f^{p_1}(f^{p_2}(q)) = f^{p_1p_2}(q). \end{aligned}$$

Now, if $f, g \in B_\phi$ and $p, q \in P$, then $(fg)^p(q) = fg(pq)$ is in $B_\phi$ because

$$f^p g^p(q) = f^p(q)g^p(q) = f(pq)g(pq),$$

which equals $fg(pq)$ by definition, so the group operation is preserved. Consider the restriction mapping $f \to f|_{\mathcal{T}}$ from $B_\phi$ to $\mathrm{Fun}(\mathcal{T}, T)$. Each function $f : \mathcal{T} \to T$ can be naturally extended to $f \in B_\phi$ by defining

$$f(zq) = f(z)^{\phi(q)}, \text{ for all } z \in \mathcal{T} \text{ and } q \in Q.$$

So the restriction mapping is surjective. Also if $f, g \in B_\phi$ such that $f \neq g$ then their restrictions are necessarily distinct because $\mathcal{T}$ is a transversal for $Q$ in $P$ so the image of every element in $P$ is determined by $(f(x))_{x \in B_\phi}$. So the mapping is injective. Now let $f, g \in B_\phi$. We have that $|_{\mathcal{T}} : fg \to (fg)|_{\mathcal{T}}$ which equals $f|_{\mathcal{T}} g|_{\mathcal{T}}$ since pointwise multiplication is defined. Thus the restriction mapping is an isomorphism. □

It follows that if $|\mathcal{T}| = |P : Q| = n$ then $B_\phi \cong T^n$, so we can define the semidirect product $G = B_\phi \rtimes P$ which is a subgroup of $\mathrm{Fun}(P, T)$. It is called the *twisted wreath product* of $T$ by $P$ with respect to $\phi$, written $T \operatorname{twr}_\phi P$, and $P$ is called the *top group*, $B_\phi$ is simply the *base group*, while $\phi$ is called the *twisting homomorphism*.

Any twisted wreath product $G$ as above has an action on its base group where the base group acts by right multiplication and the top group by the action described above. Namely, if $f \in B$ then $f$ acts on $B$ via $f : g \to fg$ and if $p \in P$ then $p$ acts on $B$ via $p : g \to g^p$. We call this action the *base group action* of the twisted wreath product.

**Lemma 4.1.2.** *The action of $G$ on its base group is quasiprimitive if and only if $\phi^{-1}(\mathrm{Inn}(T))$ is a core-free subgroup of $P$.*

A proof can be found in [22], and a discussion in [3]. Such groups are said to be of quasiprimitive type TW and they are the only quasiprimitive groups with a unique minimal normal subgroup isomorphic to $T^k$ for $k \geq 2$ which acts regularly. The action is primitive if and only if $\phi$ does not extend to a larger subgroup of $P$ and $\mathrm{Inn}(T) \leq \phi(Q)$ as shown by Baddeley in [2, Lemma 3.1], which is hard to check.

## 4.2 EXAMPLES OF TWISTED WREATH PRODUCTS

We consider a few examples of quasiprimitive twisted wreath product groups.

**Example 4.2.1.** We consider the smallest possible example for a twisted wreath product group. Let $T = A_5$, $P = A_6$ and $Q = A_5 \leq P$ such that $\phi : Q \to \mathrm{Aut}(T)$ is the identity map where we identify $T$ with inner automorphisms of $T$, which are conjugations. Since $|K : L| = 6$, we have $B \cong A_5^6$ so $G = B \operatorname{twr}_\phi P \cong A_5^6 \operatorname{twr}_\phi A_6$. Since $\mathrm{Im}(\phi)$ is not a homomorphic image of $P$, the action is primitive.

**Example 4.2.2** ( [20], Remark 2.1)**.** Let $T = A_5$, $P = S_4$ and $Q = V_4$ the Klein 4-subgroup of $P$. Let $\phi : Q \to \mathrm{Aut}(T)$ such that $\phi$ maps $(1\,2)(3\,4)$ and $(1\,3)(2\,4)$ to automorphisms of $T$ induced by conjugating by $(1\,2)$ and $(1\,2)(3\,4)$, respectively. Then $\phi^{-1}(\mathrm{Inn}(T)) = \langle(1\,2)(3\,4)\rangle$ is a core-free subgroup of $P$, so the action of $G$ is quasiprimitive of type TW.

**Example 4.2.3** ( [20], Remark 2.1)**.** Let $T = A_n$ for $n > k$, $P = S_{k+1}$ and $Q = S_k$. Let $\phi : Q \to \mathrm{Aut}(T)$ be an inclusion map from $S_k$ to the stabilizer of $n-k$ points in $S_n$. Note that $\mathrm{Aut}(T) = S_n \cong S_k \times S_{n-k}$. For $n \geq 4$, $Z(A_n) = \{1\}$, and so $\mathrm{Inn}(T) = A_n$. Now $\phi^{-1}(\mathrm{Inn}(T)) = \phi^{-1}(A_n)$ is core-free in $S_{k+1}$, since the only non-trivial normal subgroup in $S_{k+1}$ is $A_{k+1}$. So the action of $G$ is quasiprimitive of type TW. Note that $\mathrm{Im}(\phi) = S_k$ does not contain $\mathrm{Inn}(T)$ and $\mathrm{Inn}(T) \cap \mathrm{Im}(\phi) = A_n \cap S_k = A_k$ which is strictly contained in $A_n$. Thus we can extend $\phi$ to a larger subgroup of $P$, so the action is not primitive.

## 4.3 SOME USEFUL RESULTS

We now prove a few helpful results about twisted wreath product groups. In the preliminaries, we have shown that sets of functions can be written as a direct product, and this theory extends to the base group of twisted wreath product groups.

**Lemma 4.3.1.** *Let $T$ be a finite nonabelian simple group, $P$ a group with a proper subgroup $Q$ and let $\phi : Q \to T$ a homomorphism. Let $G = T \operatorname{twr}_\phi P$. Let $\mathcal{T} = \{z_1, \ldots, z_n = 1\}$ be a transversal for $Q$ in $P$, where $n = |P : Q|$, and define $T_i = \{f \in T^n : f(z_j) = 1$ for all $j \neq i\}$. Then $P$ acts on the set $\{T_1, \ldots, T_n\}$.*

*Proof.* For $f \in T_i$, $p \in P$ we have $f(z_j) = 1$ if and only if $j \neq i$. Then $f^p(z_j) = f(pz_j)$ and using cosets we can write $pz_j = z_{\sigma_p(j)}q$ for some $q \in Q$ and $\sigma_p$ is the permutation induced by $p$ on $\mathcal{T}$. It follows that

$$f(z_{\sigma_p(j)}q) = f(z_{\sigma_p(j)})^{\phi(q)} = 1 \text{ if and only if } \sigma_p(j) \neq i.$$

Thus $T_i^p = T_{\sigma_p(i)}$ for $i = 1, \ldots, n$ and $p \in P$. □

**Lemma 4.3.2.** *Let $G$ be as above. Then $Q$ is the stabilizer of $T_n$ and normalizes $T_1 \times \cdots \times T_{n-1}$.*

*Proof.* Let $q \in Q$ and $f \in T_n$ so that $f(z_j) = 1$ if and only if $j \neq n$. Then $f^q(z_j) = f(qz_j)$ and using cosets again we can write $qz_j = z_k q'$ for some $q' \in Q$ and $k \neq n$. The latter holds because if $k = n$ then $z_n = 1$ and $q' \in Q$ so $qz_j = z_n q' = q' \in Q$ and $z_j \in Q$, a contradiction. Then $f(z_k q') = f(z_k)^{\phi(q')} = 1$ as $k \neq n$ and $f \in T_n$. Hence $f^q(z_j) = 1$ for all $j \neq n$ so $f^q \in T_n$ and $Q$ is the stabilizer of $T_n$. Let $H = T_1 \times \cdots \times T_{n-1}$. We have $H \trianglelefteq G$ so $Q$ normalizes $H$. □

**Lemma 4.3.3.** *Let $G$ be as above. Then any $f \in B_\phi$ is determined by the set of images of elements of $\mathcal{T}$. Conversely, any $\tilde{f} : \mathcal{T} \to T$ can be extended to a function in $B_\phi$.*

*Proof.* By definition $B_\phi = \{f \in \mathrm{Fun}(P, T) \mid f(pq) = f(p)^{\phi(q)} \text{ for all } p \in P, q \in Q\}$. As $\mathcal{T}$ is a transversal for $Q$ in $P$, any $p \in P$ can be written as $z_j q$ for some $j \in \{1, \ldots, n\}$ and $q \in Q$. Thus $f(p) = f(z_j q) = f(z_j)^{\phi(q)}$ for any $f \in B_\phi$ and $f$ is determined by the images of elements in the transversal.

Now let $\tilde{f} \in \mathrm{Fun}(\mathcal{T}, T)$ and let $p \in P$. As $p = z_j q$ for some $j \in \{1, \ldots, n\}$ and $q \in Q$ we can define $f : P \to T$ such that $f(p) = \tilde{f}(z_j)^{\phi(q)}$. Then $f$ extends $\tilde{f}$ and $f \in B_\phi$. □

**Lemma 4.3.4.** *Let $G$ be as above. Then $T_i \cong T$ and $\langle T_1, \ldots, T_n \rangle \cong T^n$.*

*Proof.* Let $t \in T$. For $i \in \{1, \ldots, n\}$, define $f_{i,t} : \mathcal{T} \to T$ such that

$$f_{i,t} : z_j \to \begin{cases} 1 & \text{if } j \neq i \\ t & \text{if } j = i \end{cases}.$$

Then $f_{i,t}$ is well defined since $1, t \in T$ and $f_{i,t}$ is bijective because $\mathcal{T}$ is a transversal and its elements are well defined. We want to show that $\{f_{i,t} : t \in T\} = \{f \in B_\phi : f(z_j) = 1 \text{ for all } j \neq i\} = T_i$ for each $i$. By Lemma 4.3.3, $f_{i,t} \in B_\phi$. Since $f_{i,t} \in B_\phi$ and $f_{i,t}(z_j) = 1$ for all $j \neq i$, we have $\{f_{i,t} : t \in T\} = T_i$. Now we want to show that $T_i \cong T$ for all $i \in \{1, \ldots, n\}$. Let $\phi : T \to T_i$ such that $\phi(t) = f_{i,t}$. If $t_1, t_2 \in T$ and $z_j \in \mathcal{T}$ then

$$(\phi(t_1 t_2))(z_j) = f_{i,t_1 t_2}(z_j) = t_1 t_2 \text{ as } j = i,$$

whereas

$$(\phi(t_1)\phi(t_2))(z_j) = f_{i,t_1}(z_j) f_{i,t_2}(z_j) = t_1 t_2, \text{ as } j = i.$$

Hence $\phi$ is a homomorphism from $T$ to $T_i$ and it is bijective since $f_{i,t}$ is defined for each $t \in T$.

Now we want to show that $\langle T_1, \ldots, T_n \rangle \cong T^n$. We know each $T_i$ is isomorphic to $T$ so it remains to show that $T_i$ and $T_j$ commute for all $i \neq j$. Let $i \neq j$ and let $t_1, t_2 \in T$. We want to prove that $f_{i,t_1} f_{j,t_2} = f_{j,t_2} f_{i,t_1}$. We have three cases

1. Consider $k \neq i \neq j$. Then $f_{i,t_1} f_{j,t_2} : z_k \to f_{i,t_1}(z_k) f_{j,t_2}(z_k) = 1 \cdot 1 = 1$. On the other hand, $f_{j,t_2} f_{i,t_1} : z_k \to f_{j,t_2}(z_k) f_{i,t_1}(z_k) = 1 \cdot 1 = 1$. So we have equality.

2. Consider $z_i$. Then $f_{i,t_1}f_{j,t_2} : z_i \to f_{i,t_1}(z_i)f_{j,t_2}(z_i) = t_1 \cdot 1 = t_1$. On the other hand, $f_{j,t_2}f_{i,t_1} : z_i \to f_{j,t_2}(z_i)f_{i,t_1}(z_i) = 1 \cdot t_1 = t_1$. So we have equality.

3. Consider $z_j$. Then $f_{i,t_1}f_{j,t_2} : z_j \to f_{i,t_1}(z_j)f_{j,t_2}(z_j) = t_1 \cdot 1 = t_1$. On the other hand, $f_{j,t_2}f_{i,t_1} : z_j \to f_{j,t_2}(z_j)f_{i,t_1}(z_j) = 1 \cdot t_1 = t_1$. So we have equality.

Hence $T_i$ and $T_j$ commute for all $j \neq i$ and $\langle T_1, \ldots, T_n \rangle = T_1 \times \cdots \times T_n \cong T^n$. □

**Lemma 4.3.5.** *If $t \in T, z_i \in \mathcal{T}$ and $p \in P$ then $f_{i,t}^p = f_{j,t^{\phi(q)}}$, where $z_j \in \mathcal{T}$ and $q \in Q$ and $p^{-1}z_i = z_jq^{-1}$.*

*Proof.* Let $t \in T, z_i \in \mathcal{T}$ and $p \in P$. Since $\mathcal{T}$ is a transversal for $Q$ in $P$, $pz_j = z_iq$ where $z_j \in \mathcal{T}$ and $q \in Q$ are unique. Then $p^{-1}z_i = z_jq^{-1}$. It follows that

$$f_{i,t}^p(z_j) = f_{i,t}(pz_j) = f_{i,t}(z_iq) = f_{i,t}(z_i)^{\phi(q)} = t^{\phi(q)} = f_{j,t^{\phi(q)}}(z_j).$$

□

**Lemma 4.3.6.** *Let $p_1, \ldots, p_n$ be a left transversal for $Q$ in $P$ such that $p_k : k \to n$. Then $p_1^{-1}, \ldots, p_n^{-1}$ is a right transversal for $Q$ in $P$.*

*Proof.* Without loss of generality, let $p_n = 1$. Note that $p_k^{-1} : n \to k$. Suppose that $Qp_i^{-1} = Qp_j^{-1}$ and let $g \in Qp_i^{-1}$. Then $g = qp_i^{-1}$ for some $q \in Q$. So $n^g = n^{qp_i^{-1}} = n^{p_i^{-1}} = i$ since $Q$ stabilizes $n$. On the other hand, $g = q'p_j^{-1}$ for some $q' \in Q$ since $Qp_i^{-1} = Qp_j^{-1}$. Then $n^g = n^{q'p_j^{-1}} = n^{p_j^{-1}} = j$. We have a unique mapping $p_k^{-1} : n \to k$ but here the image of $n$ under $g$ is $i$ and $j$ at the same time. Thus $i = j$ and $Qp_i^{-1}$ is different from $Qp_j^{-1}$ for all $1 \leq i, j \leq n$. Therefore, $p_1^{-1}, \ldots, p_n^{-1}$ is a right transversal for $Q$ in $P$. □

We have now concluded the description of all quasiprimitive types, so we summarize their most important features in Table 1. This also concludes the necessary theory concerning groups and group actions for this thesis, and we carry on with the theory related to graphs in the next chapter.

Table 1: Quasiprimitive groups table

| **QP type** | **main properties** | **primitive** | **minimal normal subgroups** |
|---|---|---|---|
| HA | subgroups of $AGL(d,p) = \mathrm{Hol}(N)$, $N \cong C_p^d$, action on $AG(d,p)$ | yes | unique elementary abelian, regular |
| HS | subgroups of $\mathrm{Hol}(T)$, action on $T$ | yes | two, both regular and isomorphic to $T$ |
| HC | subgroups of $T^k$, action on $T^k$ | yes | two, both regular and isomorphic to $T^k$ |
| TW | $G = T \operatorname{twr}_\phi P$, $P$ a group with $Q < P$, $\phi : Q \to T$, $\phi^{-1}(\mathrm{Inn}(T)$ is core-free in $P$ | iff $\phi$ does not extend to a larger subgroup of $P$ and $\mathrm{Inn}(T) \leq \phi(Q)$ | unique, isomorphic to $T^k$, regular |
| AS | $T \leq G \leq \mathrm{Aut}(T)$ with transitive socle $T$ | iff a point stabilizer is a maximal subgroup of $G$ not containing $T$ | unique, transitive, isomorphic to $T$ |
| SD | $N = T^k \leq G \leq T^k.(\mathrm{Out}(T) \times S_k)$; $G$ is transitive by conjugation on the $k$ simple direct factors of $N$ | iff $G$ is primitive on the $k$ simple direct factors of $N$ | unique, isomorphic to $T^k$ |
| CD | $N = T^k \leq G \leq H \wr S_{k/l}$, $H$ a qp group of type SD; $G$ acts transitively by conjugation on the $k$ simple direct factors of $N$ | if $H$ is primitive | unique, isomorphic to $T^{kl}$ |
| PA | $N = T^k \leq G \leq H \wr S_k$, $H$ is qp of type AS with nonregular socle $T$, $G$ acts faithfully on a nontrivial partition, $G$ acts transitively by conjugation on the $k$ simple direct factors of $N$ | iff the partition is trivial and $H$ is primitive | unique, isomorphic to $T^k$ |

# 5 CONSTRUCTING EDGE TRANSITIVE GRAPHS

In this chapter, we first define graphs and automorphisms of graphs. The latter describe a form of symmetry on graph by preserving edge-vertex relationships, since there is no operation defined, unlike in group automorphisms. An important notion of symmetry in graphs is edge transitivity, which gives a way to permute edges in a graph. Graphs are typically studied by their automorphism groups, as a way to analyze relationships between elements in a given set. We follow the convention below for (undirected) graphs, directed graphs and digraphs:

- A *graph* (also called an *undirected graph*) is a pair $\Gamma = (V\Gamma, E\Gamma)$, where $V\Gamma$ is a set whose elements are called *vertices*, and $E\Gamma$ is a set of two-sets with two distinct vertices, $E\Gamma \subseteq \{\{x, y\} \in V\Gamma^2$ and $x \neq y\}$, whose elements are called *edges*. Two vertices are said to be *adjacent* if they form an edge. A *multigraph* is a generalization that allows multiple edges to have the same pair of endpoints.

- A *directed graph* is an ordered pair $\Gamma = (V\Gamma, E\Gamma)$ where $V\Gamma$ is a set of *vertices* and $E\Gamma \subseteq \{(x, y) : (x, y) \in V\Gamma^2$ and $x \neq y\} \subseteq V\Gamma \times V\Gamma$ a set of *edges* (also called *arcs*) which are ordered pairs of vertices. If the edge relation is antisymmetric, i.e. if $(x, y) \in E\Gamma$ then $(y, x) \notin E\Gamma$ then we have a directed graph. Conversely, if the relation is symmetric we will call it a *digraph.*

The *degree* or *valency* of a vertex is the number of edges that connect that vertex to other vertices. A vertex is *isolated* if it has valency equal to 0. A graph is said to be *regular* if all its vertices have the same valency. If the vertex set of a graph can be partitioned into two sets $X$ and $Y$ in such a way that all the vertices in each part do not share edges, then the graph is called *bipartite* and $X$ and $Y$ are the *parts of the bipartition.* If a graph is bipartite with parts $X$ and $Y$, such that all vertices in $X$ have the same degree $i$ and all vertices in $Y$ have the same degree $j$, then the graph is said to be *biregular* or simply regular of valencies $i$ and $j$.

A graph is *connected* if there exists a sequence of edges which joins any two vertices in it. Let $\Gamma$ be a connected graph with vertex set $V\Gamma$, edge set $E\Gamma$ and adjacency denoted by $\sim$. An *s-arc* in a $\Gamma$ is an $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices in $\Gamma$ such that $v_i \sim v_{i+1}$ and $v_{j-1} \neq v_{j+1}$ for each $i = 1, \ldots, s$ and $j = 1, \ldots, s - 1$. A *cycle* of length $s$ in a graph is a non-empty $s$-arc such that the only repeated vertices are the first and last vertices.

An *automorphism* of the graph $\Gamma$ is a bijection $\phi : V\Gamma \to V\Gamma$ such that $u, v \in V\Gamma$ form an edge in $\Gamma$ if and only if their images $\phi(u), \phi(v) \in V\Gamma$ also form an edge in $\Gamma$. The set of all automorphisms of a graph forms a group with respect to composition. It is called the *automorphism group* of $\Gamma$ and we denote it by $\mathrm{Aut}(\Gamma)$. A graph is *vertex transitive* if the automorphism group is transitive on the vertex set. Similarly, a graph is *edge transitive* if the automorphismm group is transitive on the edge set.

Let $G \leq \mathrm{Aut}(\Gamma)$. We say that $\Gamma$ is *locally* $(G, s)$*-arc transitive* if $\Gamma$ contains an $s$-arc and for any two $s$-arcs $\alpha$ and $\beta$ starting at the same vertex $v$, there exists an element $g \in G_v$ mapping $\alpha$ to $\beta$. Let $s \geq 2$. Then local 1-transitivity is equivalent to edge transitivity. We present three methods of constructing $G$-edge transitive graphs.

1. We build a locally $(G, s)$-arc transitive graph from a given $(G, s)$-arc transitive graph.

2. We construct arbitrary locally $(G, s)$-arc transitive, $G$-vertex intransitive graphs as coset graphs.

3. We characterize locally $(G, s)$-arc transitive graphs with a vertex of valency at most three.

## 5.1 DOUBLE COVERS

Let $\Gamma$ be a directed or undirected vertex transitive graph with vertex set $V\Gamma$ and arc set $A\Gamma$. The *standard double cover* of $\Gamma$ is the undirected graph $\overline{\Gamma}$ with a vertex set $V\Gamma \times \{1, 2\}$, and two vertices $(x, 1)$ and $(y, 2)$ are adjacent if and only if $(x, y) \in A\Gamma$. The new graph is bipartite with bipartite halves $V\Gamma \times \{i\}$ for each $i = 1, 2$.

If $G \leq \mathrm{Aut}(\Gamma)$, then $G$ also acts as a group of automorphisms of $\overline{\Gamma}$ with the action $g : (x, i) \to (x^g, i)$. If $G$ is transitive on $V\Gamma$, then $G$ has two orbits on $V\overline{\Gamma}$ and the action of $G$ on each orbit is permutationally isomorphic to the action of $G$ on $V\Gamma$. Furthermore, $G_v = G_{(v,i)}$ for each $i = 1, 2$. Then if $\Gamma$ is undirected, the action of $G_v$ on $\Gamma(v)$ is the same as the action of $G_{(v,i)}$ on $\overline{\Gamma}((v, i))$. Thus in this case, if $\Gamma$ is $G$-locally primitive, then $\overline{\Gamma}$ is also $G$-locally primitive.

Suppose again that $\Gamma$ is undirected. Then $(x, 1) \sim (y, 2)$ if and only if $(y, 1) \sim (x, 2)$. If $\Gamma$ is also connected, then for each $x, y \in V\Gamma$ there exists a path $P$ in $\Gamma$ between $x$ and $y$. This path lifts to a path in $\overline{\Gamma}$ between $(x, 1)$ and $(y, 1)$ if $P$ has even length, and to one between $(x, 1)$ and $(y, 2)$ if $P$ has odd length. There is a path between $(y, 1)$ and $(y, 2)$ if and only if $y$ is in an odd cycle in $\Gamma$. Thus for an undirected connected graph $\Gamma$, $\overline{\Gamma}$ is connected if and only if $\Gamma$ contains an odd cycle, that is, if and only if $\Gamma$ is not bipartite.

**Lemma 5.1.1.** *If $\Gamma$ is undirected and bipartite then $\overline{\Gamma}$ is disconnected, and the two components of $\overline{\Gamma}$ are both isomorphic to $\Gamma$.*

*Proof.* Let $\Gamma$ be bipartite with parts $\Delta_1$ and $\Delta_2$. Since $\Gamma$ does not contain odd cycles, we have two components $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\overline{\Gamma}$ with $\mathcal{C}_1$ containing edges $\{(x,1) \sim (y,2) : (x,y) \in E\Gamma \text{ and } x \in \Delta_1\}$ and $\mathcal{C}_2$ containing edges $\{(x,1) \sim (y,2) : (x,y) \in E\Gamma \text{ and } x \in \Delta_2\}$. We know that $G \leq \mathrm{Aut}(\Gamma)$ acts as a group of automorphisms of $\overline{\Gamma}$, with action $g : (x,i) \to (x^g, i)$. Note that $G$ preserves both components of $\overline{\Gamma}$. Let $\rho : V\overline{\Gamma} \to V\overline{\Gamma}$ such that $\rho : (x,i) \to (x, 3-i)$. Since $\Gamma$ is a graph, $\rho$ is well-defined. Let $(x,1) \sim (y,2) \in \overline{\Gamma}$. Then $\rho$ takes $(x,1)$ to $(x,2)$ and $(y,2)$ to $(y,1)$, and $(x,2) \sim (y,1)$ so $\rho$ is an automorphism of $\overline{\Gamma}$. Note that $\rho$ maps $\mathcal{C}_1$ to $\mathcal{C}_2$ since a vertex $(x,i) \in V\mathcal{C}_1$ is taken to $(x, i+1) \in \mathcal{C}_2$. Since $G$ preserves the components of $\overline{\Gamma}$ and $\rho$ interchanges them, we have $\langle G, \rho \rangle$ is transitive on $V\overline{\Gamma}$. Now let $\phi : \Gamma \to \mathcal{C}_1$ such that $\phi : z \to (z,1)$ if $z \in \Delta_1$ and $\phi : z \to (z,2)$ if $z \in \Delta_2$. For each $z \in \Gamma$ there exists a unique $(z,i) \in \mathcal{C}_i$ for $i = 1,2$, so $\phi$ is well-defined. Let $\{a,b\}$ be an edge in $\Gamma$ such that $a \in \Delta_1$ and $b \in \Delta_2$. Then as a set $\{a,b\}^\phi = \{a^\phi, b^\phi\} = \{(a,1),(b,2)\}$ which is an edge in $\mathcal{C}_1$ since $a \in \Delta_1$. Hence $\phi$ is an isomorphism from $\Gamma$ to $\mathcal{C}_1$. Since $\mathcal{C}_1^\rho = \mathcal{C}_2$, we have $\Gamma \cong \mathcal{C}_1 \cong \mathcal{C}_2$. □

**Example 5.1.2.** Let $\Gamma_1$ be the graph with $V\Gamma_1 = \{a,b,c,d\}$ where $a \sim b \sim c \sim a$ and $c \sim d$. Note that $\Gamma_1$ contains an odd cycle so $\overline{\Gamma}_1$ is connected. Figure 1 shows $\Gamma_1$ and its standard double cover.

Let $\Gamma_2$ be a path of length 3, with starting vertex $a$, middle vertices $b, c$, and ending vertex $d$. Then $\Gamma_2$ is bipartite, with $\Delta_1 = \{a,c\}$ and $\Delta_2 = \{b,d\}$. Then the corresponding standard double cover $\overline{\Gamma}_2$ contains two copies of $\Gamma_2$, namely, $\mathcal{C}_1$ which is the path $(a,1) \sim (b,2) \sim (c,1) \sim (d,2)$ and $\mathcal{C}_2$ which is the path $(a,2) \sim (b,1) \sim (c,2) \sim (d,1)$, as depicted in Figure 2.

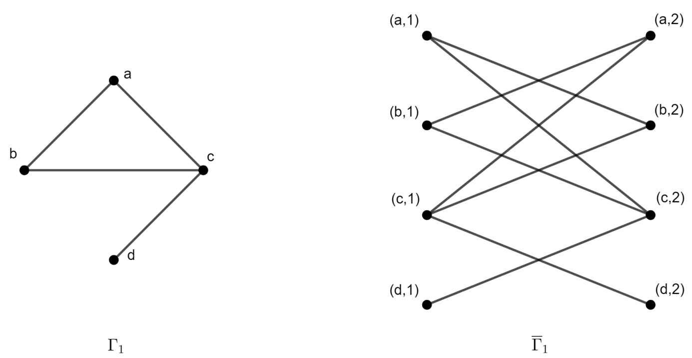



Figure 1: $\Gamma_1$ and its standard double cover $\overline{\Gamma}_1$.

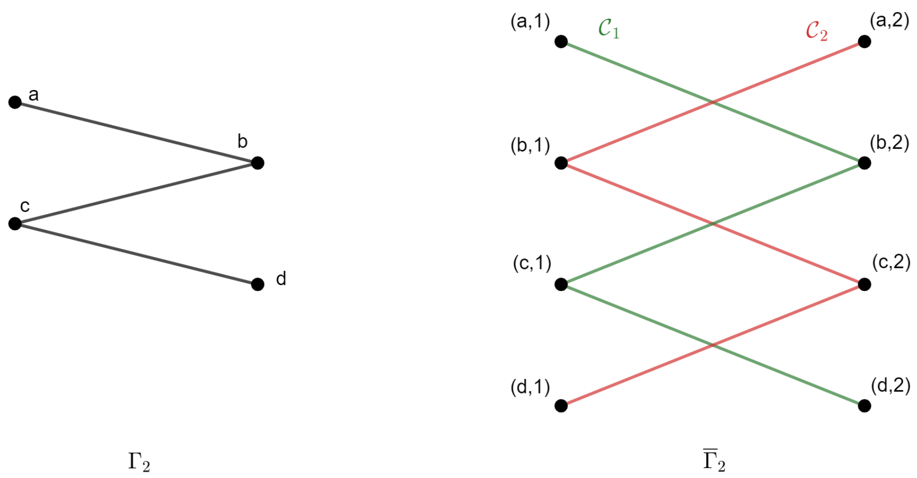



Figure 2: $\Gamma_2$ and its standard double cover $\overline{\Gamma}_2$, with red and green components isomorphic to $\Gamma_2$.

**Lemma 5.1.3.** *Let $\Gamma$ be an undirected graph. If $\Gamma$ is $(G,s)$-arc transitive, then $\overline{\Gamma}$ is locally $(G,s)$-arc transitive. In particular, there exist quasiprimitive locally $(G,2)$-arc transitive graphs of types HA, TW, AS and PA.*

*Proof.* We have seen that $G \leq \mathrm{Aut}(\Gamma)$ acts on $\overline{\Gamma}$ by $g : (x,i) \to (x^g,i)$. Suppose $\Gamma$ is $(G,s)$-arc transitive. If $((v,i),(v_1,i)\ldots,(v_s,i))$ is an $s$-arc in $\overline{\Gamma}$, then $(v,v_1,\ldots,v_s)$ is an $s$-arc in $\Gamma$, so $\overline{\Gamma}$ is locally $(G,s)$-arc transitive. In [21], Prager shows that there exist nonbipartite $(G,2)$-arc transitive graphs where $G$ is quasiprimitive of type HA, TW, AS and PA. Therefore, by taking the standard double covers of those graphs, we construct locally $(G,2)$-arc transitive graphs with $G$ of quasiprimitive type HA, TW, AS and PA on both orbits. These graphs are also vertex-transitive since we defined $\rho \in \mathrm{Aut}(\Gamma)$ by $\rho : (x,i) \to (x,3-i)$ in the proof of Lemma 5.1.1, which interchanges the parts of the bipartition. □

## 5.2 COSET GRAPHS

**Lemma 5.2.1.** *Let $\Gamma$ be a $G$-edge transitive graph without isolated vertices with $G \leq \mathrm{Aut}(\Gamma)$. If $\Gamma$ is not vertex transitive, then it is bipartite with two $G$-orbits as its parts.*

*Proof.* Let $\{u,v\} \in E\Gamma$. Let $\Delta_1 = \{u^\varphi : \varphi \in G\}$ and let $\Delta_2 = \{v^\varphi : \varphi \in G\}$. As $\Gamma$ is vertex intransitive, $\Delta_1 \cap \Delta_2 = \emptyset$. Let $x$ be an arbitrary vertex of $\Gamma$. Since $\Gamma$ has no isolated vertices, it follows that there exists $y \in V\Gamma$ such that $x \sim y$. By edge transitivity, there exists $\varphi \in G$ such that $\{u, v\}^{\varphi} = \{x, y\}$. So $x \in \Delta_1$ or $x \in \Delta_2$, that is, $V\Gamma$ is the disjoint union of $\Delta_1$ and $\Delta_2$.

Now we show that $\Delta_i$ is an independent set for $i = 1, 2$. Let $\{u, v\} \in E\Gamma$. Suppose that $\{x, y\} \in E\Gamma$ for $x, y \in \Delta_1$. Then there exists $\varphi \in G$ such that $\{u, v\}^{\varphi} = \{x, y\}$. Then either $v^{\varphi} = x$ or $v^{\varphi} = y$, so that $x \in \Delta_2$ or $y \in \Delta_2$, which is impossible as $\Delta_1 \cap \Delta_2 = \emptyset$. □

**Lemma 5.2.2.** *Let $\Gamma$ be a connected locally $(G, s)$-arc transitive graph such that $s \geq 1$ and all vertices have valency at least two. Then $G$ acts transitively on the set of edges of $\Gamma$. Furthermore, if $G$ acts intransitively on $V\Gamma$, then $\Gamma$ is a bipartite graph and the two parts of the bipartition are $G$-orbits.*

*Proof.* Local 1-arc transitivity is equivalent to edge transitivity and $s \geq 1$, so the result follows. □

**Lemma 5.2.3.** *Let $\Gamma$ be a connected locally $(G, s)$-arc transitive graph such that $s \geq 1$ and all vertices have valency at least two. Then $\Gamma$ is locally $(G, s-1)$-arc transitive.*

*Proof.* Suppose $\Gamma$ is locally $(G, s)$-arc transitive with all vertices of valency at least two. Let $\alpha = (v_0, v_1, \ldots, v_{s-1})$ and $\alpha' = (v_0, v_1', \ldots, v_{s-1}')$ be two $(s-1)$-arcs in $\Gamma$. Since every vertex has valency at least two, we can extend the $(s-1)$-arcs $\alpha$ and $\alpha'$ to $s$-arcs $\beta$ and $\beta'$ such that $\beta = (v_0, v_1, \ldots, v_s)$ and $\beta' = (v_0, v_1', \ldots, v_s')$. As $\Gamma$ is locally $(G, s)$-arc transitive, there exists $g \in G$ such that $\beta^g = \beta'$. Restricting our attention to the $(s-1)$-arcs, we have $\alpha^g = \alpha'$ with $g \in G_{v_0}$. Thus $\Gamma$ is locally $(G, s-1)$-arc transitive. □

**Lemma 5.2.4.** *Let $\Gamma$ be a graph such that all vertices have valency at least two. Then $\Gamma$ is locally $(G, 2)$-arc transitive if and only if for every vertex $v$, $G_v$ acts 2-transitively on $\Gamma(v)$.*

*Proof.* Let $\Gamma$ be a graph such that all vertices have valency at least two and let $v \in V\Gamma$. As $G$ acts locally 1-transitively on $\Gamma$ we can map an arc $(v, a)$ to an arc $(v, b)$, so we have $G_v$ transitive on $\Gamma(v)$. Let $u \in \Gamma(v)$. For every $w \in \Gamma(v) \setminus \{u\}$ there is a 2-arc $(u, v, w)$. Since $\Gamma$ is locally $(G, 2)$-arc transitive, $G_{uv}$ is transitive on $\Gamma(v) \setminus \{u\}$ so $G_v$ acts 2-transitively on $\Gamma(v)$.

Now suppose $G_v$ acts 2-transitively on $\Gamma(v)$. Let $(v, u_1, w_1)$ and $(v, u_2, w_2)$ be two 2-arcs in $\Gamma$. Then $(v, u_1, w_1)^g = (v, u_2, w_1')$ for some $g \in G_v$, $w_1' \in \Gamma(u_2) \setminus \{v\}$ by transitivity of $G_v$ on $\Gamma(v)$. Then by 2-transitivity of $G_{u_2}$ on $\Gamma(u_2)$ we can map $w_1'$ to $w_2$ by some $h \in G_{u_2 v}$. Thus $(v, u_1, w_1)^{gh} = (v, u_2, w_2)$ and $\Gamma$ is locally $(G, 2)$-arc transitive. □

**Example 5.2.5.** Let $\Gamma$ be the cube graph with eight vertices, that is, vertices of $\Gamma$ are triples with entries $0, 1$ and two vertices are adjacent if and only if they differ in exactly one coordinate. Let $G = \mathrm{Aut}(\Gamma)$. We would like to show that the cube is locally $(G, 2)$-arc transitive but not locally $(G, 3)$-arc transitive. Then $\phi_u : x \to u + x$ is an automorphism of $\Gamma$ since vertices are permuted by adding $u$ and two vertices $x, y$ differ in exactly one coordinate if and only if $u+x$ and $u+y$ differ in exactly one coordinate. Note that for any two vertices $x, y$, the map $\phi_{x-y}$ takes $y$ to $x$ so $\Gamma$ is vertex transitive. Fix vertex $(0, 0, 0)$ and consider the arc $\alpha = ((0, 0, 0), (0, 0, 1))$. We want to show that $G_{(0,0,0)}$ is transitive on $\Gamma((0, 0, 0))$, i.e. we can map $\alpha$ to the other two arcs starting at $(0, 0, 0)$, namely to $\beta = ((0, 0, 0), (0, 1, 0))$ and to $\gamma = ((0, 0, 0), (1, 0, 0))$. For $g \in S_3$, define $\bar{g} : VQ_3 \to VQ_3$ such that $\bar{g} : (x_1, x_2, x_3) \to (x_{1^g}, x_{2^g}, x_{3^g})$. Clearly $\bar{g} \in G$. For $g_1 = (23)$, we have $\bar{g_1} \in G_{(0,0,0)}$ and $\alpha^g = \beta$ and for $g_2 = (12)$, we have $\bar{g_2} \in G_{(0,0,0)}$ and $\alpha^g = \gamma$. Then $\Gamma$ is locally $(G, 1)$-arc transitive. It is obvious that $\langle g_1, g_2\rangle$ acts 2-transitively on $G_{(0,0,0)}$ so by Lemma 5.2.4, $\Gamma$ is locally $(G, 2)$-arc transitive. To show that the cube is not locally $(G, 3)$-arc transitive, we pick two 3-arcs shown in Figure 3 as follows:

$$\alpha = ((0, 0, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1)),$$

$$\beta = ((0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 0, 1)).$$

Then clearly no automorphism of $G_{(0,0,0)}$ can map $\alpha$ to $\beta$ since the ending vertex of $\alpha$ is not adjacent to the starting vertex, whereas the ending vertex of $\beta$ is adjacent to the starting vertex, and we know that automorphisms preserve neighbors.

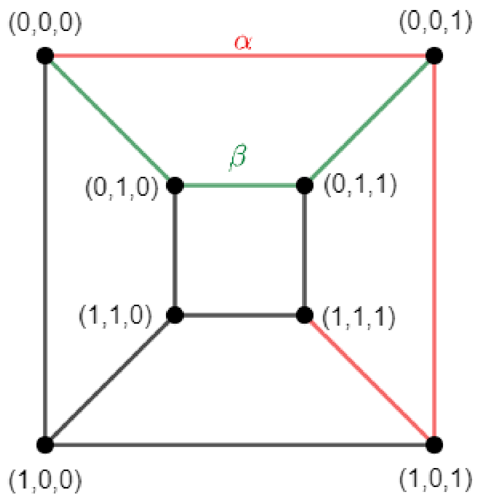



Figure 3: The cube graph $\Gamma$ and arcs $\alpha$ and $\beta$.

Let $\Gamma$ be a $G$-edge transitive graph that is not vertex transitive. As $G$ transitive on $\Delta_1$, for $v \in \Delta_1$, we may write $\Delta_1$ as the set $[G : G_v]$ of right cosets of $G_v$ in $G$ so that

$G$ acts transitively on $\Delta_1$ by right multiplication:

$$\Delta_1 = \{G_v x : x \in G\},$$

$$z : G_v x \to G_v xz, \forall z \in G.$$

Similarly for $w \in \Delta_2$,

$$\Delta_2 = \{G_w x : x \in G\},$$

$$z : G_w x \to G_w xz, \forall z \in G.$$

So, vertices of $\Gamma$ may be identified with right cosets of $G_v$ and $G_w$ in $G$.

Let $v \sim w$. Then $G_v \cap G_w = G_{\{v,w\}}$. We have a faithful action, so stabilizers are core-free. $G_v$ is transitive on $\Gamma(v)$ so neighbors of $v$ are images of $w$ under $G_v$, since $\Gamma$ is arc transitive. So neighbors of $v$ can be seen as the set $\{G_w z : z \in G_v\}$. Similarly, neighbors of $w$ are images of $v$ under $G_v$, i.e. $\{G_v z : z \in G_w\}$.

Then the adjacency relation of $\Gamma$ is given by

$$G_v x \sim G_w y \iff xy^{-1} \in G_v G_w \text{ or } yx^{-1} \in G_w G_v.$$

**Lemma 5.2.6.** *Let $\Gamma$ be a bipartite $G$-edge transitive graph, with parts $\Delta_1$ and $\Delta_2$, where $G \leq \mathrm{Aut}(\Gamma)$ and $G$ is intransitive on $V\Gamma$. Let $v \in \Delta_1$ and $w \in \Delta_2$ be adjacent. Then we may identify $\Delta_1 = [G : G_v]$ and $\Delta_2 = [G : G_w]$ such that:*

1. $G_v x \sim G_w y \iff xy^{-1} \in G_v G_w$ *or* $yx^{-1} \in G_w G_v$.

2. $\Gamma(v) = \{G_w z : z \in G_v\} = G_w G_v$ *and* $\Gamma(w) = \{G_v z : z \in G_w\} = G_v G_w$.

3. *The valencies are* $|\Gamma(v)| = |G_v : G_v \cap G_w|$ *and* $|\Gamma(w)| = |G_w : G_v \cap G_w|$.

We can thus construct edge transitive graphs from abstract groups.

**Definition 5.2.7.** Let $G$ be a group and let $L, R < G$ be such that $L \cap R$ is core-free in $G$. Let $\Delta_1 = \{Lx : x \in G\}$ and let $\Delta_2 = \{Ry : y \in G\}$. Define the bipartite graph $\Gamma = \mathrm{Cos}(G, L, R)$ such that $V\Gamma = \Delta_1 \cup \Delta_2$ and $Lx \sim Ry \iff xy^{-1} \in LR$ or $yx^{-1} \in RL$. We refer to $(L, R, L \cap R)$ as the associated *amalgam*.

**Lemma 5.2.8.** *The condition $Lx \sim Ry \iff xy^{-1} \in LR$ or $yx^{-1} \in RL$ in the coset graph $\mathrm{Cos}(G, L, R)$ is equivalent to the condition $Lx \sim Ry \iff Lx \cap Ry \neq \emptyset$.*

*Proof.* Let $\Gamma = \mathrm{Cos}(G, L, R)$ and let $\Gamma'$ be the graph with $V\Gamma' = V\Gamma$ and $Lx \sim Ry \in \Gamma' \iff Lx \cap Ry \neq \emptyset$. Let $Lx \sim Ry \in \Gamma$, that is, $xy^{-1} \in LR$ or $yx^{-1} \in RL$. Without loss of generality, suppose $xy^{-1} \in LR$. Then $xy^{-1} = ab$ or $a^{-1}x = by$ for some $a \in L$ and $b \in R$. We can write $x = aby$ so that $Lx = Laby$ and as $a \in L$, $Laby = Lby$ so $by \in Lx$. We can also write $y = (ab)^{-1}x = b^{-1}a^{-1}x$ so that $Ry = Rb^{-1}a^{-1}x$ and as $b \in R$, $Rb^{-1}a^{-1}x = Ra^{-1}x$ so $a^{-1}x \in Ry$. Combining the two statements we have $by \in Lx$ and $a^{-1}x \in Ry$ and as $by = a^{-1}x$ we conclude that $Lx \cap Ry \neq \emptyset$, i.e. $Lx \sim Ry \in \Gamma'$. Thus any edge in $\Gamma$ is also an edge in $\Gamma'$.

Now let $Lx \sim Ry \in \Gamma'$ so that $Lx \cap Ry \neq \emptyset$. Then there exists $z \in Lx \cap Ry$ such that $Lx = Lz$ and $Ry = Rz$. Then $Lx = Lz \sim Rz = Ry$ since $zz^{-1} = 1 \in LR$ so $Lx \sim Ry \in \Gamma$. We conclude that $\Gamma \cong \Gamma'$ and the conditions in the lemma are equivalent. □

Let $v \in V\Gamma$ and let $\Gamma(v)$ denote the neighborhood of $v$ in $\Gamma$. Note that $G$ acts on $\Gamma$ by right multiplication

$$\hat{g} : \Gamma \to \Gamma \text{ with } Lx \to Lxg, Ry \to Ryg, \text{ where } g \in G.$$

Since $Lx \cap Ry \neq \emptyset$ if and only if $Lxg \cap Ryg \neq \emptyset$, we have that $g$ acts as an automorphism of $\Gamma$ so we have a homomorphism $\phi$ from $G$ to $\mathrm{Aut}(\Gamma)$, defined by $\phi(g) = \hat{g}$.

Let $g \in G$. Then for $g \in G$ we have $Lxg = Lx$ if and only if $Lxgx^{-1} = L$ so $g \in L^x$. A similar argument works for $R$. This implies that every stabilizer of a vertex of $\Gamma$ is a $G$-conjugate of $L$ or $R$. Also, $G$ has two orbits on $V\Gamma$, namely $\Delta_1$ and $\Delta_2$, with representatives $L$ and $R$, respectively. Next, note that for an edge $\{Lx, Ry\}$ in $\Gamma$ there exists $z \in Lx \cap Ry$ such that $Lx = Lz$ and $Ry = Rz$. Then the stabilizer of an edge $\{Lz, Rz\}$ is $L^z \cap R^z = (L \cap R)^z$, a $G$-conjugate of $L \cap R$. The kernel of the action of $G$ on $\Gamma$ is the largest normal subgroup of $G$ in $L \cap R$, denoted $(L \cap R)_G$. If $K$ is the kernel of this action then every element of $K$ fixes all vertices and edges of $\Gamma$ and since any normal subgroup of $G$ contained in $L \cap R$ fixes every vertex of $\Gamma$, the kernel must be $(L \cap R)_G$. Since $L \cap R$ is core-free the kernel of $\phi$ is trivial so $G \leq \mathrm{Aut}(\Gamma)$.

**Example 5.2.9.** Consider $G = S_4$ and let $L = \langle (1\,3), (1\,2\,3) \rangle \leq S_4$ and $R = \langle (3\,4), (2\,3\,4) \rangle$ as subgroups of $S_4$. We would like to construct $\mathrm{Cos}(G, L, R)$. Note that $L \cong R \cong S_3$ and $L$ is the stabilizer of 4 in $S_4$ and $R$ is the stabilizer of 1 in $S_4$. Then $L \cap R$ is the stabilizer of 1 and 4 in $S_4$, so $L \cap R = \langle (2\,3) \rangle$.

We need to check that $L \cap R$ is core-free. The normal subgroups of $S_4$ are the trivial subgroup, the Klein four-group $V_4 = \{(), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$, the alternating group $A_4 = \langle (1\,2\,3), (1\,2)(3\,4) \rangle$ and $S_4$. Since $|L \cap R| = 2$, the largest normal subgroup it can contain is the trivial group, so $L \cap R$ is core-free. We now calculate right cosets for $L$ as follows

$$\begin{aligned}
L() &= \{(), (1\,3), (1\,2), (2\,3), (1\,3\,2), (1\,2\,3)\},\\
L(1\,4) &= \{(1\,4), (1\,3\,4), (1\,2\,4), (1\,4)(2\,3), (1\,3\,2\,4), (1\,2\,3\,4)\},\\
L(1\,4\,2) &= \{(1\,4\,2), (1\,3\,4\,2), (2\,4), (1\,4\,2\,3), (1\,3)(2\,4), (2\,3\,4)\},\\
L(1\,4\,3) &= \{(1\,4\,3), (3\,4), (1\,2\,4\,3), (1\,4\,3\,2), (2\,4\,3), (1\,2)(3\,4)\}.
\end{aligned}$$

Similarly, for $R$ we have

$$\begin{aligned} R() &= \{(), (2\,4), (2\,3), (3\,4), (2\,4\,3), (2\,3\,4)\}, \\ R(1\,4) &= \{(1\,4), (1\,4\,2), (1\,4)(2\,3), (1\,4\,3), (1\,4\,3\,2), (1\,4\,2\,3)\}, \\ R(1\,2\,4) &= \{(1\,2\,4), (1\,2), (1\,2\,3\,4), (1\,2\,4\,3), (1\,2)(3\,4), (1\,2\,3)\}, \\ R(1\,3\,4) &= \{(1\,3\,4), (1\,3\,4\,2), (1\,3\,2\,4), (1\,3), (1\,3\,2), (1\,3)(2\,4)\}. \end{aligned}$$

Thus $\Delta_1 = \{L(), L(1\,4), L(1\,4\,2), L(1\,4\,3)\}$ and $\Delta_2 = \{R(), R(1\,4), R(1\,2\,4), L(1\,3\,4)\}$. By Lemma 5.2.8, we have an edge $Lx \sim Ry$ if and only if $Lx \cap Ry \neq \emptyset$, and if we compare elements in the cosets above, we have that

$$\begin{aligned} L() \cap R() &= \{(), (2\,3)\}, & L() \cap R(1\,4) &= \emptyset, \\ L() \cap R(1\,2\,4) &= \{(1\,2), (1\,2\,3)\}, & L() \cap R(1\,3\,4) &= \{(1\,3\,2), (1\,3)\}, \\ L(1\,4) \cap R() &= \emptyset, & L(1\,4) \cap R(1\,4) &= \{(1\,4), (1\,4)(2\,3)\}, \\ L(1\,4) \cap R(1\,2\,4) &= \{(1\,2\,3\,4), (1\,2\,4)\}, & L(1\,4) \cap R(1\,3\,4) &= \{(1\,3\,4), (1\,3\,2\,4)\}, \\ L(1\,4\,2) \cap R() &= \{(2\,3\,4), (2\,4)\}, & L(1\,4\,2) \cap R(1\,4) &= \{(1\,4\,2)(1\,4\,2\,3)\}, \\ L(1\,4\,2) \cap R(1\,2\,4) &= \emptyset, & L(1\,4\,2) \cap R(1\,3\,4) &= \{(1\,3\,4\,2), (1\,3)(2\,4)\}, \\ L(1\,4\,3) \cap R() &= \{(3\,4), (2\,4\,3)\}, & L(1\,4\,3) \cap R(1\,4) &= \{(1\,4\,3\,2)(1\,4\,3)\}, \\ L(1\,4\,3) \cap R(1\,2\,4) &= \{(1\,2)(3\,4), (1\,2\,4\,3)\}, & L(1\,4\,3) \cap R(1\,3\,4) &= \emptyset. \end{aligned}$$

The corresponding coset graph is shown in Figure 4.

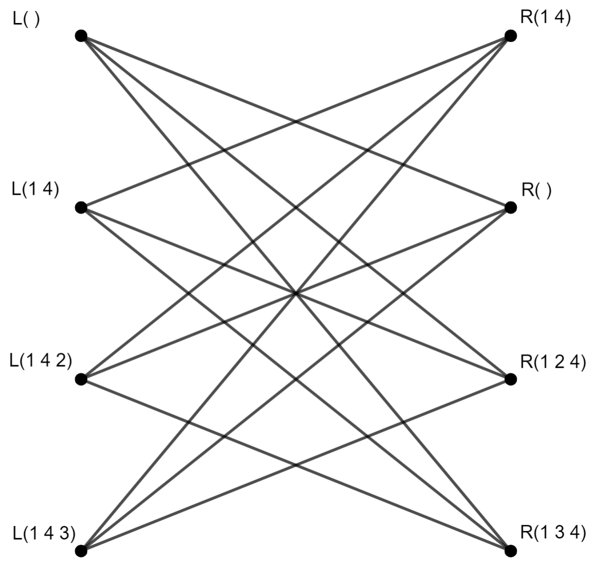



Figure 4: The coset graph $\mathrm{Cos}(S_4, \langle(1\,3), (1\,2\,3)\rangle, \langle(3\,4), (2\,3\,4)\rangle)$

**Lemma 5.2.10.** *For a group $G$ and subgroups $L, R < G$ such that $L \cap R$ is core-free in $G$, the graph $\Gamma = \mathrm{Cos}(G, L, R)$ satisfies the following properties:*

1. *$\Gamma$ is connected if and only if $\langle L, R\rangle = G$;*

2. *$G \leq \mathrm{Aut}(\Gamma)$ and $\Gamma$ is $G$-edge transitive and $G$-vertex intransitive;*

3. *$G$ acts faithfully on both $\Delta_1$ and $\Delta_2$ if and only if both $L$ and $R$ are core-free.*

*Conversely, if $\Gamma$ is $G$-edge transitive but not $G$-vertex transitive graph, and $v$ and $w$ are adjacent vertices, then $\Gamma \cong \mathrm{Cos}(G, G_v, G_w)$.*

*Proof.* Let $G$ be a group with subgroups $L$ and $R$ such that $L \cap R$ is core-free in $G$. Let $\Gamma = \mathrm{Cos}(G, L, R)$.

1. Let $G = \langle L, R\rangle$. Then for any $w \in G$, we can write $w = x_1y_1 \cdots x_ky_k$ for some $x_i \in L$ and some $y_i \in R$ such that $x_i \neq 1$ if $i \neq 1$, and $y_i \neq 1$ if $i \neq k$. We can then use the definition of cosets graphs to find a path in $\Gamma$ from $L$ to $Lw$ by multiplying with the term factor:

$$L, Ry_k, Lx_ky_k, \ldots, R(y_1 \cdots x_ky_k), L(x_1y_1 \cdots x_ky_k) = Lw.$$

   Similarly we can find a path in $\Gamma$ from $L$ to $Rw$:

$$L, R, Ly_k, Rx_ky_k, \ldots, L(y_1 \cdots x_ky_k), R(x_1y_1 \cdots x_ky_k) = Rw.$$

   Therefore we can find a path for any two vertices in $\Gamma$ so $\Gamma$ is connected.

   Now suppose $\Gamma$ is connected, and let $g \in G$. There exists a path from $Lg$ to $L$, say, $Lg, Rg_n, \ldots, Rg_3, Lg_2, Rg_1, L$. Since $Rg_1 \sim L$ we have $g_1 \in L$. Then $Lg_2 \sim Rg_1$ so $g_2 \in R^{g_1}$, which is contained in $\langle L, R\rangle$ since $g_1 \in L$ and $\langle L, R\rangle$ is closed under multiplication. We continue along the path with a similar argument and get to $g_n \in L^{g_{n-1}}$ where $g_{n-1} \in R^{g_{n-2}}$ which is contained in $\langle L, R\rangle$, so $g_n \in \langle L, R\rangle$. Finally, $g \in R^{g_n}$ and so $g \in \langle L, R\rangle$. We started with an arbitrary $g \in G$ so equality follows.

2. Let $\Delta_1 = \{Lx : x \in G\}$ and $\Delta_2 = \{Ry : y \in G\}$. For $z \in G$, let $\hat{z}$ be the permutation on $V = \Delta_1 \cup \Delta_2$ induced by $z$ which is right multiplication by $z$. Note that $G$ is intransitive on $V$ since no element of $G$ can map $L \in \Delta_1$ to $R \in \Delta_2$. Let $v$ and $w$ be vertices corresponding to $L$ and $R$ respectively. Then by definition, the set of neighbors of $v$ is $\Gamma(v) = \{Rx : x \in L\}$. For $z \in L$, the induced permutation $\hat{z}$ fixes $L$ since $L^{\hat{z}} = Lz = L$ and $\hat{z}$ fixes neighbors of $v$ since $R^{\hat{z}} = Rz \in \Gamma(v)$. As $z$ runs through $L$, $Rz$ runs through $\Gamma(v)$ so $L$ is transitive on $\Gamma(v)$. Similarly, $R$ is transitive on $\Gamma(w)$. Thus $\Gamma$ is $G$-edge transitive.

3. $G$ acts faithfully on $\Delta_1$ and $\Delta_2$ if only the identity fixes all vertices of $\Gamma$. Suppose $g$ is in the kernel of the action of $G$ on $\Delta_1$. Then $g$ stabilizes $Lh$ for all $h \in G$.

The stabilizer of $Lh$ in $G$ is $L^h$ so $g \in L^h$ for all $h \in G$. So $g \in \bigcap_{h\in G} L^h = \mathrm{core}_G(L)$. Similarly for $R$, so both are core-free if and only if $G$ acts faithfully on $\Delta_1$ and $\Delta_2$.

For the converse, let $\Gamma$ be any $G$-edge transitive graph that is not $G$-vertex transitive and let $v \sim w$. Then $\Gamma \cong \mathrm{Cos}(G, G_v, G_w)$ follows from Lemma 5.2.6 and the definition of coset graphs. □

**Lemma 5.2.11.** *For a group $G$ and subgroups $L, R < G$ such that $L \cap R$ is core-free in $G$, the graph $\Gamma = \mathrm{Cos}(G, L, R)$ satisfies the following properties:*

1. *$\Gamma$ is $G$-locally primitive if and only if $L \cap R$ is a maximal subgroup of both $L$ and $R$;*
2. *$\Gamma$ is locally $(G, 2)$-arc transitive if and only if $L$ acts 2-transitively on $[L : L \cap R]$ and $R$ acts 2-transitively on $[R : L \cap R]$;*
3. *The kernel of the action of $L$ on $\Gamma(L)$ is $\mathrm{core}_L(L \cap R)$ and the kernel of the action of $R$ on $\Gamma(R)$ is $\mathrm{core}_R(L \cap R)$.*

*Proof.* Let $G$ be a group with subgroups $L$ and $R$ such that $L \cap R$ is core-free in $G$. Let $\Gamma = \mathrm{Cos}(G, L, R)$.

1. As $G_{vw} = G_v \cap G_w = L \cap R$ is core-free and hence maximal in $G_v$ and $G_w$, we have that $G_v$ and $G_w$ act primitively on $\Gamma(v)$ and $\Gamma(w)$ respectively, so that $\Gamma$ is $G$-locally primitive.
2. Note that $LR = \bigcup Lx$ for $x \in R \setminus R \cap L$ so if $x, y \in R$ then $Lx = Ly$ if and only if $xy^{-1} \in L$ so $xy^{-1} \in L \cap R$ and $(L \cap R)x = (L \cap R)y$. Then $\Gamma(L) = \{Ry : y \in L\}$ is in bijection with $[L : L \cap R]$. By edge transitivity, the set of neighbors of $Lx$ in $\Gamma$ is the set $\{Rzx : z \in L\}$ and is also in bijection with $[L : L \cap R]$. Then the result follows from Lemma 5.2.4.
3. Let $\Gamma(L) = \{R, Rg_1, \ldots, Rg_n\}$ where $g_1, \ldots, g_n \in L$. The kernel of the action of $L$ on $\Gamma(L)$ is the intersection of $L$ with the stabilizer of each of its neighbors. We know that the stabilizer of a vertex $Rg_i$ is $R_i^g$ so we have that the kernel equals $(L \cap R) \cap (L \cap R^{g_1}) \cdots \cap (L \cap R^{g_n})$. Since $L \cap R^{g_i} = (L \cap R)^{g_i}$ we have that the kernel of $L$ acting on its neighbors is the intersection of $(L \cap R)^{g_i}$ as $g_i$ runs over $L$, which equals $\mathrm{core}_L(L \cap R)$.

□

## 5.3 LOCALLY ARC TRANSITIVE GRAPHS WITH A VERTEX OF VALENCY AT MOST THREE

If $\Gamma$ is a locally $(G, s)$-arc transitive graph with valency at least two, Lemma 5.2.3 shows that $\Gamma$ is $(G, s-1)$-arc transitive and Lemma 5.2.2 shows that if $G$ is intransitive on $V\Gamma$ then we have a bipartite graph with two $G$-orbits. If $\Gamma$ has a vertex of valency one, we have the following scenario.

**Lemma 5.3.1.** *Let $\Gamma$ be a locally $(G, s)$-arc transitive graph with $s \geq 1$ which contains a vertex of valency one. Then $\Gamma$ is a tree.*

*Proof.* Let $v$ be a vertex of valency one in $\Gamma$. Suppose $\Gamma$ contains a cycle. As $\Gamma$ is connected there exists a shortest path $\{v = w_0, w_1, \ldots, w_t\}$ such that $w_t$ is contained in a cycle $C$. Since $v$ has valency one then $t \geq 1$ and none of $w_0, \ldots, w_{t-1}$ lies on a cycle. There exists an $s$-arc that starts with $(w_{t-1}, w_t)$ and loops around $C$ finishing at a vertex $u \in C$. Let $\beta$ be the reverse of this arc, going from $u$ to $w_{t-1}$ and let $\alpha$ be the $s$-arc that agrees with $\beta$ in its first $s$ vertices but ends in a vertex of $C$ adjacent to $w_t$. As $w_{t-1} \in \beta$ and is not contained in a cycle while all vertices in $\alpha$ belong to $C$, there is no element of $G_u$ mapping $\alpha$ to $\beta$, contradicting the fact that $\Gamma$ is locally $(G, s)$-arc transitive. Hence $\Gamma$ is a tree. □

### 5.3.1 LOCALLY $(G, 2s-1)$-ARC TRANSITIVE GRAPHS

Let $\Gamma$ be a locally $(G, s)$-arc transitive graph with valency $k \geq 2$. We form a new graph $\Gamma^*$ by placing a vertex at the midpoint of each edge of $\Gamma$. As $G$ acts on $\Gamma$, $G \leq \mathrm{Aut}(\Gamma^*)$ and has two orbits on the vertices: the set $\Delta_1$ of vertices of $\Gamma$ which have valency $k$ and the set $\Delta_2$ of midpoints of edges of $\Gamma$ which have valency 2. If $\Gamma$ is a cycle then $\Gamma^*$ is also a cycle. We will show that $\Gamma^*$ is locally $(G, 2s-1)$-arc transitive and that every locally $(G, 2s-1)$-arc transitive of valency $\{2, k\}$ for $k \geq 3$ arises this way.

We define the *distance* between two vertices in a graph to be the number of edges in a shortest path connecting them. For a graph $\Gamma$ and a vertex $v$ of $\Gamma$, we denote by $\Gamma_i(v)$ the set of vertices of $\Gamma$ at distance equal to $i$ from $v$. If $\Gamma$ is a connected graph then the *distance two graph* $\Gamma^{[2]}$ of $\Gamma$ is the graph with vertex set $V\Gamma$ such that two vertices are adjacent if and only if they are at distance two in $\Gamma$.

**Lemma 5.3.2.** *If $\Gamma$ is connected and bipartite then $\Gamma^{[2]}$ has two connected components.*

*Proof.* Suppose $\Gamma$ is connected and bipartite with $V\Gamma = \Delta_1 \cup \Delta_2$. Let $v \in \Delta_1$ and $w \in \Delta_2$. Then either $v$ is adjacent to $w$ or $v$ and $w$ are at distance at least three since $\Delta_1$ is an independent set and $\Gamma$ is bipartite. Hence $v$ and $w$ belong to different components in $\Gamma^{[2]}$ because they are not at distance two.

As $\Gamma$ is connected, for any two vertices $v_i, v_j \in \Delta_1$, $i < j$, there exists a path $v_i, w_{i+1}, v_{i+2}, \dots, w_{j-1}, v_j$ and note that $w_k$'s are necessary since $\Gamma$ is bipartite. Then each $v_k$ is at distance two from $v_{k+2}$ and so they are adjacent in $\Gamma^{[2]}$, and we have a path from $v_i$ to $v_j$ in $\Gamma^{[2]}$. A similar argument works for vertices of $\Delta_2$. Therefore $\Delta_1$ and $\Delta_2$ form two connected components in $\Gamma^{[2]}$. □

**Example 5.3.3.** Let $\Gamma = K_{n,m}$ the complete bipartite graph of $n+m$ vertices. Then $\Gamma^{[2]} = K_n \cup K_m$ since vertices in the same part of the bipartition are at distance two.

If we form the distance two graph of $\Gamma^*$ as described above, the vertices of valency two form one connected component and the other connected component with vertex set $\Delta_1$ is isomorphic to $\Gamma$.

**Theorem 5.3.4.** *Let $s \geq 2$.*

1. *Let $\Gamma$ be a connected locally $(G, 2s-1)$-arc transitive of valency $\{2, k\}$ with $k \geq 3$ such that $\Gamma \neq K_{2,k}$. Let $\Delta$ be a connected component of $\Gamma^{[2]}$ containing a vertex of valency $k$. Then $V\Delta$ is the set of all vertices of $\Gamma$ of valency $k$ and $\Delta$ is $(G, s)$-arc transitive of valency $k$.*

2. *Let $\Sigma$ be a connected $(G, s)$-arc transitive of valency $k$. Then $\Sigma^*$ is a connected locally $(G, 2s-1)$-arc transitive graph. Moreover, $\Sigma^* \neq K_{2,k}$ and if $\Gamma = \Sigma^*$, the graph $\Delta$ from part 1 is equal to $\Sigma$.*

*Proof.* 1. Let $\Gamma$ be a connected locally $(G, 2s-1)$-arc transitive with $V\Gamma = \Delta_1 \cup \Delta_2$, such that $\Delta_1$ contains vertices of valency $k \geq 3$ and $\Delta_2$ contains vertices of valency two. Let $\Delta$ be a connected component of $\Gamma^{[2]}$ containing a vertex of valency $k$. As $\Gamma$ is connected, $V\Delta = \Delta_1$. Let $v \in \Delta_1$ and let $w \in \Gamma(v)$. As $w$ has valency two there exists a unique element $v(w)$ in $\Gamma(w) \setminus \{v\}$ so the map $w \to v(w)$ is a 1-1 correspondence between $\Gamma(v)$ and vertices at distance two from $v$, denoted $\Gamma_2(v)$. Then $\Delta$ has valency $k$. Now $G \leq \mathrm{Aut}(\Delta)$ and acts transitively on vertices. Let $(v_0, v_1, \dots, v_s)$ be an $s$-arc in $\Delta$. Then for each $i = 0, \dots, s-1$ there exists a unique $w_i$ such that $\Gamma(w_i) = \{v_i, v_{i+1}\}$. Then $(v_0, w_0, v_1, \dots, v_{s-1}, w_{s-1})$ is a $(2s-1)$-arc in $\Gamma$. As $\Gamma$ is locally $(G, 2s-1)$-arc transitive, $G_{v_0}$ is transitive on the set of all $s$-arcs emanating from $v_0$ in $\Delta$. Since $\Delta$ is $G$-vertex transitive, we have that $\Delta$ is $(G, s)$-arc transitive of valency $k$.

2. Let $\Sigma$ be a connected $(G, s)$-arc transitive of valency $k$ and form the graph $\Sigma^*$ by placing a new vertex at the midpoint of every edge. In its action on $\Sigma^*$, $G$ has orbits $\Delta_1$ of vertices of $\Sigma$ and $\Delta_2$ of vertices that are midpoints. Let $v_0 \in \Delta_1$ and consider the $(2s-1)$-arc $(v_0, p_0, v_1, p_1, \dots, v_{s-1}, p_{s-1})$ in $\Sigma^*$, where each $p_i$ is the midpoint of the edge $\{v_i, v_{i+1}\}$. As $\Sigma$ is $(G, s)$-arc transitive, $G_{v_0}$ acts transitively

on the set of $s$-arc in $\Sigma$ starting at $v_0$. Then $G_{v_0}$ acts transitively on the set of $(2s-1)$-arc in $\Sigma^*$ starting at $v_0$. It remains to show that $G_{p_0}$ acts transitively on the set of $(2s-1)$-arcs in $\Sigma^*$ starting at $p_0$. Let $\alpha = (p_0, v_1, p_1, \dots, p_{s-1}, v_s)$ and $\alpha' = (p_0, v_1', p_1', \dots, p_{s-1}', v_s')$ be two $(2s-1)$-arcs in $\Sigma^*$. There exist $s$-arcs $\beta = (v_0, v_1, \dots, v_s)$ and $\beta' = (v_0', v_1', \dots, v_s')$ in $\Sigma$ such that $p_i$ is the midpoint of $\{v_i, v_{i+1}\}$ and $p_i'$ is the midpoint of $\{v_i', v_{i+1}'\}$ for each $i \geq 0$. By $s$-arc transitivity, there exists $g \in G$ such that $\beta^g = \beta'$ and so $\alpha^g = \alpha'$. As $p_0$ is the midpoint of both $\{v_0, v_1\}$ and $\{v_0', v_1'\}$ we have $\{v_0, v_1\} = \{v_0', v_1'\}$, and $g \in G_{\{v_0,v_1\}} = G_{p_0}$. Thus each stabilizer of a vertex $x$ in $\Sigma^*$ acts transitively on $(2s-1)$-arcs emanating from $x$ so $\Sigma^*$ is locally $(G, 2s-1)$-arc transitive. If we let $\Gamma = \Sigma^*$, then $\Delta$ from part 1 which contains all vertices of valency $k$ is in fact equal to $\Sigma$, since the valency two vertices will form another component.

□

If $s = 1$ or $\Gamma = K_{2,k}$, the theorem does not hold. A counterexample is a "doubled 3-cycle" given in [14] and a correction to this theorem was also provided. Let us now illustrate the theorem with an example.

**Example 5.3.5.** Let $\Gamma$ be the cube graph as described in Example 5.2.5. First consider $\Gamma$ as a bipartite graph with labelling $v_0$ to $v_7$, as shown in Figure 5.

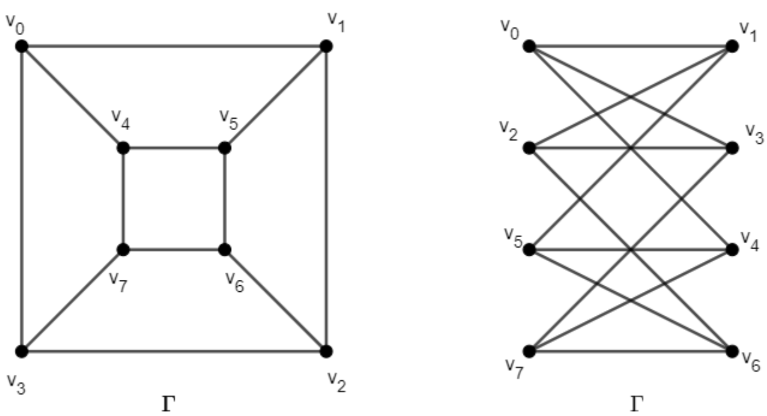
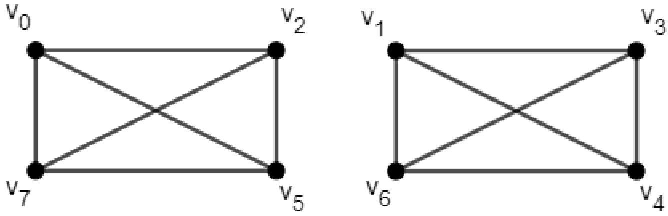



Figure 5: Cube graph drawn in two ways

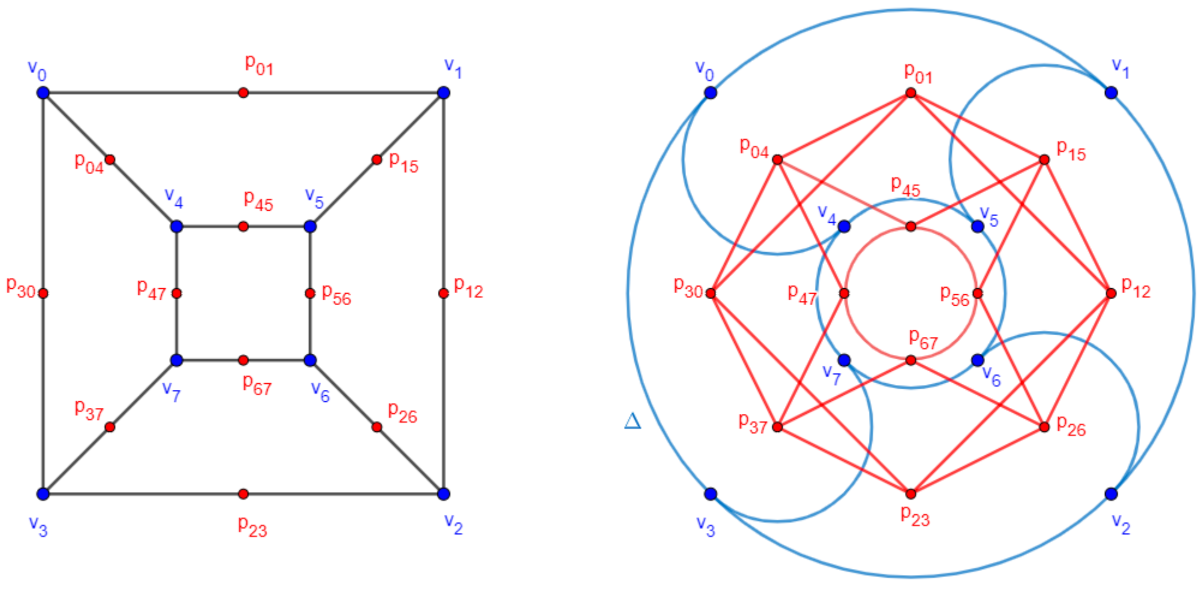
By Lemma 5.3.2, the distance two graph $\Gamma^{[2]}$ has two connected components which are formed by vertices in different parts of the bipartition. We can see this in Figure 6.




$\Gamma^{[2]}$

Figure 6: The distance two graph $\Gamma^{[2]}$ of $\Gamma$

Now consider the first drawing of $\Gamma$, and let $\Gamma^*$ be the graph obtained from $\Gamma$ by placing a vertex at the midpoint of every edge. We label the new vertices by $p_{ij}$ if $p_{ij}$ is the midpoint of the edge $\{v_i, v_j\}$. We know that $\Gamma$ is locally $(G, 2)$-arc transitive so by Theorem 5.3.4 part 2, $\Gamma^*$ is locally $(G, 3)$-arc transitive and the graph $\Delta$ formed by vertices of valency 3 in the distance two graph of $\Gamma^*$ is equal to $\Gamma$. We can see $\Gamma^*$ and the distance two graph of $\Gamma^*$, labelled $\Gamma^{*[2]}$ in Figure 7. The blue component $\Delta$ of $\Gamma^{*[2]}$ is isomorphic to $\Gamma$ and the red component is formed by the vertices which lie in the midpoints of edges of $\Gamma$.




$\Gamma^*$ $\Gamma^{*[2]}$

Figure 7: Graphs $\Gamma^*$ and $\Gamma^{*[2]}$

This also illustrates part 1 of the theorem since taking $\Gamma^*$ as a connected locally $(G, 3)$-arc transitive graph of valency $\{2, 3\}$ we get $\Delta$ from $\Gamma^{*[2]}$ as a locally $(G, 2)$-arc transitive graph of valency 3.

**Corollary 5.3.6.** *The connected locally $(G, 2s-1)$-arc transitive graphs of valency $\{2, k\}$, where $k \geq 3$ and which are not complete bipartite, are in* 1-1 *correspondence with the connected locally $(G, s)$-arc transitive graphs of valency $k$.*

**Theorem 5.3.7.** *Let $\Gamma$ be a connected locally $(G, 2s)$-arc transitive of valency $\{2, k\}$, where $k \geq 3$ and $s \geq 1$. Then $\Gamma$ is locally $(G, 2s+1)$-arc transitive.*

*Proof.* Let $\Delta_1$ and $\Delta_2$ denote the sets of vertices of valency $k$ and 2, respectively. Let $u_0 \in \Delta_2$ and let $(u_0, v_0, u_1, v_1, \ldots, u_s, v_{s+1})$ and $(u_0, v_0', u_1', v_1', \ldots, u_s', v_{s+1}')$ be two $(2s+1)$-arcs starting at $u_0$, where $v_i, v_i' \in \Delta_1$ and $u_i, u_i' \in \Delta_2$. As $\Gamma$ is $(2s)$-arc transitive, we can map $(u_0, v_0, \ldots, v_s, u_s)$ to $(u_0, v_0', \ldots, v_s', u_s')$. Since $u_s$ and $u_s'$ are vertices of $\Delta_2$ they have valency two and so they have unique neighbors outside the $(2s)$-arcs and hence $v_{s+1}$ must be mapped to $v_{s+1}'$.

All vertices in $\Gamma$ have valency at least two so $\Gamma$ is locally $(G, 2s-1)$-arc transitive by Lemma 5.2.3. Let $\Delta$ be the connected component of the distance two graph of $\Gamma$ with vertex set $\Delta_1$. Then by Theorem 5.3.4(1), $\Delta$ is locally $(G, s)$-arc transitive. Let $(u_0, v_1, u_1, v_2, \ldots, v_s, u_s)$ be a $(2s)$-arc in $\Gamma$ where each $v_i \in \Delta_1$ and $u_i \in \Delta_2$. As $\Gamma$ is locally $(G, 2s)$-arc transitive, $G_{u_0 v_1 u_1 \ldots v_s}$ acts transitively on $\Gamma(v_s) \setminus \{u_s\}$. Each vertex of $\Gamma(v_s) \setminus \{u_s\}$ is adjacent to a unique vertex of $\Gamma_2(v_s) \setminus \{v_{s-1}\}$, which are vertices at distance two from $v_s$. Then $G_{u_0 v_1 u_1 \ldots v_s}$ acts transitively on $\Gamma_2(v_s) \setminus \{v_{s-1}\}$. Since $u_0$ has valency two, there exists a unique $v_0 \in \Gamma(u_0) \setminus \{v_1\}$. Then $G_{u_0 v_1 u_1 \ldots v_s} \leq G_{v_0}$. Furthermore, $(v_0, v_1, \ldots, v_s)$ is an $s$-arc in $\Delta$ and $G_{v_0 v_1 \ldots v_s}$ acts transitively on $\Delta(v_s) \setminus \{v_{s-1}\} = \Gamma_2(v_s) \setminus \{v_{s-1}\}$. Hence $G_{v_0}$ acts transitively on the set of $(s+1)$-arcs in $\Delta$ starting at $v_0$, so $\Delta$ is $(G, s+1)$-arc transitive. As $\Gamma = \Delta^*$ it follows from Theorem 5.3.4(2) that $\Gamma$ is locally $(G, 2s+1)$-arc transitive.

□

**Lemma 5.3.8.** *Let $\Gamma$ be a connected locally $(G, s)$-arc transitive of valency $\{2, k\}$, where $k \geq 3$. Then $s \leq 13$, and this bound can be attained.*

*Proof.* By Theorem 5.3.7 we may assume that $s$ is odd. Then by Corollary 5.3.6, there exists a $\left(G, \frac{s+1}{2}\right)$-arc transitive graph of valency $k$. By the result of Weiss [31], $\frac{s+1}{2} \leq 7$. Moreover, examples of $(G, 7)$-arc transitive graphs of valency at least three are given in [18] and so by Corollary 5.3.6, locally $(G, 13)$-arc transitive graphs exist. □

## 5.4 NORMAL QUOTIENTS

**Definition 5.4.1.** A graph $\Gamma$ is *$G$-locally primitive* if for each vertex $v$, the stabilizer $G_v$ acts primitively on $\Gamma(v)$, where $\Gamma(v)$ is the set of vertices adjacent to $v$.

Let $\Gamma$ be a graph and let $G$ be its automorphism group. Suppose $G$ has a normal subgroup $N$ which acts intransitively on $V\Gamma$. We define the *quotient graph* $\Gamma_N$ to have vertex set the $N$-orbits on $V\Gamma$, and two $N$-orbits $B_1$ and $B_2$ are adjacent in $\Gamma_N$ if and only if there exist $v \in B_1$ and $w \in B_2$ such that $v$ and $w$ are adjacent in $\Gamma$. The original graph $\Gamma$ is said to be a *cover* of $\Gamma_N$ if $|\Gamma(v) \cap B_2| = 1$ for each edge $\{B_1, B_2\}$ in $\Gamma_N$ and $v \in B_1$. If $\Gamma$ is a nonbipartite $(G, s)$-arc transitive graph, $s \geq 2$, then $\Gamma$ is a cover of $\Gamma_N$ and $\Gamma_N$ is $(G/N, s)$-arc transitive as shown by Praeger [21, Theorem 4.1].

**Lemma 5.4.2.** *Let $\Gamma$ be a connected $G$-locally primitive bipartite graph with $G$-orbits $\Delta_1$ and $\Delta_2$ on $V\Gamma$ and each $|\Delta_i| > 1$. Suppose there exists $N \trianglelefteq G$ such that $N$ is intransitive on $\Delta_1$ and $\Delta_2$. Then*

1. *$\Gamma$ is a cover of $\Gamma_N$.*

2. *$N$ acts semiregularly on $V\Gamma$ and $G^{V\Gamma_N} \cong G/N$.*

3. *$\Gamma_N$ is $G/N$-locally primitive. Furthermore, if $\Gamma$ is locally $(G, s)$-arc transitive, then $\Gamma_N$ is locally $(G/N, s)$-arc transitive.*

*Proof.* 1. Let $v \in \Delta_1$ and let $B = v^N$. Choose $u \in \Gamma(v) \subseteq \Delta_2$ and set $C = u^N$. Then $C$ is a block of imprimitivity for the action of $G$ on $\Delta_2$ and hence $C \cap \Gamma(v)$ is a block of imprimitivity for the action of $G_v$ on $\Gamma(v)$. As $N_v \trianglelefteq G_v$ and $G_v$ acts primitively on $\Gamma(v)$, it follows that either $\Gamma(v) \subseteq C$ or $|\Gamma(v) \cap C| = 1$. If $\Gamma(v) \subseteq C$ then for each vertex $w \in \Delta_1$, the set $\Gamma(w)$ is contained in some $N$-orbit. thus if $B'$ is an $N$-orbit on $\Delta_1$ containing a vertex adjacent to a vertex in $C$, then $\Gamma(B') \subseteq C$. As $\Gamma$ is connected $V\Gamma = C \cup \Gamma(C)$, contradicting the intransitivity of $N$ on $\Delta_2$. Thus $|\Gamma(v) \cap C| = 1$.

2. Let $K$ be the kernel of the action of $G$ on the set of $N$-orbits on $V\Gamma$ and let $v \in V\Gamma$. Now $K_v$ fixes each $N$-orbit setwise and since distinct vertices of $\Gamma(v)$ lie in distinct $N$-orbits, we have that $K_v$ acts trivially on $\Gamma(v)$. Since $\Gamma$ is connected it follows that $K_v$ fixes all the vertices of $\Gamma$ and hence $K_v = 1$. Since this is true for all $v$, $K$ acts semiregularly on $V\Gamma$ and hence so does $N$. Furthermore, as $N \leq K$ and acts transitively on the orbits of $K$, we see that $K = N$. Then $G^{V\Gamma_N} \cong G/N$ so $G/N \leq \mathrm{Aut}(\Gamma_N)$.

3. For a vertex $v$ in the $N$-orbit $B$, the group $NG_v$ fixes $B$, contains $G_v$ and is transitive on $B$. Hence $G_B = NG_v$. Then as $N$ is the kernel of the action of $G$ on $V\Gamma_N$ and as each block in $\Gamma_N(B)$ contains exactly one vertex of $\Gamma(v)$, we have that $G_B^{\Gamma_N(B)}$ is permutationally isomorphic to $G_v^{\Gamma(v)}$ and so is primitive. Thus $\Gamma_N$ is $(G/N)$-locally primitive. Let $(B, B_1, \ldots, B_s)$ and $(B, C_1, \ldots, C_s)$ be $s$-arcs in $\Gamma_N$. Choose $v \in B$. Then there exist unique $v_i \in B_i$ and $u_i \in C_i$ such that $(v, v_1, \ldots, v_s)$ and $(v, u_1, \ldots, u_s)$ are $s$-arcs in $\Gamma$. If $\Gamma$ is locally $(G, s)$-arc transitive, then there exists $g \in G_v$ taking $(v, v_1, \ldots, v_s)$ to $(v, u_1, \ldots, u_s)$. As

the orbits of $N$ form a system of imprimitivity for $G$, it follows that $g \in G_B$ and $(B, B_1, \ldots, B_s)^g = (B, C_1, \ldots, C_s)$. Thus $\Gamma_N$ is locally $(G/N, s)$-arc transitive.

□

**Lemma 5.4.3.** *Let $\Gamma$ be a connected $G$-locally primitive bipartite graph with $G$-orbits $\Delta_1$ and $\Delta_2$ on $V\Gamma$ of sizes $n$ and $n'$, respectively. Then either $\Gamma \cong K_{n,n'}$ or $G$ is faithful on both $\Delta_1$ and $\Delta_2$.*

*Proof.* If either $n$ or $n'$ is 1, then $\Gamma = K_{n,n'}$, so assume that $n, n' \geq 2$. Let $K_i$ be the kernel of $G$ on $\Delta_i$, $i = 1, 2$. Since $G$ acts faithfully on $V\Gamma$, we know $K_1 \cap K_2 = 1$. Suppose that $K_1 \neq 1$ and note that $K_1$ acts faithfully on $\Delta_2$. Let $B$ be a nontrivial orbit of $K_1$ on $\Delta_2$ and $u$ a vertex in $B$. Let $v \in \Delta_1$ be adjacent to $u$. Since $K_1$ fixes $v$, $v$ is adjacent to every vertex in $B$. As $K_1 \leq G_v$, the orbits of $K_1$ on $\Gamma(v)$ are blocks of imprimitivity for the action of $G_v$ and since the action of $G$ is locally primitive, $\Gamma(v) = B$. This holds for all $v$ adjacent to a vertex in $B$ so as $\Gamma$ is connected, $\Gamma \cong K_{n,n'}$. The same holds if $K_2 \neq 1$. □

**Lemma 5.4.4.** *Let $\Gamma$ be a connected $G$-edge transitive but not $G$-vertex transitive graph such that $|\Gamma(u)| = 1$ for some vertex $u$. Then $\Gamma$ is a star $K_{1,k}$, and if $G$ acts faithfully on both $G$-orbits on vertices, then $k = 1$, $\Gamma = K_2$ and $G = 1$.*

*Proof.* Let $\Delta_1$ and $\Delta_2$ be the $G$-orbits on $V\Gamma$. Without loss of generality, we may assume $u \in \Delta_1$. Since $u$ has only one neighbor in $\Delta_2$ and $\Gamma$ is connected, $\Gamma$ is a star $K_{1,k}$. If $G$ is faithful on $\Delta_2$, then $G = 1$ and hence $|\Delta_1| = 1$. Thus $\Gamma = K_{1,1} = K_2$. □

**Lemma 5.4.5.** *Let $\Gamma$ be a finite connected graph with $G$-orbits $\Delta_1$ and $\Delta_2$ on $V\Gamma$ such that $G$ acts faithfully on both orbits. Suppose that every nontrivial normal subgroup $N$ of $G$ is transitive on at least one of the $\Delta_i$. Then $G$ acts quasiprimitively on at least one of its orbits.*

*Proof.* Suppose that $G$ is not quasiprimitive on either of the $\Delta_i$. Then for each $i \in \{1, 2\}$, there exists $N_i \trianglelefteq G$ such that $N_i$ is intransitive on $\Delta_i$ and transitive on $\Delta_{3-i}$. Now $N_1 \cap N_2 \trianglelefteq G$ and so if nontrivial would be transitive on at least one $\Delta_i$ by the hypothesis, however, it is a subgroup of both $N_1$ and $N_2$ so both $N_1$ and $N_2$ would be transitive on the same set, a contradiction. So $N_1 \cap N_2 = 1$ and hence $N_1 \times N_2 \trianglelefteq G$. Since each $N_i$ is transitive on $\Delta_{3-i}$, it follows that $C_{\mathrm{Sym}(\Delta_{3-i})}(N_i)$ is semiregular (see [5, Theorem 4.2A]). Thus each $N_i$ is semiregular on $\Delta_i$. Therefore $|N_1|$ divides $|\Delta_1|$ and $|\Delta_2|$ divides $|N_1|$, so $|\Delta_2|$ divides $|\Delta_1|$. A similar argument with $N_2$ shows $|\Delta_1|$ divides $|\Delta_2|$ thus $|\Delta_1| = |\Delta_2|$. Further, $|N_1| = |\Delta_1| = |N_2|$, contradicting $N_1$ being intransitive on $\Delta_1$. Hence $G$ must be quasiprimitive on at least one of the $\Delta_i$.

□

**Lemma 5.4.6.** *Let $\Gamma$ be a connected $G$-locally primitive bipartite graph with $G$-orbits $\Delta_1$ and $\Delta_2$ on $V\Gamma$. Suppose there exists $N \trianglelefteq G$ such that $N$ is transitive on $\Delta_1$ but intransitive on $\Delta_2$. Then $\Gamma_N$ is a star whose central vertex has valency the number of orbits of $N$ on $\Delta_2$. Furthermore, for each vertex $v \in \Delta_1$ and $N$-orbit $B$ in $\Delta_2$, $|B \cap \Gamma(v)| = 1$ and the vertex stabilizer $N_v$ acts trivially on $\Gamma(v)$.*

*Proof.* Let $v \in \Delta_1$ and $u \in \Delta_2$ such that $v \sim u$. Let $B = u^N$. For each $w \in B$, we have that $w = u^g$ for some $g \in N$ and $v^g \in \Gamma(u^g) = \Gamma(w)$, so that each vertex of $B$ is adjacent to some vertex in $\Delta_1$. Conversely, as $N$ acts transitively on $\Delta_1$, each vertex in $\Delta_1$ is adjacent to some vertex in $B$ so $\Gamma_N$ is a star whose central vertex has valency the number of orbits of $N$ on $\Delta_2$. The set $\Gamma(v) \cap B$ is an orbit of $N_v$ on $\Gamma(v)$ and hence is a block for $G_v$. If $\Gamma(v) \subseteq B$, then $\Gamma(v') \subseteq B$ for all $v' \in \Delta_1$ since $B$ and $\Delta_1$ are $N$-orbits, contradicting the connectivity of $\Gamma$. Hence $|\Gamma(v) \cap B| = 1$ and so $N_v$ acts trivially on $\Gamma(v)$. □

**Lemma 5.4.7.** *Let $\Gamma$ be a locally $(G, s)$-arc transitive graph such that all vertices have valency at least two and $G$ has a normal subgroup $N$ which is transitive on $\Delta_1$ but has at least three orbits on $\Delta_2$. Then $s \leq 3$.*

*Proof.* Let $B_1$ ,$B_2$, $B_3$ be three orbits of $N$ on $\Delta_2$. Let $v_0 \in B_1$ and $v_1 \in \Gamma(v_0)$. By Lemma 5.4.6, $v_1$ is adjacent to a unique vertex $v_2 \in B_2$. Let $v_3 \in \Gamma(v_2) \setminus \{v_0\}$. Then $(v_0, v_1, v_2, v_3)$ is a 3-arc in $\Gamma$. By Lemma 5.4.6, there exist $u, w \in \Gamma(v_3)$ such that $u \in B_1$ and $w \in B_3$. Then $(v_0, v_1, v_2, v_3, u)$ and $(v_0, v_1, v_2, v_3, w)$ are 4-arcs in $\Gamma$ that cannot be mapped to each other, since such a $g$ would fix $B_1$ so it could not map $u$ to $w$. Hence $s \leq 3$. □

We now state an important theorem from Burnside, which characterizes minimal normal subgroups of finite 2-transitive groups. The theorem and its proof can be found in [4, Theorem 4.3].

**Theorem 5.4.8** (Burnside's Theorem)**.** *Let $N$ be a minimal normal subgroup of a finite 2-transitive group $G$. Then $N$ is either elementary abelian and regular, or simple and primitive.*

## 5.5 QUASIPRIMITIVE ON BOTH ORBITS

We analyze the case where $G$ acts faithfully and quasiprimitively on both of its orbits.

**Theorem 5.5.1.** *Let $\Gamma$ be a finite connected $G$-locally primitive graph such that $G$ has two orbits on vertices and $G$ acts faithfully and quasiprimitively on both orbits with type $\{X, Y\}$. Then either $X = Y$, or $\{X, Y\} = \{SD, PA\}$ or $\{CD, PA\}$, and examples exist in each case. Furthermore , if $\Gamma$ is locally $(G, s)$-arc transitive with $s \geq 2$, then* *either* $X = Y \in \{HA, TW, AS, PA\}$*, or* $\{X, Y\} = \{SD, PA\}$*, and examples exist in each case.*

We prove the theorem with lemmas and propositions.

**Definition 5.5.2.** A graph $\Gamma$ is *$G$-locally quasiprimitive* if $G_v^{\Gamma(v)}$ is a quasiprimitive permutation group for every $v \in V\Gamma$.

**Lemma 5.5.3.** *Let $\Gamma$ be a connected $G$-locally quasiprimitive graph. Suppose that $\Gamma$ is bipartite and the orbits of $G$ are the bipartite halves $\Delta_1$ and $\Delta_2$. Suppose also that $G$ acts faithfully and quasiprimitively on both orbits. If $N \trianglelefteq G$, then $N^{\Delta_1}$ is regular if and only if $N^{\Delta_2}$ is regular.*

*Proof.* If there exists a vertex of valency one, then by Lemma 5.4.4, $\Gamma = K_2$ and the result is trivially true. So assume that each vertex has valency at least two. Let $N \trianglelefteq G$ and note that since $G$ is quasiprimitive and faithful on $\Delta_2$, $N$ is transitive on $\Delta_2$. Suppose that $N^{\Delta_1}$ is regular and $N^{\Delta_2}$ is not regular. Then for all $v \in \Delta_1$ we have $N_v = 1$, and there exists $u \in \Delta_2$ such that $N_u \neq 1$. Then $N_u$ acts nontrivially on $\Gamma(u)$ so

$$1 \neq N_u^{\Gamma(u)} \trianglelefteq G_u^{\Gamma(u)}.$$

Since $N_u^{\Gamma(u)}$ is normal and $G_u^{\Gamma(u)}$ is a quasiprimitive permutation group, $N_u$ acts transitively on $\Gamma(u)$. As $N$ is transitive on $\Delta_2$, $N_w$ is transitive on $\Gamma(w)$ for all $w \in \Delta_2$. Now $N$ is transitive on the edges of $\Gamma$ which implies $N_v$ acts transitively on $\Gamma(v)$, contradicting $N_v = 1$ as $v$ has at least two neighbors. □

**Lemma 5.5.4.** *Let $\Gamma$ and $G$ be as in Lemma 5.5.3 and let $N \trianglelefteq G$. If $N$ is not regular on $\Delta_1$, then $N_v^{\Gamma(v)}$ is transitive for all $v \in V\Gamma$.*

*Proof.* If there exists a vertex of valency one, then by Lemma 5.4.4, $\Gamma = K_2$ and $G = 1$ so no such $N$ exists. So assume that each vertex has valency at least two. By Lemma 5.5.3, $N$ is not regular on $\Delta_2$ either and so for all $v \in V\Gamma$ we have $N_v \neq 1$. Suppose that there exists $v \in V\Gamma$ such that $N_v^{\Gamma(v)} = 1$. Then as $\Gamma$ is connected and $G$ acts faithfully on $V\Gamma$, there exists a path $(v = v_0, v_1, \ldots, v_r)$ such that $N$ fixes $v_0, \ldots, v_{r-1}$ but not $v_r$. Now $N_v \leq N_{v_{r-1}}$ and $N_v$ moves $v_r \in \Gamma(v_{r-1})$ so

$$1 \neq N_{v_{r-1}}^{\Gamma(v_{r-1})} \trianglelefteq G_{v_{r-1}}^{\Gamma(v_{r-1})}.$$

Since $N_{v_{r-1}}^{\Gamma(v_{r-1})}$ is normal and $G_{v_{r-1}}^{\Gamma(v_{r-1})}$ is quasiprimitive, $N_{v_{r-1}}^{\Gamma(v_{r-1})}$ is transitive. $N$ is transitive on each orbit and $v_{r-1}$ is in one of them, so $N$ is transitive on the edges of $\Gamma$. This contradicts $N_v^{\Gamma(v)} = 1$ as $v$ has at least two neighbors. Hence $N_v^{\Gamma(v)} \neq 1$ and $N_v^{\Gamma(v)}$ is transitive for all $v \in V\Gamma$. □

**Proposition 5.5.5.** *Let $\Gamma$ be a finite connected $G$-locally primitive graph such that $G$ has two orbits on vertices and $G$ acts faithfully and quasiprimitively on both orbits with type $\{X, Y\}$. Then either $X = Y$, or $\{X, Y\} =\{SD, PA\}$ or $\{CD, PA\}$.*

*Proof.* Let $\Delta_1$ and $\Delta_2$ be the $G$-orbits on $V\Gamma$. Note that $G_1^{\Delta} \cong G_2^{\Delta} \cong G$. If $G$ is of type HA, AS, HC or HS on one of the $G$-orbits, then $G$ must have the same type on the other $G$-orbit due to the abstract structure and the number of minimal normal subgroups of $G$.

If $G$ is of type TW on one $G$-orbit then $G$ has a unique minimal normal subgroup $N$ isomorphic to $T^k$ for some finite nonabelian simple group $T$ and $N$ is regular on that orbit. By Lemma 5.5.3, $N$ is regular on the other $G$-orbit so we have type TW on both orbits.

Now assume $\{X, Y\} \subseteq \{\text{SD, CD, PA}\}$. It remains to show that $\{X, Y\} \neq \{\text{SD,CD}\}$. Suppose for a contradiction and without loss of generality that $G$ is of quasiprimitive type SD on $\Delta_1$ and quasiprimitive type CD on $\Delta_2$. Let $N$ be the unique minimal normal subgroup of $G$ so that $N \cong T^k$ for some finite nonabelian simple group $T$. Since $G$ acts faithfully on $\Delta_1$, by the structure of the group, we may assume that $N < G < N.(\text{Out}(T) \times S_k) < \text{Aut}(T) \wr S_k$ and for some $v \in \Delta_q$ we have $N_v = \{(t, \ldots, t) : t \in T\}$. A typical element of $G$ is of the form

$$(t_1, \ldots, t_k)(\sigma, \ldots, \sigma)\pi, \text{ where } t_i \in T, \sigma \in \text{Aut}(T) \text{ and } \pi \in S_k.$$

Now let $w \in \Gamma(v)$. As $G$ is quasiprimitive of type CD on $\Delta_2$ we have $N_w = D_1 \times \cdots \times D_l$, where each $D_i$ is a full diagonal subgroup of $T^m$ where $k = ml$ and $l \geq 2$. Thus we can write

$$D_1 = \{(t, t^{\phi_{12}}, \ldots, t^{\phi_{1m}}) : t \in T\},$$

where $\phi_{1_2}, \ldots, \phi_{1_m} \in \text{Aut}(T)$ and the other $D_i$ are conjugates of $D_1$ under elements of $G_w$. Hence we may assume that $D_i = \{(t, t^{\phi_{i2}}, \ldots, t^{\phi_{im}}) : t \in T\}$ where $\phi_{i_j} \in \text{Aut}(T)$ and $G \leq T^k.(\text{Out}(T) \times (S_m \wr S_l))$, since $N \cong T^k$ and $k = ml$. Note that $N_v \cap N_w = \{(t, \ldots, t) : t \in C\} \trianglelefteq G_v \cap G_w$ where $C$ is the centralizer of all $\phi_{i_j}$. If all the $\phi_{i_j} = 1$, then $N_v \leq N_w$ and because $G_v^{\Gamma(v)}$ is primitive we have $N_v^{\Gamma(v)} = 1$. Then by Lemma 5.5.4, $N_v = 1$ which contradicts $|N_v| = |T|$. So we may assume that at least one of the $\phi_{i_j}$ is nontrivial and $C \neq T$.

We would like to show that $G_w$ is not primitive on $\Gamma(w)$ which contradicts the $G$-local primitivity of $\Gamma$. Let $g \in G_v \cap G_w$. Then $g = (\sigma, \ldots, \sigma)\tau$ where $\sigma$ normalizes $C$. So $g$ normalizes the subgroup $A = C_1 \times \cdots \times C_k$ where each $C_i = \{(c, \ldots, c) : c \in C\} < D_i$. Let $H = \langle G_v \cap G_w, A\rangle$. Then $H = A(G_v \cap G_w)$ and $H \cap N_w = A \neq N_v \cap N_w$. Thus $G_v \cap G_w < H < G_w$, contradicting $G_v \cap G_w$ being maximal in $G_w$. Hence $\Gamma$ cannot be of type $\{\text{SD,CD}\}$. $\square$

We proved the part of Theorem 5.5.1 for the quasiprimitive action types and now we prove two stronger statements for locally $s$-arc transitive graphs.

**Lemma 5.5.6.** *There are no connected locally $(G, s)$-arc transitive with $s \geq 2$ such that $G$ acts faithfully and quasiprimitively on both orbits and of type HC or CD on one.*

*Proof.* Suppose such a graph $\Gamma$ exists and $G$ is of type HC or CD on $\Delta_1$. Then $G$ has socle $N = T_1 \times \cdots \times T_k \cong T^k$ for some finite nonabelian simple group $T$ and $k \geq 2$. Let $v \in \Delta_1$. Then there exists an integer $l \geq 2$ dividing $k$ such that $N_v = D_1 \times \cdots \times D_l$ where each $D_i$ is a full diagonal subgroup of $T_{k(i-1)/l+1} \times \cdots \times T_{ki/l}$. If $G$ is type HC then $l = k/2$ and for either HC or CD type, $G_v$ permutes the $D_i$ transitively by conjugation. As $N$ is not regular on $\Delta_1$, Lemma 5.5.4 implies that

$$1 \neq N_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}.$$

By Burnside's Theorem 5.4.8 we see that $N_v^{\Gamma(v)} \cong T$. Let $K$ denote the kernel of the action of $N_v$ on $\Gamma(v)$. Then $K \cong T^{l-1}$. Since $K \trianglelefteq N_v$ then $K$ is a product of $l-1$ of the $D_i$. As $G_v$ acts on the $D_i$ by conjugation and $G_v$ normalizes $K$, this is a contradiction. □

**Lemma 5.5.7.** *There are no connected locally $(G, s)$-arc transitive with $s \geq 2$ such that $G$ acts faithfully and quasiprimitively on both orbits of type HS or SD.*

*Proof.* Suppose such a graph $\Gamma$ exists. Then $G$ has socle $N \cong T^k$ for some finite nonabelian simple group $T$ and $k \geq 2$. We can identify $\Delta_1$ and $\Delta_2$ with the elements of $T^{k-1}$ such that the action of $N$ on $T^{k-1}$ is given by

$$(t_1, \ldots, t_k) : (a_1, \ldots, a_{k-1}) \to (t_k^{-1} a_1 t_1, \ldots, t_k^{-1} a_{k-1} t_{k-1}).$$

Now $T \cong N_v \trianglelefteq G_v$. By Lemma 5.5.4, $N_v^{\Gamma(v)} \neq 1$ and so $N_v^{\Gamma(v)} \cong T$. Since $G_v$ acts 2-transitively on $\Gamma(v)$ by Burnside's Theorem 5.4.8 the action of $N_v$ on $\Gamma(v)$ is primitive.

Let $v = (1_T, \ldots, 1_T) \in \Delta_1$. Then $N_v = \{(t, \ldots, t) : t \in T\}$. For $w \in \Gamma(v)$, we have $N_w = \{t, t^{\phi_2}, \ldots, t^{\phi_k} : t \in T\}$ with $\phi_i \in \mathrm{Aut}(T)$. Then $N_{v,w} = \{(t, \ldots, t) : t \in C_T(\phi_i) \text{ for all } i\}$. As $N_v \neq N_{v,w}$, at least one of the $\phi_i$ is nontrivial and as $N_{v,w}$ is maximal in $N_v$, we have that $N_{v,w} \cong C_T(\phi_i)$. However $G_v$ is a 2-transitive almost simple group on $\Gamma(v)$ and no such group exists where the point stabilizer of the socle is a centralizer of a (possibly outer) automorphism by [4, Section 7.4], hence no such $\Gamma$ exists. □

The final step to complete the proof of the theorem is to provide examples of $G$-locally primitive graphs which are not possible for locally $(G, s)$-arc transitive graphs, namely the types {CD, PA}, and the types $X = Y \in$ {HS, SD, HC}.

**Example 5.5.8.** Type {CD, PA} of valencies $n^2$ and $|A_{n-1}|^2$. Let $T = A_n$ for $n \geq 6$ and let $G = T^4 : (S_2 \wr S_2)$. The action of $G$ on the set of right cosets of the subgroup

$$G_v = \{(t,t,s,s) : t,s \in T\} : (S_2 \wr S_2)$$

is primitive of type $CD$. The group $G$ also acts on the set of right cosets of $G_w = A_{n-1}^4 : (S_2 \wr S_2)$ with a primitive action of type PA. Consider the graph $\Gamma = \mathrm{Cos}(G, G_v, G_w)$. Then $G$ acts primitively on its two orbits on vertices of type {CD, PA}. Now

$$G_v \cap G_w = \{(t,t,s,s) : t,s \in A_{n-1}\} : (S_2 \wr S_2)$$

is a maximal subgroup of both $G_v$ and $G_w$. Actually $G_v$ is primitive on $\Gamma(v)$ of type PA whereas $G_w$ is primitive on $\Gamma(w)$ of type CD. Thus $\Gamma$ is a $G$-locally primitive connected graph which is biregular with valencies $n^2$ and $|A_{n-1}|^2$.

**Example 5.5.9.** Type HS and SD of valency $|T : C_T(\sigma)|$. Let $T$ be a finite nonabelian simple group with automorphism $\sigma$ such that $C_T(\sigma)$ is a maximal subgroup of $T$. For example, let $T = A_n$ and let $\sigma$ be the automorphism induced by conjugation by $(1\,2)$. Let $G = T \times T$, $G_v = \{(t,t) : t \in T\}$ and $G_w = \{(t,t^\sigma) : t \in T\}$. Consider the graph $\Gamma = \mathrm{Cos}(G, G_v, G_w)$. The actions of $G$ on $\Delta_1 = [G : G_v]$ and $\Delta_2 = [G : G_w]$ are primitive of type HS. Note that $\langle G_v, G_w\rangle = G$ so $\Gamma$ is connected. Also $G_v \cap G_w = \{(t,t) : t \in C_T(\sigma)\}$ which is maximal in both $G_v$ and $G_w$. Thus $\Gamma$ is $G$-locally primitive with $G$ of type $HS$ and of valency $|T : C_T(\sigma)|$. For type SD, let $\sigma$ be of order two and let $\overline{G} = G : S_2$. Then $G_w$ also contains $(t^\sigma, t)$ for $t \in T$ since $\sigma^2 = 1$. We have $\overline{G} \leq \mathrm{Aut}(\Gamma)$ and acts primitively on both $\Delta_1$ and $\Delta_2$ with type SD. Furthermore, $\Gamma$ is also $\overline{G}$-locally primitive.

**Example 5.5.10.** Type HS and SD of valency $|T : C_T(\sigma)|^2$. Let $T$ be a finite nonabelian simple group with automorphism $\sigma$ of order two such that $C_T(\sigma)$ is a maximal subgroup of $T$. Let $G = (T^2 \times T^2) : S_2$ where $S_2$ is induced by the permutation $(1\,2)(3\,4)$ on the set $\{T_1, T_2, T_3, T_4\}$. Then $G$ has two minimal normal subgroups each isomorphic to $T^2$. Let

$$G_v = \{(t,t,s,s) : t,s \in T\} : S_2 \text{ and } G_w = \{(t,s,t^\sigma,s^\sigma) : t,s \in T\} : S_2.$$

Consider the graph $\Gamma = \mathrm{Cos}(G, G_v, G_w)$. The actions of $G$ on $\Delta_1 = [G : G_v]$ and $\Delta_2 = [G : G_w]$ are quasiprimitive of type HC. As $\langle G_v, G_w\rangle = G$, $\Gamma$ is connected. Now

$$G_v \cap G_w = \{(t,s,t,s) : t,s \in C_T(\sigma)\} : S_2$$

is maximal in $G_v$ and $G_w$ and both $G_v$ and $G_w$ are primitive permutation groups of type PA. Then $\Gamma$ is a $G$-locally primitive graph of valency $|T : C_T(\sigma)|^2$.

For type CD, let $\overline{G} = (T^2 \times T^2) : (S_2 \wr S_2)$ where $S_2 \wr S_2$ preserves the partition $\{\{1,3\},\{2,4\}\}$. Then $\overline{G} \leq \mathrm{Aut}(\Gamma)$ and $\overline{G}$ acts quasiprimitively of type CD on both $\Delta_1$ and $\Delta_2$ of quasiprimitive type CD. Furthermore, $\Gamma$ is a $G$-locally primitive graph.

## 5.6 QUASIPRIMITIVE ON ONLY ONE ORBIT

We analyze the case where $\Gamma$ is locally $(G, s)$-arc transitive and $G$ acts faithfully on both orbits, but quasiprimitively on only one of them.

**Theorem 5.6.1.** *Let $\Gamma$ be a finite locally $(G, s)$-arc transitive graph with $s \geq 2$ such that $G$ acts faithfully on both of its orbits $\Delta_1$ and $\Delta_2$ but only acts quasiprimitively on $\Delta_1$. Then the quasiprimitive action of $G$ on $\Delta_1$ is of type HA, HS, AS, PA or TW and examples exist in each case.*

We prove a lemma first before proving the theorem.

**Lemma 5.6.2.** *Let $\Gamma$ be a $G$-edge transitive connected graph such that $G$ acts faithfully on its two orbits $\Delta_1$ and $\Delta_2$ on vertices. Suppose that $G$ has a nontrivial normal subgroup $N$ such that $N_v^{\Gamma(v)} = 1$ for all $v \in \Delta_1$. If there exists $w \in \Delta_2$ such that $N_w^{\Gamma(w)} = 1$, then $N$ acts semiregularly on $V\Gamma$.*

*Proof.* As $G$ is transitive on $\Delta_2$, $N_w^{\Gamma(w)} = 1$ for all $w \in \Delta_2$. As $\Gamma$ is connected, $N_v$ fixes every element of $\Delta_1$ and $\Delta_2$. Since $G$ acts faithfully on $\Delta_1$, $N_v = 1$ so $N$ acts semiregularly on $\Delta_1$. For $w \sim v$, the stabilizer $N_w$ is contained in $N_v = 1$, so $N_w = 1$ and $N$ acts semiregularly on $\Delta_2$. □

We can now prove Theorem 5.6.1.

*Proof.* Let $\Gamma$ be a finite locally $(G, s)$-arc transitive graph with $s \geq 2$ such that $G$ acts faithfully on both of its orbits $\Delta_1$ and $\Delta_2$ but only acts quasiprimitively on $\Delta_1$. Suppose first that $G$ is quasiprimitive of type HC, SD or CD. Let $X = \mathrm{soc}(G)$. For $v \in \Delta_1$, $X_v$ is a subdirect subgroup of $X$ and as $T$ is a nonabelian finite simple group we have $X_v \cong T^r$ for some $r \geq 1$. Suppose that $X_v^{\Gamma(v)} = 1$. As $X$ does not act regularly on $\Delta_1$, Lemma 5.6.2 implies that $X_w^{\Gamma(w)} \neq 1$ for all $w \in \Delta_2$. Let $w \in \Gamma(v)$ such that $X_w$ moves $v$, so $X_w \neq X_v$. Since $X_v^{\Gamma(v)} = 1$ it follows that $X_v < X_w$ and $X_w$ is also a subdirect subgroup of $X$. Thus $X_w \cong T^l$ for some $l > r$; otherwise, $X_w = X_v$. Since $X_w^{\Gamma(w)}$ is a nontrivial normal subgroup of the 2-transitive group $G_w^{\Gamma(w)}$, by Burnside's Theorem 5.4.8, we have $X_w^{\Gamma(w)} = T$ and $X_w^{\Gamma(w)}$ is a primitive group. Thus $X_w \cong T^{r+1}$ and $(X_w)_v = X_v \cong T^r$. Since the kernel of the action of $X_w$ on $\Gamma(w)$ is contained in $(X_w)_v$, it follows that $(X_w)_v$ is equal to the kernel which implies $X_w^{\Gamma(w)}$ is regular. This contradicts the primitivity of $X_w^{\Gamma(w)}$, and we deduce that $X_v^{\Gamma(v)} \neq 1$. As $G$ is quasiprimitive on $\Delta_1$, $X$ is transitive on $\Delta_1$. Since $X_v^{\Gamma(v)} \neq 1$, Lemma 5.4.6 implies that $X$ is transitive on $\Delta_2$. Since $G$ is not quasiprimitive on $\Delta_2$, then $X = \mathrm{soc}(G)$ is not a minimal normal subgroup of $G$ and hence $G$ has type HC. Since $X_v \cong T^r$ we have $X \cong T^{2r}$ with $r \geq 2$. As $X_v^{\Gamma(v)}$ is a subgroup of the 2-transitive group $G_v^{\Gamma(v)}$, by Burnside's Theorem 5.4.8, we have $X_v^{\Gamma(v)} \cong T$. But $X_v^{\Gamma(v)}$ is a minimal normal subgroup of $G_v$ and since it acts nontrivially on $\Gamma(v)$, it acts faithfully on $\Gamma(v)$. Thus $T^r \cong X_v \cong X_v^{\Gamma(v)}$ which is a contradiction since $r \geq 2$ and we assumed $r \geq 1$. Thus $G$ is of type HA, HS, AS, PA or TW, and examples for these types exist and can be found in [13, Section 4]. □

# 6 TWISTED WREATH PRODUCT EXAMPLES

In this chapter, we describe a construction of locally 2-arc transitive stars that admit a quasiprimitive automorphism group of TW type on one orbit. We present the construction of the smallest such graph with automorphism group $A_5 \operatorname{twr}_\phi A_6$. Next, we prove that this is the only possible locally 2-arc transitive graph coming out of this group by characterizing the stabilizer of an arc.

Using the construction, we also give a conjecture for the existence of an infinite family of graphs that admit $\mathrm{PSL}(2,p) \operatorname{twr}_\phi \mathrm{ASL}(2,p)$ as an automorphism group. Computer calculations in GAP [12] prove that the conjecture holds for $p$ equal to 5 and $p$ equal to 7.

## 6.1 CHARACTERIZING EXAMPLES

**Theorem 6.1.1.** *Let $T$ be a finite nonabelian simple group. Let $P$ be a group and set $Q = P_n$ such that there exists $\phi : Q \to \mathrm{Aut}(T)$. Let $G = T \operatorname{twr}_\phi P$. Let $|P : Q| = n$ and for $i = 1, \ldots, n-1$, define $Q_i = N_Q(T_i)$. Suppose that there exists a normal elementary abelian subgroup $V_i$ in $Q_i$. Define $R_i = \{f \in T_i : f(z_i) \in V_i\}$. Then $M = R_1 \times R_2 \times \ldots \times R_{n-1}$ is normalized by $Q$.*

*Proof.* Let $\mathcal{T}$ be a transversal for $Q$ in $P$ such that $z_i : i \to n$ as in the proof of Lemma 4.3.6 and define $T_i$ as in Lemma 4.3.1. By previous lemmas in Chapter 4, we have that the $\phi$-base group $B_\phi$ has order $|T|^n$ and each $T_i \cong T$. We know that $P$ acts on the set $\{T_1, \ldots, T_n\}$ and $Q$ is the stabilizer of $T_n$. The action is conjugation so the stabilizer of $T_n$ is the set $\{g \in G : T^g = T\}$ which equals $\{g \in G : gT = Tg\} = N_P(T)$, so normalizers and stabilizers coincide and we have defined $Q$ correctly.

The subgroup $Q$ also normalizes $T_1 \times \cdots \times T_{n-1}$. For $i = 1, \ldots, n-1$ let $Q_i$ be the stabilizer in $Q$ of $T_i$ and let $V_i \trianglelefteq Q_i$ be an elementary abelian group. Let $R_i = \{f \in T_i : f(z_i) \in V_i\}$. We claim that for every $q \in Q$ we have $R_i^q = R_{i^q}$. Let $x \in V_i$ and let $p \in Q$. Then by Lemma 4.3.5 $f_{i,x}^p = f_{j,x^q}$ where $p^{-1}z_i = z_j q^{-1}$ for a unique $q \in Q$ and $z_j \in \mathcal{T}$. Now

$$i^q = i^{z_i^{-1} p z_j} = n^{p z_j} = n^{z_j} = j,$$

so $z_j = z_{i^q}$.

We know that $x$ fixes $i$ since $x \in V_i \trianglelefteq Q_i$ and $Q_i$ fixes $T_i$. Thus $x^q$ fixes $i^q$. Since $x$ is in $V_i$, which is the derived subgroup of the stabilizer of $i$, then $x$ is conjugated by $q$ into the derived subgroup of the stabilizer of $i^q$, which is $V_{i^q}$. Hence $x^q \in V_{i^q}$, and so, $f_{j,x^q} = f_{i^q,x^q}$ and $x^q \in V_{i^q}$, thus $f_{j,x^q} \in R_{i^q}$. This shows that $R_i^q = R_{i^q}$ and $p$ permutes the $R_i$. Thus $Q$ normalizes $M = R_1 \times \cdots \times R_{n-1}$. Note that $R_1 \times \cdots \times R_{n-1} = \langle R_1, \ldots R_{n-1} \rangle$ since the $R_i$ commute as they are subgroups of the $T_i$, and we have already shown that the $T_i$ commute in the proof of Lemma 4.3.4. □

**Lemma 6.1.2.** *Let $G$ and $P$ as above and let* $\mathrm{soc}(G) = N$. *If $R \leq G$ such that $Q \leq R$ in $G$ such that $R$ acts 2-transitively on $[R : Q]$ and $NR \neq G$, then $R$ is of the form $N_v \rtimes Q$, where $N_v = N \cap R$.*

*Proof.* Note that $NQ \leq NR$ since $Q \leq R$. The action of $R$ on cosets of $Q$ is 2-transitive and hence primitive, so $Q$ is maximal in $R$. Now $NQ/N \cong Q$ is maximal in $G/N \cong P$, and $NQ \leq NR$ so $NR = NQ$. Since $R \leq NQ$, by Dedekind's rule 2.1.2, we can write $R = Q(N \cap R)$. As $R \neq Q$, $N_v = N \cap R$ is non-trivial. Thus $R = Q(N \cap R) = QN_v$ and finally the equality with $N_v \rtimes Q$ follows since $N_v \trianglelefteq R$. □

**Corollary 6.1.3.** *If there exists a subgroup $V$ in $M$ such that $V$ is normalized by $Q$, $Q$ acts irreducibly on $V$ and $Q$ acts transitively on the non-zero vectors of $V$, then there exists a locally 2-arc transitive graph $\Gamma$ admitting $G$ as a group of automorphisms.*

*Proof.* Suppose $V$ is a subgroup of $M$ satisfying the above properties. Let $L = P$ and let $R = V \rtimes Q$. The action of $L$ on $[L : L \cap R]$ is chosen to be 2-transitive and the action of $R$ on $[R : L \cap R]$ is 2-transitive by Lemma 6.1.2 and the fact that $Q$ acts transitively on the non-zero vectors of $V$ (see discussion about 2-transitive affine groups in Cameron's book [4, Page 110]). Then $\Gamma = \mathrm{Cos}(G, L, R)$ is locally 2-arc transitive by Part 2 of Lemma 5.2.11. The quotient graph $\Gamma_N$ is the star $K_{1,n}$ since $N$ acts transitively on $\Delta_1 = [G : L]$, but it has $n$ orbits on $\Delta_2 = [G : R]$, as $T^n \rtimes (M \times Q) = T^n \rtimes Q$ so the orbits of $N$ on $\Delta_2$ are in bijection with cosets of $Q$ in $P$. □

## 6.2 A TWISTED WREATH STAR

Let us present a locally 2-arc transitive graph of valency $\{6, 16\}$ with amalgam $(A_6, C_2^4 : A_5, A_5)$, which admits an automorphism group of twisted wreath type.

**Construction**. Let

1. $T = A_5$ (a finite nonabelian simple group),
2. $P = A_6$,
3. $Q = A_5 \leq P$ the stabilizer of point 6 in the natural action,

4. $\phi : Q \to \mathrm{Inn}T$ (an isomorphism),
5. $G = T \operatorname{twr}_\phi P \cong A_5^6 \rtimes A_6$.

Then $G$ has a normal subgroup $N \cong T^6 = A_5^6$. The action of $G$ on the set of cosets of $L = P$ is primitive of type TW because $\phi$ does not extend to a larger subgroup of $P$ and $\mathrm{Inn}T \leq \phi(Q)$. We now apply Theorem 6.1.1, where $V_i$ equals the Klein 4-group of $Q_i \cong A_4$, since we know that $V_i \trianglelefteq Q_i$. Our computer calculations in GAP [12] show that there exists $V$ in $M$ of order $4^2$ which is normalized by $Q$ and $Q$ acts irreducibly on $V$. By Corollary 6.1.3, the graph $\Gamma = \mathrm{Cos}(G, L, R)$ where $L = P$ and $R = V \rtimes Q \cong \mathrm{ASL}(2, 4)$ is locally $(G, 2)$-arc transitive with valencies 6 and 16. This is because the action of $L$ on $[L : L \cap R]$ is equivalent to the action of $A_6$ on 6 points and the action of $R$ on $[R : L \cap R]$ is equivalent to the 2-transitive action of $\mathrm{ASL}(2, 4)$ on the $2^4$ points of the affine plane $\mathrm{AG}(2, 4)$. Note that $\Gamma$ is not locally $(G, 3)$-arc transitive as given $v = L \in \Delta_1$, $w = R \in \Delta_2$ and $w' \in \Gamma(v) \setminus \{w\}$, we have $G_{w'vw} = A_4$ and there are 15 vertices in $\Gamma(w) \setminus \{v\}$ so the stabilizer $A_4$ of order 12 cannot act transitively on 15 points.

## 6.3 CONSTRUCTION FEATURES

We now explore the features of the graph constructed in Section 6.2. We constructed a locally 2-arc transitive graph with a group of automorphisms that is quasiprimitive on only one orbit of TW type. The construction relies on the definition of the coset graphs $\mathrm{Cos}(G, G_u, G_v)$ with $G = \langle G_u, G_v \rangle$ such that $G_u$ acts 2-transitively on $[G_u : G_{uv}]$ and $G_v$ acts 2-transitively on $[G_v : G_{uv}]$. Given $G = A_5^6 \rtimes A_6$, in the example we have $G_u = P = A_6$. We may now try to find the possibilities for $G_{uv}$. Since $G_u$ acts 2-transitively on $[G_u : G_{uv}]$, there are only few possibilities. We refer to the classification of finite 2-transitive groups in [4] and conclude that $G_{uv} = A_5$ or $|G_{uv}| = 36$. Then knowing $G_{uv}$, we use Lemma 6.1.2 to calculate $G_v$. Since $G_v$ must be 2-transitive on $[G_v : G_{uv}]$, the lemma shows that $G_v = N_v \rtimes G_{uv}$, where $N_v = N \cap G_v$.

As $N_v$ is a regular minimal normal subgroup of the finite 2-transitive group $G_v$, Burnside's Theorem 5.4.8 shows that $N_v$ is elementary abelian and regular, and not simple and primitive since $G_v$ is primitive on $[G_v : G_{uv}]$. Thus $|N_v| = p^d$ for some prime $p$ and $d \in \mathbb{N}$. Note that $G_{uv}$ must act irreducibly on $N_v$, because otherwise it contradicts the minimality of $N_v$. Since 2-transitivity of a group of HA type requires transitivity on non-zero vectors, $p^d - 1$ has to divide $|G_{uv}|$. The example above has $G_{uv} = A_5$ with $p^d = 2^4$. Suppose that $|G_{uv}| = 36$. We have $|N_v| \in \{2, 2^2, 5, 7, 13, 19\}$.

Given $N_v$ of order $q = p^d$, we have that $G_v/K \cong C_q \rtimes C_{q-1}$ for some $K \trianglelefteq G_{uv}$ since $q - 1$ divides $|G_{uv}|$. The group $G_{uv}$ of order 36 is isomorphic to $(S_3 \wr S_2) \cap A_6$ which has only four normal subgroups, of orders $1, 9, 18$ and $36$, respectively.

1. If $q = 4$, then $G_v/K \cong C_4 \rtimes C_3$ which means that $G_{uv}$ has a normal subgroup $K$ of order 12, thus this case is impossible.

2. If $q = 7$, then $G_v/K \cong C_7 \rtimes C_6$ which means that $G_{uv}$ has a normal subgroup $K$ of order 3, thus this case is impossible.

3. If $q = 13$, then $G_v/K \cong C_{13} \rtimes C_{12}$ which means that $G_{uv}$ has a normal subgroup $K$ of order 3, thus this case is impossible.

4. If $q = 19$, then $G_v/K \cong C_{19} \rtimes C_{18}$ which means that $G_{uv}$ has a normal subgroup $K$ of order 2, thus this case is impossible.

This leaves cases $q = 5$ with $K$ of order 9 and $q = 2$ with $K$ of order 36, since normal subgroups of order 9 and 36 exist in $G_{uv}$. By Lemma 2.1.4, $N_v \cap K = \{1\}$ in both cases. As $K, N_v \trianglelefteq G_v$ we have $\langle N_v, K\rangle = N_v \times K$ and $N_v$ and $K$ commute. Then $N_v \leq C_N(K)$.

If $|K| = 9$, then $K \cong C_3 \times C_3$, which is an abelian group, so $Z(K) = K$ and calculations in GAP [12] show that $|C_N(K)| = 9$. As $N_v \leq C_N(K)$, the order of $N_v$ divides the order of $C_N(K)$, but 5 does not divide 9, so no subgroup $N_v$ of order 5 exists in $C_N(K)$.

If $|K| = 36$, then our calculations in GAP [12] show that $C_N(K) = \{1\}$, hence no subgroup $N_v$ of order 2 exists in $C_N(K)$ so this case is not possible either. This shows that $\mathrm{Cos}(A_5 \operatorname{twr}_\phi A_6, A_6, C_2^4 : A_5)$ is the only possible example of a locally 2-arc transitive graph with a group of automorphisms equal to $A_5 \operatorname{twr}_\phi A_6$ of quasiprimitive type TW on only one orbit.

# 7 NEW EXAMPLES

We describe a construction for locally 2-arc transitive graphs of valency $p^2$ with amalgam $(\mathrm{ASL}(2,p), \mathrm{ASL}(2,p), \mathrm{SL}(2,p))$.

## 7.1 CONSTRUCTION

Let us consider the following ingredients

1. $T = \mathrm{PSL}(2,p)$ (a finite nonabelian simple group),
2. $P = \mathrm{ASL}(2,p) \cong C_{p^2} \rtimes \mathrm{SL}(2,p)$,
3. $Q = \mathrm{SL}(2,p) \leq P$,
4. $\phi : Q \to Q/Z(Q) \leq \mathrm{Aut}T$,
5. $G = T \operatorname{twr}_\phi P \cong T^{p^2} \rtimes \mathrm{ASL}(2,p)$.

Define $T_i$ as in Lemma 4.3.1. By previous lemmas we have that the $\phi$-base group has order $|\mathrm{PSL}(2,p)|^{p^2}$ and each $T_i \cong T$. We know that $P$ acts on the set $\{T_1, \ldots, T_{p^2}\}$ and $Q$ is the stabilizer of $T_{p^2}$. The subgroup $Q$ also normalizes $T_1 \times \cdots \times T_{p^2-1}$.

For $i = 1, 2, \ldots, p^2 - 1$ let $Q_i$ be the stabilizer in $Q$ of $T_i$. Since $Q = \mathrm{SL}(2,p)$, then $Q_i$ is the stabilizer of a vector. Since stabilizers are conjugate, we may calculate the stabilizer of an arbitrary vector. Let $v = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Then $A = (a_{ij}) \in \mathrm{GL}(2,p)$ stabilizes $v$ if and only if $Av = v$ which means

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{7.1}$$

Since $A$ is invertible and $A \in \mathrm{SL}(2,p)$, $a_{22} = 1$, so the stabilizer of $v$ in $\mathrm{SL}(2,p)$ is the subgroup $\left\langle \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right\rangle$, where $a \in \mathrm{GF}(p)$. Then $Q_i \cong C_p$ for each $i$. Since $Q_i$ is a cyclic group of prime order, it does not contain any proper nontrivial normal subgroups. We thus consider $M = Q_1 \times \cdots \times Q_{p^2-1} \cong C_p^{p^2}$ which is normalized by $Q$ by Theorem 6.1.1.

Suppose there exists $X$ in $M$ of order $p^2$ which is normalized by $Q$. Then we utilize Corollary 6.1.3, with $R = X : Q \cong C_{p^2} \rtimes \mathrm{SL}(2,p) \cong P$.

Since $Q \leq P$ and $Q \leq R$, we have that $Q \leq P \cap R$ and $P \cap R$ is contained in $P$ and in $R$. Now $Q$ is maximal in $P$ since the action is 2-transitive and hence primitive, so

$P \cap R$ equals $P$ or $Q$. If $P \cap R = P$, then $P = R$ since they have the same order. Then as $R = N_v \times Q$, we have $N_v \leq P$ which is non-trivial, as $N$ is non-trivial. However, $P$ is a complement to $N$ in $G$, so $N \cap P$ is trivial. Therefore, $L \cap R = P \cap R = Q$.

The group $\mathrm{ASL}(2,p)$ acts 2-transitively on the $p^2$ points of $[\mathrm{ASL}(2,p) : \mathrm{SL}(2,p)]$, so $\Gamma = \mathrm{Cos}(G, L, R)$ is locally $(G,2)$-arc transitive and regular of valency $p^2$.

**Conjecture 7.1.1.** *For every prime $p$, with $p \geq 5$, there exists $X$ in $M$ of order $p^2$ which is normalized by $Q$.*

If $p$ equals 5 or 7, our calculations in GAP in the appendices show that the subgroup $V$ exists, and we expect that it exists for all primes $p \geq 5$. The proof of this result requires a significant amount of representation theory, so we delay it to a future project.

Assuming that the conjecture is true, the described construction provides an infinite family of locally 2-arc transitive graphs that admit a quasiprimitive group of TW type as an automorphism group.

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**Algorithm 1:** Pseudocode of the construction

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**Data:** a prime $p$

**Result:** a locally 2-arc transitive graph with a quasiprimitive group of automorphisms of TW type on only one orbit

initialization;

$T = \mathrm{PSL}(2,p)$;

$P = \mathrm{ASL}(2,p)$;

$G = T \operatorname{twr}_\phi P = (T_1 \times T_2 \times \cdots \times T_{p^2}) \rtimes P$;

$Q = N_P(T_{p^2})$;

$Q_i = N_Q(T_i)$, for $i = 1, \ldots, p^2 - 1$;

$R_i \leq T_i$ such that $R_i \cong Q_i$ and $R_i^q = R_{i^q}$, for $q \in Q$ and $i = 1, \ldots, p^2 - 1$;

$M = \langle R_1, R_2, \ldots, R_{p^2-1} \rangle \cong C_p^{p^2-1} \trianglelefteq \langle M, Q \rangle$;

**if** $V \leq M$ *of order* $p^2$ *such that* $V \trianglelefteq \langle M, Q \rangle$ **then**

  construct $R = V \rtimes Q$;

  construct the coset graph $\Gamma = \mathrm{Cos}(G, L, R)$ with amalgam $(L, R, Q)$

---

**Lemma 7.1.2.** *The constructed graph* $\Gamma = \mathrm{Cos}(G, L, R)$ *is locally* 2*-arc transitive and not locally* 3*-arc transitive.*

*Proof.* By construction, $\Gamma$ is a coset graph which is locally 2-arc transitive. By Lemma 5.4.7, $s \leq 3$ since $N$ has at least three orbits on $\Delta_2$. By definition of cost graphs, $P$ is a point stabilizer in $G$. So let $u = L \in \Gamma$, and let $v \sim u$. The stabilizer of a vector in $\mathrm{SL}(2,p)$ is isomorphic to $C_p$ as calculated in Equation 7.1.

In $\mathrm{GL}(2,p)$ the determinant need not be equal to 1 but it still has to be non-zero, so we have $p-1$ choices for $a_{22}$ (exluding 0) and $p$ choices for $a_{12}$ in Equation 7.1. So, the stabilizer of the vector $v = \begin{bmatrix}1 & 0\end{bmatrix}^T$ in $\mathrm{GL}(2,p)$ is of the form

$$\left\langle \begin{bmatrix}1 & a_{12}\\ 0 & a_{22}\end{bmatrix} : a_{12}, a_{22} \in \mathrm{GF}(p),\ a_{22} \neq 0 \right\rangle, \tag{7.2}$$

and has order $p(p-1)$ in $\mathrm{GL}(2,p)$. Let $w \sim v$ and let us find the stabilizer of $w$. Suppose $w = \begin{bmatrix}x & y\end{bmatrix}^T$, with $x, y \in \mathrm{GF}(p)$. Then

$$\begin{bmatrix}1 & a_{12}\\ 0 & a_{22}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}x + a_{12}y\\ a_{22}y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}. \tag{7.3}$$

The second equation yields $a_{22} = 1$ since 1 is the unique multiplicative identity in $\mathrm{GF}(p)$. The first equation yields $a_{12}y = 0$, and $\mathrm{GF}(p)$ is a field so it has no divisors of zero. This shows that either $a_{12} = 0$ or $y = 0$. If $y = 0$, then $w$ is in the vector subspace spanned by $x$, and the stabilizer of this subspace is given in Equation 7.2 since we must fix all vectors of the form $\begin{bmatrix}x & 0\end{bmatrix}^T$. If $a_{12} = 0$, then we have the identity matrix. In either case, the stabilizer is of order at most $p^2 - p$. There are $p^2 - 3$ vertices in $\Gamma(w) \setminus \{v\}$, so the stabilizer of a 2-arc in $\Gamma$ cannot be transitive on the set of neighbors of $w$. □

## 7.2 THE AUTOMORPHISM GROUPS OF THE NEW GRAPHS

Let us summarize some information about the construction in Chapter 7.

**Theorem 7.2.1.** *The graph* $\Gamma = \mathrm{Cos}(G, L, R)$ *constructed in Chapter 7 has the following properties:*

1. $\Gamma$ *is bipartite with* $\Delta_1 = [G : L]$ *and* $\Delta_2 = [G : R]$.

2. $|V\Gamma| = 2|\Delta_1| = 2|T|^{p^2} = 2p^{p^2}\left(\frac{p^2-1}{2}\right)^{p^2}$.

3. $\Gamma$ *is regular of valency* $p^2$.

4. $\Gamma$ *is locally* $(G, 2)$*-arc transitive but not locally* $(G, 3)$*-arc transitive where* $G = \mathrm{PSL}(2, p) \operatorname{twr}_\phi \mathrm{ASL}(2, p)$.

*Proof.* The first three properties follow from the coset graph construction as shown in Lemma 5.2.11 and the last property is proved in Lemma 7.1.2. □

Let $A = \mathrm{Aut}(\Gamma)$. We know that $G \leq A$. Define $A^+$ to be the stabilizer of $\Delta_1$ in $A$. Since $A^+$ stabilizes $\Delta_1$, it stabilizes $\Delta_2$ as well. As $G$ is vertex intransitive, we have $G \leq A^+$, and $G$ is locally 2-arc transitive, so $A^+$ is locally $s$-arc transitive for some $s \geq 2$. We would like to answer the following questions:

1. Is $A^+$ quasiprimitive on $\Delta_1$, on $\Delta_2$, on neither or on both?

2. Are $A^+$ and $G$ equal?

If $A^+$ is quasiprimitive on at least one orbit, then since $s \geq 2$, these are all the possible cases for the action of $A^+$:

1. By Theorem 5.6.1, $A^+$ acts quasiprimitively only on one orbit. The quasiprimitive type is HA, HS, AS, PA or TW.

2. By Theorem 5.5.1 and adopting notation, $A^+$ acts quasiprimitively on both orbits of type $X = Y$, where $X$ is of type HA, TW, AS or PA, or $\{X, Y\}$ ={SD, PA}.

We now explore and rule out some possibilities for $A^+$, which follow from properties of the graph.

Consider first the case where $A^+$ acts quasiprimitively only on one orbit. In [14], Giudici, Li and Praeger further characterize locally $s$-arc transitive graphs with a star normal quotient, namely, the ones where we have a quasiprimitive action only on one orbit.

1. The HS case is fully determined in [14, Construction 3.2] and suitable choices for the construction are shown in [14, Table 1.1]. There are five infinite families that come out of this construction with possible valencies $q^2+1$ and $q$ for $q=2^l$ and $l \geq 3$ odd, $q^3+1$ and $q$ for $q=3^l$ and $l \geq 3$ odd, $q^3+1$ and $q$ for $q \geq 3$, $(q^d-1)/(q-1)$ and $q^{d-1}$, $q+1$ and $q$ for $q \geq 4$. It is clear that valencies for each part of the bipartition are different, so this case is impossible.

2. The HA case is also completely determined by [14, Construction 4.1], where the order of $\Delta_1$ equals the order of a $d$-dimensional vector space over $\mathrm{GF}(q)$, while the order of $\Delta_2$ equals the number of $m$-dimensional vector spaces of $V$ which are permuted by the complement to the vector space in the group. It follows that $\Delta_1$ and $\Delta_2$ have different orders, so this case cannot happen either.

3. The AS case is described in [14, Theorem 1.3] and [14, Table 1.2], where it is shown that the socle $T$ is $\mathrm{PSL}(n,q)$, $n \geq 3$, $\mathrm{PSU}(n,q)$, $n \geq 3$, $P\Omega^+(8,q)$, $P\Omega^+(8,q)$ with $q$ odd, $E_6(p^f)$ or ${}^2E_6(p^f)$. The number of orbits on $\Delta_2$ is denoted by $k$. The table shows that $k$ is either an odd prime, equal to 3, equal to 4, divides $\gcd(n,q-1)$ or divides $\gcd(n,q+1)$. In our examples, $k$ equals $p^2$ so $k$ never satisfies these criteria, and we rule out the AS case.

4. The PA case is described in [14, Example 5.1] which is also discussed in the global analysis [13, Example 4.3]. It constructs a family of locally 3-arc transitive graphs of valencies $k$ and $n$ with amalgam

$$(S_{n-1} \wr S_k, S_n \times (S_{n-1} \wr S_{k-1}, S_{n-1}^k : S_{k-1}).$$

The vertex set consists of all $k$-tuples from a set $\Omega$ of size $n$. Since we have constructed regular graphs, we would have $k=n$ and the number of vertices equals $n^n$. Since $T=\mathrm{PSL}(2,p)$, we have $|T| = p\frac{p^2-1}{2}$.

Let $(m)_q$ denote the highest power of $q$ that divides $m$. We have $(|T|^{p^2})_p = p^2$, since $p$ does not divide $p^2-1$. Also, $(n^n)_p = (n)_p \cdot n$. If $(n)_p = a$ then $n = p^a r$ where $\gcd(p,r)=1$. Then

$$p^2 = (|T|^{p^2})_p = (n^n)_p = a \cdot p^a \cdot r,$$

which implies that $r=1$ since otherwise $r$ would divide $p^2$. However, if $r=1$, then $n=p^a$ which is a contradiction because $\gcd(p^2-1,n)>1$. Hence this case is also not possible.

5. Except Example 6.2, no other examples of locally 2-arc transitive graphs that admit a quasiprimitive action of TW type on only one orbit are presented in the available literature. If $A^+$ acts in such a way, the best case scenario is that $A^+=G$ and our construction is new, so we do not attempt to rule this case out.

We have ruled out all but one case where $A^+$ acts quasiprimitively on only one orbit, so we now consider the case where $A^+$ acts quasiprimitively on both orbits. Let us first examine the possible type {SD, PA}.

In [15], Giudici, Li and Praeger give a complete classification of locally $s$-arc transitive graphs where the action of the automorphism group is quasiprimitive of type {SD, PA}. More precisely, [15, Theorem 1.1] describes the general construction. Parts (4) and (5) of this theorem state that $v \in \Delta_1$ has $(q^n-1)/(q-1)$, $q^3+1$, $q^3+1$ or $q^2+1$ neighbors, depending on $T$, while $w \in \Delta_2$ has $q^d$ neighbors. The two valencies are different, so this case is impossible since our constructed graph is regular.

If $A^+$ acts quasiprimitively of the same type on each orbit, we have the four possibilities HA, AS, TW or PA.

The HA case can also be ruled out immediately since the size of the vertex set of the graph must be a prime power [16].

The AS case with $s = 2$ where the group is of Ree type has been completely classified in [7] via three infinite families of locally 2-arc transitive graphs. It is shown in [7, Theorem 1.1] that all vertex-intransitive locally 2-arc transitive graphs admitting a Ree simple group $G = Ree(q)$ where $q = 3^{2n+1} \geq 27$ either belong to these families (which are described in [7, Table 1]) or they arise as standard double covers of connected $(G, 2)$-arc transitive graphs classified in [8]. The possible valencies for standard double covers given in [7, Theorem 1.1] are 4, 8, $3^e$ with $e \geq 1$ and $e$ divides $2n+1$ or $3^e > 3$, which clearly do not match the valencies of our constructed graphs. Further, [7, Table 1] lists valencies 4, 7, and 8 for the infinite families, so we can rule out these possibilities. In [8, Theorem 1.1], for $q = 3^{2n+1}$, $n \geq 1$, all connected $(G, 2)$-arc transitive graphs admitting a Ree simple group are classified and the possible valencies are 3, 4, 8 or $3^l$ with $l \mid (2n+1)$, which again can be ruled out.

The AS case for locally $s$-arc transitive graphs with $s \geq 2$ that admit a group of Suzuki type has been worked out in [27]. The classification is summarized in [27, Theorem 1.2], where it is shown that any such graph is either vertex transitive and classified in [6], it is the standard double cover of a graph in [6], or the graph is bipartite and various conditions are given for the valencies in each part. The possible valencies are either different or odd primes, which is not the case with our construction. Also, in [6] the valency is shown to be an odd prime or a power of 2, thus we can dismiss this case entirely.

In [17], the AS case where the group is one of 14 sporadic simple groups has been determined, and can be ruled out via valencies that are pointed out in [17, Tables 4-5]. Actually, we can prove more for our construction. Since $p \geq 5$, $p \equiv 1$ or $-1 \pmod 3$. Then $p^2 - 1 \equiv 0 \pmod 3$, which shows that 3 divides the order of $|T|$ and thus $3^{p^2}$ divides the order of $G$. Hence, we refer to the orders of sporadic groups available in [32, 33], and we conclude that $G$ does not fit into any of the sporadic groups, so we can rule them out completely.

Baddeley [3] gave a characterization of 2-arc transitive graphs that admit a quasiprimitive group of twisted wreath type. The construction is done in terms of admissible maps and Cayley graphs [3, Construction 4.1]. The vertex set of the graph is identified with the base group, so it has size $|S|^k$ for some simple group $k$ and $k > 1$, the vertex set of our graph would need to have size $|T|^{p^2}$. By Theorem 7.2.1 we know that this is not the case for the group $G$, but if $A^+$ satisfies this condition, then

$$2|T|^{p^2} = |S|^k, \tag{7.4}$$

for some $k > 1$ and $S$ a simple group. Here $k > 1$ because the quasiprimitive action is of twisted wreath type. Since $p$ is a prime at least 5, we have $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$, so $\frac{p^2-1}{2}$ is even and we can write it as $2^m n$ for some natural numbers $m$ and $n$ with $\gcd(2, n) = 1$ and $m \geq 1$. By rearranging Equation 7.4, we have

$$2^{mp^2+1} p^{p^2} n^{p^2} = |S|^k.$$

Note that $p^2 - 1 = 2^{m+1} n$ so $p$ cannot divide $n$ for $p \geq 5$. Since $S$ is a finite nonabelian simple group, its order is even by the Feit-Thompson Theorem [9, 10], so $(|S|)_2 = 2^a$ for some natural number $a$. Then $2^{ak} = (|S|^k)_2 = 2^{mp^2+1}$. Therefore, alongside the fact that $p$ is a prime, $k$ must equal $p$ or $p^2$ but neither $p$ nor $p^2$ divide $mp^2 + 1$, so this case is impossible.

The rest of the AS cases, the TW and PA cases have not been characterized any further in available literature so we do not have sufficient information to rule them out. Our twisted wreath product group embedded in a bigger twisted wreath product group or almost simple group which acts on the same graph seems infeasible and thus leads us to believe that the construction is new.

We cannot compute the full automorphism group of these graphs yet, thus this problem remains open. We summarize this discussion as follows.

**Theorem 7.2.2.** *Let* $\Gamma$ *be the graph* $\mathrm{Cos}(G, L, R)$ *constructed in Chapter 7. Then one of the following holds:*

1. $\Gamma$ *is a vertex transitive graph, and hence* $(\mathrm{Aut}(\Gamma), 2)$*-transitive.*
2. $\Gamma$ *is a standard double cover of a graph from Part 1.*
3. $\mathrm{Aut}(\Gamma)$ *is not quasiprimitive on either orbit of* $\Gamma$*.*
4. $\mathrm{Aut}(\Gamma)$ *is quasiprimitive on only one orbit of* $\Gamma$ *of twisted wreath type.*
5. $\mathrm{Aut}(\Gamma)$ *is quasiprimitive on both orbits of* $\Gamma$ *of almost simple type (excluding groups of Ree type Ree*$(q)$*, where* $q = 3^{2n+1}$ *for* $n \geq 1$*, groups of Suzuki type and sporadic groups), or product action type.*

# 8 CONCLUSION

In this thesis we thoroughly described locally $s$-arc transitive graphs which admit automorphism groups of quasiprimitive type. Since the graphs are edge-transitive but vertex-intransitive, the vertex set is split into two orbits, called the bipartition parts. If any two arcs of length $s$ emanating from a single vertex $v$ can be mapped to each other by some automorphism which fixes $v$, then the graph is said to be locally $s$-arc transitive.

If the group acting on a graph has a nontrivial normal subgroup that is intransitive on both bipartition parts, then the graph arises as a cover of a smaller locally $s$-arc transitive graph. This means that we can only consider the case where $G$ acts quasiprimitively on at least one of the parts of the bipartition, thus we gave a detailed description of the classification of quasiprimitive groups by following Praeger's [21] O'Nan-Scott type theorem.

Giudici, Li and Praeger [13] initiated a global analysis of these graphs and their properties. We gave an overview of this analysis and proved Theorem 5.5.1 and Theorem 5.6.1 which classify possible quasiprimitive actions on locally arc transitive graphs. If the action is quasiprimitive on both orbits, then the possible types are HA, TW, AS or PA, or {SD, PA}. If the action is quasiprimitive on only one orbit, then the possible types are HA, HS, AS, PA or TW.

We then focused on the twisted wreath case. We described a construction and stated a conjecture for the existence of an infinite family of locally 2-arc transitive graphs which admit a group of automorphisms that acts quasiprimitively of twisted wreath type on only one orbit. The groups of automorphisms that the graphs admit are $\mathrm{PSL}(2,p)\ \mathrm{twr}_\phi\ \mathrm{ASL}(2,p)$, where $p$ is a prime and $p \geq 5$. The conjecture was verified for $p = 5$ and $p = 7$ using GAP calculations, which can be found in the appendices.

We finally discussed possible automorphism groups of these graphs. We examined the stabilizer of the bipartition and posed a few questions which are useful to characterize the automorphism group. Many cases were ruled out but the problem of calculating the full automorphism group of these graphs remains open for future research.

# 9 DALJŠI POVZETEK V SLOVENSKEM JEZIKU

V magistrski nalogi natančno opišemo $s$-ločno tranzitivne grafe, ki imajo kvaziprimitivno grupo avtomorfizmov. Ker ti grafi niso vozliščno tranzitivni, vendar so povezavno tranzitivni, je njihova množica vozlišč razdeljena na dve orbiti. Za graf pravimo, da je lokalno $s$-ločno tranzitiven, če za poljubna loka dolžine $s$, ki izhajata iz vozlišča $v$ obstaja avtomorfizem, ki loka med seboj preslika in fiksira vozlišče $v$. Giudici, Li in Praeger [13] v svojem delu začnejo z globalno analizo teh grafov in njihovih lastnosti. V magistrski nalogi predstavimo analizo teh grafov in se nato osredotočimo na primer zasukanega venčnega tipa.

V 2. poglavju podamo osnovne definicije iz teorije grup, predstavimo razne produkte grup in definiramo podstavek (*angl.* socle), ki je koristno orodje v O'Nan-Scottovem izreku (glej O'Nana, Scotta [25] in Aschbacherja [1]), ki klasificira primitivne grupe in v izreku o O'Nan-Scottovem tipu (glej Praeger [21]), ki klasificira kvaziprimitivne grupe.

Če ima grupa, ki deluje na graf, netrivialno normalno podgrupo, ki je netranzitivna na obeh orbitah vozlišč, je graf pokritje manjšega lokalno $s$-ločno tranzitivnega grafa, zato obravnavamo samo primer, ko $G$ deluje kvaziprimitivno na vsaj enega od obeh delov. V 3. poglavju podrobno opišemo klasifikacijo kvaziprimitivnih grup, ki sledi Praegerjevi obravnavi. Ti tipi so sestavljeni iz podgrup holomorfov Abelovih preprostih (*angl.* holomorph of an abelian group) ali sestavljenih grup (*angl.* holomorph of a simple group), podgrup zasukanega venčnega produkta, skoraj preprostih grup (*angl.* almost simple group), preprostih in sestavljenih diagonalnih grup in grup produktov delovanj(*angl.* product action group).

V 4. poglavju podamo natančen oris primera zasukanega venčnega tipa, kjer tudi dokažemo nekaj rezultatov, ki jih uporabljamo skozi celotno nalogo. Naš cilje je podati poglobljen pregled teh grup, saj so le te zapletene ter zato manj raziskane kot druge kvaziprimitivne grupe. Informacije o kvaziprimitivnih grupah povzamemo v tabeli 1.

V 5. poglavju navedemo nekaj osnovnih definicij iz področja teorije grafov in opišemo postopek konstrukcije povezavno tranzitivnih grafov. Glavne konstrukcije vključujejo standardna dvojna pokritja in kosetne grafe. Karakteriziramo lokalno ločno tranzitivne grafe s stopnjo vozlišč največ tri in ponazorimo nekaj primerov v zvezi z dvo-razdaljnimi grafi in z grafi pridobljenimi iz predhodnih, tako da na sredino vsake povezave dodamo dodatno vozlišče. Nato opišemo postopek pridobivanja kvocientinh grafov glede na normalno podgrupo grupe avtoorfizmov, ki deluje netranzitivno na množico vozlišč. Ta metoda se uporablja za analizo primerov, ko grupa avtorfizmov grafa deluje kvazimprimitivno na vsaj eni orbiti vozlišč grafa. Izreka 5.5.1 in 5.6.1 povzemata možne vrste kvaziprimitivnih delovanj, odvisno od tega, ali grupa deluje na obe orbiti ali samo na eno. Če je delovanje kvaziprimitivno na obeh orbitah, so možne vrste HA, TW, AS ali PA ali {SD, PA}. Če je delovanje kvaziprimitivno samo na eni orbiti, so možne vrste HA, HS, AS, PA ali TW.

V 6. poglavju podamo konstrukcijo lokalno 2 -ločno tranzitivnih grafov, ki premorejo grupo avtomorfizmov, ki kvaziprimitivno deluje le na eno orbito zasukanega venčnega tipa. Za natančno predstavitev konstrukcije najprej dokažemo izrek 6.1.1 in lemo 6.1.2. V izreku 6.1.1 dokažemo, da če je $G = T \operatorname{twr}_{\phi} P = (T_1 \times \cdots T_n) \rtimes P$ in če obstaja normalna elementarna Abelova podgrupa $V_i$ v normalizatorju vsake grupe $T_i$ znotraj normalizatorja $T_n$, potem $Q$ normalizira $R_1 \times \cdots \times R_{n-1}$, kjer je $R_i$ podgrupa $T_i$, izomorfna $V_i$. V lemi 6.1.2 karakteriziramo podgrupe $R$ od $G$, ki delujejo 2-tranzitivno na $[R : Q]$. Z uporabo izreka 6.1.1 in leme 6.1.2 dokažemo posledico 6.1.3, ki predstavlja konstrukcijo lokalno 2-ločno tranzitivnega grafa, čigar grupa avtomorfizmov je $G$.

V 7. poglavju podamo domnevo o obstoju neskončne družine takih grafov in jo preverimo za dva primera, z uporabo računskega programa v jeziku GAP, ki je priložen v prilogah. Domneve ne dokažemo, saj se opira na teorijo predstavitev (*angl.* representation theory). Tako domneva ostaja odprta za prihodnja raziskovanja. Grafi predstavljeni v tem poglavju premorejo $\mathrm{PSL}(2, p) \operatorname{twr}_{\phi} \mathrm{ASL}(2, p)$ kot grupo avtorfizmov in so $p^2$-regularni. Postopek pridobivanja grafov je prikazan v algoritmu 1. Dokažemo tudi, da ti grafi niso lokalno 3-ločno tranzitivni (glej lemo 7.1.2). V razdelku 7.1 razpravljamo o možnih podgrupah avtomorfizmof teh grafov. Študiramo stabilizator dvodelne množice vozlišč in postavimo nekaj vprašanj, ki so koristna za karakterizacijo grupe avtoorfizmov. Številni primere lahko izključimo, vendar je problem izračuna celotne grupe avtorfizmov teh grafov odprt za prihodnje raziskave.

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# Appendices

# APPENDIX A GAP code

## A.1 $G_{uv} = (S_3 \wr S_2) \cap A_6 \leq A_5 \operatorname{twr}_\phi A_6$

```
g:=AlternatingGroup(6);;
h:=Stabiliser(g,1);;
hom1:=FactorCosetAction(g,h);;
k:=Stabiliser(h,2);;
hom2:=FactorCosetAction(h,k);;
p:=Group(KuKGenerators(g,hom1,hom2));

A5 := AlternatingGroup(5);;
A6 := AlternatingGroup(6);;
W := WreathProduct(A5,A6);;
P := AsSubgroup(W, p);; # P inside A_5^6 wr A_6
G := WreathProduct(SymmetricGroup(3), SymmetricGroup(2));;
Guv := Intersection(G, AlternatingGroup(6));;

# now we try to find Guv inside P
L:=[];;
for subp in AllSubgroups(P) do
       if IsIsomorphicGroup(Guv, subp) then Add(L,subp); fi; od; # all
    Guv_i in P

for g in L do
  Display(Size(Centralizer(Socle(W), g))); od; # centralizer of Guv_i
   in socW -> trivial

L2 := [];;
for group in L do
  for subgp in AllSubgroups(group) do
    if Size(subgp) = 9 and IsNormal(group, subgp) then Add(L2, subgp);
    fi; od; od;
for g in L2 do
  Display(Size(Centralizer(Socle(W), g))); od; # centralizer of K in
   Guv_i in socW -> order 9
```

## A.2 $G = A_5 \operatorname{twr}_\phi A_6$

```
A5 := AlternatingGroup(5);;
A6 := AlternatingGroup(6);;
W := WreathProduct(A5,A6);; # wreath product

g:=AlternatingGroup(6);;
h:=Stabiliser(g,1);;
hom1:=FactorCosetAction(g,h);;
k:=Stabiliser(h,2);;
hom2:=FactorCosetAction(h,k);;
p:=Group(KuKGenerators(g,hom1,hom2)); # p as P in the twisted wreath

P := AsSubgroup(W, p);; # P inside A_5^6 wr A_6

J := [];;
for k in [1..30] do
  if k mod 5 = 1 then Add(J,k); fi; od;

Ti := [];; # T_i is the group in position i, each isomorphic to A_5
for i in J do
  g1 := (i, i + 1, i + 2);
  g2 := (i, i + 1, i + 2, i + 3, i + 4);
  Add(Ti, Group(g1, g2));
od;

Q := Normalizer(P, Ti[6]);;
Size(Q); # 60 = |A_5|

Qi := [];;

for t in Ti{[1..5]} do
  Add(Qi, Normalizer(Q, t)); od; # each Qi is isomorphic to A_4

# define Ri as the subgp of Ti iso to Vi, normalized by Qi

Ri := [];; # Ri=Qi in Ti
for i in [1..5] do
  for subgp in AllSubgroups(Ti[i]) do
  if (IsNormal(Qi[i], subgp) and Size(subgp)=4) then
Add(Ri, subgp); fi; od; od;

gens_of_ri := [];;
for r in Ri do
  gen:=GeneratorsOfGroup(r);
  Add(gens_of_ri, gen[1]);
```

```
  Add(gens_of_ri, gen[2]);
od; # list with all gens of Ri

M := Group(gens_of_ri);;

MQ := Group(Concatenation(GeneratorsOfGroup(Q), GeneratorsOfGroup(M)))
    ;;

# in Magma, asked for NormalSubgroups

MQ :=  PermutationGroup<30 | (1,5)(2,3),(1,3)(2,5),(6, 9)(8,10),(6, 8)
    (9,10),(11,15)(12,14),(11,12)(14,15),(16,19)(17,18),(16,17)(18,19)
    ,(22,24)(23,25),(22,25)(23,24),(2,5,3)(6,24,11)(7,21,13)(8,22,14)
    (9,25,12)(10,23,15)(16,19,17)(26,27,28),(1,2,5)(6,24,18)(7,21,20)
    (8,25,19)(9,23,17)(10,22,16)(12,14,15)(27,28,29),(1,6,17)(2,10,18)
    (3,8,19)(4,7,20)(5,9,16)(11,12,15)(22,23,24)(27,29,30)>;

#NormalSubgroups(MQ);

# R0 of order 16:
R0 := Group( (1, 2)(3, 5)(11, 15)(12, 14)(16, 19)(17, 18)(22, 25)(23,
    24),(1, 5)(2, 3)(6, 9)(8, 10)(11, 14)(12, 15)(22, 25)(23, 24),(6,
    8)(9, 10)(11, 12)(14, 15)(16, 18)(17, 19)(22, 25)(23, 24), (6, 9)
    (8, 10)(11, 15)(12, 14)(16, 17)(18, 19)(22, 24)(23, 25));;
IsNormal(Q,R0); #true

QR := Group(Concatenation(GeneratorsOfGroup(Q), GeneratorsOfGroup(R0))
    );;

phiQRtoQ:=FactorCosetAction(QR,Q);
Transitivity(Image(phiQRtoQ)); # returns 2
```

## A.3 $G = A_5 \operatorname{twr}_\phi \mathrm{ASL}(2,5)$

```
LoadPackage("SONATA");; # to check isomorphism of groups

g1 := (2, 3, 5, 4)(6, 16, 21, 11)(7, 18, 25, 14)(8, 20, 24, 12)(9, 17,
    23, 15)(10, 19, 22, 13);;
g2 := (2, 10, 21)(3, 14, 16)(4, 18, 11)(5, 22, 6)(7, 9, 17)(8, 13, 12)
   (15, 25, 23)(19, 20, 24);;
g3 :=  (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18,
    19, 20)(21, 22, 23, 24, 25);;
K := Group(g1,g2,g3);; # isomorphic to ASL(2,5)
T := AlternatingGroup(5);; # A_5
W := WreathProduct(T, K);; # W = T twr K

h1 := (1, 4)(2, 3)(6, 11, 21, 16)(7, 12, 22, 17)(8, 13, 23, 18)(9, 14,
    24, 19)(10, 15, 25, 20)(26, 80, 101, 52)(27, 76, 102, 55)(28, 77,
    103, 54)(29, 78, 104, 51)(30, 79, 105, 53)(31, 90, 121, 70)(32,
   86, 122, 66)(33, 87, 123, 67)(34, 88, 124, 68)(35, 89, 125, 69)
   (36, 100, 116, 60)(37, 96, 117, 56)(38, 97, 118, 57)(39, 98, 119,
   58)(40, 99, 120, 59)(41, 82, 115, 72)(42, 85, 113, 75)(43, 83,
   111, 73)(44, 84, 112, 74)(45, 81, 114, 71)(46, 95, 106, 65)(47,
   91, 107, 61)(48, 92, 108, 62)(49, 93, 109, 63)(50, 94, 110, 64);;
h2 := (3, 4, 5)(6, 46, 101)(7, 47, 102)(8, 48, 103)(9, 49, 104)(10,
   50, 105)(11, 66, 76)(12, 67, 77)(13, 68, 78)(14, 69, 79)(15, 70,
   80)(16, 86, 55)(17, 87, 54)(18, 88, 51)(19, 89, 53)(20, 90, 52)
   (21, 106, 26)(22, 107, 27)(23, 108, 28)(24, 109, 29)(25, 110, 30)
   (31, 42, 81)(32, 45, 82)(33, 41, 83)(34, 43, 84)(35, 44, 85)(36,
   61, 56)(37, 62, 57)(38, 63, 58)(39, 64, 59)(40, 65, 60)(71, 121,
   113)(72, 122, 114)(73, 123, 115)(74, 124, 111)(75, 125, 112)(91,
   96, 116)(92, 97, 117)(93, 98, 118)(94, 99, 119)(95, 100, 120);;
h3 := (1, 6, 14, 19, 21)(2, 7, 13, 18, 22)(3, 8, 12, 17, 23)(4, 9, 11,
    16, 24)(5, 10, 15, 20, 25)(26, 33, 37, 41, 46)(27, 31, 39, 42,
   47)(28, 32, 36, 44, 49)(29, 34, 40, 45, 50)(30, 35, 38, 43, 48)
   (51, 59, 64, 68, 71)(52, 56, 65, 67, 72)(53, 57, 62, 69, 73)(54,
   60, 63, 66, 74)(55, 58, 61, 70, 75)(76, 85, 90, 91, 98)(77, 84,
   86, 93, 100)(78, 81, 88, 94, 99)(79, 83, 89, 92, 97)(80, 82, 87,
   95, 96)(101, 106, 115, 117, 123)(102, 107, 113, 119, 121)(103,
   109, 112, 116, 122)(104, 110, 114, 120, 124)(105, 108, 111, 118,
   125);;

P := AsSubgroup(W, Group(h1, h2, h3));; # P \cong K inside TW

J := [];;
for k in [1..125] do
  if k mod 5 = 1 then Add(J,k); fi; od;
```

```
Ti := [];; # T_i is the group in position i, each isomorphic to A_5
for i in J do
  g1 := (i, i + 1, i + 2);
  g2 := (i, i + 1, i + 2, i + 3, i + 4);
  Add(Ti, Group(g1, g2));
od;

Q := Normalizer(P, Ti[25]);;
Size(Q); # 120 = |SL(2,5)|

Qi := [];;
for t in Ti{[1..24]} do
  Add(Qi, Normalizer(Q, t)); od; # each Qi is cyclic of order 5

Ri := [];; # Ri=Qi in Ti
for i in [1..24] do
  for subgp in AllSubgroups(Ti[i]) do
  if (IsNormal(Qi[i], subgp) and Size(subgp)=5) then
Add(Ri, subgp); fi; od; od;


# create group S:= group <R1, ..., R24>

gens_of_ri := [];;
for r in Ri do
  gen:=GeneratorsOfGroup(r);
  Add(gens_of_ri, gen[1]); od; # list with all gens of Ri

S := Group(gens_of_ri);;

M := Group(Concatenation(GeneratorsOfGroup(Q), GeneratorsOfGroup(S)))
   ;;


# we move on to Magma to find generators of M
```

```
# from Magma
M:=PermutationGroup<125|(1,2,3,5,4)(6,39,70,100,105)(7,40,66,96,101)
    (8,36,67,97,102)(9,37,68,98,103)(10,38,69,99, 104)
    (11,73,107,45,77)(12,74,108,41,78)(13,75,109,43,79)
    (14,71,110,44,80)(15,72,106,42,76)(16,83,47,114,54)
    (17,84,48,115,51)(18,85,49,111,53)(19,81,50,112,52)
    (20,82,46,113,55)(21,119,90,60,30)(22,120,86,56,26)
    (23,116,87,57,27)(24,117,88,58,28)(25,118,89,59,29)
    (31,34,32,35,33)(61,63,65,62,64)(91,93,95,92,94)
    (121,124,122,125,123),  (1,6,101,92,89,117)(2,8,105,94,90,118)
    (3,9,102,91,87,119)(4,7,104,95,88,120)(5,10,103,93,86,116)
    (11,99,58,82,22,40)(12,96,59,85,24,39)(13,98,56,81,23,38)
    (14,97,57,83,21,37)(15,100,60,84,25,36)(16,68,50,80,27,75)
    (17,67,47,76,30,71)(18,70,48,78,29,72)(19,69,46,79,26,73)
    (20,66,49,77,28,74)(31,45,108,64,55,111)(32,44,109,63,54,112)
    (33,41,106,62,53,115)(34,43,107,61,52,113)(35,42,110,65,51,114)
    (121,124,125), (1,3,4,2,5), (6,9,7,8,10), (11,14,12,13,15),
    (16,18,19,17,20), (21,24,23,25,22), (26,29,27,28,30),
    (31,32,33,34,35), (36,38,39,37,40), (41,44,45,42,43),
    (46,47,49,50,48), (51,52,53,54,55),  (56,59,57,58,60),
    (61,62,63,64,65), (66,67,69,70,68), (71,74,75,72,73),
    (76,79,78,80,77), (81,84,82,83,85), (86,89,87,88,90),
    (91,92,93,94,95), (96,97,99,100,98),  (101,102,104,105,103),
    (106,109,108,110,107), (111,114,113,115,112),
    (116,117,119,120,118)>;
N:=NormalSubgroups(M : OrderEqual:=25);
X:=N[1]`subgroup;
X.1;
X.2;
# back to GAP
X:=Group((1, 2, 3, 5, 4)(6, 10, 8, 7, 9)(16, 20, 17, 19, 18)(21, 25,
    24, 22, 23)(31, 35, 34, 33, 32)(36, 39, 40, 38, 37)(41, 42, 44,
    43, 45)(46, 47, 49, 50, 48)(51, 53, 55, 52, 54)(56, 58, 59, 60,
    57)(61, 65, 64, 63, 62)(71, 73, 72, 75, 74)(76, 77, 80, 78, 79)
    (86, 89, 87, 88, 90)(91, 94, 92, 95, 93)(96, 100, 97, 98, 99)(101,
     102, 104, 105, 103)(106, 108, 107, 109, 110)(111, 115, 114, 112,
    113)(116, 118, 120, 119, 117), (1, 5, 2, 4, 3)(11, 15, 13, 12, 14)
    (16, 17, 18, 20, 19)(21, 23, 22, 24, 25)(26, 27, 30, 29, 28)(31,
    33, 35, 32, 34)(36, 40, 37, 39, 38)(46, 48, 50, 49, 47)(56, 59,
    57, 58, 60)(61, 63, 65, 62, 64)(66, 70, 67, 68, 69)(71, 74, 75,
    72, 73)(76, 80, 79, 77, 78)(81, 84, 82, 83, 85)(91, 95, 94, 93,
    92)(96, 99, 98, 97, 100)(101, 102, 104, 105, 103)(106, 108, 107,
    109, 110)(111, 115, 114, 112, 113)(116, 118, 120, 119, 117));; # X
     is of order 25

# <X,Q> as a subgroup of W
```

```
X_Q := Group(Concatenation(GeneratorsOfGroup(Q), GeneratorsOfGroup(X))
   );;
IsSubgroup(W, X_Q);

phiXQtoQ:=FactorCosetAction(X_Q,Q);
Transitivity(Image(phiXQtoQ)); # returns 2

Intersection(P, X_Q) = Q; # true

PQ := Group(Concatenation(GeneratorsOfGroup(X_Q), GeneratorsOfGroup(P)
   ));;
W = PQ; # true so the graph is connected
```

### A.4 $G = \mathrm{PSL}(2,7) \operatorname{twr}_{\phi} \mathrm{ASL}(2,7)$

```
g1 := (2, 4, 3, 7, 5, 6)(8, 36, 29, 43, 15, 22)(9, 39, 31, 49, 19, 27)
    (10, 42, 33, 48, 16, 25)(11, 38, 35, 47, 20, 23)(12, 41, 30, 46,
    17, 28)(13, 37, 32, 45, 21, 26)(14, 40, 34, 44, 18, 24);;
g2 := (2, 14, 43)(3, 20, 36)(4, 26, 29)(5, 32, 22)(6, 38, 15)(7, 44,
    8)(9, 13,37)(10, 19, 30)(11, 25, 23)(12, 31, 16)(17, 18, 24)(21,
    49, 45)(27, 42,46)(28, 48, 39)(33, 35, 47)(34, 41, 40);;
g3 := (1, 2, 3, 4, 5, 6, 7)(8, 9, 10, 11, 12, 13, 14)(15, 16, 17, 18,
    19, 20, 21)(22, 23, 24, 25, 26, 27, 28)(29, 30, 31, 32, 33, 34,
    35)(36, 37, 38, 39, 40, 41, 42)(43, 44, 45, 46, 47, 48, 49);;
A := Group(g1,g2,g3);;
T := Group( [ (3,5,4)(6,8,7), (1,8,2)(5,7,6) ] );; #PSL(2,7)
W:=WreathProduct(T,A);;

g:=A;;
h:=Stabiliser(g,1);;
hom1:=FactorCosetAction(g,h);;
k:=Normaliser(h,SylowSubgroup(h,7));;
hom2:=FactorCosetAction(h,k);;
p:=Group(KuKGenerators(g,hom1,hom2));; # ASL(2,7) as the point
    stabilizer in TW

Q:=Subgroup(p,[p.1,p.2]); # build Q using ASL(2,7)
Orbits(Q);

A8:=AsSubgroup(SymmetricGroup(392),AlternatingGroup(8));
IsNormal(Q, A8);
for subgp in ConjugateSubgroups(A8, T) do
  if IsNormal(Q, subgp) then Display(subgp); fi; od; # try to find T
    in A8
T := Group( [ (3,5,4)(6,8,7), (1,8,2)(5,7,6) ] )
WW := Group(Concatenation(GeneratorsOfGroup(T), GeneratorsOfGroup(p)))
    ;; ## we build WW as the group T wr p because p does not sit
    inside W when using KuK Generators function
n:=NormalClosure(WW, T); # the smallest normal subgroup of WW
    containing n
Size(Intersection(n, p));

Ti := ConjugateSubgroups(WW, T);;
T2 := Ti[2];;
Q2:=Normalizer(Q,T2);;
for subgp in AllSubgroups(T2) do
  if (IsNormal(Q2, subgp) and Size(subgp)=7) then R2 := subgp;
fi;od;
```

```
Ri:=ConjugateSubgroups(Q, R2);;
gens_of_ri := [];;
for r in Ri do
      gen:=GeneratorsOfGroup(r);
      Add(gens_of_ri, gen[1]); od; # list with all gens of Ri
S:=Group(gens_of_ri);;
M := Group(Concatenation(GeneratorsOfGroup(Q), GeneratorsOfGroup(S)))
   ;;
Size(M);
GeneratorsOfGroup(M);

# Magma
M := PermutationGroup<392 |(  3,  5,  4)(  6,  8,  7)(  9, 89, 33, 57,
    41, 17)( 10, 90, 34, 58, 42, 18)( 11, 93, 37, 61, 45, 19)( 12,
   91, 35, 59, 43, 20)( 13, 92, 36, 60, 44, 21)( 14, 96, 40, 64, 48,
   22) ( 15, 94, 38, 62, 46, 23)( 16, 95, 39, 63, 47, 24)( 25,
   49,111,130, 98, 71)( 26, 50,108,131, 99, 68)( 27, 51,105,129, 97,
   65)( 28, 52,112,133,101, 72)( 29, 53,109,134,102, 69)( 30,
   54,106,132,100, 66)( 31, 55,110,135,103, 70)( 32, 56,107,136,104,
   67)( 73,305,163, 83,291,155)( 74,306,166, 81,289,153)( 75,307,162,
    82,290,154)( 76,308,165, 86,294,158)( 77,309,168, 84,292,156)(
   78,310,161, 85,293,157)( 79,311,164, 87,295,159)
   (80,312,167,88,296,160)(113,313,337,177,205,213)
   (114,314,338,178,202,210)(115,315,339,181,203,211)
   (116,316,340,179,207,215)(117,317,341,180,206,214)
   (118,318,342,184,201,209)(119,319,343,182,208,216)
   (120,320,344,183,204,212)(121,300,377,385,353,227)
   (122,303,383,390,358,229)(123,302,384,391,359,231)
   (124,298,380,387,355,226)(125,299,378,386,354,225)
   (126,304,379,389,357,228)(127,297,381,388,356,230)
   (128,301,382,392,360,232)(137,345,189,193,169,257)
   (138,346,186,194,170,258)(139,347,190,195,171,259)
   (140,348,187,196,172,260)(141,349,191,197,173,261)
   (142,350,188,198,174,262)(143,351,185,199,175,263)
   (144,352,192,200,176,264)(145,329,246,370,362,276)
   (146,330,242,372,364,275)(147,331,241,375,367,274)
   (148,332,243,369,361,273)(149,333,248,371,363,280)
   (150,334,247,374,366,279)(151,335,245,373,365,278)
   (152,336,244,376,368,277)(217,236,323,283,265,249)
   (218,235,321,281,271,250)(219,234,322,282,268,251)
   (220,233,326,286,266,252)(221,240,
```

(222,239,325,285,269,254)
(223,238,327,287,270,255)(224,237,328,288,267,256),( 1, 3, 2)( 4, 6, 5)( 9, 74, 27)( 10, 75, 25)( 11, 78, 28)( 12, 73, 26)( 13, 77, 30)( 14, 80, 32)( 15, 76, 29)( 16, 79, 31)
( 17,113, 51)( 18,114, 53)( 19,115, 50)( 20,116, 49)( 21,117, 55)( 22,118, 54)( 23,119, 56)( 24,120, 52)( 33,177, 97)( 34,178,102)( 35,179, 98)( 36,180,103)( 37,181, 99)( 38,182,104)
( 39,183,101)( 40,184,100)( 41,193,105)( 42,194,106)( 43,195,107)( 44,196,108)( 45,197,109)( 46,198,110)( 47,199,111)( 48,200,112)( 57, 81,129)( 58, 82,130)( 59, 83,131)( 60, 84,132)
( 61, 85,133)( 62, 86,134)( 63, 87,135)( 64, 88,136)( 65, 89,137)( 66, 90,138)( 67, 91,139)( 68, 92,140)( 69, 93,141)( 70, 94,142)( 71, 95,143)( 72, 96,144)(121,364,233)(122,361,234)
(123,362,235)(124,367,236)(125,365,237)(126,363,238)(127,368,239)(128,366,240)(145,377,282)(146,378,283)(147,379,281)(148,380,285)(149,381,286)(150,382,284)(151,383,287)(152,384,288)
(153,247,289)(154,245,291)(155,242,293)(156,248,296)(157,246,295)(158,243,290)(159,244,292)(160,241,294)(161,276,311)(162,278,305)(163,275,310)(164,277,309)(165,273,307)(166,279,306)
(167,274,308)(168,280,312)(169,257,324)(170,263,323)(171,260,322)(172,258,321)(173,264,328)(174,261,327)(175,262,326)(176,259,325)(185,249,346)(186,250,348)(187,251,347)(188,252,351)
(189,253,345)(190,254,352)(191,255,350)(192,256,349)(201,212,303)(202,209,300)(203,210,297)(204,215,304)(205,213,301)(206,211,298)(207,216,302)(208,214,299)(217,372,225)(218,375,228)
(219,370,227)(220,371,230)(221,374,232)(222,369,226)(223,373,229)(224,376,231)(265,387,331)(266,385,330)(267,386,335)(268,390,332)(269,388,336)(270,389,333)(271,391,329)(272,392,334)
(313,337,360)(314,342,353)(315,338,356)(316,343,359)(317,339,355)(318,344,358)(319,341,354)(320,340,357),(58,59,61,63,60,64,62),(10,12,11,16,13,14,15),(34,35,37,39,36,40,38),
(18,20,19,24,21,22,23),(90,91,93,95,92,96,94),(42,43,45,47,44,48,46),(305,310,311,309,312,308,307),(290,291,293,295,292,296,294),(314,316,315,320,317,318,319),(201,208,202,207,203,204,206),
(170,171,173,175,172,176,174),(346,347,349,351,348,352,350),(130,131,133,135,132,136,134),(25,26,28,31,30,32,29),( 98, 99,101,103,100,104,102),(49,50,52,55,54,56,53),(66,67,69,71,68,72,70),
(106,107,109,111,108,112,110),(273,278,275,276,277,280,274),(241,243,245,242,246,244,248),(353,359,356,357,355,358,354),(297,304,298,303,299,300,302),(321,325,327,323,322,328,326),
(249,251,256,252,250,254,255),(82,83,85,87,84,88,86),(73,78,79,77,80,76,75),(178,179,181,183,180,184,182),(114,116,115,120,117,118,119),(138,139,141,143,140,144,142),(194,195,197,199,196,
200,198),(161,164,168,167,165,162,163),(154,155,157,159,156,160,158)

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,(338,340,339,344,341,342,343),(209,216,210,215,211,212,214)
    ,(258,259,261,263,260,264,262),(185,187,192,188,186,190,191
 ),(121,123,127,126,124,122,125),(385,391,388,389,387,390,386)
    ,(265,268,267,266,271,269,270),(233,235,239,238,236,234,237)
    ,(361,365,364,362,368,363,367),(329,336,333,331,332,335,330),
(217,219,224,220,218,222,223),(281,285,287,283,282,288,286)
    ,(145,152,149,147,148,151,146),(369,373,372,370,376,371,375)
    ,(225,227,231,230,228,226,229),(377,384,381,379,380,383,378)>;
N:=NormalSubgroups(M : OrderEqual:=49);
X:=N[1]`subgroup;
X.1;
X.2;


n1 := (25, 29, 32, 30, 31, 28, 26)(49, 52, 54, 53, 50, 55, 56)(66, 68,
     67, 72, 69, 70,71)(73, 78, 79, 77, 80, 76, 75)(82, 86, 88, 84,
    87, 85, 83)(98, 104, 103, 99, 102, 100, 101)(106, 111, 110, 109,
    112, 107, 108)(114, 118, 120, 116,119, 117, 115)(121, 127, 124,
    125, 123, 126, 122)(130, 131, 133, 135, 132, 136, 134)(138, 143,
    142, 141, 144, 139, 140)(145, 148, 152, 151, 149, 146,147)(154,
    159, 158, 157, 160, 155, 156)(161, 165, 164, 162, 168, 163,   167)
    (170, 174, 176, 172, 175, 173, 171)(178, 181, 180, 182, 179, 183,
    184)(185, 190, 188, 187, 191, 186, 192)(194, 196, 195, 200, 197,
    198, 199)(201, 207, 206, 202, 204, 208, 203)(209, 216, 210, 215,
    211, 212,214)(217, 222, 220, 219, 223, 218, 224)(225, 229, 226,
    228, 230, 231,227)(233, 236, 235, 234, 239, 237, 238)(241, 245,
    246, 248, 243, 242,244)(249, 251, 256, 252, 250, 254, 255)(258,
    261, 260, 262, 259, 263,264)(265, 266, 270, 267, 269, 268, 271)
    (273, 280, 276, 278, 274, 277,275)(281, 287, 282, 286, 285, 283,
    288)(290, 293, 292, 294, 291, 295,296)(297, 303, 302, 298, 300,
    304, 299)(305, 308, 309, 310, 307, 312,311)(314, 317, 316, 318,
    315, 319, 320)(321, 326, 328, 322, 323, 327, 325)(329, 330, 335,
    332, 331, 333, 336)(338, 343, 342, 341, 344, 339, 340)(346, 347,
    349, 351, 348, 352, 350)(353, 355, 359, 358, 356, 354,357)(361,
    365, 364, 362, 368, 363, 367)(369, 370, 375, 372, 371, 373,376)
    (377, 384, 381, 379, 380, 383, 378)(385, 390, 389, 391, 386, 387,
    388);;


n2 := (10, 13, 12, 14, 11, 15, 16)(18, 23, 22, 21, 24, 19, 20)(25, 26,
     28, 31, 30, 32,29)(34, 35, 37, 39, 36, 40, 38)(42, 45, 44, 46,
    43, 47, 48)(49, 56, 55, 50,53, 54, 52)(58, 63, 62, 61, 64, 59, 60)
    (66, 71, 70, 69, 72, 67, 68)(73, 79,80, 75, 78, 77, 76)(82, 88,
    87, 83, 86, 84, 85)(90, 96, 95, 91, 94, 92,93)(98, 101, 100, 102,
    99, 103, 104)(106, 108, 107, 112, 109, 110, 111)(114,120, 119,
    115, 118, 116, 117)(121, 127, 124, 125, 123, 126, 122)(130,
    134,136, 132, 135, 133, 131)(138, 142, 144, 140, 143, 141, 139)
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   (154, 160, 159,155, 158, 156, 157)(161, 168, 165, 163, 164, 167,
   162)(170, 173, 172, 174,171, 175, 176)(178, 180, 179, 184, 181,
   182, 183)(185, 191, 190, 186, 188,192, 187)(194, 195, 197, 199,
   196, 200, 198)(201, 208, 202, 207, 203, 204,206)(209, 211, 216,
   212, 210, 214, 215)(233, 236, 235, 234, 239, 237,238)(241, 248,
   244, 246, 242, 245, 243)(249, 252, 255, 256, 254, 251,250)(258,
   259, 261, 263, 260, 264, 262)(265, 266, 270, 267, 269, 268,271)
   (273, 278, 275, 276, 277, 280, 274)(290, 295, 294, 293, 296,
   291,292)(297, 298, 299, 302, 304, 303, 300)(305, 312, 310, 308,
   311, 307,309)(314, 319, 318, 317, 320, 315, 316)(321, 322, 325,
   328, 327, 326,323)(329, 330, 335, 332, 331, 333, 336)(338, 344,
   343, 339, 342, 340,341)(346, 352, 351, 347, 350, 348, 349)(353,
   358, 357, 359, 354, 355,356)(361, 365, 364, 362, 368, 363, 367)
   (385, 390, 389, 391, 386, 387, 388);;

N:= Group( n1, n2 );;

N_Q := Group(Concatenation(GeneratorsOfGroup(Q), GeneratorsOfGroup(N))
   );;
IsSubgroup(WW, N_Q);

phiNQtoQ:=FactorCosetAction(N_Q,Q);
Transitivity(Image(phiNQtoQ)); # returns 2

Intersection(p, N_Q) = Q; #true
PQ := Group(Concatenation(GeneratorsOfGroup(N_Q), GeneratorsOfGroup(p)
   ));;
WW = PQ;
```