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FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA

(DOCTORAL THESIS)

KUBIČNI VOZLIŠČNO TRANZITIVNI GRAFI, KI PREMOREJO AVTOMORFIZEM Z MAJHNIM ŠTEVILOM ORBIT

(CUBIC VERTEX-TRANSITIVE GRAPHS HAVING AN AUTOMORPHISM WITH FEW ORBITS)

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SOMENTOR: IZR. PROF. DR. ADEMIR HUJDUROVIĆ

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Abstract

A graph admitting a cyclic group of automorphisms with k orbits on vertices of equal size is called a k-multicirculant. Cubic k-multicirculants have been extensively studied in the context of vertex- and arc-transitive graphs, and several classification results exist. The case where the orbits on vertices have different sizes is significantly less studied and understood, in part due to the lack of appropriate combinatorial and algebraic tools. While k-multicirculants are usually studied through the theory of voltage graphs and regular covers, no equivalent tools exist for the case of G-orbits of unequal size. We introduce the notion of a generalised voltage graph and the associated generalised cover construction, which allows us to construct graphs admitting a group of automorphisms G with no symmetry constrains. This new concept can be seen as a generalisation of the notion of a voltage graph, and several fundamental results of the theory of voltage graphs generalise nicely to this wider context. We focus in particular on those generalised voltage graphs that yield cubic covering graphs admitting a cyclic group of automorphisms. We introduce a secondary construction, the cyclic generalised cover construction, which we use to study cubic graphs admitting a cyclic group with few orbits on vertices, and thus of relatively large order. In particular, we classify all cubic vertex-transitive graphs Γ admitting a cyclic group of automorphisms G with an orbit of size $n/3$ or greater. We do this in three steps. We first completely classify cubic vertex-transitive tricirculants. We then extend this result and classify all cubic vertex-transitive graphs admitting a cyclic group of automorphism with at most 3 orbits on vertices, and we show that all such graphs admit a k-multicirculant automorphism with $k \leq 3$. Finally, we classify all cubic vertex-transitive graphs Γ admitting a cyclic group of automorphisms G with an orbit of size $|V(\Gamma)|/3$ or greater. Moreover, we show that all such graphs belonging to all but one infinite class admit a k-multicirculant automorphism for some $k \in \{1, 2, 3\}$.

Math. Subj. Class. (2020): 05C25, 05C60, 05C75, 05E18, 20B25.

Keywords: Cubic graph, vertex-transitive graph, voltage graph, graph cover, multicirculant, polycirculant, cyclic group.

Povzetek

Grafi, ki premorejo ciklične grupe avtomorfizmov z k enako velikimi orbitami na vozliščih, se imenujejo k-multicirkulanti. Kubični k-multicirkulanti so bili deležni že številnih raziskav znotraj konteksta vozliščno- in ločno-tranzitivnih grafov, zato zanje obstaja že več klasifikacij. Primer, ko so orbite na vozliščih različnih velikosti, je bil v preteklosti deležen manj pozornosti, delno tudi zaradi pomanjkanja primernih orodij za študij teh grafov. Čeprav za študij k-multicirkulantov uporabljamo teorijo napetostnih grafov, za grafe z G-orbitami različnih velikosti v literaturi ni zaslediti primerljive teorije. V tem raziskovalnem delu vpeljemo koncept posplošenega napetostnega grafa in koncept konstrukcije posplošenega krova, s katerima lahko konstruiramo grafe, ki dopuščajo podgrupo grupe avtomorfizmov G brez kakršnih koli simetrijskih omejitev. Ta nova koncepta predstavljata posplošitev teorije napetostnih grafov, več temeljnih rezultatov te teorije pa se tudi dobro posploši na širši kontekst posplošenih napetostnih grafov. V pričujočem delu se osredotočamo specifično na tiste posplošene napetostne grafe, ki porodijo kubične krovne grafe s ciklično grupo avtomorfizmov. Za namen preučevanja kubičnih grafov, ki dopuščajo ciklične grupe z manjšim številom orbit na vozliščih in ki so torej relativno višjega reda, vpeljemo tudi sekundarni koncept ciklične posplošene konstrukcije krovnih grafov. Vse kubične vozliščno-tranzitivne grafe Γ, ki dopuščajo ciklične grupe avtomorfizmov G z orbito velikosti $n/3$ ali več, klasificiramo v treh korakih. Sprva klasificiramo kubične vozliščno-tranzitivne tricirkulante, nato pa ta rezultat razširimo v klasifikacijo vseh kubičnih vozliščno-tranzitivnih grafov, ki dopuščajo ciklične grupe avtomorfizmov z največ 3 orbitami na vozliščih. Pokažemo tudi, da vsi takšni grafi dopuščajo k-multicirkulantni avtomorfizem s $k \leq 3$. V zadnjem koraku klasificiramo vse kubične vozliščno-tranzitivne grafe Γ, ki dopuščajo ciklično grupo avtomorfizmov G z orbito velikosti $|\mathrm{V}(\Gamma)|/3$ ali več. Hkrati pokažemo tudi, da vsi takšni grafi, z izjemo grafov, ki spadajo v specifičen neskončen razred, dopuščajo k-multicirkulantni avtomorfizem za nekatere $k \in \{1, 2, 3\}$.

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Ključne besede: Kubični graf, vozliščno-tranzitivni graf, napetostni graf, krovni graf, multicirkulant, policirkulant, ciklična grupa.

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Chapter 1

Introduction

Graphs exhibiting a considerable degree of symmetry are widely studied as their structure is to a great extent determined by the action of a group of automorphisms. Specific classes of vertex-transitive graphs Γ are often studied in terms of the action of a specific type of vertex-transitive group of automorphisms G. The case where G is not necessarily vertex-transitive, but its action still possesses a certain degree of transitivity are also of interest. For example, graphs admitting a cyclic group $G \leq \mathrm{Aut}(\Gamma)$ that partitions $\mathrm{V}(\Gamma)$ into k orbits of equal size, are called k-multicirculants and have been widely studied in the context of highly symmetric graphs (see [2, 22, 26, 34, 36, 45, 54], for example). The case where the orbits of G are not all of equal size is far less studied and understood, which is partly due to the lack of appropriate tools to study such graphs: while k-multicirculant are usually studied using the theory of ordinary voltage graphs, as defined by Gross and Tucker [32], (see, for example, [22, 35, 32, 40, 54, 67]), no analogous theory for the case of G-orbits of unequal size exists in the literature. This makes evident the need to develop new algebraic and combinatorial tools to study graphs admitting a group of automorphism G with laxer symmetry conditions.

The two main objectives of this dissertation are then the following: the first is, broadly speaking, the study and classification of cubic vertex-transitive graphs admitting a cyclic group of automorphism with an orbit on vertices of relatively large size (relative to the order of the graph). The second goal is the development of new tools to construct graphs admitting a group of automorphisms isomorphic to a given group. We aim to generalize the theory of voltage graphs to study graphs with no symmetry constrains.

We study cubic k-multicirculants, with $k \in \{1, 2, 3\}$ in Chapter 3, where we give an overview of cubic vertex-transitive circulants (1-multicirculants) and bicirculants (2-multicirculants), and completely classify cubic vertex-transitive tricirculants (3-multicirculants). We show that a cubic vertex-transitive tricirculant is either isomorphic to the Tutte-Coxeter graph or the truncated tetrahedron, or has order $6m$ for some $m \in \mathbb{Z}$ and is isomorphic to the prism $\mathrm{P}_{3\mathrm{m}}$, the Möbius ladder $\mathrm{M}_{3\mathrm{m}}$, the graph $\mathrm{X}(m)$ (defined in Section 3.2.2) or $\mathrm{Y}(m)$ (defined in Section 3.2.3); see Theorem 3.2.1.

In Chapter 4, we introduce the notion of a generalised voltage graph and the associated generalised cover construction, which can be regarded as a generalisation of the derived cover of an voltage graph, that allows us to construct and study graphs admitting a group of automorphisms G that is not necessarily semiregular (in fact, we impose no conditions on G). A *generalised voltage graph* is a quadruple $(\Delta, G, \omega, \zeta)$, where Δ is a graph, G is an arbitrary group, ω is a mapping assigning a subgroup of G to every vertex and arc of Δ and ζ is a mapping assigning elements of G to every arc of Δ. Given a generalised voltage graph $(\Delta, G, \omega, \zeta)$, we can construct an associated covering graph $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$, called the *generalised cover* of $(\Delta, G, \omega, \zeta)$, in much the same way a regular cover is constructed from an voltage graph. We show that a graph Γ with a group of automorphism $G \leq \mathrm{Aut}(\Gamma)$ is isomorphic to the generalised cover $\mathrm{GenCov}(\Delta/G, G, \omega, \zeta)$ where Γ/G is the quotient of Γ by G and ω and ζ are appropriately chosen functions (see Theorem 4.1.3). Many well-known results from the theory of voltage graphs generalise nicely to the wider theory of generalised voltage graphs, making it possible to characterise structural properties of a covering graph $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ in terms of the function ω and ζ (see for instance, Theorems 4.4.3, 4.5.4 and 4.6.3), as well as constructing some 'naturally occurring' automorphisms (see Lemmas 4.3.1 and 4.3.3).

In Chapter 5, we study generalised voltage graphs $(\Delta, G, \omega, \zeta)$ where the group G is cyclic. The fact that cyclic groups have a very particular structure can be exploited to simplify to a great extend the results of Chapter 4, and devise a method for constructing graphs admitting a cyclic group $G \leq \mathrm{Aut}(\Gamma)$, from a special kind of labelled quotient graph. A labelled graph is a pair (Δ, λ) where Δ is a graph and λ is a function, called a *labelling*, assigning an integer value to every arc of Δ. For a graph Γ and a cyclic group $G \leq \mathrm{Aut}(\Gamma)$, one can consider the labelled quotient graph $(\Gamma/G, \lambda_G)$, where Γ/G is the quotient of Γ by G and λ_G is a labelling that encodes information about the action of G on the arcs of Γ. We show that the graph Γ can be reconstructed from its labelled quotient $(\Gamma/G, \lambda_G)$ along with two integer valued functions ι and ζ encoding information about the G-orbits of Γ (see Theorem 5.2.3). We call the quadruple $(\Gamma/G, \lambda_G, \iota, \zeta)$ a *cyclic generalised voltage graph* and we say Γ is a *cyclic generalised cover* of the labelled graph $(\Gamma/G, \lambda_G)$. If in addition Γ is cubic, we say $(\Gamma/G, \lambda_G, \iota, \zeta)$ is a *ccv-graph* (short for cubic cyclic voltage graph) and Γ is a *ccv-cover* of the labelled graph $(\Gamma/G, \lambda_G)$. This means every cubic graph Γ admitting a cyclic group $G \leq \mathrm{Aut}(\Gamma)$ is a *ccv*-cover of a labelled graph (Δ, λ) and can be reconstructed from it. In the last sections of Chapter 5, we characterize all labelled graphs (Δ, λ) that admit a *ccv*-cover (Lemma 5.3.2), and we determine a set of necessary conditions for a *ccv*-cover of (Δ, λ) to be vertex-transitive (Theorem 5.4.14).

In Chapter 6 we extend the classification of cubic tricirculants to encompass all cubic graphs admitting a cyclic group of automorphism with at most 3 orbits on vertices, possibly of unequal size. We show that each such graph can be constructed as a *ccv*-cover of one of six labelled graphs (see Theorem 6.2.1) and that it is isomorphic to a k-multicirculant for some $k \in \{1, 2, 3\}$.

In Chapter 7, we study the set $\mathcal{G}$ of cubic vertex-transitive graphs Γ admitting a cyclic group of automorphisms G with an orbit of size $|\mathrm{V}(\Gamma)|/3$ or greater. To classify the elements of $\mathcal{G}$, we determine a set of necessary conditions for a labelled graph (Δ, λ) to admit a *ccv*-cover that belongs to $\mathcal{G}$

(see Theorem 7.2.1) and denote by $\mathcal{Q}^*$ be the set of all labelled graphs satisfying these conditions. Further, we show that if Γ is a cubic vertex-transitive and $G \leq \mathrm{Aut}(\Gamma)$ is cyclic and admits an orbit of size $|\mathrm{V}(\Gamma)|/k$ for some integer k, then G has at most $6k-5$ orbits on vertices (Theorem 7.1.4). In particular, this implies that the set $\mathcal{Q}^*$ is finite. An exhaustive computer search shows that $|\mathcal{Q}^*| = 363$. We resort to the results of Chapters 5 and 6 to determine which of the elements of $\mathcal{Q}^*$ admit a vertex-transitive *ccv*-cover belonging to $\mathcal{G}$. We show that a graph in $\mathcal{G}$ is a *ccv*-covers of one of seven labelled graphs (see Theorem 7.2.22 and Figure 7.2.7). Moreover, the *ccv*-covers of six of these labelled graphs are all k-multicirculants for some $k \in \{1,2,3\}$. The seventh labelled graph, denoted Δ_{36} and studied in Section 7.2.3, admits an infinite family of vertex-transitive *ccv*-covers such that $k = 6$ is the smallest integer for which a member of this family is a k-multicirculant. We also show in Chapter 7 that with the exception of $K_{3,3}$, the order of an automorphism of a vertex-transitive cubic graph Γ equals the length of its largest orbit on vertices (see Theorem 7.1.1). Then, since $K_{3,3}$ is a circulant, the set $\mathcal{G}$ is precisely the set of cubic vertex-transitive graphs Γ admitting an automorphism of order $|\mathrm{V}(\Gamma)|/3$ or greater.

All the original results presented in this PhD dissertation are contained in the following research papers:

- P. Potočnik, M. Toledo, Classification of cubic vertex-transitive tricirculants, *Ars Math. Contemp.* **18** (2020) 1–31.
- P. Potočnik, M. Toledo, Finite cubic graphs admitting a cyclic group of automorphisms with at most three orbits on vertices, Discrete Math. Accepted.
- P. Potočnik, M. Toledo, Generalised voltage graphs. Submitted. *arXiv:1910.08421.*
- P. Potočnik, M. Toledo, G. Verret, Cubic vertex-transitive graphs admitting automorphisms of large order. In preparation.

Chapter 2

Preliminary concepts

2.1 Groups

Throughout this dissertation, all groups are assumed to be finite, unless otherwise specified. If G is a group and X is a non-empty set, a *(right) action* of G on X is a mapping $\phi : (X, G) \to X$ such that:

1. $\phi(x, 1_G) = x$ for all $x \in X$,
2. $\phi(x, gh) = \phi(\phi(x, g), h)$.

We denote by x^g the image of (x, g) under ϕ. For $x \in X$ we define the *stabiliser* of x under G, denoted G_x, as the set $G_x := \{g \in G \mid x^g = x\}$ and the *orbit* of x under G as the set $x^G := \{x^g \mid g \in G\}$. Once an action has been specified, the orbits of G on X are called G-orbits. For an element $g \in G$ an orbit of g is an orbit of the cyclic group $\langle g \rangle$ generated by g.

We say that the action of G on X is *transitive* if for any two $x, y \in X$ there exists $g \in G$ such that $x^g = y$, and we say it is *semiregular* if $x^g \neq x$ for all $x \in X$ and all $g \in G \setminus \{1_G\}$. That is, an action is semiregular if the stabilizer of every $x \in X$ is trivial. An action is *regular* if it is both semiregular and transitive. Observe that if G acts regularly on X, then for any two $x, y \in X$ there exists a unique $g \in G$ such that $x^g = y$. An action is said to be *faithful* if only the neutral element 1_G of G fixes all elements of X.

2.2 Graphs

A *graph* is an ordered 4-tuple $(V, D; \mathrm{beg}, \mathrm{inv})$ where D and $V \neq \emptyset$ are disjoint finite sets of *darts* and *vertices*, respectively, $\mathrm{beg} \colon D \to V$ is a mapping which assigns to each dart x its *initial vertex* $\mathrm{beg}\ x$, and $\mathrm{inv} \colon D \to D$ is an involution which interchanges every dart x with its *inverse dart*, also denoted by x^{-1}. The *final vertex* of a dart x, denoted $\mathrm{end}\, x$, is the initial vertex of of its inverse x^{-1}. That is, $\mathrm{end}\, x = \mathrm{beg}\, x^{-1}$. We say two vertices u and v are adjacent, and we write $u \sim v$, if there exists a dart x such that $\mathrm{beg}\, x = u$ and $\mathrm{end}\, x = v$. Informally, we may think of a dart as a "half edge" rooted at its beginning vertex with the other "half" of the edge as its inverse.

If Γ is a graph, we let $\mathrm{V}(\Gamma)$ and $\mathrm{D}(\Gamma)$ denote the vertex- and dart-set of Γ, respectively, and we let beg_Γ and inv_Γ be the corresponding beginning and inverse functions. We will often omit the subscripts in beg_Γ and inv_Γ and simply write beg and inv if there is no possibility of ambiguity.

The orbits of inv are called *edges*. The edge containing a dart x is called a *semiedge* if $\mathrm{inv}\, x = x$, a *loop* if $\mathrm{inv}\, x \neq x$ while $\mathrm{beg}\,(x^{-1}) = \mathrm{beg}\, x$, and is called a *link* otherwise. The *endvertices of an edge* are the initial vertices of the darts contained in the edge. Two darts x and y are *parallel* if $\mathrm{beg}\, x = \mathrm{beg}\, y$ and $\mathrm{beg}\, x^{-1} = \mathrm{beg}\, y^{-1}$. Two edges are parallel if they have the same endvertices. When we present a graph as a drawing, the links are drawn in the usual way as a line between the points representing its endvertices, a loop is drawn as a closed curve at its unique endvertex and a semiedge is drawn as a pendant segment attached to its unique endvertex.

A *simple graph* is a graph with no loops, semi-edges or parallel edges. With this definition, a simple graph corresponds to the 'classical notion' of an undirected graph (a non-empty set of vertices V along with a set of unordered pairs of elements of V). Indeed a graph Γ in the classical sense can be represented as a graph (with the definition given in the above paragraphs) by letting $V = \mathrm{V}(\Gamma)$ be the vertex-set of Γ, letting $D = \{(u,v)\colon u, v \in \mathrm{V}(\Gamma), u \sim_\Gamma v\}$ be the dart-set of Γ, and by letting $\mathrm{beg}(u,v) = u$ and $\mathrm{inv}(u,v) = (v,u)$. In the case of a simple graph, the concept of a dart corresponds precisely to that of an arc.

The *neighbourhood* of a vertex v is defined as the set of darts that have v for its initial vertex and the *valence* of v is the cardinality of the neighbourhood. A graph is k-valent if all vertices have valence k.

A *walk* of length n in a graph is a sequence of darts $W = (x_1, x_2, \ldots, x_n)$ such that $\mathrm{beg}\, x_{i+1} = \mathrm{end}\, x_i$ for all $i \in \{1, \ldots, n-1\}$. We say W is *closed* if $\mathrm{end}\, x_n = \mathrm{beg}\, x_1$, in which case we say it is *based* at $\mathrm{beg}\, x_1$. We say W is *reduced* if $x_{i+1} \neq x_i^{-1}$ for all $i \in \{1, \ldots, n-1\}$. A *path* is a reduced walk where $\mathrm{beg}\, x_i \neq \mathrm{beg}\, x_j$ for all $i \neq j$ and a *cycle* is a closed path. A cycle of length n is also called an n-cycle. A *tree* is a connected graph without cycles. We say W is a uv-walk if $\mathrm{beg}\, x_1 = u$ and $\mathrm{end}\, x_n = v$. A graph is *connected* if there is a uv-walk for any two vertices u and v. A connected, 3-valent simple graph is called a *cubic graph*.

For a walk $W = (x_1, x_2, \ldots, x_n)$, we define the inverse of W as the walk $W^{-1} = (x_n^{-1}, x_{n-1}^{-1}, \ldots, x_1^{-1})$. If $W_1 = (x_1, x_2, \ldots, x_n)$ and $W_2 = (y_1, y_2, \ldots, y_m)$ are two walks such that $\mathrm{end}\, x_n = \mathrm{beg}\, y_1$, then we define the concatenation of W_1 and W_2 as $W_1W_2 = (x_1, \ldots, x_n, y_1, \ldots, y_m)$. Note that if W is a uv-walk then W^{-1} is a vu-walk.

A *morphism* between two graphs $(\mathrm{V}, \mathrm{D}, \mathrm{beg}, \mathrm{inv})$ and $(\mathrm{V}', \mathrm{D}', \mathrm{beg}', \mathrm{inv}')$ is a function $f\colon \mathrm{V} \cup \mathrm{D} \to \mathrm{V}' \cup \mathrm{D}'$ such that $f(\mathrm{V}) \subseteq \mathrm{V}'$, $f(\mathrm{D}) \subseteq D'$ and $(\mathrm{inv}\, f(x)) = f(\mathrm{inv}\, x)$ and $(\mathrm{beg}\, f(x)) = f(\mathrm{beg}\, x)$. That is, a morphism is an incidence-preserving mapping taking vertices to vertices and darts to darts. Note that the condition $(\mathrm{beg}\, f(x)) = f(\mathrm{beg}\, x)$ implies that f is completely defined by its restriction to the set of darts D. We say that a graph morphism f is an *epimorphism* if it is surjective, and an *isomorphism* if it is bijective. An isomorphism of a graph onto itself is called an *automorphism*. The set of all automorphisms of a graph Γ is a group with the operation of function composition, which we denote by $\mathrm{Aut}(\Gamma)$.

For a graph Γ and a subgroup $G \leq \mathrm{Aut}(\Gamma)$, we say Γ is G-vertex-transitive, G-edge-transitive or G-arc-transitive if the respective action of G on the vertices, edges or darts of Γ is transitive. We generally omit the prefix "G" when G is the full automorphism group of the graph, and simply say Γ is vertex-, edge- or arc-transitive.

2.2.1 Common graphs

Several well-known graphs and families of graphs appear frequently in this dissertation. We formally define them here for convenience.

For a positive integer n, the *prism* $\mathrm{P_n}$ is the cubic graph of order $2n$ with vertices $\{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ and edges of the form u_iv_i and $u_iu_{i\pm1}$, for $i \in \{1, \dots, n\}$, where the addition is taken modulo n.

For a positive integer n, the *Möbius ladder* $\mathrm{M_n}$ is the cubic graph of order $2n$ with vertices $\{u_1, \dots, u_{2n}\}$ and edges of the form u_iu_{i+n} and $u_iu_{i\pm1}$, for $i \in \{1, \dots, n\}$, where the addition is taken modulo $2n$

For positive integers n and k, the *generalised Petersen graph* $\mathrm{GP}(n, k)$ is the cubic graph of order $2n$ with vertices $\{u_1, \dots, u_n\} \cup \{v_1, \dots, v_n\}$ and edges of the form u_iv_i, $u_iu_{i\pm1}$ and v_iv_{i+k}, for $i \in \{1, \dots, n\}$, where the addition is taken modulo n. The graph $\mathrm{GP}(5, 2)$ is generally referred to simply as the Petersen graph. The graphs $\mathrm{GP}(8, 3)$, $\mathrm{GP}(10, 2)$ and $\mathrm{GP}(10, 3)$ are often called the Möbius-Kantor graph, the dodecahedron and the Desargues graph, respectively.

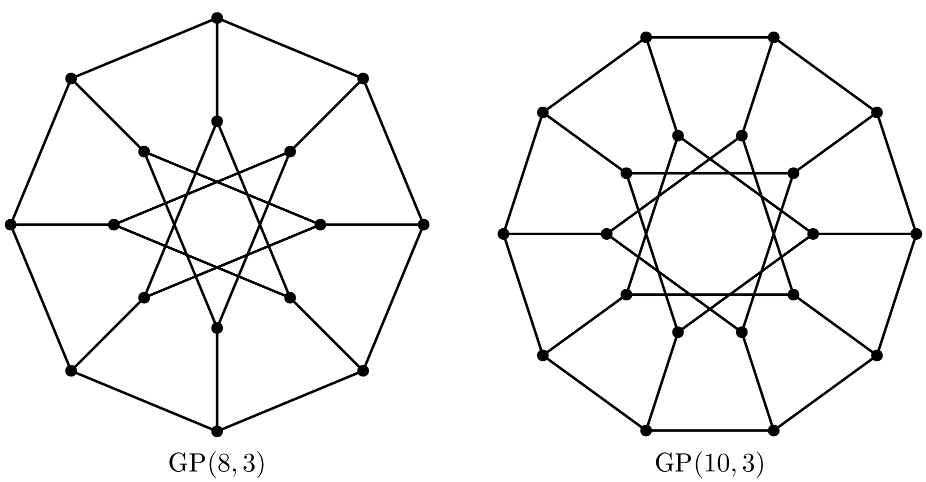


Figure 2.2.1: The Möbius-Kantor graph and the Desargues graph.

The Pappus graph is the arc-transitive cubic graph of order 18 with vertices $\{u_1, \dots, u_6\} \cup \{v_1, \dots, v_6\} \cup \{w_1, \dots, w_6\}$ and edges of the form $u_iu_{i\pm1}$, u_iv_i, $v_iw_{i\pm1}$ and w_iw_{i+3}, where the addition is taken modulo 6. It is the incidence graph of the Pappus configuration.

The Heawood graph is the arc-transitive cubic graph of order 14 with vertices $\{u_1, \dots, u_{14}\}$ and edges of the form $u_iu_{i\pm1}$ for $i \in \{1, \dots, 14\}$ and u_iu_{i+5} for every even $i \in \{1, \dots, 14\}$.

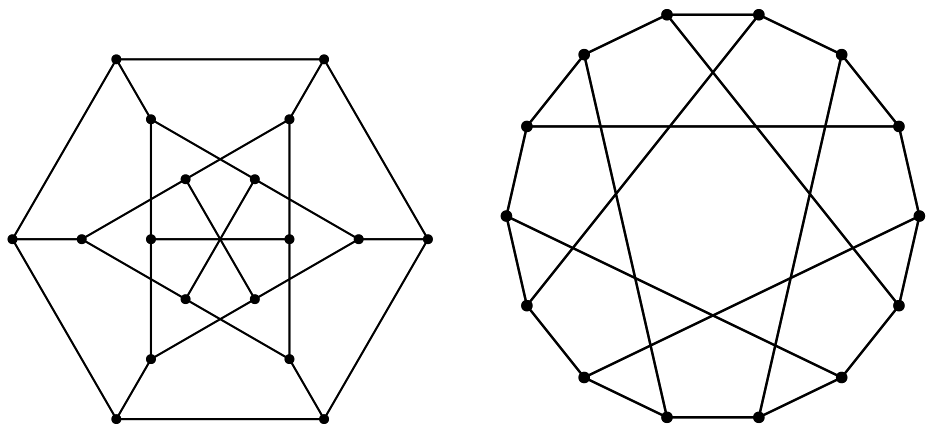


Figure 2.2.2: The Pappus graph and the Heawood graph.

For a group G and a subset $S \subseteq G$ such that $s \in S$ implies $s^{-1} \in S$, we define the *Cayley graph* $\mathrm{Cay}(G, S)$ as the simple graph with vertex-set G where $g \sim h$ if and only if $h = gs$ for some $s \in S$.

2.3 Voltage graphs

We now present some of the basic definitions and results pertaining to the theory of voltage graphs. For further details and the complete proofs of the results in this section, we refer the reader to [30, 32, 31].

Definition 2.3.1. Let Δ be a graph and G be a group. Let $\zeta\colon \mathrm{D}(\Delta) \to G$ be a mapping such that for all $x \in \mathrm{D}(\Delta)$ we have $\zeta(x^{-1}) = \zeta(x)^{-1}$. We say that the triple (Δ, G, ζ) is a *voltage graph* and we call the mapping ζ a *voltage assignment*.

Definition 2.3.2. Let (Δ, G, ζ) be a voltage graph. Let Γ be the graph defined by:

- $\mathrm{V}(\Gamma) = \mathrm{V}(\Delta) \times G$
- $\mathrm{D}(\Gamma) = \mathrm{D}(\Delta) \times G$
- $\mathrm{beg}_\Gamma(x, g) = (\mathrm{beg}_\Delta\, x, g)$
- $\mathrm{inv}_\Gamma(x, g) = (\mathrm{inv}_\Delta\, x, \zeta(x)g)$

Then Γ is called the *derived cover* of (Δ, G, ζ) and is denoted by $\mathrm{Cov}(\Delta, G, \zeta)$.

We say a graph Γ is a *regular cover* of a graph Δ if Γ is isomorphic to the derived cover of a voltage graph (Δ, G, ζ) for some group G and voltage assignment ζ.

Let (Δ, G, ζ) be a voltage graph and let $\Gamma := \mathrm{Cov}(\Delta, G, \zeta)$. For $x \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$ the set $\{(x, g) \mid g \in G\}$ is called the *fibre* above x and is denoted $\mathrm{fib}(x)$. The group G acts like a group of automorphisms by right multiplication on the second coordinate of the vertices and darts of $\mathrm{Cov}(\Delta, G, \zeta)$. Moreover, the orbits of G on vertices and darts are precisely the fibres of $\mathrm{Cov}(\Delta, G, \zeta)$. The action of G is regular on the fibre of each vertex and dart and is semiregular on the vertices and on the darts of $\mathrm{Cov}(\Delta, G, \zeta)$. For the sake of simplicity, we will write v_g instead of (v, g) for a vertex of Γ, and x_g instead of (x, g) for a dart of Γ.

Given a graph Γ and a group $G \leq \mathrm{Aut}(\Gamma)$ we define the quotient Γ/G to be the graph $(V', D'; \mathrm{beg}', \mathrm{inv}')$ where V' and D' are the sets $\mathrm{V}(\Gamma)/G$ and $\mathrm{D}(\Gamma)/G$ of orbits of G on V and D, respectively, and beg' and inv' are defined by $\mathrm{beg}'(x^H) = (\mathrm{beg}\, x)^G$ and $\mathrm{inv}'(x^H) = (\mathrm{inv}\, x)^G$ for every $x \in \mathrm{D}(\Gamma)$. The mapping $\pi_G \colon \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to V' \cup D'$ that maps a vertex or a dart x to its G-orbit x^G is then an epimorphism of graphs, called *the quotient projection with respect to* G. If G acts semiregularly on $\mathrm{V}(\Gamma)$ (that is, if the vertex-stabiliser G_v is trivial for every vertex $v \in \mathrm{V}(\Gamma)$), then the quotient projection π_G is a *regular covering projection* and in particular, it preserves the valence of the vertices.

The following theorem states one of the fundamental facts in the theory of graph covers, explaining the relation between a graph and its quotient by a semiregular group of automorphisms.

Theorem 2.3.3. *Let Γ be a simple graph, let $G \leq \mathrm{Aut}(\Gamma)$ be a group of automorphisms acting semiregularly on $\mathrm{V}(\Gamma)$ and let Γ/G be the quotient of Γ by G. Then there exists a voltage assignment $\zeta \colon \mathrm{D}(\Gamma) \to G$ such that $\Gamma \cong \mathrm{Cov}(\Gamma/G, G, \zeta)$.*

Let (Δ, G, ζ) be a voltage graph and let $\Gamma = \mathrm{Cov}(\Gamma/G, G, \zeta)$. If $W = (x_1, x_2, \ldots, x_n)$ is a uv-walk in Δ, then a *lift* of W based at a vertex $u_i \in \mathrm{fib}(u)$ is a walk $\overline{W} = (y_1, y_2, \ldots, y_n)$ beginning at u_i such that the projection $\pi(\overline{W}) := (\pi(y_1), \pi(y_2), \ldots, \pi(y_n))$ is equal to W. We define the *net voltage* of W as the product (in G) $\zeta(x_n)\zeta(x_{n-1}) \ldots \zeta(x_1)$.

Theorem 2.3.4. *Let (Δ, G, ζ) be a voltage graph, let $\Gamma = \mathrm{Cov}(\Delta, G, \zeta)$ and let W be a uv-walk in Δ for some $u, v \in \Delta$. Then the following hold:*

1. *For every $u_i \in \mathrm{fib}(u)$ there exists a unique lift of W based at u_i in Γ;*
2. *A lift of W is reduced if and only if W is reduced;*
3. *A lift of W is closed if and only if W is closed and has trivial net voltage.*

For a voltage graph (Δ, G, ζ) and for a spanning tree T of Δ, we say that the voltage assignment ζ is T-normalised if $\zeta(x) = 1_G$ for all darts x lying on T. A well-known result by Gross and Tucker [32] states that if (Δ, G, ζ) is a voltage graph and T is a spanning tree of Δ, then there exists a T-normalised voltage assignment ζ' such that $\mathrm{Cov}(\Delta, G, \zeta) \cong \mathrm{Cov}(\Delta, G, \zeta')$. This, together with Theorem 2.3.3 above gives us the following result.

Theorem 2.3.5. *Let Γ be a simple graph and $G \leq \mathrm{Aut}(\Gamma)$ be a group of automorphisms acting semiregularly on $\mathrm{V}(\Gamma)$. Let T be a spanning tree of the quotient graph Γ/G. Then there exists a T-normalised voltage assignment $\zeta \colon \mathrm{D}(\Gamma) \to G$ such that $\Gamma \cong \mathrm{Cov}(\Gamma/G, G, \zeta)$.*

Lemma 2.3.6. *Let (Δ, G, ζ) be a voltage graph where ζ is T-normalised for a spanning tree T of Δ. Then* $\mathrm{Cov}(\Delta, G, \zeta)$ *is connected if and only if* $G = \langle \zeta(x) \mid x \in \mathrm{D}(\Delta)\rangle$.

Lemma 2.3.7. *Let (Δ, G, ζ) be a voltage graph. Then* $\mathrm{Cov}(\Delta, G, \zeta)$ *is a simple graph if and only if the following holds:*

1. $\zeta(x) \neq \zeta(y)$ *for any two parallel darts x and y;*
2. $\zeta(x) \neq 1_G$ *whenever* $\mathrm{beg}\, x = \mathrm{end}\, x$.

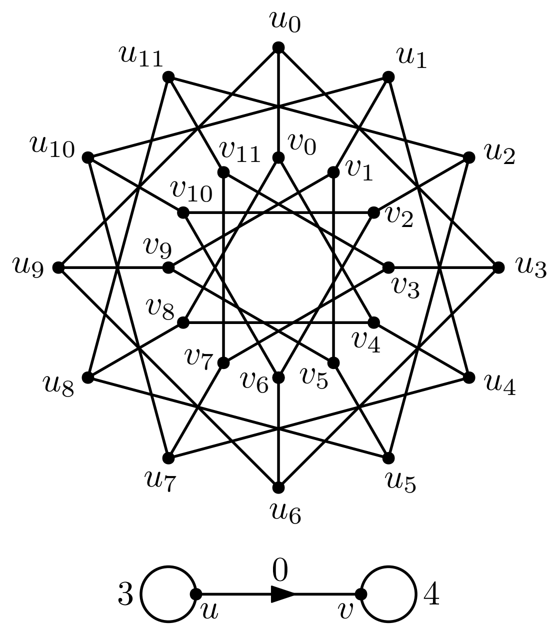


Figure 2.3.1: The generalised Petersen graph $\mathrm{GP}(12, 3, 4)$ as the derived cover of the 'handcuffs' graph with a voltage assignment on the cyclic group $\mathbb{Z}_{12}$

When presenting a voltage graph (Δ, G, ζ) in a figure, we will indicate the voltage assignment ζ using the following convention. If e is a link with endvertices u and v, we will draw an arrowhead directed from u to v, and write a group element $r \in G$ next to it, to indicate that the dart x underlying this link and beginning at u (and ending at v) has voltage r. Its inverse x^{-1} then has voltage r^{-1}. If a link lacks an arrowhead it is assumed that both darts underlying it have trivial voltage. Similarly, we will write a group element $r \in G$ next to a loop or a semi-edge to indicate that one of the darts underlying this loop or semi-edge has voltage r while its inverse has voltage r^{-1}.

Lemma 2.3.8. *Let (Δ, G, ζ) be a voltage graph and let $\varphi : G \to G$ be a group automorphism. Then $\varphi \circ \zeta$ is a voltage assignment and* $\mathrm{Cov}(\Delta, G, \zeta) \cong \mathrm{Cov}(\Delta, G, \varphi \circ \zeta)$.

Chapter 3

Cubic multicirculants

A connected simple graph Γ admitting a cyclic group of automorphisms G having k orbits of vertices of equal size larger than 1 is called a *k-multicirculant* and a generator of G is then called a *k-multicirculant automorphism* of Γ. In particular, 1-, 2- and 3-multicirculants are generally called circulants, bicirculants and tricirculants, respectively (see [2, 22, 26, 34, 36, 45, 54], for example). A famous and longstanding *polycirculant conjecture* claims that every vertex-transitive graph and digraph is a k-multicirculant for some k (see [6, 42, 43, 44]). It is known that every cubic vertex-transitive graph has a k-multicirculant automorphisms [46] and that no fixed k exists such that every cubic vertex-transitive graph is a k-multicirculant [69].

It is particularly interesting to study conditions under which vertex-transitive graphs admit k-multicirculant automorphisms of large order (and thus relatively small k); see, for example, [7, 69]. On the other hand, existence of a k-multicirculant automorphism with small k often restricts the structure of a vertex-transitive graph to the extent that allows a complete classification. This is usually achieved by making use of the theory of voltage graphs. For fixed integers k and s, we can construct the set of all s-valent k-multicirculants by determining the possible quotient graphs Γ/G, where Γ is an s-valent k-multicirculant and $G \leq \mathrm{Aut}(\Gamma)$ is a cyclic group generated by a k-multicirculant automorphism, and constructing their regular covers (see for instance [22, 35, 54]).

Lemma 3.0.1. *For $k \in \mathbb{N}$, a simple graph Γ is a k-multicirculant if and only if it has order nk for some $n \in \mathbb{Z}$ and is isomorphic to the derived cover of a voltage graph $(\Delta, \mathbb{Z}_n, \zeta)$ such that:*

1. *Δ is a connected graph on k vertices with no parallel semi-edges.*
2. *$\zeta(x) \neq \zeta(y)$ for any two parallel darts $x, y \in \mathrm{D}(\Delta)$.*
3. *ζ is T-normalised for some spanning tree T of Δ.*

Moreover, Γ is s-valent if and only if Δ is s-valent.

Proof. By definition, the derived cover of a voltage graph with a non-trivial cyclic voltage group G must be a k-multicirculant graph, as G acts as a semiregular group of automorphisms with all G-orbits of equal size. For the converse, suppose Γ is a k-multicirculant for some k. Then there exists a cyclic group $G \leq \mathrm{Aut}(\Gamma)$ acting semiregularly on $\mathrm{V}(\Gamma)$, with k orbits of equal size. Then the graph Γ/G is connected and has order k and by Theorem 2.3.5 there exists a T-normalised voltage assignment $\zeta\colon \mathrm{D}(\Gamma/G) \to G$ such that $\Gamma \cong \mathrm{Cov}(\Gamma/G, G, \zeta)$. Item (2) follows at once from Lemma 2.3.7. Now if a dart x underlies a semi-edge, then $x = x^{-1}$ and so $\zeta(x) = \zeta(x^{-1}) = \zeta(x)^{-1}$. That is the voltage of a dart underlying a semi-edge must be an involution in G. Since a cyclic group can have at most one involution, then two parallel semi-edges must have the same voltage, which contradicts Lemma 2.3.7 and the simplicity of Γ. It follows that Γ/G has no parallel semi-edges. $\square$

3.1 Cubic circulants and bicirculants

We devote the following paragraphs to giving an overview of cubic vertex-transitive circulants and bicirculants. While the former can be easily be shown to be isomorphic to Möbius ladders and prisms, the latter were classified in [54].

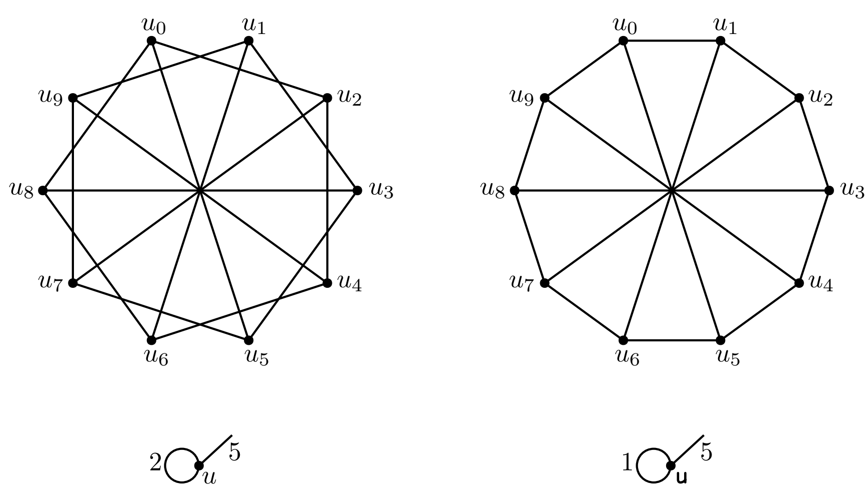


Figure 3.1.1: The prism P_5 (left) and the Möbius ladder M_5 (right) as the derived graph of a 3-valent voltage graph on a single vertex with voltage group $\mathbb{Z}_{10}$.

It follows from Lemma 3.0.1 that a cubic vertex-transitive circulant Γ of order n is isomorphic to $\mathrm{Cov}(\Delta, \mathbb{Z}_n, \zeta)$ where Δ is the 3-valent graph consisting of a single vertex u incident to a loop $\{x, x^{-1}\}$ and a semi-edge $\{y\}$. Since y underlies a semi-edge, $\zeta(y) = \frac{n}{2}$ and since Γ is connected, then by Lemma 2.3.6 $\gcd(\zeta(x), \frac{n}{2}) = 1$. That is $\gcd(\zeta(x), n) \in \{1, 2\}$. Since there is an automorphism of $\mathbb{Z}_n$ mapping ζ to $\gcd(\zeta, n)$, we can assume by Lemma 2.3.8 that $\zeta(x) \in \{1, 2\}$. It is an easy exercise to see that $\mathrm{Cov}(\Delta, \mathbb{Z}_n, \zeta)$ is isomorphic to the Möbius ladder $\mathrm{M}_{n/2}$ when $\zeta(x) = 1$, or to the prism $P_{n/2}$ when $\zeta(x) = 2$ (and $\frac{n}{2}$ is odd).

By the same token, a cubic vertex-transitive bicirculant must be the regular cover of a trivalent, connected graph on 2 vertices and with no parallel semi-edges. It is straightforward to see that the graphs I, H and T of Figure 3.1.2

are the only graphs with these properties. For a positive integer m and group elements $r, s \in \mathbb{Z}_m$, let $\mathrm{I}(m; r, s)$ be the derived cover of the voltage graph $(I, \mathbb{Z}_m, \zeta)$ where the graph I and the voltage assignment ζ are depicted in Figure 3.1.2. Define $\mathrm{H}(m; r, s)$ and $\mathrm{F}(m; r)$ similarly. The characterisation theorem below follows from Propositions 3 and 4 of [54].

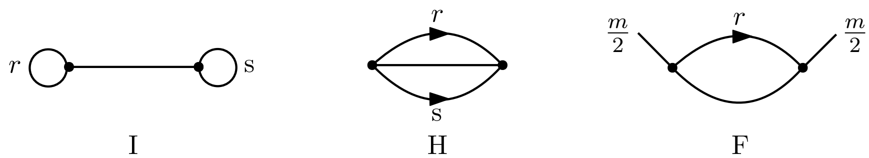


Figure 3.1.2: The three possible quotients of a cubic bicirculant by a bicirculant automorphism.

Theorem 3.1.1. *[54, Propositions 3 and 4] A cubic vertex-transitive bicirculant is isomorphic to one of the following:*

1. *$I(m; r, 1)$ with $m \geq 3$, $r^2 \equiv \pm 1 \pmod{m}$, or $m = 10$ and $r = 2$;*
2. *$H(m; r, s)$ with $m \geq 3$, $r \neq s$, $\gcd(m, r, s) = 1$;*
3. *$F(m; r)$ with $m \geq 2$ and $r = 1$, or $m/2$ is odd and $r = 2$.*

The graphs in item (1) correspond to the family of vertex-transitive generalised Petersen graphs. Indeed, for $m \geq 3$ and $r \in \{1, \ldots, m-1\}$ we have $\mathrm{I}(m; r, 1) \cong \mathrm{GP}(m, r)$. In particular the graph $\mathrm{I}(10; 2, 1)$ is isomorphic to the dodecahedron graph $\mathrm{GP}(10, 2)$.

The graphs $\mathrm{H}(m; r, s)$ from item (2) are the cubic cyclic Haar graphs (see [33]), isomorphic to the Cayley graph $\mathrm{Cay}(D_m; \{\tau, \rho^r\tau, \rho^s\tau\})$ where $D_m = \langle \rho, \tau \mid \rho^m, \tau^2, (\rho\tau)^2 \rangle$ is the dihedral group of order $2m$.

A graph $F(m; r)$ from item (3) is isomorphic to the Möbius ladder M_m if $r = 1$ or to the prism P_m when $r = 2$.

3.2 Cubic tricirculants

The study of cubic k-multicirculants with $k \geq 3$ has been limited mostly to arc-transitive graphs. In particular, cubic arc-transitive tricirculants were completely classified in [35] and, shortly after, this was extended in [22] to encompass cubic arc-transitive 4- and 5-multicirculants as well.

It is shown in [35] that only 4 cubic arc-transitive tricicrulants exist: $K_{3,3}$, the Pappus Graph, the Tutte-Coxeter graph (see Figure 3.2.3) and a graph on 54 vertices (isomorphic to the graph $\mathrm{Y}(9)$ of defined in Section 3.2.3). Meanwhile, a cubic arc-transitive 4-multicirculant is either one of 26 exceptional graphs or belongs to one of two infinite families, both arising as covers of a single 3-valent graph on 4 vertices. There are only 2 cubic arc-transitive 5-multicirculants.

This work culminated in a beautiful paper [26], where it was shown that for all square-free values of k coprime to 6 there exist only finitely many cubic arc-transitive k-multicirculants.

The more general case of vertex-transitive k-multicirculants that are not necessarily arc-transitive is much less studied, mainly due to the lack of powerful group theoretic tools, that are available only in the arc-transitive case. While cubic arc-transitive k-multicirculants are sparse, data suggest that their vertex-transitive counter parts are much more abundant.

The goal of this section is the study and classification of cubic vertex-transitive tricirculants. For convenience, we state here the classification theorem for cubic vertex-transitive tricirculants, which summarizes some of the main results of the section. The contents and results in the remainder of this chapter are taken almost verbatim from [57].

Theorem 3.2.1. *A cubic graph Γ is a vertex-transitive tricirculant if and only if the order of Γ is $6k$ for some positive integer k and one of the following holds:*

1. *Γ is of type 1, $k > 1$, k is odd and Γ is isomorphic to the graph $X(k)$, described in Section 3.2.2 (Definition 3.2.5), or Γ is isomorphic to the truncated tetrahedron $\mathrm{Tr}(\mathrm{K}_4)$ (see Figure 3.2.3).*
2. *Γ is of type 2, $k > 1$, k is odd and Γ is isomorphic to the graph $Y(k)$, described in Section 3.2.3 (Definition 3.2.29)*
3. *Γ is of type 3 and is isomorphic to the prism P_{3k} with k odd, or Γ is isomorphic to the Möbius ladder M_{3k}.*
4. *Γ is of type 4, $k = 5$ and Γ is isomorphic to the Tutte-Coxeter graph (see Figure 3.2.3).*

Lemma 3.2.2. *Up to isomorphism there are precisely four non-isomorphic cubic graphs on three vertices with no parallel semi-edges, namely the graphs $\Delta_{T_1}, \Delta_{T_2}, \Delta_{T_3}$ and Δ_{T_4} depicted in Figure 3.2.1.*

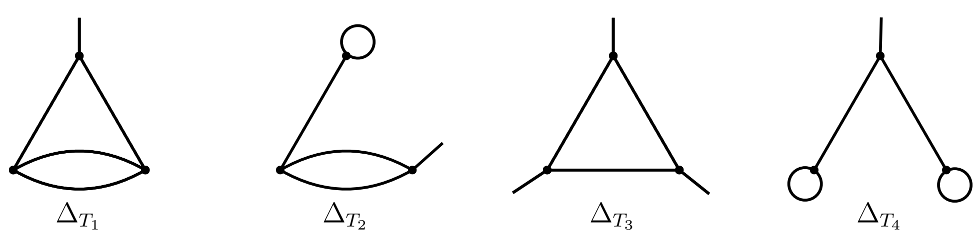


Figure 3.2.1: The four connected 3-valent graphs on three vertices without parallel semi-edges

Consider the voltage assignments $\zeta_i \colon \Delta_{T_i} \to \mathbb{Z}_{2k}$ given in Figure 3.2.2 where the value $\zeta_i(x)$ is written next to the drawing of each dart $x \in \mathrm{D}(\Delta_{T_i})$.

Lemma 3.2.3. *Let Γ be a cubic tricirculant, let ρ be a corresponding tricirculant automorphism and let n be the order of ρ. Then $n = 2k$ for some positive integer k and there exist elements $r, s \in \mathbb{Z}_n$ and $i \in \{1, 2, 3, 4\}$ such that $\Gamma \cong \mathrm{Cov}(\Delta_{T_i}, \mathbb{Z}_{2k}, \zeta_i)$ where $\zeta_i \colon \mathrm{D}(\Delta_{T_i}) \to \mathbb{Z}_{2k}$ is the voltage assignment defined by Figure 3.2.2.*

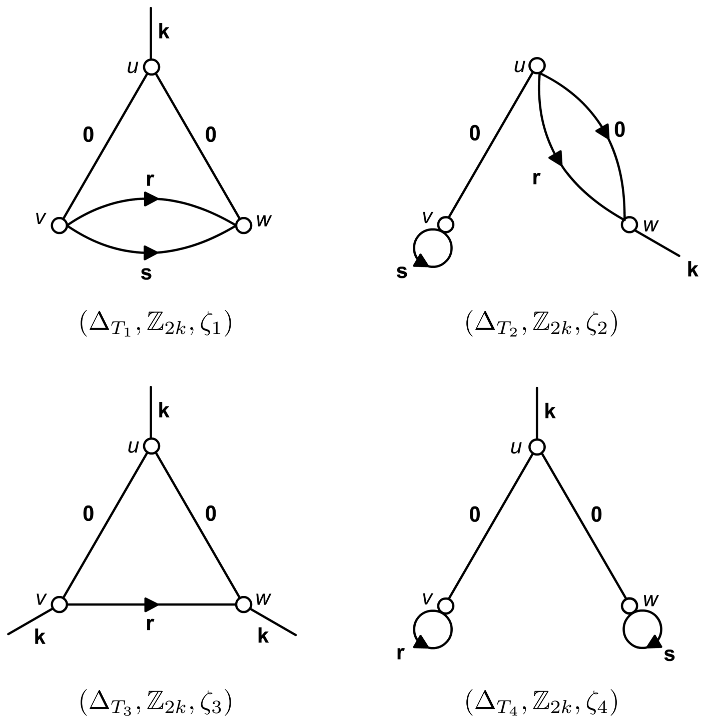


Figure 3.2.2: The voltage assignments giving rise to cubic tricirculants.

Proof. Note first that the quotient $\Gamma/\langle\rho\rangle$ is a graph with three vertices of valency 3, one for each orbit under the action of $\langle\rho\rangle$. Since ρ is a semiregular automorphism of Γ of order n, the group $\langle\rho\rangle$ is isomorphic to $\mathbb{Z}_n$ and acts semiregularly on $\mathrm{V}(\Gamma)$. By Theorem 2.3.3, $\Gamma \cong \mathrm{Cov}(\Gamma/\langle\rho\rangle, \mathbb{Z}_n, \zeta)$ for some voltage assignment $\zeta\colon \mathrm{D}(\Gamma/\langle\rho\rangle \to \mathbb{Z}_n$. Note that by definition of voltage assignments, it follows that $\zeta(x) = -\zeta(x^{-1})$ for every $x \in \mathrm{D}(\Gamma/\langle\rho\rangle)$, implying that if x is a semi-edge, then $\zeta(x)$ is an element of order at most 2 in $\mathbb{Z}_n$, and since Γ has no semi-edges, $\zeta(x)$ must in fact have order 2 in $\mathbb{Z}_n$. Since every graph with three vertices in which every vertex has valence 3 contains at least one semi-edge, it follows that $n = 2k$ for some positive integer k, and moreover, $\zeta(x) = k$ for every semi-edge x of $\Gamma/\langle\rho\rangle$. Since Γ has no parallel edges, this also implies that every vertex of $\Gamma/\langle\rho\rangle$ is the initial vertex of at most one semi-edge. By Lemma 3.2.2, $\Gamma/\langle\rho\rangle \cong \Delta_{T_i}$ for some $i \in \{1,2,3,4\}$ and we may thus assume that $\Gamma \cong \mathrm{Cov}(\Delta_{T_i}, \zeta_i)$ for some $i \in \{1,2,3,4\}$ and some voltage assignment $\zeta_i\colon \Delta_{T_i} \to \mathbb{Z}_n$. Finally, in view of Theorem 2.3.5, we may assume that $\zeta_i(x) = 0$ for every edge x belonging to a chosen spanning tree of Δ_{T_i}. In particular, ζ_i can be chosen as shown in Figure 3.2.2. $\square$

A cubic tricirculant isomorphic to $\mathrm{Cov}(\Delta_{T_i}, \mathbb{Z}_{2k}, \zeta_i)$, $i \in \{1,2,3,4\}$, is said to be of *Type* i and is denoted by $T_i(k,r,s)$, if $i \neq 3$, or $T_3(k,r)$. From Lemma 3.2.3 we get the following characterization of connected cubic tricirculants.

Theorem 3.2.4. *A cubic graph* Γ *is a tricirculant if and only if* $n = 6k$ *and* *it is isomorphic to* $T_i(k, r, s)$ *or* $T_3(k, r)$*, for some* $i \in \{1, 2, 4\}$ *and* $r, s \in \mathbb{Z}_{2k}$*. Furthermore,* $T_i(k, r, s)$ *is connected if and only if* $\gcd(2k, k, r, s) = 1$ *while* $T_3(k, r)$ *is connected if and only if* $\gcd(2k, k, r) = 1$*.*

Note that in principle, a cubic tricirculant could be of more than one type. However, each vertex-transitive cubic tricirculant is of Type i for exactly one $i \in \{1, 2, 3, 4\}$, with the exception of the triangular prism P_3 which is both of Type 1 as well as of Type 3.

The remainder of this Chapter devoted to the analysis of the tricirculants graphs arising from the voltage assignments ζ_i, and in particular, to determining sufficient and necessary conditions for vertex-transitivity. We will consider the graphs of order at most 48 separately in Section 3.2.1. As for the graphs of larger order, the analysis is divided in 4 separate sections, one for each type of cubic tricirculant. In each, we will characterize vertex-transitivity for the given type as well as other structural properties.

3.2.1 Graphs of small order

To make the classification of cubic vertex-transitive tricirculants as general and as neat as possible, we will treat cubic tricirculant graphs of small order separately. Since a cubic tricirculant must have order $6k$, for some positive k, we present, in Table 3.1, the complete list of all cubic vertex-transitive tricirculants of order at most 48 obtained from the census [56] of cubic vertex-transitive graphs. Notice that with the exception of the truncated tetrahedron, $\mathrm{Tr}(K_4)$, of order 12 and the Tutte-Coxeter graph, TC, of order 30 (see Figure 3.2.3), all vertex-transitive cubic tricirculants of order at most 48 are isomorphic to either $\mathrm{X}(k)$, $\mathrm{Y}(k)$, P_k (the prism of order $2k$) or M_k (the Möbius ladder of order $2k$) for some integer k.

Order	Type	Graph	Order	Type	Graph	Order	Type	Graph
6	1,3	P_3	18	3	M_9	36	3	M_{18}
6	3	M_3	24	3	M_{12}	42	1	$\mathrm{X}(7)$
12	1	$\mathrm{Tr}(K_4)$	30	1	$\mathrm{X}(5)$	42	2	$\mathrm{Y}(7)$
12	3	M_6	30	2	$\mathrm{Y}(5)$	42	3	P_{21}
18	1	$\mathrm{X}(3)$	30	3	P_{15}	42	3	M_{21}
18	2	$\mathrm{Y}(3)$	30	3	M_{15}	48	3	M_{24}
18	3	P_9	30	4	TC			

Table 3.1: Graphs of small order

The graphs $K_{3,3}$, $\mathrm{Y}(3)$ (the Pappus graph) and the Tutte-Coxeter graph are arc-transitive. The fourth arc-transitive cubic tricirculant, denoted F054A in [5], does not appear in Table 3.1, as it has 54 vertices; it corresponds to the graph $\mathrm{Y}(9)$ defined in Section 3.2.3. The four aforementioned graphs are the only edge-transitive cubic tricirculants. Indeed, a cubic graph that is both vertex- and edge-transitive is automatically arc-transitive [72]. A graph that is edge-transitive but not vertex-transitive is called a semi-symmetric graph, and must have two orbits of vertices under its full automorphism group [21]. It follows that no tricirculant can be semi-symmetric, as its tricirculant automorphism would mix the two vertex orbits, making the graph vertex-transitive.

For the remainder of this paper, all cubic tricirculants will have order at least 54.

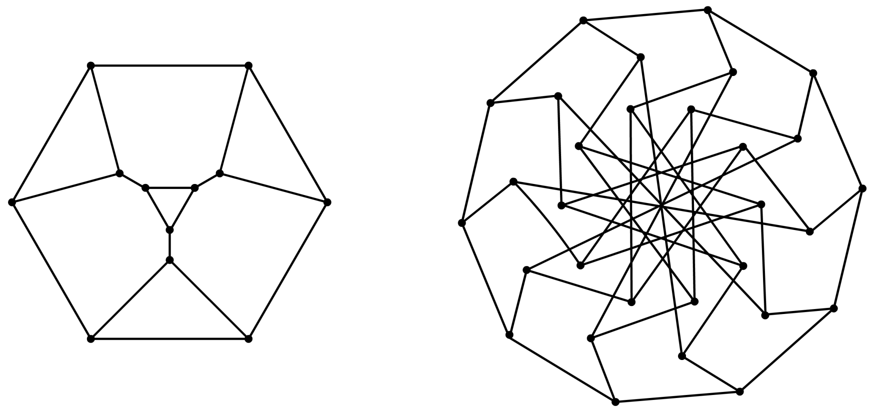


Figure 3.2.3: The truncated tetrahedron $Tr(K_4)$ and the Tutte-Coxeter graph.

3.2.2 Type 1

Let $k \geq 9$ be an integer, and let r and s be two distinct elements of $\mathbb{Z}_{2k}$. Recall that $T_1(k, r, s)$ is the covering graph arising from (Δ_{T_1}, ζ_1) shown in Figure 3.2.2; for convenience, we repeat the drawing here (see Figure 3.2.4).

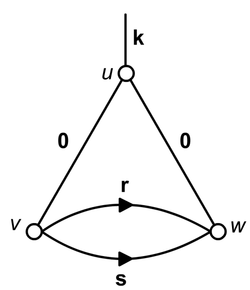


Figure 3.2.4: The voltage assignment of Δ_{T_1} giving rise to the graph $T_1(k, r, s)$.

By the definition of a derived cover, the vertices of $T_1(k, r, s)$ are then the pairs (x, i) where $x \in \{u, v, w\}$ is a vertex of Δ_{T_1} and $i \in \mathbb{Z}_{2k}$. Note that $T_1(k, r, s)$ is connected if and only if k, r and s generate $\mathbb{Z}_{2k}$, that is, if and only if $\gcd(k, r, s) = 1$.

To simplify notation, we write u_i instead of (u, i) and similarly for v_i and w_i. Then $U = \{u_0, u_1, ..., u_{2k-1}\}$, $V = \{v_0, v_1, ..., v_{2k-1}\}$ and $W = \{w_0, w_1, ..., w_{2k-1}\}$ are the respective fibres of vertices u, v and w, and the edge-set of $T_1(k, r, s)$ is then the union of the sets

$$\begin{array}{rcl} E_K & = & \{u_i u_{i+k} : i \in \mathbb{Z}_{2k}\}, \\ E_0 & = & \{u_i v_i : i \in \mathbb{Z}_{2k}\} \cup \{u_i w_i : i \in \mathbb{Z}_{2k}\}, \\ E_R & = & \{v_i w_{i+r} : i \in \mathbb{Z}_{2k}\}, \\ E_S & = & \{v_i w_{i+s} : i \in \mathbb{Z}_{2k}\}. \end{array}$$

Definition 3.2.5. For an odd integer $k > 1$, let

$$r^* = \begin{cases} \frac{k+3}{2}, & \text{if } k \equiv 1 \pmod 4 \\ \frac{k+3}{2} + k, & \text{if } k \equiv 3 \pmod 4 \end{cases}$$

and let $\mathrm{X}(k) = T_1(k, r^*, 1)$.

We can now state the main theorem of this section.

Theorem 3.2.6. *Let $k \geq 9$ and let Γ be a connected cubic tricirculant of Type 1 with $6k$ vertices. Then the following holds:*

1. *Γ is vertex-transitive if and only if it is isomorphic to $\mathrm{X}(k)$ with k odd.*
2. *If Γ is not vertex-transitive, then it has two vertex orbits under its full automorphism group, one twice the size of the other.*

The next theorem gives more information about the graph $\mathrm{X}(k)$.

Theorem 3.2.7. *Let k be an odd integer, $k > 1$. Then the following holds:*

1. *$\mathrm{X}(k)$ is a bicirculant if and only if $3 \nmid k$, in which case it is isomorphic to the generalized Petersen graph $\mathrm{GP}(3k, k + (-1)^\alpha)$ where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod 3$.*
2. *$\mathrm{X}(k)$ has arc-type $2 + 1$.*

The rest of the section is devoted to the proof of Theorems 3.2.6 and 3.2.7. Clearly, as a tricirculant of Type 1, Γ is isomorphic to $T_1(k, r, s)$ for some integer k and elements $r, s \in \mathbb{Z}_{2k}$ (see Lemma 3.2.3). We shall henceforth assume that $\Gamma = T_1(k, r, s)$ for some $k \geq 9$ and that Γ is connected. For a symbol X from the set of symbols $\{0, R, S, K\}$, edges in E_X will be called edges of type X, or simply X-edges.

Define ρ as the permutation given by $\rho(u_i) = u_{i+1}$, $\rho(v_i) = v_{i+1}$ and $\rho(w_i) = w_{i+1}$, and note that ρ is a tricirculant automorphism of $T_1(k, r, s)$. Further, observe that

$$T_1(k, r, s) \cong T_1(k, s, r), \tag{3.2.1}$$

and if $a \in \mathbb{Z}$ is such that $\gcd(2k, a) = 1$, then

$$T_1(k, ar, as) \cong T_1(k, r, s). \tag{3.2.2}$$

Lemma 3.2.8. *Let k be an odd integer, $k > 1$, and let r^* and $\mathrm{X}(k)$ be as in Definition 3.2.5. Then the graph $\mathrm{X}(k)$ is vertex-transitive.*

Proof. Recall that $\mathrm{X}(k) \cong T_1(k, r^*, 1)$. Define ϕ as the mapping given by:

$$\begin{array}{llll} u_i \mapsto w_{i-r^*+2}, & w_i \mapsto v_{i-2r^*+2}, & v_i \mapsto u_{i-r^*+2} & \text{if } i \text{ is even;} \\ u_i \mapsto v_{i+r^*-2}, & v_i \mapsto w_{i+2r^*-2}, & w_i \mapsto u_{i+r^*-2} & \text{if } i \text{ is odd.} \end{array}$$

That ϕ is indeed an automorphism follows from the fact that k is odd, r^* is even and the congruence $k+3-2r^* \equiv 0$ holds. Since ϕ transitively permutes the three $\langle\rho\rangle$-orbits U, V and W, the group $\langle\rho,\phi\rangle$ acts transitively on the vertices of $T_1(k,r^*,1)$. □

Lemma 3.2.9. *For $k > 1$, $\mathrm{X}(k)$ has arc-type $2+1$.*

Proof. Let G be the full automorphism group of $\mathrm{X}(k)$ and consider the automorphism φ interchanging u_i with u_{-i} and v_i with w_{-i}, $i \in \mathbb{Z}_{2k}$. Note that φ is an element of G_{u_0}, the stabiliser of u_0. Moreover, φ fixes u_k but interchanges v_0 with w_0. That is, φ fixes one of the neighbours of u_0 while interchanging the remaining two. This implies that the arc-type of $\mathrm{X}(k)$ is either 3, when $\mathrm{X}(k)$ is arc-transitive, or $2+1$. However, there are no arc-transitive graphs of Type 1. The result follows. □

Remark 3.2.10. The automorphism given in the proof above is an automorphism of $\Gamma \cong T_1(k,r,s)$, from which we see that if Γ is not vertex-transitive, then $V \cup W$ is a vertex orbit under its full automorphism group $\mathrm{Aut}(\Gamma)$. Then Γ must have two vertex orbits: U and $V \cup W$. This shows that item (2) of Theorem 3.2.6 holds.

Now, recall that Γ is connected and equals $T_1(k,r,s)$, for some $k \geq 9$, $r,s \in \mathbb{Z}_{2k}$. We may also assume that Γ is vertex-transitive, however, for some of the results in this section vertex-transitivity is not needed. When possible, we will not assume the graph to be vertex-transitive, but rather only to have some weaker form of symmetry, that we will define in the following paragraphs.

A simple graph Γ is said to be *c-vertex-regular*, for some $c \geq 3$, if there are the same number of c-cycles through every vertex of Γ.

For a dart x of Γ and a positive integer c denote by $\epsilon_c(x)$ the number of c-cycles that contain x. For a vertex v of Γ, let $\{x_1,x_2,x_3\}$ be the set of darts beginning at v ordered in such a way that $\epsilon_c(x_1) \leq \epsilon_c(x_2) \leq \epsilon_c(x_3)$. The triplet $(\epsilon_c(x_1),\epsilon_c(x_2),\epsilon_c(x_3))$ is then called the *c-signature* of v. If for $c \geq 3$ all vertices in Γ have the same c-signature, then we say Γ is *c-cycle-regular* and the signature of Γ is the signature of any of its vertices. Note that for every $c \geq 3$, a vertex-transitive graph is necessarily c-cycle-regular, and every c-cycle-regular graph is c-vertex-regular.

If c equals the girth of the graph, then following [60], a c-cycle-regular graph will be called *girth-regular*.

Lemma 3.2.11. *If Γ is 4-vertex-regular, then neither r nor s equals k.*

Proof. Suppose $r = k$. Since Γ has no parallel edges, we see that $s \neq k$. Observe that (u_0,v_0,w_r,u_r) and (u_0,w_0,v_{-r},u_{-r}) are the only 4 cycles of Γ through u_0.

Meanwhile, there exists a unique 4-cycle through v_0, namely (v_0,w_r,u_r,u_0), which contradicts Γ being 4-vertex-regular. Hence $r \neq k$, and since $T_1(k,s,r) \cong T_1(k,r,s)$ (see (3.2.1)), this also shows that $s \neq k$. □

For the sake of simplicity denote a dart in Δ_{T_1} starting at vertex a, pointing to vertex b and having voltage x by $(ab)_x$. Recall that formally a walk in Δ_{T_1} is a sequence of darts $(x_1, x_2, ..., x_n)$, for instance $((vw)_r, (wu)_0, (uu)_k)$ is a walk of length 3. However, since Γ is a simple graph a walk in Γ will be denoted as a sequence of vertices, as it is normally done.

Lemma 3.2.12. *If Γ is 8-cycle-regular, then neither r nor s equals 0.*

Proof. Suppose $r = 0$. Since Γ has no parallel edges, we see that $s \neq 0$. Suppose there is an 8-cycle C in Γ. Such a cycle, when projected to Δ_{T_1}, yields a reduced closed walk ω in Δ_{T_1} whose ζ_1-voltage is 0. Note that ω cannot trace three darts with voltage 0 consecutively, as this would lift into a 3-cycle contained in C. This implies ω necessarily visits the dart $(vw)_s$ or its inverse at least once. Furthermore, since $\gcd(k, s) = 1$ and $8 < 9 \leq k$, ω must trace $(wv)_{-s}$ as many times as it does $(vw)_s$. By observing Figure 3.2.4, the reader can see that if ω traces the dart $(vw)_s$, then it must also trace the semi-edge $(uu)_k$ before tracing $(wv)_{-s}$, as ω cannot trace three consecutive darts with voltage 0. Since ω has net voltage 0, it necessarily traces $(uu)_k$ an even number of times. Moreover, ω must trace a dart in $\{(uv)_0, (vu)_0, (uw)_0, (wu)_0\}$ immediately before and immediately after tracing $(uu)_k$. Hence, ω must visit the set $\{(vw)_s, (wv)_{-s}\}$ at least twice; the semi-edge $(uu)_k$ at least twice; and the set $\{(uv)_0, (vu)_0, (uw)_0, (wu)_0\}$ at least 4 times. This already amounts to 8 darts, none of which is $(vw)_r$ or its inverse. Therefore, no 8-cycle in Γ visits an R-edge. However, for $X \neq R$ there is at least one 8-cycle through every X-edge, as $(u_0, v_0, w_s, u_s, u_{s+k}, w_{s+k}, v_k, u_k)$ and $(u_0, w_0, v_{-s}, u_{-s}, u_{k-s}, v_{k-s}, w_k, u_k)$ are 8-cycles in Γ. This contradicts our hypothesis of Γ being 8-cycle-regular. The proof when $s = 0$ is analogous. □

Now, observe that $(u_0, u_k, v_k, w_{k+r}, u_{k+r}, u_r, w_r, v_0, u_0)$ is a cycle of length 8 in Γ starting at u_0. In what follows we will study the 8-cycle structure of Γ to determine conditions for vertex-transitivity. Recall that each 8-cycle in Γ projects into a closed walk of length 8 having net voltage 0 in Δ_{T_1}. We can thus, provided we are careful, determine how many 8-cycles pass through any given vertex of Γ by counting closed walks of length 8 and net voltage 0 in the quotient Δ_{T_1}. Note that only closed walks that are reduced will lift into cycles of Γ. Therefore, we may safely ignore non-reduced walks and focus our attention exclusively on reduced ones. Define $\mathcal{W}_8$ as the set of all reduced closed walks of length 8 in Δ_{T_1}.

Every element of $\mathcal{W}_8$ having net voltage 0 lifts into a closed walk of length 8 in Γ. In principle, the latter walk might not be a cycle of Γ. However, we will show that this never happens in our particular case. For this, it suffices to show that Γ does not admit cycles of length 4 or smaller, as any closed walk of length 8 that is not a cycle is a union of smaller cycles, one of which will have length smaller or equal to 4.

Lemma 3.2.13. *Γ has girth at least 5.*

Proof. Suppose Γ has a 3-cycle. Since Γ is vertex-transitive, this would mean that there is a reduced closed walk with net voltage 0 of length 3 in Δ_{T_1} that visits u. From Figure 3.2.4 we get that such a closed walk must be either $((uv)_0, (vw)_r, (wu)_0)$, $((uv)_0, (vw)_s, (wu)_0)$ or one of their inverses. This would imply that either $r = 0$ or $s = 0$, a

Label	Net voltage	Starting at u	Starting at v
I	0	12	10
II	$\pm(2r)$	8	8
III	$\pm(2s)$	8	8
IV	$\pm(r+s)$	8	4
V	$\pm(r-s)$	8	4
VI	$\pm(k+2r-s)$	12	10
VII	$\pm(k+2s-r)$	12	10
VIII	$\pm(k+3r-2s)$	4	6
IX	$\pm(k+3s-2r)$	4	6
X	$\pm(3r-s)$	4	6
XI	$\pm(3s-r)$	4	6
XII	$\pm(2r-2s)$	4	6
XIII	$\pm(4r-4s)$	0	2

Table 3.2: Net voltages of closed walks of length 8 in Δ_{T_1}.

contradiction. Similarly, the existence of 4-cycles in Γ would imply there is a closed walk with net voltage 0 of length 4 visiting u. The only such walks are $((uv)_0, (vw)_r, (wu)_0, (uu)_k))$, $((uu)_k, (uv)_0, (vw)_r, (wu)_0)$, $((uv)_0, (vw)_s, (wu)_0, (uu)_k))$, $((uv)_0, (vw)_s, (wu)_0, (uu)_k))$ and their inverses. In all eight cases, we have that $r = k$ or $s = k$, contradicting Lemma 3.2.11. $\square$

Now, let N be the set of the net voltages of walks in $\mathcal{W}_8$, expressed as linear combinations of k, r and s, where we view r and s as indeterminants over $\mathbb{Z}_{2k}$. For instance, the closed walk $((uv)_0, (vw)_r, (wv)_{-s}, (vw)_r, (wu)_0, (uv)_0, (vw)_r, (wu)_0)$ has net voltage $3r - s$. For an element $\nu \in N$, let $\mathcal{W}(\nu)$ be the set of walks in $\mathcal{W}$ with net voltage ν and for a vertex x of Δ_{T_1}, let $\mathcal{W}_8^x(\nu)$ be the set of walks in $\mathcal{W}(\nu)$ starting at x.

For each $\nu \in N$, we have computed the number of elements in $\mathcal{W}_8^u(\nu)$ and in $\mathcal{W}_8^v(\nu)$. The result is displayed in Table 3.2. Notice that, if $\omega \in \mathcal{W}_8(\nu)$, then $\omega^{-1} \in \mathcal{W}_8^x(-\nu)$. This means ω has voltage 0 if and only if ω^{-1} does too. For this reason, and since we are interested in walks with net voltage 0, we have grouped walks with net voltage ν along with their inverses having net voltage $-\nu$. This computation is straightforward but somewhat lengthy. For this reason and to avoid human error, we have done it with the help of a computer programme written in SAGE [70].

From the first Row of Table 3.2 we know that there are always 12 walks starting at u in $\mathcal{W}_8$ that have net voltage 0, regardless of the values of r and s. Similarly, there are always 10 such walks starting at v. Since Γ is vertex-transitive, the number of walks in $\mathcal{W}_8$ having net voltage 0 (after evaluating r and s in $\mathbb{Z}_{2k}$) starting at u and those starting at v must be the same. It follows that, for some $\nu \in N$ with $|\mathcal{W}_8^v(\nu)| > |\mathcal{W}_8^u(\nu)|$, the equation $\nu \equiv 0 \pmod{2k}$ holds when we evaluate r and s in $\mathbb{Z}_{2k}$. We thus see that at least one expression in VIII–XIII in Table 3.2 is congruent to 0 modulo $2k$. In fact, if Γ is vertex-transitive, then at most one of these expressions can be congruent to 0.

Lemma 3.2.14. *If Γ is 4-vertex-regular and 8-cycle-regular, then exactly one of the following equations modulo $2k$ holds:*

$$3s - 2r + k \equiv 0 \tag{3.2.3}$$
$$3r - 2s + k \equiv 0 \tag{3.2.4}$$
$$3r - s \equiv 0 \tag{3.2.5}$$
$$3s - r \equiv 0 \tag{3.2.6}$$
$$4r - 4s \equiv 0 \tag{3.2.7}$$

Proof. We will slightly abuse notation and, for a label $x \in \{\mathrm{I}, \mathrm{II}, ..., \mathrm{XIII}\}$, we will refer by x to the congruence equation modulo $2k$ obtained by making the expression labelled x in Table 3.2 congruent to 0. For instance, the equation $2r \equiv 0$ will be referred to as II.

First note that if II or III holds then $r = k$ or $s = k$, contradicting Lemma 3.2.11. If V holds then (v_0, w_r, v_0) is a 2-cycle in Γ, which is not possible. Similarly, XII implies the existence of a 4-cycle in Γ, contradicting Lemma 3.2.13. Now, if XIII holds, then either $2r - 2s \equiv 0$, which is not possible, or $2r - 2s + k \equiv 0$, which gives (4.8).

If VI holds, then neither X nor XI can hold as this would imply $2r - 2s \equiv 0$ (subtracting IV from X or from XI, respectively). Therefore, IV excludes X and XI. If IV and I are the only equations to hold, we would have 6 more elements of $\mathcal{W}_8$ through u than through v. This means that if IV holds then necessarily VII, IX and XII also hold. However, XII implies $2r-2s+k \equiv 0$ and subtracting this from VIII we get $r = 0$, in contradiction with Lemma 3.2.12. Hence, VII, IX and XII cannot all hold at the same time and so IV can never hold.

Suppose VI holds, then one of the equations in $\{\mathrm{VIII}, \mathrm{IX}, \mathrm{X}, \mathrm{XI}, \mathrm{XIII}\}$ must also hold. Note that VI and VIII imply III; VI and IX imply V; VI and X imply $r = k$; VI and XIII imply $s = k$. Therefore, by Lemma 3.2.11, only XI can hold. But VI and XI imply VII, and so we would have 40 elements of $\mathcal{W}_8$ starting at u but only 36 starting at v. This would contradict Γ being 8-cycle-regular. Thus VI cannot hold. An analogous reasoning, where the roles of r and s are interchanged, shows that VII cannot hold. We have shown that the only equations that can hold are in $\{\mathrm{VIII}, \mathrm{IX}, \mathrm{X}, \mathrm{XI}, \mathrm{XIII}\}$. However, if two or more of these equations hold, then we would have more elements of $\mathcal{W}_8$ through v than we would have through u. It follows that exactly one equation in $\{\mathrm{VIII}, \mathrm{IX}, \mathrm{X}, \mathrm{XI}, \mathrm{XIII}\}$ holds. □

Lemma 3.2.14 tells us, for each of the 5 possible equations, exactly which walks in $\mathcal{W}_8$ will lift to 8-cycles and so we can count exactly how many 8-cycles pass through a given vertex or edge of Γ. For instance, if $3s - 2r + k \equiv 0$, then there are 16 walks in $\mathcal{W}_8$ through u that will lift to an 8-cycle (see Table 3.2). Since a closed walk in $\mathcal{W}_8$ and its inverse lift to the same cycle in Γ, we see that there are 8 cycles of length 8 through every vertex in Γ. Moreover, there are exactly 5 cycles of length 8 through every edge of type R or 0 while there are 6 such cycles through each edge of type K or S. It follows that the 8-signature of a vertex x of Γ is $(5, 5, 6)$ whenever $3s - 2r + k \equiv 0$ and thus, in this case, Γ is 8-cycle-regular with 8-signature $(5, 5, 6)$. Table 3.3 shows the number of 8-cycles through each vertex and through each edge, depending on its type, for each of the 5 cases described in Lemma 3.2.14.

Congruence	*0*-edge	R-edge	S-edge	K-edge	8-signature
$3s-2r+k$	5	5	6	6	$(5,5,6)$
$3r-rs+k$	5	6	5	6	$(5,5,6)$
$3r-s$	6	6	4	4	$(4,6,6)$
$3s-r$	6	4	6	4	$(4,6,6)$
$4r-4s$	4	4	4	4	$(4,4,4)$

Table 3.3: Number of 8-cycles through each edge-type of $T_1(k,r,s)$

We will now show that equations 3.2.5, 3.2.6 and 3.2.7 cannot hold when Γ is vertex-transitive. This will be proved in Lemmas 3.2.16, 3.2.17 and 3.2.18. But first we need to show a result about the subgraph induced by the *0*- and R-edges.

Recall that that $\Gamma = T_1(k,r,s)$ with $k \geq 9$ and that it is connected. Henceforth we also assume that Γ is vertex-transitive. Denote by $\Gamma_{0,R}$ the subgraph that results from deleting the edges of type S and K from Γ. Equivalently, $\Gamma_{0,R}$ is the subgraph induced by *0*- and R-edges. We see that $\Gamma_{0,R}$ is 2-valent and that it has $\gcd(2k,r)$ connected components, each of which is a cycle of length $6k/\gcd(2k,r)$.

Since $\Gamma_{0,R}$ only has edges of type *0* and R, any two vertices in the same connected components must have indices that differ in a multiple of r. It is not hard to see that, indeed, each connected component consist precisely of all those vertices whose indices are congruent modulo $\gcd(2k,r)$. We now prove an auxiliary result that will be used both in the case where (3.2.5) as well as in the case where (3.2.3) or (3.2.4) holds.

Suppose that no automorphism of Γ maps an S-edge to a *0*-edge. Since Γ is vertex-transitive, there is an automorphism that maps a vertex from V to a vertex in U. Such an automorphism then maps an S-edge to a K-edge. Since all S-edges as well as all the K-edges are in the same orbit, this then implies that $E_S \cup E_K$ forms a single edge-orbit of $\mathrm{Aut}(\Gamma)$.

Lemma 3.2.15. *Suppose that no automorphism of Γ maps an S-edge to a 0-edge, then $\Gamma_{0,R}$ is disconnected, and the set of connected components induce a block system for the vertex-set of Γ.*

Proof. By the discussion above the lemma, $E_S \cup E_K$ forms a single edge-orbit in Γ. Now note that $\mathrm{Aut}(\Gamma)$ also acts as a group of automorphism on $\Gamma_{0,R}$ and that this action is transitive, since the edges removed consist of an edge-orbit of Γ. It follows that connected components of $\Gamma_{0,R}$ form a block system for the vertex set of Γ. Now, suppose $\Gamma_{0,R}$ consists of a single cycle of length $6k$, $(w_0, u_0, v_0, w_r, u_r, v_r \dots v_{(2k-1)r})$. Notice that the vertex antipodal to u_0 in this cycle is u_{kr} which is the same as u_k, as r is odd. We see that edges of type K in Γ join vertices that are antipodal in $\Gamma_{0,R}$, while S-edges do not. Since this $6k$-cycle is a block of imprimitivity, K-edges can only be mapped into K-edges and thus U is an orbit of $\mathrm{Aut}(\Gamma)$, contradicting the assumption that Γ is vertex-transitive. Hence, $\Gamma_{0,R}$ is disconnected. $\square$

Lemma 3.2.16. *If $3r - s \equiv 0 \pmod{2k}$, $k \geq 9$, then $T_1(k,r,s)$ is not vertex-transitive.*

Proof. Let $\Gamma = T_1(k,r,s)$ where $3r - s \equiv 0 \pmod{2k}$. A computer assisted counting shows that S-edges and K-edge have exactly 4 distinct 8-cycles passing through them, while *0*- and R-edges have 6 such cycles (see Table 3.3). It follows that any automorphism sending a vertex $u \in U$ to a vertex $v \in V$ must necessarily map the K-edge incident to u into the S-edge incident to v. Thus $E_S \cup E_K$ is an orbit of edges under the action of $\mathrm{Aut}(\Gamma)$.

Since we are assuming that $3r - s \equiv 0$ we get that any number dividing k and r must also divide s, but since $\gcd(k,r,s) = 1$, we have that $\gcd(k,r) = 1$ and then $\gcd(2k,r) \in \{1,2\}$. In light of Lemma 3.2.15, $\Gamma_{0,R}$ is disconnected, implying $\gcd(2k,r) \neq 1$ and therefore $\gcd(2k,r) = 2$. From the congruence $3r - s \equiv 0$ we also get that r and s must have the same parity thus making s even and k odd. Recall that each connected component of $\Gamma_{0,R}$ consists of all the vertices whose indices are congruent modulo $\gcd(2k,r)$, so $\Gamma_{0,R}$ has two connected components: one containing all the vertices with even index, and the other containing those with odd index. Since s is even and k is odd, S-edges in Γ join vertices with same parity, while K-edges join vertices with distinct parity. We know that each of the two connected components of $\Gamma_{0,R}$ is a block of imprimitivity of Γ, an so any automorphism of Γ must either preserve the parity of all indices, or of none at all. It follows that no automorphism can send S-edges into K-edges, and therefore no automorphism can send a vertex in V to a vertex in U. We conclude Γ cannot be vertex-transitive. □

Lemma 3.2.17. *If $3s - r \equiv 0 \pmod{2k}$, $k \geq 9$, then $T_1(k,r,s)$ is not vertex-transitive.*

Proof. Set $\Gamma = T_1(k,r,s)$ and $\Gamma' = T_1(k,r',s')$, where $r' = s$ and $s' = r$. Notice that the triplet (k,r',s') satisfies equation (3.2.5) and so, by Lemma 3.2.16, Γ' cannot be vertex-transitive. By observation (3.2.1), $\Gamma \cong \Gamma'$. Therefore Γ is not vertex-transitive. □

Lemma 3.2.18. *If $4r - 4s \equiv 0 \pmod{2k}$, $k \geq 9$, then $T_1(k,r,s)$ is not vertex-transitive.*

Proof. Let $\Gamma = T_1(k,r,s)$, with $4r - 4s \equiv 0 \pmod{2k}$ and $k \geq 9$. From the congruence $4r-4s \equiv 0$, we see that either $2r-2s \equiv 0$ or $2r-2s+k \equiv 0$, but the former contradicts Lemma 3.2.13. Hence $2r - 2s + k \equiv 0$ and the closed walk $((uv)_0, (vw)_r, (wv)_{-s}, (vw)_r, (wv)_{-s}, (vu)_0, (uu)_k)$ in Δ_{T_1} lifts to a 7-cycle in Γ. We will show that Γ cannot be 7-vertex-regular. Following a similar procedure as the one used to obtain Table 3.2, we have counted all closed walks of length 7 in Δ_{T_1} starting at u and those starting at v, and we have computed their net voltage. The result is displayed in Table 3.4.

It is plain to see that if, in addition to I, any one of the voltages in Rows III–VI is congruent to 0, then either $r = 0$, $r = k$, $s = 0$ or $s = k$, contradicting Lemmas 3.2.11 or 3.2.12. We see that only the voltages in Row I or II can hold, and thus Γ is not 7-vertex-regular and hence cannot be vertex-transitive. □

In what follow we deal with the case where (3.2.3) holds, that is, when $3s - 2r + k \equiv 0 \pmod{2k}$ and $k \geq 9$. From Table 3.3 we see that S-edges and K-edge have exactly 6 distinct 8-cycles passing through them, while *0*- and R-edges have only 5. In particular, no automorphism of Γ maps an S-edge to

Label	Net voltage	Starting at u	Starting at v
I	$\pm(k+2r-2s)$	8	10
II	$\pm(k+r+s)$	12	8
III	$\pm(k+2r)$	6	4
IV	$\pm(k+2s)$	6	4
V	$\pm(3r-2s)$	2	6
VI	$\pm(3s-2r)$	2	6

Table 3.4: Net voltages of closed walks of length 7 in Δ_{T_1}.

a *0*-edge, implying that $E_S \cup E_K$ is an edge-orbit of Γ (see the discussion above Lemma 3.2.15).

Lemma 3.2.19. *Suppose that* $3s-2r+k \equiv 0 \pmod{2k}$*,* $k \geq 9$*. If* e *is an edge in* $E_K \cup E_S$*, then the endpoints of* e *belong to different connected components of* $\Gamma_{0,R}$*.*

Proof. Suppose a K-edge e has both of its endpoints in the same connected component, C, of $\Gamma_{0,R}$. Then, because Γ is vertex-transitive and C is a block of imprimitivity, both of the endpoints of any K-edge must belong to the same connected component of $\Gamma_{0,R}$. This means that the subgraph induced by *0*-edges, R-edges and K-edges is disconnected. Recall that K-edges and S-edges belong to a single edge-orbit in Γ and thus, for every S-edge, e', there exists $\phi \in \mathrm{Aut}(\Gamma)$ such $\phi(e) = e'$. Since ϕ also acts as an automorphism on $\Gamma_{0,R}$, it follows that the endpoints of $\phi(e)$ are contained in the component $\phi(C)$, thus making Γ a disconnected graph, which is a contradiction. □

Lemma 3.2.20. *Suppose that* $3s-2r+k \equiv 0 \pmod{2k}$*,* $k \geq 9$*. Then the subgraph* $\Gamma_{0,R}$ *has an even number of connected components.*

Proof. Let e be a K-edge whose endpoints are in two different connected components C_1 and C_2. Let $\rho^r \in \mathrm{Aut}(\Gamma)$ be an "r-fold" rotation. That is, ρ^r adds r to the index of every vertex. It is plain to see that ρ^r fixes the connected components of $\Gamma_{0,R}$ set-wise and that it send K-edges into K-edges. Moreover, since it is transitive on $U \cap C_1$, we have that all K-edges having an endpoint in C_1 will have the other endpoint in C_2. Hence K-edges "pair up" connected components of $\Gamma_{0,R}$. The result follows. □

Proposition 3.2.21. *If* $T_1(k,r,s)$ *is connected and vertex-transitive, then the following statements (or the analogous three statements obtained by interchagning* r *and* s*) hold:*

1. $3s-2r+k \equiv 0 \pmod{2k}$
2. k *and* s *are odd and* $\gcd(k,s) = 1$
3. r *is even and* $\gcd(k,r) = \{1,3\}$.

Proof. First, note that item (1) follows from the combination of Lemma 3.2.14 and Lemmas 3.2.16, 3.2.17 and 3.2.18.

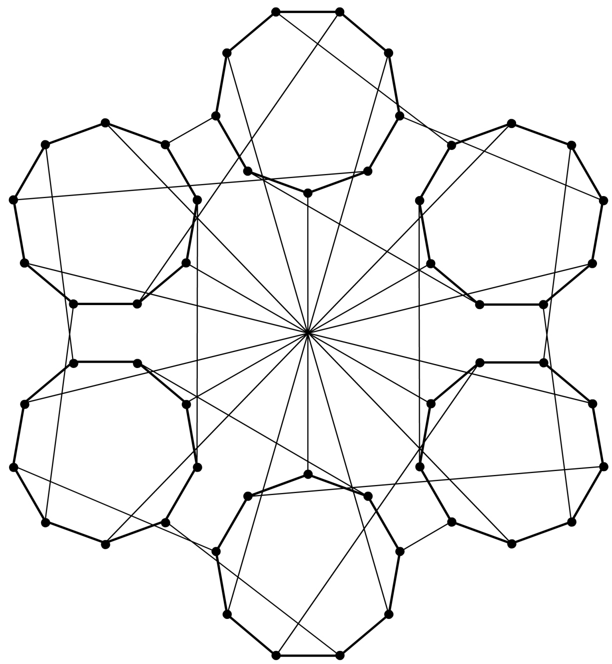


Figure 3.2.5: $T_1(9,6,1)$. Bold edges correspond to 0- and R-edges.

We will continue by proving (3). Set $d = \gcd(k,r)$, so that $r = dr'$ and $k = dk'$ for some integers k' and r'. Since $3s - 2r + k \equiv 0 \pmod{2k}$, we see that, for some odd integer a:

$$3s = d(ak' + 2r')$$

By connectedness, $\gcd(k,r,s) = 1$, and hence $\gcd(d,s) = 1$ implying that d divides 3. Hence $d = \gcd(k,r) \in \{1,3\}$. Moreover, since $\Gamma_{0,R}$ has $\gcd(2k,r)$ connected components and this number must be even by Lemma 3.2.20, we see that $\gcd(2k,r) \in \{2,6\}$, and therefore r is even.

Now, from the congruence $3s - 2r + k \equiv 0 \pmod{2k}$ we see that s and k are either both even, or both odd. Since $\gcd(2k,r,s) = 1$ and r is even, s must necessarily be odd and then k is also odd. Set $d' := \gcd(k,s)$ so that $s = d's'$ and $k = d'k'$ for some integers s' and k'. Again, from equation (3.2.3) we obtain

$$2r = d'(ak' + 3s')$$

for some odd integer a. Thus $d' \in \{1,2\}$, but since s is odd, we see that $\gcd(k,s) = 1$. This proves item (2) and completes the proof. $\square$

Remark 3.2.22. Conditions (1)–(3) (or their analogues) in Proposition 3.2.21 are also sufficient to prove vertex-transitivity of a connected graph $T_1(k,r,s)$. This will be shown after the proof on Lemma 3.2.24. Moreover, observe that if k and s satisfy condition (2) of Lemma 3.2.21, then $\gcd(2k,1) = 1$ and $T_1(k,r,s)$

is isomorphic to $T_1(k, rs^{-1}, 1)$ where s^{-1} is the multiplicative inverse of s in $\mathbb{Z}_{2k}$. Hence, in what follows we may assume that $s = 1$.

Lemma 3.2.23. *Let k be an odd number and set $s = 1$. There exists a unique element $r \in \mathbb{Z}_{2k}$ such that conditions (1) and (3) of Proposition 3.2.21 are satisfied. This unique r equals r^* from Definition 3.2.5.*

Proof. Suppose that such r exists. From condition (1) of Lemma 3.2.23, we get $r = (3 - ak)/2$ for some odd integer a. Since r must be a positive integer smaller than $2k$, it follows that $a \in \{-1, -3\}$. Hence $(3+k)/2$ and $(3+3k)/2 = (3+k)/2+k$ are the only possible candidates for r. Since k is odd, the integers $(3 + k)/2$ and $(3 + k)/2 + k$ have different parity. Set r to be whichever one of these two expressions is even. Note that any odd number that divides both r and k must divide 3 and that this is true whether r equals $(3 + k)/2$ or $(3 + k)/2 + k$. Thus condition (3) of Lemma 3.2.23 is satisfied and r equals r^* from Definition 3.2.5. □

We have just proved that once an odd $k \geq 9$ is prescribed, then there is at most one graph $T_1(k, r, 1)$ that is connected and vertex-transitive. The following lemma generalizes this to an arbitrary value of s.

Lemma 3.2.24. *Let $k \geq 9$ be an odd integer and let $T_1(k, r, s)$ be vertex-transitive. Then $T_1(k, r, s) \cong \mathrm{X}(k)$.*

Proof. Recall that $\mathrm{X}(k) \cong T_1(k, r^*, 1)$, where r^* is as in Definition 3.2.5. Since $T_1(k, r, s)$ is connected and vertex-transitive, conditions (1)–(3) of Proposition 3.2.21 hold. In particular, s is relatively prime to $2k$. Then $T_1(k, r, s) \cong T_1(k, s^{-1}r, 1)$, where s^{-1} is the multiplicative inverse of s in $\mathbb{Z}_{2k}$. By Lemma 3.2.23, $T_1(k, s^{-1}r, 1) = T_1(k, r^*, 1)$. □

Notice that in the previous proof vertex-transitivity is only used to ensure that the graph $T_1(k, r, s)$ satisfies conditions (1)–(3) of Proposition 3.2.21, and from there $T_1(k, r, s)$ is shown to be isomorphic to $T_1(k, r^*, 1)$. Since we know from Lemma 3.2.8 that $T_1(k, r^*, 1)$ is vertex-transitive, we have that conditions (1)–(3) of Proposition 3.2.21 are not only necessary, but also sufficient for vertex-transitivity, and thus Proposition 3.2.21 may be regarded a characterization theorem.

Remark 3.2.25. For $k \geq 9$, $\mathrm{X}(k)$ has girth 8. The proof of this fact is straightforward but lengthy and for this reason we decide to omit it. Note as well that $\mathrm{X}(k)$ is not a bipartite graph as every connected component of $\mathrm{X}(k)_{0,R}$ is a cycle of odd length.

We have now completed the proof of Theorem 3.2.6. Item (1) follows from Lemmas 3.2.8 and 3.2.24 while item (2) follows from Remark 3.2.10. Now that we have a characterization for vertex-transitivity, we would like to know when a cubic vertex-transitive tricirculant of Type 1 is also a bicirculant. Note that the only cubic vertex-transitive circulants are prisms and Möbius ladders, which have girth 4. It follows from Lemma 3.2.13 that no cubic vertex-transitive tricirculant of Type 1 is a circulant.

Lemma 3.2.26. *Let $\Gamma = T_1(k,r,s)$ be connected and vertex-transitive. If $3 \nmid k$, then Γ is a bicirculant.*

Proof. Let $\varphi : V(\Gamma) \to V(\Gamma)$ be the mapping given by:

$$\begin{array}{llll} u_i \mapsto v_i, & v_i \mapsto w_{i+r}, & w_i \mapsto u_i & \text{if } i \text{ is even;} \\ u_i \mapsto w_{i+k+s}, & v_i \mapsto u_{i+k+s}, & w_i \mapsto v_{i+k+s-r} & \text{if } i \text{ is odd.} \end{array}$$

It can be readily seen that φ is indeed a graph automorphism. Moreover, since r is even and both k and s are odd, φ preserves the parity of the index of all vertices. In fact, it is plain to see that φ is the product of the following two disjoint permutation cycles ϕ_1 and ϕ_2 of length $3k$:

$$\begin{array}{lcl} \phi_0 & = & (u_0, v_0, w_r, u_r, v_r, w_{2r}, u_{2r}, v_{2r}, ..., u_{(k-1)r}, v_{(k-1)r}, w_0); \\ \phi_1 & = & (u_k, w_s, v_{k+2s-r}, u_{k+r}, w_{r+s}, v_{k+2s}, u_{k+2r} ... v_{k+2s+(k-2)r}). \end{array}$$

Hence, Γ is a bicirculant. □

Lemma 3.2.27. *If $3|k$, then* X(k) *is not a bicirculant.*

Proof. Suppose X(k) admits a semiregular automorphism φ having two orbits, O_1 and O_2 of size $3k$. Following the notation in [54], X(k) must be isomorphic to $\mathrm{H}(3k,i,j)$, $\mathrm{I}(3k,i,j)$ or $\mathrm{F}(3k,i)$ for some $i,j \in \mathbb{N}$. Since X(k) is not bipartite, it cannot be isomorphic to $\mathrm{H}(3k,i,j)$. Further, if $\mathrm{X(k)} \cong \mathrm{F(3k,i)}$ then each of its orbits under φ would admit a matching, which is not possible because $3k$ is odd. Suppose $\mathrm{X(k)} \cong \mathrm{I(3k,i,j)}$ for some $i,j \in \mathbb{N}$. It is known that vertex-transitive I-graphs are generalized Petersen graphs and as such, one of the two orbits, say O_1, consists of a single cycle having, in this case, length $3k$. Recall that the connected components of $\mathrm{X(k)}_{\mathit{0},\mathrm{R}}$ are blocks of imprimitivity. Since $\gcd(2k,r) = 6$, there are 6 such blocks, each of size k. It follows that no two adjacent vertices of O_1 are in the same block. On the other hand, it is plain to see that any 2-path in X(k) has at least two vertices in the same block (see Figure 3.2.5). We conclude that X(k) cannot be a bicirculant. □

Corollary 3.2.28. *If $T_1(k,r,s)$ is vertex-transitive with $3 \nmid k$, then $T_1(k,r,s) \cong \mathrm{GP}(3k, k+(-1)^\alpha)$ where $\alpha \in \{1,2\}$ and $\alpha \equiv k \pmod 3$.*

Proof. Let $T_1(k,r,s)$ be vertex-transitive and suppose $k \equiv 1 \pmod 3$. Let $\varphi = \phi_0\phi_1$ be the automorphism described in the proof of Lemma 3.2.26. We will show that u_k is adjacent to $\varphi^{(k+1)}(u_k)$. Notice that $\varphi^{(k+1)}(u_k) = v_{k+2s-r+\frac{k-1}{3}r}$, but $k+2s-r \equiv r-s$ so we may write $\varphi^{(k+1)}(u_k) = v_{r-s+\frac{k-1}{3}r}$. Recall that $\gcd(2k,3) = 1$ and r is even. From the congruence $k+2r-3s \equiv 0$ we see that:

$$\begin{array}{rcl} 2r-3s & \equiv & 3k \\ 3r-3s+r(k-1) & \equiv & 3k \\ r-s+r(k-1)/3 & \equiv & k \end{array}$$

So that $\varphi^{(k+1)}(u_k) = v_k$, which is adjacent to u_k. The case when $k \equiv 2 \pmod 3$ can be proved with the use of a similar argument. □

This completes the proof of Theorem 3.2.7. Item (1) follows from Lemma 3.2.26 and Corollary 3.2.28, while item (2) is proved in Lemma 3.2.9.

3.2.3 Type 2

Let $k \geq 9$ be an integer, and let $r, s \in \mathbb{Z}_{2k}$. Recall that $T_2(k, r, s)$ is the derived graph of Δ_{T_2} with the normalized voltage assignment for $\mathbb{Z}_{2k}$ shown in Figure 3.2.6.

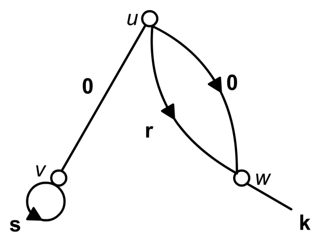


Figure 3.2.6: The voltage assignment of Δ_{T_2} giving rise to the graph $T_2(k, r, s)$.

Then, with the same notation as in Section 3.2.2, $U = \{u_0, u_1, ..., u_{2k-1}\}$, $V = \{v_0, v_1, ..., v_{2k-1}\}$ and $W = \{w_0, w_1, ..., w_{2k-1}\}$ are the respective fibres of vertices u, v and w in Δ_{T_2}. The set of edges of $T_2(k, r, s)$ can be expressed as the union $E_K \cup E_R \cup E_S \cup E_0$ where:

$$\begin{array}{rcl} E_K & = & \{w_i u_{i+k} : i \in \mathbb{Z}_{2k}\} \\ E_0 & = & \{u_0 v_0 : i \in \mathbb{Z}_{2k}\} \cup \{u_0 w_0 : i \in \mathbb{Z}_{2k}\} \\ E_R & = & \{u_i w_{i+r} : i \in \mathbb{Z}_{2k}\} \\ E_S & = & \{v_i v_{i+s} : i \in \mathbb{Z}_{2k}\}. \end{array} \tag{3.2.8}$$

Similarly as with Type 1 cubic tricirculants, we see that every cubic tricirculant of Type 2 is isomorphic to $T_2(k, r, s)$ for an appropriate choice of k and $r, s \in \mathbb{Z}_{2k}$.

Definition 3.2.29. For an odd integer $k > 1$, let $\Upsilon(k) = T_2(k, 2, 1)$.

Theorem 3.2.30. *Let $k \geq 9$ and let Γ be a connected cubic tricirculant of Type 2 with $6k$ vertices. Then the following holds:*

1. *Γ is vertex-transitive if and only if it is isomorphic to $\Upsilon(k)$ for some odd integer k.*
2. *If Γ is not vertex-transitive, then it has three distinct vertex orbits under its full automorphism group.*

The next theorem gives more information about the graph $\Upsilon(k)$.

Theorem 3.2.31. *For an odd integer $k > 1$ the following holds:*

1. *$\Upsilon(k)$ a bicirculant if and only if $3 \nmid k$, in which case it is isomorphic to the graph $H(6k, 1, \alpha k + 2)$ where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod 3$ (see [54]).*

2. $\mathrm{Y}(k)$ *is the underlying graph of the toroidal map* $\{6,3\}_{\alpha,3}$ *where* $\alpha = \frac{1}{2}(k-3)$.

3. $\mathrm{Y}(k)$ *has arc-type* $2+1$.

We devote the remainder of this section to proving Theorems 3.2.30 and 3.2.31.

Remark 3.2.32. Note that $T_2(k,r,s)$ and $T_2(k,r,-s)$ are in fact the exact same graph. Then, we can safely assume $s < k$.

Lemma 3.2.33. *For an odd integer* $k > 1$, $\mathrm{Y}(k)$ *is vertex-transitive.*

Proof. Since $\mathrm{Y}(k) = T_2(k,r,s)$, it suffices to provide an automorphism of $T_2(k,2,1)$ that mixes the sets U, V and W. Let φ be the mapping given by:

$$\begin{array}{llll} u_i \mapsto v_{i+1}, & v_i \mapsto u_{i+1}, & w_i \mapsto v_i & \text{if } i \text{ is even;} \\ u_i \mapsto w_{i+2+k}, & v_i \mapsto w_{i+2}, & w_i \mapsto u_{i+k} & \text{if } i \text{ is odd.} \end{array}$$

Now consider a vertex $u_i \in U$ with i even. Observe that φ maps u_i to v_{i+1} and that the neighbourhood $N(u_i) := \{v_i, w_i, w_{i+2}\}$ is mapped to $\{u_{i+1}, v_i, v_{i+2}\}$, which is precisely the neighbourhood of v_{1+i}. That is, φ maps the neighbourhood of any vertex u_i into the neighbourhood of its image, when i even. The reader can verify the remaining cases and see that φ is indeed a graph automorphism. □

Lemma 3.2.34. *For an odd integer* $k > 9$, $\mathrm{Y}(k)$ *has arc-type* $2+1$.

Proof. Let G be the full automorphism group of $\mathrm{Y}(k)$, $k > 9$, and consider the automorphism φ interchanging w_i with w_{-i}, u_i with u_{-i-2} and v_i with v_{-i-2}, $i \in \mathbb{Z}_{2k}$. Note that $\varphi \in G_{w_0}$ and that it fixes w_k while interchanging u_0 with u_{-2}. Since no $\mathrm{Y}(k)$ with $k > 9$ is arc-transitive, it follows that its arc-type is $2+1$. □

For the rest of this section, let $k \geq 9$ be an integer, let $r, s \in \mathbb{Z}_{2k}$ and suppose $\Gamma = T_2(k,r,s)$ is vertex-transitive. In what follows we will show that Γ is isomorphic to $\mathrm{Y}(k)$. As with cubic tricirculants of Type 1, the strategy will be to count reduced closed walks in the quotient Δ_{T_2}. Observe that the walk $((wu)_0, (uw)_r, (ww)_k, (wu)_{-r}, (uw)_0, (ww)_k)$ starting at w is a closed walk of length 6 having net voltage 0, regardless of what r and s evaluate to in $\mathbb{Z}_{2k}$. As was done with graphs of Type 1, we computed all closed walks in Δ_{T_2} along with their net voltages. Again, it would be convenient if closed walks of length 6 with net voltage 0 always lift into 6-cycles. For this it suffices to show that the girth of Γ is at least 4.

Lemma 3.2.35. Γ *has no cycles of length* 3.

Proof. Suppose to the contrary, that Γ contains a triangle T. By observing Figure 3.2.6 we can see that all three vertices of T belong to V or they belong to $U \cup W$. Since Γ is vertex-transitive, there is at least one triangle through every vertex in W. We can thus assume without loss of generality that all three

Label	Net voltage	Starting at u	Starting at v	Starting at w
I	0	2	0	4
II	$\pm(k-s)$	8	8	8
III	$\pm(k-r-s)$	4	4	4
IV	$\pm(k+r-s)$	4	4	4
V	$\pm(3r)$	2	0	2
VI	$\pm(2r)$	2	0	4
VII	$\pm(6s)$	0	2	0
VII	$\pm(r-2s)$	4	6	2
IX	$\pm(r+2s)$	4	6	2

Table 3.5: Net voltages of closed walks of length 6 in Δ_{T_2}.

vertices of T are in $U \cup W$. This implies that $r = k$ and further, that there are 2 distinct triangles through every vertex in W: the lifts of $((wu)_0, (uw)_r, (ww)_k)$ and $((wu)_{-r}, (uw)_0, (ww)_k)$. However, since the subgraph induced by V is 2-valent, there is at most one triangle through every vertex in V, contradicting the vertex-transitivity of Γ. $\square$

Note that since Γ is simple, $s \neq k$ and if $r = k$, Γ would contain a triangle. Thus the following corollary holds.

Corollary 3.2.36. *Neither r nor s equals k.*

Now, let $\mathcal{W}_6$ be the set of all walks of length 6 in Δ_{T_2} and let N be the set of net voltages of elements of $\mathcal{W}_6$, expressed in terms of k, r and s. It follows from Lemma 3.2.35 that walks in $\mathcal{W}_6$ with net voltage 0 will lift into 6-cycles, and not just closed walks. Table 3.2.3 shows, for each element n of N, how many closed walks with net voltage n start at each vertex of Δ_{T_2}.

From Row I of Table 3.2.3 we see that there are at least 4 walks $\omega \in \mathcal{W}_6$ starting at w. Therefore, at least one expression from Rows II–IX must be congruent to 0 (mod $2k$). However, a careful inspection of Table 3.2.3 shows that if neither of the expressions in Rows VIII and IX are congruent to 0 (mod $2k$), then there are at least 2 more walks in $\mathcal{W}_6$ with net voltage 0 starting at w than there are those starting at v. This means that if Γ is vertex-transitive, then one of the following two equations modulo $2k$ hold:

$$r \equiv 2s \tag{3.2.9}$$

$$r \equiv -2s \tag{3.2.10}$$

Suppose (3.2.9) holds. By Remark 3.2.32 we can in fact write $r = 2s$. Then the mapping $\phi : T_2(k, 2s, s) \to T_2(k, -2s, -s)$ given by $x_i \mapsto x_{-i}$, for all $x \in U \cup V \cup W$, is a graph isomorphism. But, again by Remark 3.2.32, $T_2(k, -2s, -s) = T_2(k, -2s, s)$ so that any graph $T_2(k, r, s)$ with $r \equiv 2s$ is isomorphic to a graph $T_2(k, r', s)$ satisfying $r' \equiv -2s$. We can therefore limit our analysis to the case when (3.2.9) holds.

Let $\Gamma = T_2(k, r, s)$ be connected and vertex-transitive, and suppose $r \equiv 2s$. Note that any number dividing both k and s must also divide r, and since by connectedness of Γ $\gcd(k, r, s) = 1$ we see that $\gcd(k, s) = 1$ and $\gcd(2k, s) \in \{1, 2\}$. We will show that $\gcd(2k, s)$ cannot be 2.

Lemma 3.2.37. *If* Γ *is vertex-transitive,* $\gcd(2k, s) = 1$ *and* $\gcd(2k, r) = 2$.

Proof. In order to get a contradiction, suppose $\gcd(2k, s) = 2$. Then s is even and k is odd, as $\gcd(k, s) = 1$. Furthermore, the 2-valent subgraph of Γ induced by the set V is the union of two k-cycles. We will show that these are in fact the only two k-cycles in Γ which will imply that Γ cannot be vertex-transitive. Suppose there is a k-cycle C' that visits a vertex not in V. Define $\Gamma[U \cup W]$ as the subgraph of Γ induced by the set $U \cup W$. Since r is even and k is odd, it is not difficult to see that $\Gamma[U \cup W]$ is bipartite, with sets $\{w_i : i \text{ is even}\} \cup \{u_i : i \text{ is odd}\}$ and $\{w_i : i \text{ is odd}\} \cup \{u_i : i \text{ is even}\}$. Therefore no k-cycle of Γ is contained in $\Gamma[U \cup W]$. This means that C' visits at least one vertex in each of sets U, V and W.

Now, C' projects onto a closed walk C in Δ_{T_2} that has net voltage 0. Define x_S as the number of times C traces the dart $(vv)_s$ minus the number of times it traces $(vv)_{-s}$, so that $x_S s$ is the total voltage contributed to C by darts in $\{(vv)_{\pm s}\}$. Define x_R similarly, and define x_K as the number of times C traces the semi-edge $(ww)_k$. We obtain the following congruence modulo $2k$:

$$x_R r + x_S s + x_K k \equiv 0.$$

Recall that both r and s are even, making both $x_R r$ and $x_S s$ even. It follows that $x_K k$ is even, but since k is odd, x_K must be even, and hence $x_K k \equiv 0 \pmod{2k}$. We thus have

$$x_R r + x_S s \equiv 0$$

and since $r \equiv 2s \pmod{2k}$,

$$\begin{aligned} x_R 2s + x_S s &\equiv 0, \\ s(2x_R + x_S) &\equiv 0. \end{aligned}$$

From this, and because $\gcd(k, s) = 1$, we see that $2x_R + x_S = 0$ or $2x_R + x_S \geq k$. To see that $2x_R + x_S$ cannot equal 0, define A as the set of darts in Δ_{T_2} with voltage 0 or r and observe that C must visit A an even number of times. Since C has odd length, C visits $\{(vv)_{\pm s}\} \cup \{(ww)_k\}$ an odd number of times. Then, since x_K is even, x_S must be odd. But $2x_R$ is even so $2x_R + x_S \neq 0$. We thus have

$$2x_R + x_S \geq k.$$

Now, notice that if C traces the dart $(uw)_r$, then it must immediately trace a dart in $\{(wu)_0, (ww)_k\}$. This means that C traces a dart in the subgraph of Δ_{T_2} induced by $U \cup W$ at least $2x_R$ times. Then $2x_R + x_S < k$, since C has length k and it must visit $(vu)_0$ at least once. We thus have $k > 2x_R + x_S \geq k$, a contradiction. The result follows. □

Lemma 3.2.38. *If* $T_2(k, r, s)$ *is vertex-transitive,* k *is odd.*

Proof. Suppose to the contrary that k is even. Since s is odd, we have $sk \equiv k \pmod{2k}$, and thus $s(2 \cdot \frac{1}{2})k \equiv k$. But k is even, so we can rewrite that expression as $2s \cdot \frac{k}{2} \equiv k$. Recall that $r \equiv 2s$ and then $r\frac{k}{2} \equiv k$.

Now, consider the walk of length $k + 1$ in Δ_{T_2} that starts in w, traces the semi-edge $(ww)_k$ once and then traces the 2-path $((wu)_r, (uw)_0)$ exactly $\frac{k}{2}$ times.

Note that this walk is closed and has net voltage $\frac{k}{2}r + k \equiv 0$. Furthermore, it is easy to see that it lifts into a $(k+1)$-cycle in Γ and that it does not visit any vertex in V.

Since Γ is vertex-transitive, there must be a $(k+1)$-cycle C' through each vertex in V. Note that the subgraph of Γ induced by V is a single $2k$-cycle, and thus C' must visit all three set U, V and W. It follows that C' projects into a walk C of length $k+1$ having net voltage 0 and that it visits all three vertices of Δ_{T_2}. Furthermore, since $k+1$ is odd, the number of times C visits $\{(vv)_{\pm s}\}$ must be of different parity than the number of times it traces $(ww)_k$. Define x_S, x_R and x_K as in the proof of Lemma 19. Hence

$$x_R r + x_S s + x_K k \equiv 0.$$

Now, r and k are even, and so $x_R r$ and $x_K k$ are also even. It follows that $x_S s$ is even, but since s is odd, s_S is necessarily even. This means C visits $\{(vv)_{\pm s}\}$ an even number of times. It follows that C traces $(ww)_k$ and odd number of times, that is, x_K is odd. We thus have

$$\begin{aligned} x_R r + x_S s + k &\equiv 0, \\ x_R r + x_S s &\equiv k, \end{aligned}$$

and since $r \equiv 2s$,

$$s(2x_R + x_S) \equiv k.$$

This implies $2x_R + x_S \geq k$, as s and k are relatively prime. However, $2x_R + x_R \leq k-1$ since C has length $k+1$ and C visits the set $\{(vu)_0, (uv)_0\}$ at least twice. This contradiction arises from the assumption that k is even. We conclude that k is odd. □

Lemma 3.2.39. *For an odd $k > 9$, if $T_2(k,r,s)$ is vertex-transitive then $T_2(k,r,s) \cong \mathrm{Y}(k)$.*

Proof. Let $T_2(k,r,s)$ be vertex-transitive. Then, by Lemmas 3.2.38 and 3.2.37 k is an odd integer, $\gcd(2k, s) = 1$ and $r \equiv 2s$. Denote by s^{-1} the multiplicative inverse of s in $\mathbb{Z}_{2k}$. Now, define φ as the mapping between $T_2(k,2,1)$ and $T_2(k,r,s)$ given by $x_i \mapsto x_{s^{-1}i}$, for $x \in U \cup V \cup W$. It is plain to see that φ is the desired isomorphism. Therefore $T_2(k,r,s) \cong T_2(k,2,1) = \mathrm{Y}(k)$. □

This, together with Lemma 3.2.33, proves the first claim of Theorem 3.2.30. We now proceed to show that a tricirculant of type 2 that is not vertex-transitive must have 3 distinct orbits on vertices.

Lemma 3.2.40. *Let Γ be a tricirculant of Type 2. If Γ is not vertex-transitive, then it has 3 distinct vertex orbits.*

Proof. We will slightly abuse notation and say $\Gamma = T_2(k,r,s)$ for some $k \in \mathbb{Z}$ and $r, s \in \mathbb{Z}_{2k}$. Let $\varphi \in \mathrm{Aut}(\Gamma)$ and let $u_i \in U$. Suppose that φ maps u_i to some vertex $v_j \in V$. Then φ must map the neighbourhood of u_i, $\{v_i, w_i, w_{i+r}\}$ to $\{u_j, v_{j-s}, v_{j+s}\}$, the neighbourhood of v_j. It is straightforward to see that φ must map a vertex in W to a vertex in V. That is, φ "mixes" the set V with the sets U and W implying that Γ is vertex-transitive, a contradiction.

Similarly, if φ maps u_i to some $w_j \in W$ then it must map $\{v_i, w_i, w_{i+r}\}$ to $\{w_{j+k}, u_j, u_{j-r}\}$. Thus φ maps v_i into either U or W contradicting again Γ not being vertex-transitive. This shows that the set U is a single vertex orbit of Γ under $\mathrm{Aut}(\Gamma)$. An analogous argument shows that V is a single vertex orbit and therefore Γ has three distinct orbits on vertices: U, V and W. □

The proof of Theorem 3.2.30 is now complete. In the following paragraphs we will give further details about the structure of a vertex-transitive tricirculant of type 2. In particular, we will show that $\mathrm{Y}(k)$ can be seen as a map on the torus and we will give sufficient and necessary conditions for $\mathrm{Y}(k)$ to be a bicirculant. This will prove items (1) and (2) of Theorem 3.2.31.

Proposition 3.2.41. *Let $k \geq 3$ be an odd integer, then $\mathrm{Y}(k)$ admits an embedding on the torus with hexagonal faces, yielding a map of type $\{6,3\}_{\alpha,3}$, where $\alpha = \frac{1}{2}(k-3)$ (see [13] for details about the notation).*

Proof. First, notice that vertices in $U \cup W$ with even index form a cycle of length $2k$, $C_1 = (w_0, u_0, w_2, u_2, ..., w_{2k-2}, u_{2k-2})$. Likewise, vertices in $U \cup W$ with odd index form a $2k$-cycle, C_2. The vertices in V form a third cycle of length $2k$, C_3.

Now, $w_i w_{i+k} \in E$ for all $i \in \mathbb{Z}_{2k}$, and $i + k$ has different parity than i. This means each vertex of C_1 of the form w_i is adjacent, through a K-edge, to a vertex of C_2. Further, each vertex of the form u_i in C_1 is adjacent to a vertex in C_3, namely v_i. Note that the vertices of C_2 that are not adjacent to a vertex of C_1 are precisely those of the form u_i (with i odd), and that $u_i v_i \in E$ for all odd i. That is, every other vertex in C_2 has a neighbour in C_3.

Therefore, we can think of $\mathrm{Y}(k)$ as three stacked $2k$-cycles C_1, C_2 and C_3 where every other vertex of C_i has a neighbour in C_{i-1} while each of the remaining vertices has a neighbour in C_{i+1}, where $i \pm 1$ is computed modulo 3 (see Figure 3.2.7 for a detailed example). From here it is clear that $T_2(k, 2, 1)$ can be embedded on a torus, tessellating it with $3k$ hexagons. It can be readily verified that this yields the map $\{6,3\}_{\alpha,3}$, where $\alpha = \frac{1}{2}(k-3)$, following the notation in [13]. □

Corollary 3.2.42. *For $k \geq 3$, $\mathrm{Y}(k)$ has girth 6 and is bipartite.*

Observe that $\mathrm{Y}(9)$ corresponds to the map $\{6,3\}_{3,3}$, which is a regular map (see Chapter 8.4 of [13]). It follows that $T_2(9, 2, 1)$ is arc-transitive, making it one of the four possible cubic arc-transitive tricirculant graphs (called $F054A$ in [35], following Foster's notation).

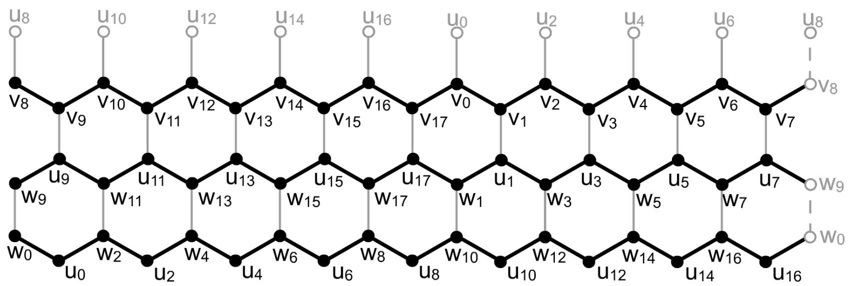


Figure 3.2.7: The graph Y(9). The subgraph induced by bold edges has three connected components that correspond, from bottom to top, to the cycles C_1, C_2 and C_3 described in the proof of Proposition 3.2.41.

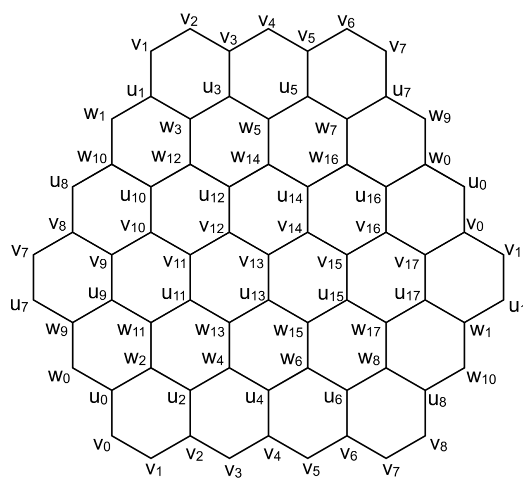


Figure 3.2.8: A drawing of Y(9) as the map on the torus $\{6,3\}_{3,3}$.

Lemma 3.2.43. *If $k > 1$ is an odd integer such that $3 \nmid k$, then* $\mathrm{Y}(k)$ *is a bicirculant.*

Proof. Consider the mapping φ defined by:

$$\begin{array}{llll} u_i \mapsto w_{i+2+k}, & v_i \mapsto w_{i+2}, & w_i \mapsto u_{i+k}, & \text{if } i \text{ is even;} \\ u_i \mapsto v_{i+1}, & v_i \mapsto u_{i+1}, & w_i \mapsto v_i, & \text{if } i \text{ is odd.} \end{array}$$

For a vertex $u_i \in U$, we have $\varphi^l(u_i) \in U$ if and only if $3|l$. That is, if we start at U, every third iteration of φ lands us in U again. Moreover, we have $\varphi^3(u_i) = u_{i+3+k}$. In general $\varphi^{3l}(u_i) = u_{i+3l+lk}$. Hence $\varphi^{3l}(u_i) = u_i$ if and only if $3l + lk \equiv 0$. If $3 \nmid k$, then $l = k$ is the smallest value for l that satisfies $3l + lk \equiv 0$. It follows that the orbit of u_i has size $3k$. It is plain to see that the vertices not in this orbit form an orbit of size $3k$ on their own, namely, the orbit of u_{i+1}. Hence $\mathrm{Y}(k)$ is a bicirculant and is isomorphic to the graph

H$(3k, 1, \alpha k+2)$, where $\alpha \in \{1, 2\}$ and $\alpha \equiv k \pmod 3$ (see [54] for details). It is worthwhile to mention that if 3 does divide k, then φ has 6 orbits of size k. □

Lemma 3.2.44. *If $k > 1$ is an odd integer such that $3 \mid k$, then* Y(k) *is not a bicirculant.*

Proof. Let k be an odd integer divisible by 3. To make this proof more succinct, we will assume that Y(k) is not arc-transitive, that is, that $k > 9$. The graphs Y(3) and Y(9) can be verified to not be bicirculants individually. By Lemma 3.2.34, Y(K) has arc-type $2 + 1$ and the set $E' = \{u_i v_i \mid \in \mathbb{Z}_{2k}\} \cup \{w_i w_{k+1} \mid i \in \mathbb{Z}_{2k}\}$ is an orbit of edges under the full automorphism group of Y(k). Then each connected component of Y$(k) - E'$ is a block of imprimitivity for Aut(Y(k)). Furthermore Y$(k) - E'$ has three connected components that correspond precisely to the three $2k$-cycles C_i, $i \in \{1, 2, 3\}$, in the proof of Proposition 3.2.41. For convenience, we will relabel the vertices of Y(k) as follows: for each $i \in \{1, 2, 3\}$, let V$(C_i) = \{v_{i,j} \mid j \in \mathbb{Z}_k\}$, where $v_{i,j}$ is adjacent to $v_{i,j-1}$, $v_{i,j+1}$ and $v_{i+1,j-1}$. It is straightforward to see that this labelling is always possible.

Now, suppose Y(k) admits a bicirculant automorphism ρ. Since a ρ-orbit has size $3k$, and every $2k$-cycle C_i is a block of imprimitivity, ρ must cyclically permute them. Without loss of generality $\rho(C_i) = C_{i+1}$. Then, ρ is determined, up to two possibilities, by its action on a single vertex. Indeed, if $\rho(v_{i,0}) = v_{i+1,a}$ then the two neighbours of $v_{i,0}$ in C_i must be mapped to the neighbours of $\rho(v_{i,0})$ in C_{i+1}. In other words, the set $\{v_{i,-1}, v_{i,1}\}$ is bijectively mapped to $\{v_{i,a-1}, v_{i,a+1}\}$ in one of two possible ways, and this completely determines the action of ρ. In short, for all $i \in \{1, 2, 3\}$ and $j \in \mathbb{Z}_{2k}$ one of the following holds:

1. $\rho(v_{i,j}) = v_{i+1,j+a}$,

2. $\rho(v_{i,j}) = v_{i+1,-j+a}$.

If (2) holds, then $\rho^3(v_{i,j}) = v_{i,-j+a}$ and so $\rho^6(v_{i,j}) = v_{i,j}$. That is, ρ has order 6, which implies that Y(k) has order 12 and hence $k = 2$, which contradicts k being odd and divisible by 3. If (1) holds, then $\rho^3(v_{i,j}) = v_{i,j+3a}$ and since $3 \mid k$, a vertex $v_{1,j}$ can only be mapped to a vertex $v_{1,b}$ by an element of $\langle \rho \rangle$ if $j \equiv b \pmod 3$. This implies that ρ has at least 3 orbits on vertices, contradicting that it is a bicirculant automorphism. The result follows. □

Theorem 3.2.31 now follows from Proposition 3.2.41 and Lemmas 3.2.34, 3.2.43 and 3.2.44.

3.2.4 Type 3

Let $k \geq 9$ be an integer, and let $r \in \mathbb{Z}_{2k}$. Recall that $T_3(k, r)$ is the derived graph of Δ_{T_3} with the normalized voltage assignment for $\mathbb{Z}_{2k}$ shown in Figure 3.2.9.

Let $U = \{u_0, u_1, ..., u_{2k-1}\}$, $V = \{v_0, v_1, ..., v_{2k-1}\}$ and $W = \{w_0, w_1, ..., w_{2k-1}\}$ be the respective fibers of vertices u, v and w in Δ_{T_3}. It is clear that any cubic tricirculant of Type 3 is isomorphic to $T_3(k, r)$ for some k and r.

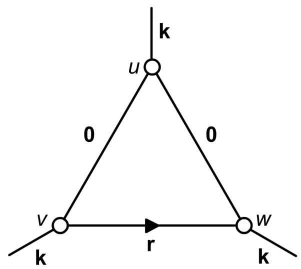


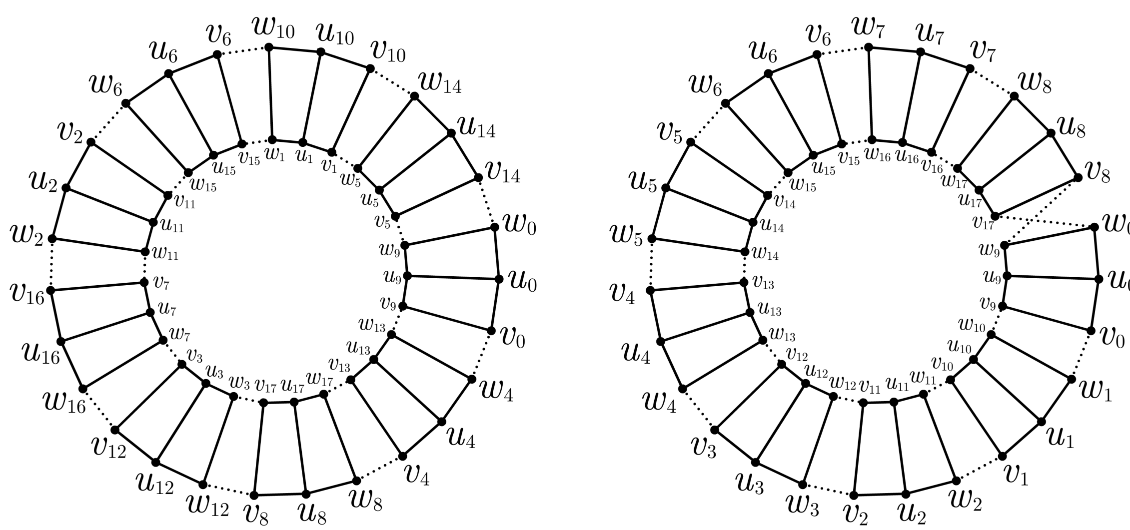
Figure 3.2.9: The voltage assignment of Δ_{T_3} giving rise to the graph $T_3(k, r)$.

Theorem 3.2.45. *If $T_3(k, r)$ is connected, then it is a circulant isomorphic to either a prism or a Möbius ladder.*

Proof. First, recall that $T_3(k, r)$ is connected if and only if $\gcd(k, r) = 1$. Then $\gcd(2k, r) \in \{1, 2\}$, depending on whether r is even or odd.

If r is odd, then the subgraph induced by *0*- and R-edges is a single cycle of length $6k$, $(w_0, u_0, v_0, w_r, u_r, v_r, ..., w_{2k-r}, u_{2k-r}, v_{2k-r}, w_0)$. It is straighforward to see that K-edges join antipodal vertices in this cycle. Hence, in this case $T_3(k, r)$ is a Möbius ladder.

If r is even, then the graph induced by *0*- and R-edges is the union of two disjoint cycles of length $3k$: one consisting of all the vertices with even index, and the other consisting on those with odd index. Observe that K-edges connect these two cycles creating a prism. □



$T_3(9, 4)$ $T_3(9, 1)$

Figure 3.2.10: A prism and a Möbius ladder on 54 vertices.

3.2.5 Type 4

Let k be a positive integer, and let r and s be two distinct integers in $\mathbb{Z}_{2k}$. Recall that $T_4(k, r, s)$ is the derived graph of Δ_{T_4} with the normalized voltage assignment for $\mathbb{Z}_{2k}$ shown in Figure 3.2.11. Let $U = \{u_0, u_1, ..., u_{2k-1}\}$, $V =$

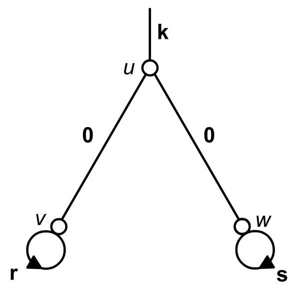


Figure 3.2.11: The voltage assignment of Δ_{T_4} giving rise to the graph $T_4(k, r, s)$.

$\{v_0, v_1, ..., v_{2k-1}\}$ and $W = \{w_0, w_1, ..., w_{2k-1}\}$ be the respective fibers of vertices u, v and w in Δ_{T_4}. Then, the set of edges of $T_4(k, r, s)$ can be expressed as the union $E_K \cup E_R \cup E_S \cup E_0$ where:

$$\begin{array}{rcl} E_K & = & \{u_i u_{i+k} : i \in \mathbb{Z}_{2k}\} \\ E_0 & = & \{u_i v_i : i \in \mathbb{Z}_{2k}\} \cup \{u_i w_i : i \in \mathbb{Z}_{2k}\} \\ E_R & = & \{v_i v_{i+r} : i \in \mathbb{Z}_{2k}\} \\ E_S & = & \{w_i w_{i+s} : i \in \mathbb{Z}_{2k}\} \end{array}$$

For $X \in \{0, R, S, K\}$, edges in E_X will be called edges of type X, or simply X-edges. Similarly as with tricirculants of Types 1, 2 and 3, every cubic tricirculant of type 4 with $6k$ vertices is isomorphic to $T_4(k, r, s)$ for an appropriate choice of r and s.

Remark 3.2.46. Note that for any k, r and s the following isomorphism holds:

$$T_4(k, r, s) \cong T_4(k, -r, s) \cong T_4(k, r, -s) \cong T_4(k, s, r)$$

Theorem 3.2.47. *There are no cubic vertex-transitive tricirculants of Type 4 of order greater or equal to 54.*

The rest of this section is devoted to prove Theorem 3.2.47 as well as to characterise Type 4 tricirculants with 2 and 3 vertex orbits. We will assume henceforth that $k \geq 9$ and thus the order of $T_4(k, r, s)$ is at least 54.

Lemma 3.2.48. *If $T_4(k, r, s)$ is vertex-transitive, then $r \neq s$ and $r \neq -s$.*

Proof. Suppose that $r = s$ and consider the graph $\Gamma := T_4(k, r, r)$, with $k \geq 9$ and $1 \leq r \leq k - 1$. Observe that $(v_0, v_r, u_r, w_r, w_0, u_0, v_0)$ is a 6-cycle of Γ that does not contain any K-edge but does contain edges of all other types. For Γ to be vertex-transitive, there must be at least one 6-cycle through a K-edge; otherwise K-edges conform a single edge-orbit and thus U is a single vertex-orbit. However, such a cycle would quotient down to a closed walk of length 6 in Δ_{T_4} having voltage 0 and tracing the semi-edge $(uu)_k$. By observing Figure 3.2.11 we see that any closed walk of length 6 visiting $(uu)_k$ has net voltage $k \pm 3r$. That is, if Γ is vertex-transitive, then $3r \equiv k \mod 2k$. Observe that under these conditions, Γ is connected if and only if $r = 1$ and thus making $k = 3$, contradicting that $k \geq 9$. This shows that $r \neq s$ and in view of Remark 3.2.46, $r \neq -s$. □

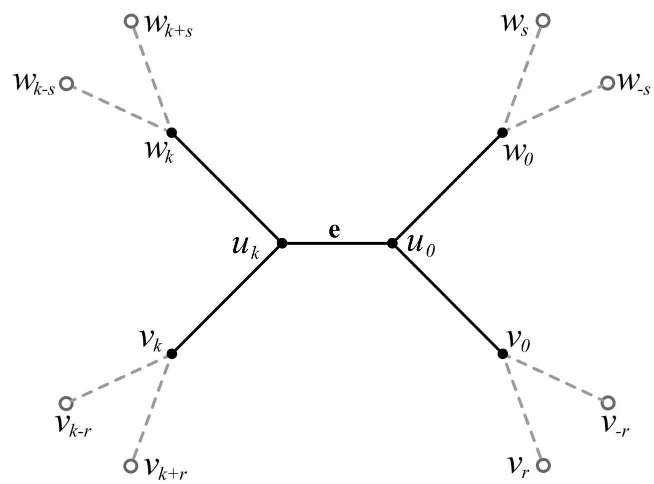


Figure 3.2.12: Neighbourhood of u_0u_k. The subgraph H shown in solid edges and vertices.

Lemma 3.2.49. *Let e be a K-edge of $T_4(k, r, s)$ and let a be an integer greater than 4. If C is an cycle of length a containing e then there exists another cycle of length a, C', such that the intersection $C \cap C'$ contains a 3-path whose middle edge is e.*

Proof. Without loss of generality, we can assume $e = u_0u_k$. Define H as the subgraph of Γ induced by vertices at distance at most one from e, and let C be a k-cycle through e (see Figure 3.2.12). Observe that C intersects H in a 3-path, P, whose middle edge is e. Now, let ϕ be the mapping that acts by multiplying the index of each vertex by -1; that is ϕ maps u_i into u_{-i}, v_i into v_{-i}, and w_i to w_{-i}. Observe that ϕ is in fact an automorphism of Γ. Moreover, ϕ fixes each vertex and each edge of H. In particular, it fixes P. Therefore, $\phi(C)$ is a k-cycle through e and P lies in the intersection of C and $\phi(C)$. To see that $\phi(C)$ is different from C, observe that ϕ interchanges the two vertices in each of the following sets: $\{v_{k+r}, v_{k-r}\}$, $\{v_r, v_{-r}\}$,$\{w_{k+s}, w_{k-s}\}$ and $\{w_s, w_{-s}\}$ (white vertices in Figure 3.2.12). It is clear that C can visit at most one vertex in each of these four sets, and if C visits one vertex in one of these sets, then $\phi(C)$ must visit the other. □

Lemma 3.2.50. *If $T_4(k, r, s)$ is vertex-transitive, then in $\mathbb{Z}_{2k}$ one of the equalities (A1)–(A4) below and one of the equalities (B1)–(B4) below hold:*

(A1)	$k+2r+s=0$	(B1)	$k+2s+r=0$
(A2)	$k-2r+s=0$	(B2)	$k-2s+r=0$
(A3)	$r+3s=0$	(B3)	$s+3r=0$
(A4)	$r-3s=0$	(B4)	$s-3r=0.$

Proof. Suppose $T_4(k,r,s)$ is vertex-transitive. We will first show that one of the equalities (A1)–(A4) holds in $\mathbb{Z}_{2k}$. The rest of the lemma will follow from the fact that $T_4(k,r,s) \cong T_4(k,s,r)$. Since $T_4(k,r,s)$ is vertex-transitive, the edge-neighbourhood of any vertex in U can be mapped by an automorphism to the edge-neighbourhood of any vertex in V. In particular, either there exists an automorphism mapping the K-edge u_0u_k to the *0*-edge u_0v_0 or there is an automorphism mapping u_0u_k to the R-edge v_0v_r. Therefore, the property described in Lemma 3.2.49 should also hold for u_0v_0 or for v_0v_r (or possibly both). We will see what this means in terms of k, r and s.

Suppose that the property described in Lemma 3.2.49 holds for u_0v_0. Observe that there is an 8-cycle $C = (u_0, v_0, v_{-r}, u_{-r}, u_{k-r}, v_{k-r}, v_k, u_k, u_0)$ through u_0v_0 (see Figure 3.2.13). Then there must be another 8-cycle C' whose intersection with C contains the 3-path (v_{-r}, v_0, u_0, u_k), but no 4-path containing this 3-path. This in turns implies that some vertex in $\{v_{-3r}, u_{-2r}\}$ is adjacent to a vertex in $\{w_{k-s}, w_{k+s}\}$. Since no vertex in V is adjacent to a vertex in W, we have that either $u_{-2r}w_{k-s} \in E$ or $u_{-2r}w_{k+s} \in E$. This implies $2r \equiv k+s$ or $2r \equiv k-s$, and so (A1) or (A2) holds.

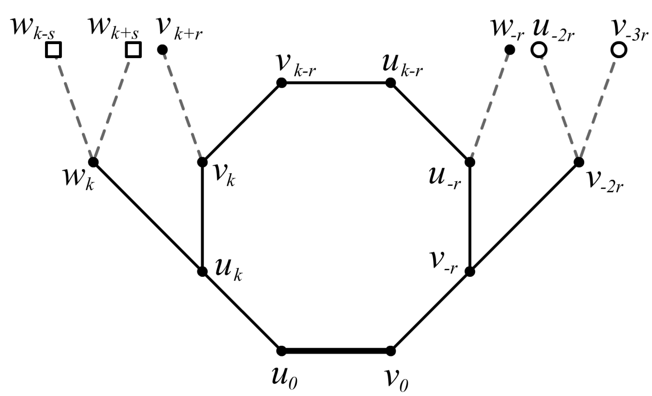


Figure 3.2.13: A part of the graph $T_4(k,r,s)$ containing u_0v_0.

If, on the other hand, the property described in Lemma 3.2.49 holds for v_0v_r, then one vertex in $\{w_{r+s}, w_{r-s}\}$ must be adjacent to a vertex in $\{w_s, w_{-s}\}$ (see Figure 3.2.14), implying that one of the following expressions must be equal to 0 in $\mathbb{Z}_{2k}$: $r+3s$, $r+s$, $r-3s$, $r-s$. However, in view of Lemma 3.2.48, $r+s \neq 0$ and $r-s \neq 0$. Hence (A3) or (A4) holds. $\square$

We are now ready to prove Theorem 3.2.47. Let $k \geq 9$ be an integer and suppose $T_4(k,r,s)$ is vertex-transitive. Then, by Lemma 3.2.50, one of the equalities (A1)–(A4) and one of the equalities (B1)-(B4) must hold. If, for instance both (A1) and (B1) hold, then by adding them we see that $r = -s$,

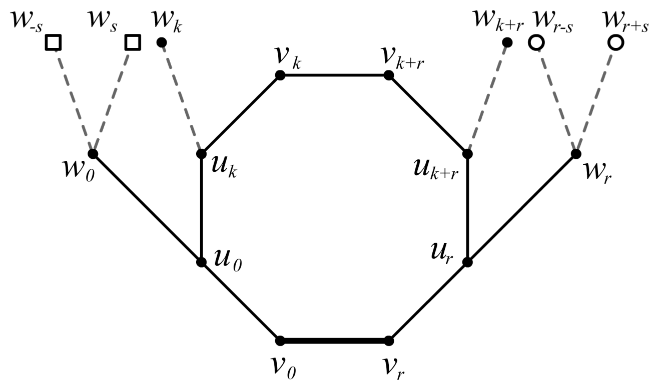


Figure 3.2.14: A subgraph containing v_0v_r.

which contradicts Lemma 3.2.48. Similarly, we get that $r \pm s$ if both equalities in any of the following pair hold: (A1) and (B2), (A2) and (B2), (A3) and (B3). If (A1) and (B4), or (A2) and (B4) hold, then $k = r$, which contradicts the simplicity of Γ. If (A1) and (B3), (A2) and (B3), or (a3) and (B4) hold, then either $k = 5$ or $\gcd(k, r, s) \neq 1$. It is readily seen that we get a contradiction in each of the 16 possible cases that arise from Lemma 3.2.50. We conclude that a connected $T_4(k, r, s)$ cannot be vertex-transitive if $k \geq 9$.

Lemma 3.2.51. *Let $k \geq 9$ and $T_4(k, r, s)$ be connected. Let $\alpha = \gcd(2k, r)$, $R = r/\alpha$, $S = s/\alpha$. Then $T_4(k, r, s)$ has two orbits on vertices if and only if $\alpha \in \{1, 2\}$ and $(SR^{-1})^2 \equiv \pm 1 \pmod{2k/\alpha}$, where R^{-1} is the multiplicative inverse of R modulo $2k/\alpha$ (or the equivalent statement obtained from interchaging r and s).*

Proof. For a graph Γ of type 4, let Γ^* be the graph obtained from Γ by deleting all vertices in U (along with the edges incident to them) and making v_i adjacent to w_i, $i \in \mathbb{Z}_{2k}$. Then Γ^* is isomorphic to the bicirculant I-graph $I(2k, r, s)$ of order $4k$. Moreover, each automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ induces an automorphism $\varphi^* \in \mathrm{Aut}(\Gamma^*)$ in a natural way. Denote by $\mathrm{Aut}^*(\Gamma^*)$ the group of all automorphism of Γ^* induced by an automorphism of Γ.

Now suppose Γ has two orbits on vertices. Since K-edges can only be mapped to K-edges, these two orbits must necessarily be U and $V \cup W$. That is, $\mathrm{Aut}(\Gamma)$ acts transitively on $V \cup W$ and so $\mathrm{Aut}^*(\Gamma^*)$ is transitive on the vertices of Γ^*.

If $\Gamma^* = I(2k, r, s)$ is connected then it must be a vertex-transitive generalised Petersen graph [54]. That is, without loss of generality, the graph induced by V is a $2k$-cycle, and thus $\gcd(2k, r) = 1$. Let r^{-1} be the multiplicative inverse of r modulo $2k$ and let $\Lambda = T_4(k, 1, sr^{-1})$. Then $\Lambda \cong \Gamma = T_4(k, r, s)$. Moreover, Λ^* is isomorphic to the generalised Petersen graph $GP(2k, sr^{-1})$, and since it is vertex-transitive we have $(sr^{-1})^2 \equiv \pm 1 \pmod{2k}$ [23].

If $\Gamma^* = I(2k, r, s)$ is disconnected, then it must have two connected components. Indeed, since Γ is connected we have $\gcd(2k, k, r, s) = 1$, but $I(2k, r, s)$ is disconnected if and only if $\gcd(2k, r, s) \neq 1$. Then necessarily $\gcd(2k, r, s) = 2$ and both r and s are even while k is odd. Each connected component is isomorphic to $I(k, r/2, s/2)$. An analogous argument as the one used in the connected case shows that $\gcd(k, r/2) = 1$, (and since k is odd, $\gcd(k, r) = 1$ and $\gcd(2k, r) = 2$) and $(SR^{-1})^2 = ((s/2)(r/2)^{-1})^2 \equiv \pm 1 \pmod{k}$.

For the reverse implication let $k \geq 9$ and $r, s \in \mathbb{Z}_{2k}$. Suppose $\alpha = \gcd(2k, r) \in \{1, 2\}$ and $(SR^{-1})^2 \equiv \pm 1 \pmod{2k/\alpha}$, where R^{-1} is the multiplicative inverse of R modulo $2k/\alpha$. Let $\Gamma = T_4(k, r, s)$. Define $\varphi : \Gamma \to \Gamma$ as follows:

$$u_i \mapsto u_{(SR^{-1})i}, \quad v_i \mapsto w_{(SR^{-1})i}, \quad w_i \mapsto v_{(SR^{-1})i} \quad \text{for all } i \in \mathbb{Z}_{2k}$$

Observe that $SR^{-1}r \equiv s \pmod{2k}$ and $sSR^{-1} \equiv \pm r \pmod{2k}$. From here, it is routine to check that φ is an automorphism of Γ. We conclude that $V \cup W$ is a vertex orbit of Γ. $\square$

Chapter 4

Generalised voltage graphs

Graphs (and mathematical objects in general) that posses non-trivial symmetry have many nice features, one of them being that they allow a more compact description, which not only saves space for storage but also enables more efficient analysis of the graph. For instance, as we have seen in Chapters 2 and 3 a graph Γ admitting a semiregular group of automorphisms G can be reconstructed from its quotient Γ/G by using the regular cover construction from a voltage graph. Let us mention another two typical examples of this phenomenon.

Suppose that Γ is a connected graph and G a group of automorphisms of Γ. If G acts transitively on the arcs (ordered pairs of adjacent vertices) of Γ, then Γ can be reconstructed as a coset-graph from the group G, the vertex-stabiliser G_v and an element $a \in G$ where a is an element swapping the vertex v with a neighbour of v; see [9, Section 1.2] or [50], for example. Describing arc-transitive graphs as coset graphs is a standard method often used in the classification results (see [14, 18, 38, 52, 75], to name a few) as well as a way to store a graph in a database (see [56]).

Now, suppose a group of automorphisms G acts regularly on the vertices of Γ. Then Γ can be reconstructed as the Cayley graph $\mathrm{Cay}(G, s)$ where $S \subseteq G$ is the set of all elements of G mapping a prescribed vertex v to each of its neighbours. That is, for a fixed $v \in \mathrm{V}(\Gamma)$ we have $S = \{g \in G \mid v^g \sim v\}$.

In the case when G is an arbitrary group of automorphisms of Γ (not necessarily semiregular or arc-transitive) no similar method which encodes a complete information about Γ has been described in the literature so far even though the usefulness of such a potential method has been discussed on several occasions (for example in [39]). The aim of this Chapter is to present such a method, which generalises the theory of voltage graphs, and some other well-known graph constructions such as the aforementioned coset graph and Cayley graph constructions.

The contents in the remainder of this chapter are for the most part taken verbatim from [59]

4.1 Generalised voltage graphs

In what follows, we first introduce all the necessary notation and concepts needed to present the generalised cover construction and then state the main results of the chapter.

For a group G, we let $S(G)$ denote the set of all subgroups of G and for $g, h \in G$, we let $h^g := g^{-1}hg$ be the conjugate of h by g. For a subgroup $H \leq G$ and an element $g \in G$, we let $H^g = \{h^g \mid h \in H, g \in G\}$ be the conjugate of H by g.

We can now present the construction which generalises that of the derived covering graph introduced in the classical work of Gross and Tucker [30, Section 2.1.1].

Definition 4.1.1. Let Δ be a connected graph, let G be a group, and let $\omega\colon \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta) \to S(G)$ and $\zeta\colon \mathrm{D}(\Delta) \to G$ be two functions such that the following hold for all $x \in \mathrm{D}(\Delta)$:

$$\omega(x) \leq \omega(\operatorname*{beg}_{\Delta} x); \tag{4.1.1}$$

$$\omega(x) = \omega(x^{-1})^{\zeta(x)}; \tag{4.1.2}$$

$$\zeta(x^{-1})\zeta(x) \in \omega(x). \tag{4.1.3}$$

We then say that the quadruple $(\Delta, G, \omega, \zeta)$ is a *generalised voltage graph* and we call the functions ω and ζ a *weight function* and a *voltage assignment*, respectively.

Definition 4.1.2. Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph and let Γ be the graph defined by:

- $\mathrm{V}(\Gamma) = \{(v, \omega(v)g) \mid g \in G, v \in \mathrm{V}(\Delta)\}$;
- $\mathrm{D}(\Gamma) = \{(x, \omega(x)g) \mid g \in G, x \in \mathrm{D}(\Delta)\}$;
- $\mathrm{beg}_\Gamma(x, \omega(x)g) = (\mathrm{beg}_\Delta(x), \omega(\mathrm{beg}_\Delta x)g)$;
- $\mathrm{inv}_\Gamma(x, \omega(x)g) = (\mathrm{inv}_\Delta x, \omega(\mathrm{inv}_\Delta x)\zeta(x)g)$.

Then Γ is called the *generalised cover* arising from $(\Delta, G, \omega, \zeta)$ and is denoted by $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$.

One should of course check that the functions beg_Γ and inv_Γ from Definition 4.1.2 are well defined and that inv_Γ is indeed an involution on $\mathrm{D}(\Gamma)$. We do that in Lemma 4.1.5 in Section 4.1.1. Two explicit examples illustrating this construction are presented in Example 4.1.6.

We now formulate a result (Theorem 4.1.3 below) which can be seen as the main motivation for the introduction of the generalised voltage graphs.

Theorem 4.1.3. *Let Γ be a graph and let $G \leq \mathrm{Aut}(\Gamma)$. Then there exist functions $\zeta\colon \mathrm{D}(\Gamma/G) \to G$ and $\omega\colon \mathrm{V}(\Gamma/G) \cup \mathrm{D}(\Gamma/G) \to S(G)$, such that $(\Gamma/G, G, \omega, \zeta)$ is a generalised voltage graph and Γ is isomorphic to the associated generalised covering graph* $\mathrm{GenCov}(\Gamma/G, G, \omega, \zeta)$.

The above theorem is proved in Section 4.2. In fact, there we state and prove a more detailed version (see Thereom 4.2.1), where more information is given on the weight function ω and the generalised voltage assignment ζ.

Let us now consider three special cases of generalised covers corresponding to the three constructions mentioned at the beginning of this chapter . If G acts semiregularly on $\mathrm{V}(\Gamma)$ then the weights $\omega(x)$ for $x \in \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ appearing in Theorem 4.1.3 are all trivial (as explicitly stated in Theorem 4.2.1). Consequently the generalised voltage assignment ζ satisfies the condition $\zeta(x^{-1}) = \zeta(x)^{-1}$ for all $x \in \mathrm{D}(\Gamma)$ and can thus be viewed as a voltage assignment as defined in [30]. The generalised covering graph $\mathrm{GenCov}(\Gamma, G, \omega, \zeta)$ then coincides with the derived graph $(\Gamma/G)^{\zeta}$ as defined in [30, Section 2.1.1] and Theorem 4.1.3 can be viewed as a generalisation of the classical result [30, Theorem 2.2.2] of Gross and Tucker.

The second special case which we want to point out is when G acts transitively on the arcs and vertices of Γ. In this case the quotient Γ/G is a vertex with a single semi-edge attached to it. Let v be an arbitrary vertex of Γ and let x be a dart emanating from v. As can be deduced from Theorem 4.2.1 the weight function ω appearing in Theorem 4.1.3 is given by $\omega(v^G) = G_v$ and $\omega(x) = G_x$. Furthermore $\zeta(x^G)$ is an element $a \in G$ swapping x with its inverse x^{-1}. As one can easily see, the generalised covering construction with $(\Gamma/G, G, \omega, \zeta)$ is then essentially the same as the coset graph construction (see [9, Section 1.2], for example).

Finally, suppose that G acts locally arc-transitively (that is, for every vertex v the stabiliser G_v acts transitively on the arcs emanating from v) but intransitively on the vertices of Γ. For the sake of simplicity we also assume that Γ has no isolated vertices (that is, that every vertex has an arc emanating from it). These graphs have been extensively studied (see [24] for a nice overview of the topic). One can easily see that then G acts transitively on the edges and that it has precisely two orbits on the vertices. In particular, the quotient Γ/G is isomorphic to K_2 (the graph with two vertices and a single edge connecting them). Let u and v be two adjacent vertices of Γ and let x be a dart with $\operatorname{beg} x = v$ and $\operatorname{end} x = u$. Then, in view of Theorem 4.2.1, the function ω featuring in Theorem 4.1.3 satisfies $\omega(v^G) = G_v$, $\omega(u^G) = G_u$ and $\omega(x^G) = \omega((x^{-1})^G) = G_u \cap G_v$. Furthermore the function ζ can be chosen to be trivial on both darts of Γ/G. The generalised covering construction applied to such a data is then essentially the same as the construction now known as the bicoset construction (see [17, Definition 2.1], or see [24], where this construction is called the *coset graph construction*, or see [29, p. 380] for a historical reference, where the resulting graph was called the *graph of the completion* of an amalgam).

When analysing the connectivity properties of the generalised covers (see Section 4.5), we obtained, as a byproduct, Theorem 4.5.6, which also generalises some well-known facts from algebraic graph theory. We do not state Theorem 4.5.6 here, as we have not yet set the necessary terminology to do so. Instead, we state a simpler but interesting special case.

Proposition 4.1.4. *Let Γ be a graph, let $G \le \mathrm{Aut}(\Gamma)$ be such that Γ/G is a tree and let Γ' be a subgraph of Γ that is mapped isomorphically to T by the quotient map. Then, Γ is connected if and only if $G = \langle G_v \mid v \in \mathrm{V}(\Gamma') \rangle$.*

A well-known special case of the above proposition occurs when the quotient Γ/G is isomorphic to K_2. As we described above, Γ is then isomorphic to the bicoset graph arising from G and the stabilisers G_v and G_u of two adjacent vertices. Proposition 4.1.4 then claims that Γ is connected if and only if $G = \langle G_v, G_u \rangle$ (see [24, Lemma 3.7]).

Another well-known special case of Theorem 4.5.6 occurs when G acts transitively on the arcs of Γ. As mentioned above, the quotient is a vertex with a single semi-edge attached and the generalised cover isomorphic to Γ is essentially the coset graph $\mathrm{Cos}(G_v, a)$ where a is an element of G inverting an arc emanating from a vertex v. Theorem 4.5.6 then asserts that Γ is connected if and only if $G = \langle G_v, a \rangle$ (see [9, Lemma 2.1]).

Section 4.2 is devoted to the proof of Theorem 4.1.3 In Section 4.3 we investigate the action of G on the generalised cover $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and we show some natural isomorphisms between generalised covers. In Section 4.4 we give a generalisation of a well-known result stating that every voltage assignment can be thought of as being trivial on a prescribed spanning tree of a voltage graph. Section 4.5 is devoted to a characterisation of connectivity of a generalised cover and we prove a more general version of Proposition 4.1.4. In Section 4.6 we give necessary and sufficient conditions for a generalised cover to be a simple graph.

4.1.1 Generalised covering graphs

We will now discuss the generalised covering graphs. We first prove that Definition 4.1.2 indeed yields a graph.

Lemma 4.1.5. *Assume the notion from Definition 4.1.2. Then the functions* beg_Γ *and* inv_Γ *are well defined and* $\mathrm{inv}_\Gamma \mathrm{inv}_\Gamma X = X$ *for every dart* X *of* Γ.

Proof. Suppose that for some dart $x \in \mathrm{D}(\Delta)$ and $g, h \in G$, we have $\omega(x)g = \omega(x)h$. Then $gh^{-1} \in \omega(x) \leq \omega(\mathrm{beg}_\Delta x)$, hence $\omega(\mathrm{beg}\, x)g = \omega(\mathrm{beg}\, x)h$. This shows that the value $\mathrm{beg}_\Gamma(x, \omega(x)g)$ is independent of the choice of the representative g of the coset $\omega(x)g$ and hence beg_Γ is a well-defined function on $\mathrm{D}(\Gamma)$.

Similarly $(\zeta(x)g)(\zeta(x)h)^{-1} = (gh^{-1})^{\zeta(x)^{-1}} \in \omega(x)^{\zeta(x)^{-1}} = \omega(x^{-1})$. Therefore $\omega(x^{-1})\zeta(x)g = \omega(x^{-1})\zeta(x)h$, implying that inv_Γ is a well-defined function on $\mathrm{D}(\Gamma)$.

Finally let us show that inv_Γ is an involution. Let $X := (x, \omega(x)g)$ be an arbitrary dart of Γ. Then

$$\mathrm{inv}_\Gamma \mathrm{inv}_\Gamma X = \mathrm{inv}_\Gamma(\mathrm{inv}_\Gamma(x, \omega(x)g)) = \mathrm{inv}_\Gamma(x^{-1}, \omega(x^{-1})\zeta(x)g) =$$

$$= (x, \omega(x)\zeta(x^{-1})\zeta(x)g) = (x, \omega(x)g).$$

□

To illustrate the generalised covering construction we provide two simple examples, both with the base graph Δ being the complete graph on two vertices and the group G being the symmetric group S_6.

Example 4.1.6. Consider the symmetric group $\mathcal{S}_6$ and set $\sigma = (123)(546)$ and $\rho = (23)(45)$. Let $G = \langle \sigma, \rho \rangle$, $H = \langle \rho \rangle$ and $K = \langle \rho\sigma \rangle$. Let Δ be a graph consisting of two vertices u and v joined by a single edge between them with $\operatorname{beg} x = u$ and $\operatorname{beg} y = v$. Let ω be a weight function for Δ given by $\omega(u) = H$, $\omega(v) = K$, $\omega(x) = \omega(y) = 1$. Let ζ be the voltage assignment given by $\zeta(x) = \sigma$ and $\zeta(y) = \sigma^2$. Then $(\Delta, G, \omega, \zeta)$ is a voltage graph and its generalised cover is isomorphic to three pairs of parallel edges (see Figure 4.1.1, right). In contrast, the generalised cover of $(\Delta, G, \omega, \zeta')$ where ζ assigns trivial voltage to both darts x and y is isomorphic to a cycle of length 6 (see Figure 4.1.1, left).

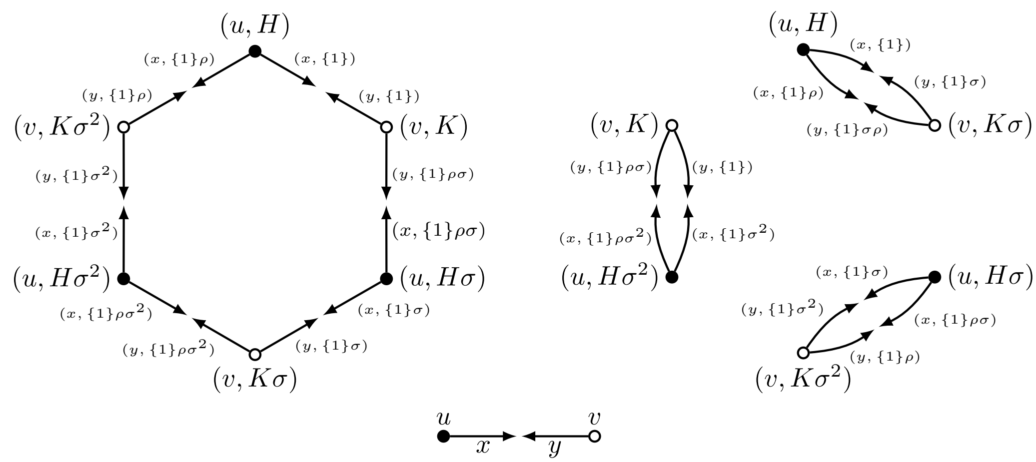


Figure 4.1.1: The generalised covers of two generalised voltage graphs on K_2.

Remark 4.1.7. The following formula for the end of a dart in a generalised cover $\Gamma = \operatorname{GenCov}(\Delta, G, \omega, \zeta)$ will be used often in calculations:

$$\operatorname{end}_\Gamma(x, \omega(x)g) = \operatorname{beg}_\Gamma(x^{-1}, \omega(x^{-1})\zeta(x)g) = (\operatorname{end}_\Delta x, \omega(\operatorname{end}_\Delta x)\zeta(x)g).$$

Lemma 4.1.8. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph and let $x_0 \in \mathrm{D}(\Delta)$ be a dart such that $x_0 \neq x_0^{-1}$. Then* $\operatorname{GenCov}(\Delta, G, \omega, \zeta) = \operatorname{GenCov}(\Delta, G, \omega, \overline{\zeta})$ *where*

$$\overline{\zeta}(x) = \begin{cases} \zeta(x) & \text{if } x \neq x_0^{-1}; \\ \zeta(x_0)^{-1} & \text{if } x = x_0^{-1}. \end{cases}$$

Proof. Let us begin by showing that $(\Delta, G, \omega, \overline{\zeta})$ is a well defined generalised voltage graph. For this we need to show that ω and $\overline{\zeta}$ satisfy (4.1.2). Since $(\Delta, G, \omega, \zeta)$ is a generalised voltage graph, we have $\omega(x) = \omega(x^{-1})^{\zeta(x)} = \omega(x^{-1})^{\overline{\zeta}(x)}$ whenever $x \neq x_0$. It remains to show that $\omega(x_0^{-1}) = \omega(x_0)^{\overline{\zeta}(x_0^{-1})}$. From (4.1.2) we have $\omega(x_0) = \omega(x_0^{-1})^{\zeta(x_0)}$ and then $\omega(x_0^{-1}) = \omega(x_0)^{\zeta(x_0)^{-1}}$. Since $\zeta(x_0)^{-1} = \overline{\zeta}(x_0^{-1})$ we see that $\omega(x_0^{-1}) = \omega(x_0)^{\overline{\zeta}(x_0^{-1})}$. Therefore $(\Delta, G, \omega, \overline{\zeta})$ is well defined.

Now, let $\Gamma := \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and $\overline{\Gamma} := \mathrm{GenCov}(\Delta, G, \omega, \overline{\zeta})$. Observe that $\mathrm{V}(\Gamma) = \mathrm{V}(\overline{\Gamma})$, $\mathrm{D}(\Gamma) = \mathrm{D}(\overline{\Gamma})$ and $\mathrm{beg}_\Gamma = \mathrm{beg}_{\overline{\Gamma}}$. Moreover, inv_Γ agrees with $\mathrm{inv}_{\overline{\Gamma}}$ in all darts except, possibly, in those lying in $\mathrm{fib}(x_0^{-1})$. Hence, it suffices to show that $\mathrm{inv}_\Gamma(x_0^{-1}, \omega(x_0^{-1})g) = \mathrm{inv}_{\overline{\Gamma}}(x_0^{-1}, \omega(x_0^{-1})g)$ for all $g \in G$. In other words, that $(x_0, \omega(x_0)\zeta(x_0^{-1})g) = (x_0, \omega(x_0)\overline{\zeta}(x_0^{-1})g)$. From (4.1.3) we have $\zeta(x_0^{-1}) \in \omega(x_0)\zeta(x_0)^{-1}$, and then $\omega(x_0)\zeta(x_0^{-1}) = \omega(x_0)\zeta(x_0)^{-1} = \omega(x_0)\overline{\zeta}(x_0^{-1})$. Therefore, $\mathrm{inv}_\Gamma = \mathrm{inv}_{\overline{\Gamma}}$ and $\Gamma = \overline{\Gamma}$. □

Remark 4.1.9. In view of Lemma 4.1.8 we see that, without changing the generalised cover $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$, we can always modify ζ in such a way that $\zeta(x)^{-1} = \zeta(x^{-1})$ holds for every dart x not underlying a semiedge.

We finish this section by a number of general remarks about the generalised covering graphs. In what follows we assume that $(\Delta, G, \omega, \zeta)$ is a generalised voltage graph and that $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$.

Remark 4.1.10. If $x \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$, then the set $\{(x, \omega(x)g) : g \in G\} \subseteq \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ is called the *fibre above* x and is denoted by $\mathrm{fib}(x)$. Note that the mapping $\varphi\colon \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to V(\Delta) \cup \mathrm{D}(\Delta)$ defined by $\varphi(x, \omega(x)g) := x$ for every $(x, \omega(x)g) \in V(\Gamma) \cup \mathrm{D}(\Gamma)$ is a graph epimorphism, which we shall call *the generalised covering projection associated with* $(\Delta, G, \omega, \zeta)$.

Remark 4.1.11. For $u \in \mathrm{V}(\Delta)$ let $\iota(u) = |G : \omega(u)|$ and observe that $\iota(u) = |\mathrm{fib}(u)|$. Similarly for $x \in \mathrm{D}(\Delta)$ such that $\mathrm{beg}\, x = u$, let

$$\lambda(x) = |\omega(u) : \omega(x)|. \tag{4.1.4}$$

Observe that $\iota(u)\lambda(x) = |G : \omega(x)| = |\mathrm{fib}(x)|$. Moreover, for $\tilde{u} \in \mathrm{fib}(u)$ we have $|\Gamma(\tilde{u}) \cap \mathrm{fib}(x)| = \lambda(x)$ and hence the valence of $\tilde{u}$ in Γ is $\sum_{x \in \Delta(u)} \lambda(x)$.

Remark 4.1.12. If for every vertex $u \in \mathrm{V}(\Delta)$ there exist a constant c_u such that $\lambda(x) = c_u$ for every dart $x \in \Delta(u)$, then the generalised covering projection $\varphi\colon \Gamma \to \Delta$ is *a branched covering* as defined [47, 48].

4.2 Reconstruction

This section is devoted to the proof of Theorem 4.1.3, stated in Section 4.1. We will state and prove a slightly more detailed version of the theorem. For this, we will need to define the notion of a transversal.

For a group G acting on a set Ω we let Ω/G denote the set of orbits of this action; that is $\Omega/G = \{x^G : x \in \Omega\}$. A subset of Ω which contains precisely one element from each orbit in Ω/G is called a *transversal* of Ω/G.

Theorem 4.2.1. *Let Γ be a graph and let $G \leq \mathrm{Aut}(\Gamma)$. Let $\mathcal{T}_V$ be a transversal of $\mathrm{V}(\Gamma)/G$ and let $\mathcal{T}_D$ be a transversal of $\mathrm{D}(\Gamma)/G$ such that $x \in \mathcal{T}_D$ implies $\mathrm{beg}\, x \in \mathcal{T}_V$ (note that such a pair of transversals always exists). Set $\mathcal{T} := \mathcal{T}_D \cup \mathcal{T}_V$. Then every vertex or dart of Γ/G can be written uniquely as x^G with*

$x \in \mathcal{T}$. For x in $\mathcal{T}_D$ let $\iota(x)$ be the unique element of $\mathcal{T}_D$ such that $x^{-1} \in \iota(x)^G$. Let $\zeta : \mathrm{D}(\Gamma/G) \to G$ be a function such that:

$$\iota(x)^{\zeta(x^G)} = x^{-1} \text{ for every } x \in \mathcal{T}_D. \tag{4.2.1}$$

Define $\omega : \mathrm{V}(\Gamma)/G \cup \mathrm{D}(\Gamma)/G \to S(G)$ by letting

$$\omega(x^G) := G_x \quad \text{for every} \quad x \in \mathcal{T}.$$

Then the quadruple $(\Gamma/G, G, \omega, \zeta)$ is a generalised voltage graph and there exists an isomorphism between Γ and $\mathrm{GenCov}(\Gamma/G, G, \omega, \zeta)$ which maps every G-orbit on $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ bijectively to a fibre of the corresponding generalised covering graph $\mathrm{GenCov}(\Gamma/G, G, \omega, \zeta)$.

Proof. Since we work with three distinct graphs in this proof, namely Γ, Γ/G and $\Theta := \mathrm{GenCov}(\Gamma/G, G, \omega, \zeta)$, we will be very careful not to confuse the corresponding V, D, beg and inv operators. In particular, we let $(V', D', \mathrm{beg}', \mathrm{inv}') := \Gamma/G$ and reserve the shorthand notation x^{-1} for $\mathrm{inv}_\Gamma x$ only when $x \in \mathrm{D}(\Gamma)$. Further, for $x \in \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$, we let $\overline{x} := x^G$ denote the G-orbit of x; note that then $\overline{x} \in V' \cup D'$.

We will now show that the quadruple $(\mathrm{V}(\Gamma)/G, \mathrm{D}(\Gamma)/G, \omega, \zeta)$ satisfies conditions (4.1.1)–(4.1.3). For the remainder of this proof, consider a dart $x \in \mathcal{T}_D$. By definition of the quotient graph it follows that the darts $\overline{x}$ and $\overline{\iota(x)}$ are mutually inverse; that is

$$\mathrm{inv}' \, \overline{x} = \overline{\iota(x)} \quad \text{and} \quad \mathrm{inv}' \, \overline{\iota(x)} = \overline{x}. \tag{4.2.2}$$

Moreover, by definition of ζ it also follows that $\iota(x) = (x^{-1})^{\zeta(\overline{x})^{-1}}$ and since ζ is an automorphism, we have $\iota(x)^{-1} = x^{\zeta(\mathrm{inv}' \, \overline{x})}$ and so

$$x^{\zeta(\overline{\iota(x)})} = \iota(x)^{-1}. \tag{4.2.3}$$

Similarly, by (4.2.1) we have

$$(\iota(x)^{-1})^{\zeta(\overline{x})} = x. \tag{4.2.4}$$

Using formulae (4.2.2), (4.2.3) and (4.2.4), we may conclude that

$$x^{\zeta(\mathrm{inv}' \, \overline{x})\zeta(\overline{x})} = x^{\zeta(\overline{\iota(x)})\zeta(\overline{x})} = (\iota(x)^{-1})^{\zeta(\overline{x})} = x.$$

In particular, $\zeta(\mathrm{inv}' \, \overline{x})\zeta(\overline{x}) \in G_x$, and since $\omega(\overline{x}) = G_x$, we see that the condition (4.1.3) is satisfied.

Let us now show that the condition (4.1.2) is satisfied. Observe first that the stabilisers $G_{\iota(x)}$ and $G_{\iota(x)^{-1}}$ of mutually inverse darts y and y^{-1} are equal and that for any $z \in \mathrm{D}(\Gamma)$ and any $g \in \mathrm{Aut}(\Gamma)$, the conjugate $(G_z)^g$ of the stabiliser G_z equals the stabiliser G_{z^g}. We thus see that

$$\omega(\mathrm{inv}' \, \overline{x})^{\zeta(\overline{x})} = \omega(\overline{\iota(x)})^{\zeta(\overline{x})} = (G_{\iota(x)})^{\zeta(\overline{x})} = (G_{\iota(x)^{-1}})^{\zeta(\overline{x})} =$$

$$G_{(\iota(x)^{-1})^{\zeta(\overline{x})}} = G_x = \omega(\overline{x}),$$

showing that the condition (4.1.2) is satisfied. Finally, since $G_x \leq G_{\mathrm{beg}\,x}$ for any dart $x \in \mathrm{D}(\Gamma)$, we see that (4.1.1) holds. Therefore $(\Gamma/G, G, \omega, \zeta)$ is a generalised voltage graph.

We will now show that the corresponding generalised cover Θ is isomorphic to the graph Γ. By definition of the generalised cover and the function ω, we see that $\mathrm{V}(\Theta) = \{(\overline{v}, G_v g) : v \in \mathcal{T}_V, g \in G\}$ and $\mathrm{D}(\Theta) = \{(\overline{x}, G_x g) : x \in \mathcal{T}_D, g \in G\}$. Let $\varphi : \mathrm{V}(\Theta) \cup \mathrm{D}(\Theta) \to \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ be given by

$$\varphi(\overline{x}, G_x g) = x^g \ \text{ for } \ x \in \mathcal{T}.$$

To see that φ is well defined, suppose that for some $x \in \mathcal{T}$, we have $G_x g = G_x g'$. It follows that $g'g^{-1} \in G_x$ and so $x^g = x^{g'}$. Then $\varphi(\overline{x}, G_x g) = x^g = x^{g'} = \varphi(\overline{x}, G_x g')$. Hence φ is well defined. Since the converse of every implication in the preceding lines holds, this also shows that φ is injective. Since φ is clearly surjective, it remains to be shown that $\mathrm{beg}_\Gamma \varphi(X) = \varphi(\mathrm{beg}_\Theta X)$ and $\mathrm{inv}_\Gamma \varphi(X) = \varphi(\mathrm{inv}_\Theta X)$ for every dart $X \in \mathrm{D}(\Theta)$. Let $X \in \mathrm{D}(\Theta)$. Then $X = (\overline{x}, G_x g)$ for some $x \in \mathcal{T}_D$. Then

$$\mathrm{beg}_\Gamma \varphi(X) = \mathrm{beg}_\Gamma \varphi(\overline{x}, G_x g) = \mathrm{beg}_\Gamma(x^g) = (\mathrm{beg}_\Gamma x)^g.$$

On the other hand

$$\varphi(\mathrm{beg}_\Theta(\overline{x}, G_x g)) = \varphi(\mathrm{beg}'\,\overline{x}, \omega(\mathrm{beg}'\,\overline{x})g) = \varphi(\overline{(\mathrm{beg}\,x)}, G_{\mathrm{beg}\,x} g) = (\mathrm{beg}\,x)^g.$$

Hence $\mathrm{beg}_\Gamma \varphi(X) = \varphi(\mathrm{beg}_\Theta X)$, as claimed. Further, we see that

$$\mathrm{inv}_\Gamma \varphi(X) = \mathrm{inv}_\Gamma \varphi(\overline{x}, G_x g) = \mathrm{inv}_\Gamma(x^g) = (\mathrm{inv}_\Gamma x)^g.$$

Recall now the definition of the dart $\iota(x)$ and the fact that $\mathrm{inv}'\,\overline{x} = \overline{\iota(x)}$. Then

$$\varphi(\mathrm{inv}_I(\overline{x}, G_x g)) = \varphi(\mathrm{inv}'\,\overline{x}, \omega(\mathrm{inv}'\,\overline{x})\zeta(\overline{x})g) = \varphi(\overline{\iota(x)}, G_{\iota(x)}\zeta(\overline{x})g) =$$

$$\iota(x)^{\zeta(\overline{x})g} = (\mathrm{inv}_\Gamma x)^g.$$

Note the the last equality follows from (4.2.4) while the second to last from the fact that $\iota(x) \in \mathcal{T}_D$. We have thus shown that $\mathrm{inv}_\Gamma \varphi(X) = (\mathrm{inv}_\Gamma x)^g = \varphi(\mathrm{inv}_\Theta X)$. We conclude that φ is an isomorphism and thus $\Gamma \cong \Theta$, as claimed. Moreover, φ clearly maps fibres in Θ to G-orbits on Γ. This finishes the proof of Theorem 4.1.3. $\square$

Remark 4.2.2. Assume the notation introduced in the statement and the proof of the above theorem. Consider a dart $X \in \Gamma/G$ and let x be the unique element in $\mathcal{T}_D$ such that $X = x^G$. Then $\zeta(X)$ is defined to be an element g of G which maps x^{-1} to the unique element $y \in \mathcal{T}$ such that $(x^{-1})^G = y^G$ (note that there is some freedom in the choice of such an element g). In particular, if $x^{-1} \in \mathcal{T}_D$, then one can assume that $\zeta(X)$ is trivial.

For a subgroup G and a subgroup $H \leq G$ we let $\mathrm{core}_G(H) = \cap_{g \in G} H^g$ denote the *core* of H in G. For the reasons that will become apparent in Remark 4.2.4 below and in Proposition 4.3.2, we introduce the following property of generalised voltage graphs:

Definition 4.2.3. A generalised voltage graph $(\Delta, G, \omega, \zeta)$ is said to be *faithful* provided that

$$\mathrm{core}_G\big(\bigcap_{x\in \mathrm{D}(\Delta)} \omega(x)\big) = 1. \tag{4.2.5}$$

Remark 4.2.4. The generalised voltage graph $(\Gamma/G, G, \omega, \zeta)$ mentioned in the statement of Theorem 4.1.3 is in fact faithful. Indeed: Using the notation introduced in the proof above, observe that $\{x^g : x \in \mathcal{T}_D, g \in G\} = D$. Therefore

$$\mathrm{core}_G\big(\bigcap_{X\in \mathrm{D}(\Gamma/G)} \omega(X)\big) = \mathrm{core}_G\big(\bigcap_{x\in\mathcal{T}_D} G_x\big) = \bigcap_{x\in\mathcal{T}_D, g\in G} (G_x)^g = \bigcap_{x\in D(\Gamma)} G_x,$$

which is trivial since G acts faithfully on $\mathrm{D}(\Gamma)$.

4.3 Automorphisms of generalised covers

This section is devoted to the study of the automorphisms of the generalised covering graph Γ arising from a given generalised voltage graph $(\Delta, G, \omega, \zeta)$. The first lemma and the proposition that follows it show that the group G acts in a natural way on Γ, as one would of course expect.

Lemma 4.3.1. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph, let $h \in G$ and let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$. Then the permutation $\bar{h}$ of $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ defined by $(x, \omega(x)g)^{\bar{h}} := (x, \omega(x)gh)$ for every $x \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$ and $g \in G$ is an automorphism of Γ.*

Proof. We leave it to the reader to check that $\bar{h}$ is indeed a permutation of $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$. To see that $\bar{h}$ is an automorphism of Γ, observe that:

$$\begin{aligned}
(\mathrm{beg}(x, \omega(x)g))^{\bar{h}} &= (\mathrm{beg}\, x, \omega(\mathrm{beg}\, x)g)^{\bar{h}} \\
&= (\mathrm{beg}\, x, \omega(\mathrm{beg}\, x)gh) \\
&= \mathrm{beg}(x, \omega(x)gh) \\
&= \mathrm{beg}((x, \omega(x)g)^{\bar{h}}),
\end{aligned}$$

and

$$\begin{aligned}
((x, \omega(x)g)^{-1})^{\bar{h}} &= (x^{-1}, \omega(x^{-1})\zeta(x)g)^{\bar{h}} \\
&= (x^{-1}, \omega(x)^{-1}\zeta(x)gh) \\
&= (x, \omega(x)gh)^{-1} \\
&= ((x, \omega(x)g)^{\bar{h}})^{-1}.
\end{aligned}$$

Therefore, $\bar{h}$ is a graph morphism from Γ to Γ as it commutes with beg and inv, and thus an automorphism of Γ □

Recall that we call the generalised voltage graph $(\Delta, G, \omega, \zeta)$ is faithful provided that the core of the group $\cap_{x\in \mathrm{D}(\Delta)}\omega(x)$ in G is trivial (see Definition 4.2.3). We now prove a result that explains the choice of the word "faithful".

Proposition 4.3.2. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph. Then the mapping $\Phi : G \to \mathrm{Aut}(\Gamma)$, $\Phi(h) = \bar{h}$, is a homomorphism of groups, which is injective if and only if the generalised voltage graph is faithful.*

Proof. Let K be the kernel of Φ. Observe that for $h \in G$, $(x, \omega(x)g)^{\bar{h}} = (x, \omega(x)g)$ if and only if $h^{g^{-1}} \in \omega(x)$ or, equivalently, $h \in \omega(x)^g$. Hence, $h \in K$ if and only if

$$h \in \bigcap_{x \in \mathrm{D}(\Gamma) \cup \mathrm{V}(\Gamma)} \bigcap_{g \in G} \omega(x)^g.$$

However, since $\omega(x) \leq \omega(\mathrm{beg}\, x)$ for all $x \in \mathrm{D}(\Delta)$, we see that $\omega(x)^g \leq \omega(\mathrm{beg}\, x)^g$ and thus

$$\bigcap_{x \in \mathrm{D}(\Gamma) \cup \mathrm{V}(\Gamma)} \bigcap_{g \in G} \omega(x)^g = \bigcap_{x \in \mathrm{D}(\Gamma)} \bigcap_{g \in G} \omega(x)^g = \bigcap_{g \in G} \Big(\bigcap_{x \in \mathrm{D}(\Gamma)} \omega(x) \Big)^g. \tag{4.3.1}$$

Therefore, K is trivial if and only if the right side of (4.3.1) is trivial, that is, if and only if the generalised voltage graph is faithful. □

The next lemma and the corollaries that follow it establish another natural source of automorphisms of the generalised covering graph.

Lemma 4.3.3. *Let $(\Delta, G, \omega, \zeta)$ and $(\Delta', G', \omega', \zeta')$ be two generalised voltage graphs, and let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and $\Gamma' = \mathrm{GenCov}(\Delta', G', \omega', \zeta')$. Suppose that there exist a graph isomorphisms $\varphi : \Delta \to \Delta'$ and a group isomorphism $f : G \to G'$ such that $f(\omega(x)) = \omega'(\varphi(x))$ for every $x \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$ and $f(\zeta(x)) = \zeta'(\varphi(x))$ for every $x \in \mathrm{V}(\Delta)$. Let $\Phi : \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to \mathrm{V}(\Gamma') \cup \mathrm{D}(\Gamma')$ be given by*

$$\Phi \colon (x, \omega(x)g) \mapsto (\varphi(x), \omega'(\varphi(x))f(g)) \text{ for every } x \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta).$$

Then Φ is an isomorphism of graphs between Γ and Γ'.

Proof. We first show that the image of Φ does not depend on the choice of representatives of a coset $\omega(x)g$ and that Φ is injective. Suppose that $\omega'(\varphi(x))f(g)) = \omega'(\varphi(x))f(h))$ for some $h, g \in G$. Then $f(\omega(x))f(g) = f(\omega(x))f(h)$ and hence, since f is a group isomorphism, $\omega(x)g = \omega(x)h$. This shows that Φ is a well-defined function on $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ and that it is injective. Surjectivity of Φ follows directly from the fact that both φ and f are surjective. It remains to see that Φ intertwines the functions beg and beg$'$ as well as inv and inv$'$. Note that

$$\Phi(\mathrm{beg}(x, \omega(x)g)) = \Phi(\mathrm{beg}\, x, \omega(\mathrm{beg}\, x)g) = (\varphi(\mathrm{beg}\, x), f(\omega(\mathrm{beg}\, x))f(g)).$$

But φ is a graph isomorphism, therefore

$$\varphi(\mathrm{beg}\, x) = \mathrm{beg}'(\varphi(x)) \text{ and } f(\omega(\mathrm{beg}\, x)) = \omega'(\varphi(\mathrm{beg}\, x)) = \omega'(\mathrm{beg}'(\varphi(x))).$$

Hence

$$\Phi(\mathrm{beg}(x, \omega(x)g)) = (\mathrm{beg}'(\varphi(x)), \omega(\mathrm{beg}'(\varphi(x))f(g)).$$

On the other hand,

$$\mathrm{beg}'(\Phi(x,\omega(x)g)) = \mathrm{beg}'(\varphi(x), f(\omega(x))f(g)) = (\mathrm{beg}'(\varphi(x)), \omega'(\mathrm{beg}'(\varphi(x)))f(g)).$$

Therefore, $\Phi(\mathrm{beg}(x,\omega(x)g)) = \mathrm{beg}'\,\Phi(x,\omega(x)g)$, as required. Similarly,

$$\Phi((x,\omega(x)g)^{-1}) = \Phi(x^{-1},\omega(x^{-1})\zeta(x)g) = (\varphi(x^{-1}), f(\omega(x^{-1}))f(\zeta(x)g)).$$

But $f(\omega(x^{-1}))f(\zeta(x)g)) = \omega'(\varphi(x^{-1}))f(\zeta(x))f(g)) = \omega'(\varphi(x)^{-1})\zeta'(\varphi(x))f(g)$. Thus,

$$\Phi((x,\omega(x)g)^{-1}) = (\varphi(x^{-1}), \omega'(\varphi(x)^{-1})\zeta'(\varphi(x))f(g)).$$

On the other hand,

$$(\Phi(x,\omega(x)g))^{-1} = (\varphi(x),\omega'(\varphi(x))f(g))^{-1} = (\varphi(x^{-1}),\omega'(\varphi(x)^{-1})\zeta'(\varphi(x))f(g)).$$

This shows that $\Phi(x,\omega(x))^{-1} = (\Phi(x,\omega(x)g)^{-1})$ and concludes the proof that Φ is an isomorphism of graph Γ and Γ'. □

This result has two immediate but useful consequences.

Corollary 4.3.4. *Suppose $(\Delta, G, \omega, \zeta)$ is a generalised voltage graph and f an automorphism of the group $\mathrm{Aut}(G)$. Then*

$$\mathrm{GenCov}(\Delta, G, \omega, \zeta) \cong \mathrm{GenCov}(\Delta, G, f\circ\omega, f\circ\zeta).$$

Proof. Set $\varphi := \mathrm{id}$, $\omega' := f\circ\omega$, $\zeta' := f\circ\zeta$ and apply Lemma 4.3.3. □

Corollary 4.3.5. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph, let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$, and let $\varphi \in \mathrm{Aut}(\Delta)$ and $f \in \mathrm{Aut}(G)$ be such that $\omega(x^\varphi) = f(\omega(x))$ for every $x \in \mathrm{V}(\Delta)\cup\mathrm{D}(\Delta)$ and $\zeta(x^\varphi) = f(\zeta(x))$ for every $x \in \mathrm{V}(\Delta)$. Then the permutation Φ given by*

$$(x,\omega(x)g)^\Phi = (x^\varphi, \omega(x^\varphi)f(g)) \text{ for every } x \in \mathrm{V}(\Delta)\cup\mathrm{D}(\Delta)$$

is an automorphism of Γ.

Remark 4.3.6. Both the automorphisms arising from Lemma 4.3.1 as well as those from Corollary 4.3.5 preserve the partition of the vertices and darts of the generalised covering graph into fibres. What is more, the automorphisms arising from Lemma 4.3.1 preserve each fibre set-wise. In the case of (non-generalised) covering graphs, there exists a nice characterisation of such automorphisms of the covering graph; see [40], for example. In the case of generalised covering graphs, such a nice characterisation seems to be out of the reach.

4.4 Normalised voltages

It this section we first show that the generalised voltage assignment ζ can always be assumed to be trivial on the darts of any fixed spanning tree of the base graph Δ. This is the analogue of a well-known result in the theory of regular (non-generalised) voltage graphs (see Theorem 2.3.5 or [32, p. 91], for example).

Lemma 4.4.1. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph and let $x \in \mathrm{D}(\Delta)$ such that $\operatorname{beg} x \neq \operatorname{end} x$. Then there exists a voltage assignment ζ_x assigning the trivial element 1_G to x and a weight function ω_x such that $\operatorname{GenCov}(\Delta, G, \omega, \zeta) \cong \operatorname{GenCov}(\Delta, G, \omega_x, \zeta_x)$.*

Proof. Let $v := \operatorname{end} x$ and note that by assumption, $\operatorname{beg} x \neq v$. Let $\operatorname{beg} := \operatorname{beg}_\Delta$. Now define $\zeta_x : \mathrm{D}(\Delta) \to G$ and $\omega_x : \mathrm{D}(\Delta) \cup \mathrm{V}(\Delta) \to S(G)$ as follows:

$$\zeta_x(y) = \begin{cases} \zeta(y)\zeta(x) & \text{if } \operatorname{beg} y = v; \\ \zeta(x)^{-1}\zeta(y) & \text{if } \operatorname{end} y = v; \\ \zeta(y) & \text{otherwise.} \end{cases}$$

$$\omega_x(z) = \begin{cases} \omega(z)^{\zeta(x)} & \text{if } v \in \{z, \operatorname{beg} z\}; \\ \omega(z) & \text{otherwise.} \end{cases}$$

Let us show that $(\Delta, G, \omega_x, \zeta_x)$ is a generalised voltage graph. Clearly $\omega_x(y) \leq \omega_x(\operatorname{beg} y)$ for all $y \in \mathrm{D}(\Delta)$, so condition (4.1.1) holds.

Let us show that (4.1.2) holds, that is, that $\omega_x(y) = \omega_x(y^{-1})^{\zeta_x(y)}$ for every $y \in \mathrm{D}(\Delta)$. If $\operatorname{beg} y = v$, then $\omega_x(y^{-1})^{\zeta_x(y)} = \omega(y^{-1})^{\zeta(y)\zeta(x)} = \omega(y)^{\zeta(x)} = \omega_x(y)$. Suppose now that $\operatorname{end} y = v$. Since $\operatorname{beg} y^{-1} = v$, we see that $\omega_x(y^{-1}) = \omega(y^{-1})^{\zeta(x)}$. Therefore $\omega_x(y^{-1})^{\zeta_x(y)} = \omega_x(y^{-1})^{\zeta(x)^{-1}\zeta(y)} = \omega(y^{-1})^{\zeta(y)}$. Finally, if neither $\operatorname{beg} y$ nor $\operatorname{end} y$ equals v, then $\omega_x(y^{-1})^{\zeta_x(y)} = \omega(y^{-1})^{\zeta(y)} = \omega(y) = \omega_x(y)$. We thus see that (4.1.2) holds for every $y \in \mathrm{D}(\Delta)$.

Now, let us show that the condition (4.1.3) is satisfied, that is, that for every $y \in \mathrm{D}(\Delta)$, we have $\zeta_x(y^{-1})\zeta_x(y) \in \omega_x(y)$. If $\operatorname{beg} y = v$, then $\zeta_x(y^{-1})\zeta_x(y) = \zeta(x)^{-1}\zeta(y^{-1})\zeta(y)\zeta(x)$. But $\zeta(y^{-1})\zeta(y) \in \omega(y)$, therefore $\zeta_x(y^{-1})\zeta_x(y) \in \omega(y)^{\zeta(x)}$, and since $\omega(y)^{\zeta(x)} = \omega_x(y)$, we see that $\zeta_x(y^{-1})\zeta_x(y) \in \omega_x(y)$. If $\operatorname{end} y = v$, then $\zeta_x(y^{-1})\zeta_x(y) = \zeta(y^{-1})\zeta(x)\zeta(x)^{-1}\zeta(y) = \zeta(y^{-1})\zeta(y)$. Since $\zeta(y^{-1})\zeta(y) \in \omega(y)$ and $\omega(y) = \omega_x(y)$, we see that $\zeta(y^{-1})\zeta(y) \in \omega_x(y)$. Finally, if neither $\operatorname{beg} y$ nor $\operatorname{end} y$ equals v, then $\zeta_x(y^{-1})\zeta_x(y) = \zeta(y^{-1})\zeta(y) \in \omega(y) = \omega_x(y)$. This concludes the proof of (4.1.3) and shows that $(\Delta, G, \omega_x, \zeta_x)$ is a generalised voltage graph.

Now let $\Gamma := \operatorname{GenCov}(\Delta, G, \omega, \zeta)$ and $\Gamma' := \operatorname{GenCov}(\Delta, G, \omega_x, \zeta_x)$. Define $\varphi : \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to \mathrm{V}(\Gamma') \cup \mathrm{D}(\Gamma')$ by

$$\varphi((z, \omega(z)g)) = \begin{cases} (z, \omega_x(z)\zeta(x)^{-1}g) & \text{for } z \in \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma),\ v \in \{z, \operatorname{beg} z\}; \\ (z, \omega_x(z)g) & \text{for } z \in \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma),\ v \notin \{z, \operatorname{beg} z\}. \end{cases}$$

We first show that φ is well defined and injective. Let $a, b \in G$ and let $z \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$. We need to show that $\varphi(z, \omega(z)a) = \varphi(z, \omega(z)b)$ if only if $(z, \omega(z)a) = (z, \omega(z)b)$. If $v \notin \{z, \operatorname{beg} z\}$, then $\varphi(z, \omega(z)a) = (z, \omega_x(z)a) = (z, \omega(z)a)$ and $\varphi(z, \omega(x)b) = (z, \omega_x(z)b) = (z, \omega(z)b)$ and the claim follows. Assume now that $v \in \{z, \operatorname{beg} z\}$. Note that by the definition of ω_x, it then follows that $\omega_x(z) = \omega(z)^{\zeta(x)}$. Hence $ab^{-1} \in \omega(z)$ holds if and only if $ab^{-1} \in \omega_x(z)^{\zeta(x)^{-1}}$ holds, and therefore $\omega(z)a = \omega(z)b$ holds if and only if $\omega_x(z)^{\zeta(x)^{-1}}a = \omega_x(z)^{\zeta(x)^{-1}}b$ holds if and only if $\omega_x(z)\zeta(x)^{-1}a = \omega_x(z)\zeta(x)^{-1}b$ holds, and thus if and only if $\varphi(z, \omega(x)a) = \varphi(z, \omega(x)b)$ holds. We have thus shown that φ is well defined and injective.

To prove surjectivity of φ, take an arbitrary $(z, \omega_x(z)g) \in \mathrm{V}(\Gamma') \cup \mathrm{D}(\Gamma')$. If $v \notin \{z, \operatorname{beg} z\}$, then $(z, \omega_x(z)g) = \varphi(z, \omega(z)g)$ and thus $(z, \omega_x(z)g)$ is in the image of φ. Similarly, if $v \in \{z, \operatorname{beg} z\}$, then $(z, \omega_x(z)g) = \varphi(z, \omega(z)\zeta(x)^{-1}g)$, proving that φ is surjective.

It remains to show that φ intertwines the functions $\operatorname{beg}_\Gamma$ and $\operatorname{beg}_{\Gamma'}$ as well as the functions $\operatorname{inv}_\Gamma$ and $\operatorname{inv}_{\Gamma'}$. Let us first show that φ intertwines $\operatorname{beg}_\Gamma$ and $\operatorname{beg}_{\Gamma'}$. If $\operatorname{beg} y = v$, then:

$$\begin{aligned}
\varphi(\operatorname{beg}_\Gamma(y, \omega(y)g)) &= \varphi((\operatorname{beg} y, \omega(\operatorname{beg} y)g)) \\
&= (\operatorname{beg} y, \omega_x(\operatorname{beg} y)\zeta(x)^{-1}g) \\
&= \operatorname{beg}_{\Gamma'}(y, \omega_x(y)\zeta(x)^{-1}g) \\
&= \operatorname{beg}_{\Gamma'}(\varphi(y, \omega(y)g)).
\end{aligned}$$

On the other hand, if $\operatorname{beg} y \neq v$, then:

$$\begin{aligned}
\varphi(\operatorname{beg}_\Gamma(y, \omega(y)g)) &= \varphi((\operatorname{beg} y, \omega(\operatorname{beg} y)g)) \\
&= (\operatorname{beg} y, \omega_x(\operatorname{beg} y)g) \\
&= \operatorname{beg}_{\Gamma'}(y, \omega_x(y)g) \\
&= \operatorname{beg}_{\Gamma'}(\varphi(y, \omega(y)g)).
\end{aligned}$$

Let us now show that φ intertwines
$\operatorname{inv}_\Gamma$ and $\operatorname{inv}_{\Gamma'}$. If $\operatorname{end} y = v$, then:

$$\begin{aligned}
\varphi((y, \omega(y)g)^{-1}) &= \varphi((y^{-1}, \omega(y^{-1})\zeta(y)g)) \\
&= (y^{-1}, \omega_x(y^{-1})\zeta(x)^{-1}\zeta(y)g) \\
&= (y^{-1}, \omega_x(y^{-1})\zeta_x(y)g) \\
&= (y, \omega_x(y)g)^{-1} \\
&= \varphi(y, \omega(y)g)^{-1}.
\end{aligned}$$

Similarly, if $\operatorname{beg} y = v$, then:

$$\begin{aligned}
\varphi((y, \omega(y)g)^{-1}) &= \varphi((y^{-1}, \omega(y^{-1})\zeta(y)g)) \\
&= (y^{-1}, \omega_x(y^{-1})\zeta(y)g) \\
&= (y^{-1}, \omega_x(y^{-1})\zeta_x(y)\zeta(x)^{-1}g) \\
&= (y, \omega_x(y)\zeta(x)^{-1}g)^{-1} \\
&= \varphi(y, \omega(y)g)^{-1}.
\end{aligned}$$

Finally, if $v \notin \{\operatorname{beg} y, \operatorname{end} y\}$, then:

$$\begin{aligned}
\varphi((y, \omega(y)g)^{-1}) &= \varphi((y^{-1}, \omega(y^{-1})\zeta(y)g)) \\
&= (y^{-1}, \omega_x(y^{-1})\zeta(y)g) \\
&= (y^{-1}, \omega_x(y^{-1})\zeta_x(y)g) \\
&= (y, \omega_x(y)g)^{-1} \\
&= \varphi(y, \omega(y)g)^{-1}.
\end{aligned}$$

Therefore φ is an isomorphism and $\Gamma \cong \Gamma'$, as claimed. $\square$

Remark 4.4.2. The voltage assignments ζ and ζ_x of Lemma 4.4.1 coincide in all darts of Δ, except in those whose initial vertex is v.

Theorem 4.4.3. *Let $(\Delta, G, \omega, \zeta)$ be a voltage graph and let T be a spanning tree of Δ. Then there exists a voltage assignment ζ' and a weight function ω' such that $\zeta'(x) = 1_G$ for all $x \in \mathrm{D}(T)$ and $\mathrm{GenCov}(\Delta, G, \omega, \zeta) \cong \mathrm{GenCov}(\Delta, G, \omega', \zeta')$.*

Proof. Let $(\Delta, G, \omega, \zeta)$ be a voltage graph. We will show that for any subgraph T of $(\Delta, G, \omega, \zeta)$ that is isomorphic to a tree, there is a voltage assignment ζ^T and a weight function ω^T such that ζ^T is trivial on all darts of T and $\mathrm{GenCov}(\Delta, G, \omega, \zeta) \cong \mathrm{GenCov}(\Delta, G, \omega^T, \zeta^T)$. We proceed by induction on the size (the number of edges) of T. Note that if T has only one edge, then the result follows at once from Lemma 4.4.1 and Remark 4.1.9. Now, suppose the result holds for any tree of size k, $k < |\mathrm{V}(\Delta)| - 1$. Let T be a spanning tree of Δ and notice that T has $|\mathrm{V}(\Delta)| - 1$ edges. Let v be a leaf (a vertex of valency 1) of T and let u be the unique vertex of T adjacent to v. Then there is a unique dart x in T such that $\operatorname{beg} x = u$ and $\operatorname{end} x = v$. Let T' be the tree obtained by deleting the vertex v from T. Then, by induction hypothesis there is a weight function $\omega^{T'}$ and a voltage assignment $\zeta^{T'}$ such that $\zeta^{T'}$ is trivial on all dart in T' and $\mathrm{GenCov}(\Delta, G, \omega, \zeta) \cong \mathrm{GenCov}(\Delta, G, \omega^{T'}, \zeta^{T'})$. By Lemma 4.4.1, there exists a weight function ω^T and a voltage assignment ζ^T such that $\zeta^T(x) = 1_G$ and $\mathrm{GenCov}(\Delta, G, \omega^{T'}, \zeta^{T'}) \cong \mathrm{GenCov}(\Delta, G, \omega^T, \zeta^T)$. Moreover, by Remark 4.4.2, $\zeta^T(y) = \zeta^{T'}(y)$ for all $y \in \mathrm{D}(T) \setminus \{x, x^{-1}\}$. Therefore, ζ^T is trivial on all darts of T and $\mathrm{GenCov}(\Delta, G, \omega, \zeta) \cong \mathrm{GenCov}(\Delta, G, \omega^{T'}, \zeta^{T'}) \cong \mathrm{GenCov}(\Delta, G, \omega^T, \zeta^T)$. $\square$

Remark 4.4.4. Let T be a spanning tree of a graph Δ. A generalised voltage graph $(\Delta, G, \omega, \zeta)$ where $\zeta(x) = 1$ for every dart of T is said to be *T-normalised*. In light of Theorem 4.4.3 we can always assume a voltage graph to be T-normalised for a prescribed spanning tree T. This will often prove to be useful as it makes calculations simpler.

Remark 4.4.5. Observe that the isomorphism between $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and $\mathrm{GenCov}(\Delta, G, \omega_x, \zeta_x)$ provided in the proof of Lemma 4.4.1 mapped fibres of the first generalised cover to the fibres of the second generalised cover. Consequently, also the isomorphism between the graphs $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and $\mathrm{GenCov}(\Delta, G, \omega', \zeta')$ in Theorem 4.4.3 can be chosen to map fibres to fibres.

4.5 Connectivity

The aim of this section is a characterisation of connectivity of the generalised covering graph $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ in terms of the generalised voltage graph $(\Delta, G, \omega, \zeta)$. The main result of the section, Theorem 4.5.4, is a generalisation of Lemma 2.3.6.

Lemma 4.5.1. *Let* $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ *and let* $(u, \omega(u)g)$ *be a vertex of* Γ. *Then the set of neighbours of* $(u, \omega(u)g)$ *is the set* $\{(\mathrm{end}_\Delta x, \omega(\mathrm{end}_\Delta x)zg) \mid z \in \zeta(x)\omega(u), \mathrm{beg}_\Delta x = u\}$.

Proof. Notice that the neighbourhood of $(u, \omega(u)g)$ is precisely $\{\mathrm{end}_\Gamma(x, \omega(x)h) \mid \mathrm{beg}(x, \omega(x)h) = (u, \omega(u)g)\}$. Now, $\mathrm{beg}_\Gamma(x, \omega(x)h) = (u, \omega(u)g)$ if and only $\mathrm{beg}_\Delta x = u$ and $h \in \omega(u)g$. Since $\mathrm{end}_\Gamma(x, \omega(x)h) = (\mathrm{end}_\Delta x, \omega(\mathrm{end}_\Delta x)\zeta(x)h)$, we see that the neighbourhood of $(u, \omega(u)g)$ consists of all vertices of the form $(\mathrm{end}_\Delta x, \omega(\mathrm{end}_\Delta x)zg)$ where $z \in \zeta(x)\omega(u)$ and $\mathrm{beg}_\Delta x = u$. □

Let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and let $W = (x_0, x_1, ..., x_{n-1})$ be a walk of length n in Δ. Let $v_i = \mathrm{beg}_\Delta(x_i)$, for $i \in \{0, ..., n-1\}$, and let $v_n = \mathrm{end}(x_{n-1})$. We define the *voltage of* W, denoted $\zeta(W)$, as

$$\zeta(W) = \prod_{i=0}^{n-1} \zeta(x_{n-1-i})\omega(v_{n-1-i}). \tag{4.5.1}$$

Notice that the voltage of a dart is an element of G whereas the voltage of a walk is a subset of G, even if such walk consists of a single dart.

Let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and let $\varphi\colon \Gamma \to \Delta$ be the associated generalised covering projection. If $(x_0, x_1, \ldots, x_{n-1})$ is a walk in Γ, then clearly its *projection* $(\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_{n-1}))$ is a walk in Δ.

Lemma 4.5.2. *Let* $(\Delta, G, \omega, \zeta)$ *be a generalised voltage graph with generalized cover* Γ *and let* W *be a uv-walk of length* n *in* Δ. *If* $\overline{\mathcal{W}}$ *is the set of walks of length* n *in* Γ *that start at the vertex* $(u, \omega(u))$ *and project to* W, *then the set of final vertices of walks in* $\overline{\mathcal{W}}$ *is precisely* $\{(v, \omega(v)z) : z \in \zeta(W)\}$.

Proof. We proceed by induction on the length of W. If W is a walk of length 1, then the result follows at once from Lemma 4.5.1. Suppose Lemma 4.5.2 holds for all walks of length at most n, and let $W = (x_0, x_1, ..., x_n)$. Let $v_i = \mathrm{beg}\, x_i$ for all $i \in \{0, ..., n\}$ and let $v_{n+1} = \mathrm{end}\, x_n$. Denote by $\overline{\mathcal{W}}$ the set of walks of length $n+1$ in Γ that project to W. Let $W' = (x_0, x_1, ..., x_{n-1})$ and let $\overline{\mathcal{W}'}$ the set of walks of length n in Γ that project to W'. Define F and F' as the set of final vertices of $\overline{\mathcal{W}}$ and $\overline{\mathcal{W}'}$, respectively. Then

$$F = \{\mathrm{end}(x, \omega(x)h) \mid \mathrm{beg}(x, \omega(x)h) \in F'\}.$$

But $F' = \{(v_n, \omega(v_n)z) \mid z \in \zeta(W')\}$ since W' has length n. Then $\mathrm{beg}(x, \omega(x)h) \in F'$ if and only if $h \in \omega(v_n)\zeta(W')$. Thus

$$\begin{aligned} F &= \{\mathrm{end}(x, \omega(x)h) \mid h \in \omega(v_n)\zeta(W')\} \\ &= \{(v_{n+1}, \omega(v_{n+1})\zeta(x_n)h) \mid h \in \omega(v_n)\zeta(W')\} \\ &= \{(v_{n+1}, \omega(v_{n+1})z) \mid z \in \zeta(x_n)\omega(v_n)\zeta(W')\} \\ &= \{(v_{n+1}, \omega(v_{n+1})z) \mid z \in \zeta(W)\}. \end{aligned}$$

The result follows. □

Lemma 4.5.3. *Let $(\Delta, G, \omega, \zeta)$ be a voltage graph where ζ is T-normalized for some spanning tree T of Δ. Let $\mathcal{W}$ be the set of closed walks in Δ based at some vertex $v_0 \in \mathrm{V}(\Delta)$. Let $A = \langle \zeta(x) \mid x \in \mathrm{D}(\Delta) \setminus \mathrm{D}(T) \rangle$ and let $B = \langle \omega(v) \mid v \in \mathrm{V}(\Delta) \rangle$. Then $\langle \zeta(W) \mid W \in \mathcal{W} \rangle = \langle A, B \rangle$.*

Proof. First, note that by Equality (4.5.1), it follows that $\zeta(W) \in \langle A, B \rangle$ for all $W \in \mathcal{W}$. Hence $\langle \zeta(W) \mid W \in \mathcal{W} \rangle \leq \langle A, B \rangle$.

We next show that $B \leq \langle \zeta(W) \mid W \in \mathcal{W} \rangle$. Let $v \in \mathrm{V}(\Delta)$ and denote by P the unique v_0v-path in T. Further, let $\overline{P} = (x_0, x_1, \cdots, x_{n-1})$ be the closed walk $P \cdot P^{-1}$ based at v_0, obtained by concatenating the path P and its inverse path P^{-1}. Note that $v = \mathrm{beg}(x_m)$ for $m = n/2$. Since every dart lying on T has trivial voltage, $\zeta(\overline{P}) = \prod_{i=0}^{n-1} \omega(\mathrm{beg}\, x_{n-1-i})$ and so $\omega(v) \subseteq \zeta(\overline{P}) \subseteq \{\zeta(W) \mid W \in \mathcal{W}\}$. This shows that $B \leq \langle \zeta(W) \mid W \in \mathcal{W} \rangle$, as claimed.

Now, let $x \in \mathrm{D}(\Delta)$. If x lies on T, then $\zeta(x) = 1_G \in \{\zeta(W) \mid W \in \mathcal{W}\}$. Suppose now $x \notin \mathrm{D}(T)$. Then, there is a unique shortest closed walk F_x based at v_0 such that x is the only dart in F_x not contained in T. Since every dart of F_x, except possibly x, has trivial voltage, Equation (4.5.1) implies that $\zeta(x) \in \zeta(F_x)$. Since F_x is a closed walk based at v_0, it follows that $\zeta(x) \in \{\zeta(W) \mid W \in \mathcal{W}\}$, and thus $A \leq \langle \zeta(W) \mid W \in \mathcal{W} \rangle$. □

Theorem 4.5.4. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph where ζ is T-normalized for some spanning tree T of Δ. Let $A = \langle \zeta(x) \mid x \in \mathrm{D}(\Delta) \setminus \mathrm{D}(T) \rangle$ and let $B = \langle \omega(v) \mid v \in \mathrm{V}(\Delta) \rangle$. Then* $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ *is connected if and only if $G = \langle A, B \rangle$.*

Proof. Let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ and let $u, v \in \mathrm{V}(\Delta)$. Since Δ is connected, for every $\tilde{u} \in \mathrm{fib}(u)$, there exists a $\tilde{u}\tilde{v}$-walk in Γ for some $\tilde{v} \in \mathrm{fib}(v)$; in other words, $\mathrm{fib}(v)$ can be reached from any vertex of Γ. Then, Γ is connected if and only if for every vertex $(v, \omega(v)h) \in \mathrm{fib}(v)$, there is a walk connecting $(v, \omega(v))$ to $(v, \omega(v)h)$. We will show that the latter happens if and only if $\langle A, B \rangle = G$.

Suppose $(v, \omega(v))$ can reach any other vertex in $\mathrm{fib}(v)$ by means of a walk. Such a walk projects in Δ to a closed walk based at v. Let $\mathcal{W}$ be the set of closed walks in Δ based at v and let $\overline{\mathcal{W}}$ be the set of all walks of Γ that project to a walk in $\mathcal{W}$. Define $\zeta_{\mathcal{W}} := \{x \in \zeta(W) | W \in \mathcal{W}\}$. By Lemma 4.5.2, the end vertex of any walk in $\overline{\mathcal{W}}$ is $(v, \omega(v)g)$ for some $g \in \zeta_{\mathcal{W}}$. Then $\zeta_{\mathcal{W}}$ intersects every coset of $\omega(v)$, and since $\omega(v) \subseteq \zeta_{\mathcal{W}}$ we see that $G = \langle \zeta_{\mathcal{W}} \rangle = \langle A, B \rangle$, where the second equality follows from Lemma 4.5.3.

For the reverse implication, suppose $G = \langle \zeta_{\mathcal{W}} \rangle = \langle A, B \rangle$. Consider a vertex $(v, \omega(v)g) \in \mathrm{fib}(v)$. Then $g = g_1 g_2 \ldots g_k$ where $g_i \in \zeta_{\mathcal{W}}$ and thus $g_i \in \zeta(W_i)$ for some walk $W_i \in \mathcal{W}$, for all $i \in \{1, \ldots, k\}$. Clearly the concatenation $W_k W_{k-1} \ldots W_1$ is a walk in $\mathcal{W}$. Furthermore

$$g = g_1 g_2 \ldots g_k \in \zeta(W_1)\zeta(W_2) \ldots \zeta(W_k) = \zeta(W_k W_{k-1} \ldots W_1).$$

Therefore, there is a walk W' in Γ connecting $(v, \omega(v))$ to $(v, \omega(v)g)$ that projects to $W_k W_{k-1} \ldots W_1$. □

We conclude this section with an interesting application of the above theorem, whose special cases, when G acts transitively on the arcs of Γ or on the edges but not on the arcs, are well-known and often-used facts in the algebraic graph theory.

Let Γ be a graph, let $G \in \mathrm{Aut}(\Gamma)$, let $\mathcal{T}$ be a transversal of the action of G on $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$, let $\mathcal{T}_V = \mathcal{T} \cap \mathrm{V}(\Gamma)$, let $\mathcal{T}_D = \mathcal{T} \cap \mathrm{D}(\Gamma)$, and let $\mathcal{T}_D^\circ = \{x \in \mathcal{T}_D : x^{-1} \in \mathcal{T}_D\}$. Further, let $\mathrm{beg}\,|_{\mathcal{T}_D}$ and $\mathrm{inv}\,|_{\mathcal{T}_D^\circ}$ be the restrictions of the functions beg_Γ and inv_Γ to $\mathcal{T}_D$ and $\mathcal{T}_D^\circ$, respectively. We say that $\mathcal{T}$ is *a connected transversal of G on Γ* provided that the following holds: (1) $\mathrm{beg}_\Gamma\, x \in \mathcal{T}_V$ for every $x \in \mathcal{T}_D$; (2) the graph $\Gamma[\mathcal{T}] := (\mathcal{T}_V, \mathcal{T}_D^\circ, \mathrm{beg}\,|_{\mathrm{D}(\Gamma)}, \mathrm{inv}\,|_{\mathrm{D}(\Gamma)})$ is connected.

Lemma 4.5.5. *Let Γ be a graph and let $G \le \mathrm{Aut}(\Gamma)$. Then there exists a connected transversal of G on Γ if and only if the quotient graph Γ/G is connected.*

Proof. Let $\Delta := \Gamma/G$. If Δ is connected, then it contains a spanning tree T. By Theorem 4.2.1, Γ is isomorphic to $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ for some weight function ω and generalised voltage assignment ζ. Moreover, the isomorphism between Γ and $\Gamma' := \mathrm{GenCov}(\Delta, G, \omega, \zeta)$ can be chosen to map G-orbits on Γ to fibres of Γ'. Furthermore, by Theorem 4.4.3, Γ' is isomorphic to $\Gamma'' := \mathrm{GenCov}(\Delta, G, \omega', \zeta')$, where ζ' is T-normalised. Again, in view of Remark 4.4.5, this isomorphism can be chosen to map fibres to fibres, and hence there is an isomorphism Φ between Γ and Γ'' which maps G-orbits on Γ to fibres of Γ''.

Let $X_V = \{(v, \omega'(v)) \mid v \in \mathrm{V}(T)\}$ and $X_D = \{(x, \omega'(x)) \mid x \in \mathrm{D}(T)\}$. Let $\mathrm{beg}\,|_X$ and $\mathrm{inv}\,|_X$ be respective restrictions of $\mathrm{beg}_{\Gamma''}$ and $\mathrm{inv}_{\Gamma''}$ to X_D. Note that, since ζ' is T-normalised, $(x, \omega'(x)) \in X_D$ implies $(x, \omega'(x))^{-1} \in X_D$. It follows that $X := (X_V, X_D, \mathrm{beg}\,|_X, \mathrm{inv}\,|_X)$ is a subgraph of Γ''. Moreover, X is isomorphic to T. Now, since no two elements of $X_V \cup X_D$ belong to the same fibre of Γ'', X can be extended to a set $\mathcal{T}$ containing exactly one element of each fibre of Γ''. Finally, since Φ maps G-orbits on Γ to fibres of Γ'', we see that $\Phi^{-1}(\mathcal{T})$ is a connected transversal of G on its action on Γ.

For the converse, observe that if $\mathcal{T}$ is a connected transversal, then its image under the quotient map is a connected spanning subgraph of Γ/G.

□

Theorem 4.5.6. *Let Γ be a graph, let $G \le \mathrm{Aut}(\Gamma)$ and let $\mathcal{T}$ be a connected transversal of G on Γ. Let $\mathcal{T}_V$, $\mathcal{T}_D$ and $\mathcal{T}_D^\circ$ be as in the preceding paragraph. For a dart $x \in \mathcal{T}_D \setminus \mathcal{T}_D^\circ$ let $a_x \in G$ be such that $(x^{-1})^{a_x} \in \mathcal{T}_D$. Then Γ is connected if and only if*

$$G = \langle \bigcup_{v \in \mathcal{T}_V} G_v \cup \{a_x : x \in \mathcal{T}_D \setminus \mathcal{T}_D^\circ\} \rangle.$$

Proof. Let $\Delta := \Gamma/G$ and define $\zeta : \mathrm{D}(\Delta) \to G$ by

$$\zeta(x^G) = \begin{cases} 1_G & \text{if } x \in \mathcal{T}_D^\circ; \\ (a_x)^{-1} & \text{if } x \in \mathcal{T}_D \setminus \mathcal{T}_D^\circ. \end{cases}$$

As in Thoerem 4.2.1, for $x \in \mathcal{T}_D$ let $\iota(x)$ be the unique element of $\mathcal{T}_D$ such that $x^{-1} \in \iota(x)^G$, and let $\omega : \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta) \to S(G)$ be defined by

$$\omega(x^G) := G_x \quad \text{for every} \quad x \in \mathcal{T}.$$

We will first show that ζ satisfies the condition (4.2.1) stated in Theorem 4.2.1; that is, we will show that $\iota(x)^{\zeta(x^G)} = x^{-1}$ for every $x \in \mathcal{T}_D$. First suppose that $x \in \mathcal{T}_D \setminus \mathcal{T}_D^\circ$. Then $\iota(x)^{\zeta(x^G)} = \iota(x)^{(a_x)^{-1}}$ which by the definition of a_x equals x^{-1}. On the other hand, if $x \in \mathcal{T}_D^\circ$, then $\iota(x) = x^{-1}$ and hence $\iota(x)^{\zeta(x^G)} = x^{-1}$. Therefore ζ satisfies condition the (4.2.1) stated in Theorem 4.2.1. By Theorem 4.2.1 it follows that $(\Delta, G, \omega, \zeta)$ is a generalised voltage graph and $\Gamma \cong \mathrm{GenCov}(\Delta, G, \omega, \zeta)$.

Let $\Gamma[\mathcal{T}]$ be as in the paragraph preceding Theorem 4.5.6. Since $\mathrm{V}(\Gamma[\mathcal{T}]) = \mathcal{T}_V$ is a transversal of the action of G on $\mathrm{V}(\Gamma)$, it follows that the image Δ' of $\Gamma[\mathcal{T}]$ under the quotient projection $\Gamma \to \Delta$ is a spanning subgraph of Δ. Moreover, since $\Gamma[\mathcal{T}]$ is connected, Δ' is connected and hence it contains a spanning tree T of Δ. However, ζ is trivial on all the darts of Δ', implying that ζ is T-normalised.

Let $A = \langle \zeta(X) \mid X \in \mathrm{D}(\Delta) \setminus \mathrm{D}(T) \rangle$ and $B = \langle \omega(v) \mid v \in \mathrm{V}(\Delta) \rangle$, as in Theorem 4.5.4. Observe that $A = \langle \zeta(x^G) \mid x \in \mathcal{T}_D \setminus \mathcal{T}_D^\circ \rangle = \langle (a_x)^{-1} \mid x \in \mathcal{T}_D \setminus \mathcal{T}_D^\circ \rangle$. Similarly, $B = \langle G_v \mid v \in \mathcal{T}_V \rangle$. The result now follows from Theorem 4.5.4. $\square$

Note that Proposition 4.1.4 is the special case of Theorem 4.5.6 when the quotient graph Γ/G is a tree.

4.6 Simplicity

In this section we explore the simplicity of generalised covers. Lemmas 4.6.1 and 4.6.2 give necessary and sufficient conditions on a generalised voltage graph for its generalised cover to have, respectively, a pair of parallel darts or a self-inverse dart. Since existence of a pair of parallel darts is equivalent to existence of a pair of parallel edges or a loop, and existence of a self-inverse dart is equivalent to existence of a semi-edge, these two lemmas yield a characterisation of simple generalised covers, given in Theorem 4.6.3.

Lemma 4.6.1. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph and let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$. Then there is a pair of parallel darts in Γ if and only if $\zeta(y)h\zeta(x)^{-1} \in \omega(\mathrm{end}_\Delta(x))$ for some $x, y \in \mathrm{D}(\Delta)$ and $h \in \omega(\mathrm{beg}_\Delta(x))$ such that $\mathrm{beg}_\Delta(x) = \mathrm{beg}_\Delta(y)$, $\mathrm{end}_\Delta(x) = \mathrm{end}_\Delta(y)$, and either $x \neq y$, or $x = y$ but $h \notin \omega(x)$.*

Proof. For simplicity, we shall use the abbreviations beg and end for beg_Δ and end_Δ, respectively, throughout this proof.

Suppose there is a pair of parallel darts in Γ. Since $\mathrm{Aut}(\Gamma)$ acts transitively on each dart-fibre, we may assume without loss of generality that one of these two darts is of the form $(x, \omega(x))$ for some $x \in \mathrm{D}(\Delta)$. Then the other dart in the pair is of the form $(y, \omega(y)h)$ where

$$\mathrm{beg}_\Gamma(x, \omega(x)) = \mathrm{beg}_\Gamma(y, \omega(y)h) \quad \text{and} \quad \mathrm{end}_\Gamma(x, \omega(x)) = \mathrm{end}_\Gamma(y, \omega(y)h).$$

The first equality implies that $\mathrm{beg}(x) = \mathrm{beg}(y)$ and $\omega(\mathrm{beg}(x)) = \omega(\mathrm{beg}(y))h$. Similarly, from the second equality, we deduce that $\mathrm{end}(x) = \mathrm{end}(y)$ and $\omega(\mathrm{end}(x))\zeta(x) = \omega(\mathrm{end}(y))\zeta(y)h$. In particular, $\zeta(y)h\zeta(x)^{-1} \in \omega(\mathrm{end}(x))$.

Since $(x, \omega(x)) \neq (y, \omega(y)h)$, we also see that either $x \neq y$, or $x = y$ but $h \notin \omega(x)$.

Conversely, if $\zeta(y)h\zeta(x)^{-1} \in \omega(\mathrm{end}(x))$ for some $x, y \in \mathrm{D}(\Delta)$ and $h \in \omega(\mathrm{beg}(x))$ such that $\mathrm{beg}(x) = \mathrm{beg}(y)$, $\mathrm{end}(x) = \mathrm{end}(y)$, and either $x \neq y$, or $x = y$ but $h \notin \omega(x)$, then we see that the darts $(x, \omega(x))$ and $(y, \omega(y)h)$ are distinct. Furthermore, $\omega(\mathrm{end}(x))\zeta(x) = \omega(\mathrm{end}(x))\zeta(y)h = \omega(\mathrm{end}(y))\zeta(y)h$, from which we see that $\mathrm{end}_\Gamma(x, \omega(x)) = \mathrm{end}_\Gamma(y, \omega(y)h)$. Finally, since $h \in \omega(\mathrm{beg}(x)) = \omega(\mathrm{beg}(y))$ we see that $\omega(\mathrm{beg}(x)) = \omega(\mathrm{beg}(y))h$ and so $\mathrm{beg}_\Gamma(x, \omega(x)) = \mathrm{beg}_\Gamma(y, \omega(y)h)$. Therefore $(x, \omega(x))$ and $(y, \omega(y)h)$ are parallel darts of Γ. □

Lemma 4.6.2. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph and let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$. Then Γ has a semi-edge if and only if there is $x \in \mathrm{D}(\Delta)$ such that $\zeta(x) \in \omega(x)$.*

Proof. Recall that a graph has a semi-edge if and only if it has a self-inverse dart. Since $\mathrm{Aut}(\Gamma)$ is transitive on each fibre, we may assume that if such a dart exists, then it is of the form $(x, \omega(x))$ for some $x \in \mathrm{D}(\Delta)$. Recall that the inverse of this dart is $(x^{-1}, \omega(x^{-1})\zeta(x))$. Hence the dart $(x, \omega(x))$ is self-inverse if and only if $x = x^{-1}$ and $\zeta(x) \in \omega(x)$. □

The following characterisation of generalised voltage graph that yield simple generalised covers is a direct consequence of Lemmas 4.6.1 and 4.6.2.

Theorem 4.6.3. *Let $(\Delta, G, \omega, \zeta)$ be a generalised voltage graph and let $\Gamma = \mathrm{GenCov}(\Delta, G, \omega, \zeta)$. Then Γ is a simple graph if and only if for all darts $x \in \mathrm{D}(\Delta)$ all of the following conditions hold:*

1. $\zeta(x)h\zeta(x)^{-1} \notin \omega(\mathrm{end}_\Delta(x))$ *for all* $h \in \omega(\mathrm{beg}_\Delta(x)) \setminus \omega(x)$;
2. $\zeta(y)h\zeta(x)^{-1} \notin \omega(\mathrm{end}_\Delta(x))$ *for all darts y, $y \neq x$, which are parallel to x in Δ and for all* $h \in \omega(\mathrm{beg}_\Delta(x))$;
3. $\zeta(x) \notin \omega(x)$ *if* $x = x^{-1}$.

Chapter 5

Cyclic generalised voltage graphs

In this chapter we introduce the notion of a cyclic generalised voltage graph, which can be regarded as a special case, where the voltage group is a cyclic group, of the less restrictive generalised voltage graph. A cyclic generalised voltage graph is a quadruple $(\Delta, \lambda, \iota, \zeta)$ where Δ is a connected graph and λ, ι and ζ are integer-valued functions. As is expected, every cyclic generalised voltage graph has an associated covering graph, admitting a cyclic group of automorphisms (see Definitions 5.1.1 and 5.1.2). We will define these notion in detail and lay out the associated terminology in Section 5.1.

In Section 5.2, we show the connection between the new concept of a cyclic generalised voltage graphs and the generalised voltage graph of Chapter 4. Every cyclic generalised voltage graph $(\Delta, \lambda, \iota, \zeta)$ is equivalent to a generalised voltage graph $(\Delta, \mathbb{Z}_n, \omega, \zeta')$ in the sense that the order n of $\mathbb{Z}_n$ and the functions ω and ζ' can be reconstructed from the integer-valued functions λ, ι and ζ, and vice versa (see Theorem 5.2.2). Moreover, the covers of $(\Delta, \lambda, \iota, \zeta)$ and $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ are isomorphic graphs. The theory of cyclic voltage graph introduced in this section is then a reinterpretation of the general theory of generalised voltage graph and many of the results stated in this section are, simply put, translations of more general results to the language of cyclic generalised voltage graphs. We chose to introduce new terminology (instead of relying on the terminology pertaining to generalised voltage graphs) because, while the theory of generalised voltage graphs deals for the most part in group theoretical terms, the new language introduced in this chapter allows us to state the equivalent results (for the particular case of cyclic groups) in terms of much simpler notions like integer congruence and divisibility, making this theory much more intuitive.

By the end of the chapter, in Sections 5.3 and 5.4, we will focus in the particular class of cyclic generalised voltage graphs (which we call *ccv*-graphs) whose cover is a cubic graph, and we will determine a series of necessary conditions for their covers to be vertex-transitive. The contents of this chapter first appeared in [58] with the exception of the material in Section 5.4.

5.1 Cyclic generalised covers

We now introduce the notions of a *cyclic generalised voltage graph* and *generalised cyclic cover*. We start with formal definitions. Later in the section we explain how to depict the cyclic generalised voltage graphs in the most economical manner, and provide an illuminating example at the end of Section 5.1.

5.1.1 Formal definitions and basic results

Definition 5.1.1. A pair (Δ, λ) is a *dart labelled graph* provided that Δ is a finite connected graph and $\lambda\colon \mathrm{D}(\Delta) \to \mathbb{N}$ is a function. If for a dart labelled graph (Δ, λ) there exists a function $\iota\colon \mathrm{V}(\Delta) \to \mathbb{N}$ such that

$$\lambda(x)\iota(\operatorname{beg} x) \quad = \quad \lambda(x^{-1})\iota(\operatorname{end} x), \tag{5.1.1}$$

then we say that (Δ, λ) is *extendable*. If, in addition, $\zeta\colon \mathrm{D}(\Delta) \to \mathbb{Z}$ is a function such that

$$\zeta(x^{-1}) \quad \equiv \quad -\zeta(x) \pmod{\lambda(x)\iota(\operatorname{beg} x)} \tag{5.1.2}$$

for every dart $x \in \mathrm{D}(\Delta)$, then we say that the quadruple $(\Delta, \lambda, \iota, \zeta)$ is a *cyclic generalised voltage graph*.

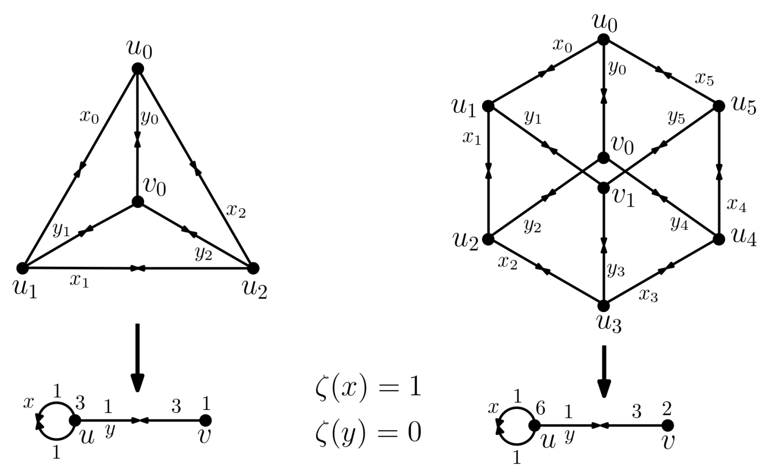


Figure 5.1.1: K_4 and the cube graph Q_3 as cyclic generalised covers of a cyclic generalised graph with two vertices. Mutually inverse darts are depicted as arrows pointing at each other. The number next to a dart indicate its λ-value while the number next to a vertex indicate its ι-value.

Definition 5.1.2. Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph and let Γ be the graph defined by:

- $\mathrm{V}(\Gamma) = \{v_i \mid v \in \mathrm{V}(\Delta), i \in \mathbb{Z}_{\iota(v)}\}$;
- $\mathrm{D}(\Gamma) = \{x_i \mid x \in \mathrm{D}(\Delta), i \in \mathbb{Z}_{\lambda(x)\iota(\operatorname{beg} x)}\}$;

- $\mathrm{beg}_\Gamma(x_i) = (\mathrm{beg}_\Delta\, x)_i$;
- $\mathrm{inv}_\Gamma(x_i) = (\mathrm{inv}_\Delta\, x)_{i+\zeta(x)}$.

Then Γ is called the *cyclic generalised cover* arising from $(\Delta, \lambda, \iota, \zeta)$ and is denoted by $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$.

For a given vertex $v \in \mathrm{V}(\Delta)$, the set $\{v_i \mid i \in \mathbb{Z}_{\iota(v)}\}$ is called the fibre above v and is denoted by $\mathrm{fib}(v)$. Similarly, for a dart $x \in \mathrm{D}(\Delta)$, we call $\{x_i \mid i \in \mathbb{Z}_{\lambda(x)\iota(\mathrm{beg}\, x)}\}$ the fibre above x, and denote it $\mathrm{fib}(x)$. The function $\pi : \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$ mapping each $x_i \in \mathrm{fib}(x)$ to x, where $x \in \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$, is called the *generalised covering projection.*

Of course one should verify that $\Gamma := \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$ is a well-defined graph. That is, that the function inv_Γ is well defined and $\mathrm{inv}_\Gamma \mathrm{inv}_\Gamma\, x = x$ for all darts x. Observe that condition (5.1.1) guarantees that the fibres $\mathrm{fib}(x)$ and $\mathrm{fib}(x^{-1})$ of two mutually inverse darts of Δ have the same size. Furthermore, it follows from condition (5.1.2) that $\mathrm{inv}_\Gamma \mathrm{inv}_\Gamma\, x = x$. Hence inv_Γ is a well-defined function and Γ is a graph.

Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph and let $v \in \mathrm{V}(\Delta)$. We define the λ-valency of v by

$$\mathrm{val}_\lambda(v) := \sum_{x \in \Delta(v)} \lambda(x), \tag{5.1.3}$$

where $\Delta(v)$ denotes the (dart) neighbourhood of v. Now, let $x \in \mathrm{D}(\Delta)$, $u = \mathrm{beg}\, x$ and consider the cyclic generalised cover $\Gamma := \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$. From Definition 5.1.2 we see that a dart $x_j \in \mathrm{fib}(x)$ emanates from a vertex $u_i \in \mathrm{fib}(u)$ if and only if $j \equiv i \pmod{\iota(u)}$. Clearly, there are exactly $\lambda(x)$ elements of $\mathbb{Z}_{\lambda(x)\iota(u)}$ congruent to i modulo $\iota(u)$. Then, since $|\mathrm{fib}(x)| = \lambda(x)\iota(u)$, we see that $\lambda(x)$ is the number of darts in $\mathrm{fib}(x)$ that are incident to a given vertex in $\mathrm{fib}(u)$. In particular this implies that for a vertex $u_i \in \mathrm{fib}(u)$ we have $\mathrm{val}(u_i) = \mathrm{val}_\lambda(u)$.

Lemma 5.1.3 below, which follows almost directly from Definition 5.1.2, gives in rather simple terms the adjacency rule for the cover of a cyclic generalised voltage graph, and will prove to be quite useful in the following pages.

Lemma 5.1.3. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph let $u, v \in \mathrm{V}(\Delta)$ and let $\Gamma = \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$. Two vertices $u_i \in \mathrm{fib}(u)$ and $v_j \in \mathrm{fib}(v)$ are adjacent if and only if*

$$j - i \equiv \zeta(x) \pmod{\gcd(\iota(u), \iota(v))} \tag{5.1.4}$$

for some $x \in \mathrm{D}(\Delta)$ such that $\mathrm{beg}_\Delta\, x = u$ and $\mathrm{end}_\Delta\, x = v$.

Proof. Observe that u_i is adjacent to v_j if and only if there exist a dart $x \in \mathrm{D}(\Delta)$ with $\mathrm{beg}_\Delta\, x = u$ and $\mathrm{end}_\Delta\, x = v$, and a dart $x_\ell \in \mathrm{fib}(x)$ such that $\mathrm{beg}_\Gamma\, x_\ell = u_i$ and $\mathrm{end}_\Gamma\, x_\ell = v_j$. By Definition 5.1.2, we see that

$$\mathrm{beg}_\Gamma\, x_\ell = (\mathrm{beg}_\Delta\, x)_\ell = u_\ell\,;$$

$$\operatorname{end}_\Gamma x_\ell = \operatorname{beg}_\Gamma(\operatorname{inv}_\Gamma x_\ell) = \operatorname{beg}_\Gamma((\operatorname{inv}_\Delta x)_{\ell+\zeta(x)}) = (\operatorname{end}_\Delta x)_{\ell+\zeta(x)} = v_{\ell+\zeta(x)}.$$

Recall that the indices at u and v should be considered as elements of $\mathbb{Z}_{\iota(u)}$ and $\mathbb{Z}_{\iota(v)}$, respectively. Hence u_i is adjacent to v_j if and only if $i \equiv \ell \pmod{\iota(u)}$ and $j \equiv \ell + \zeta(x) \pmod{\iota(v)}$ for some integer ℓ. The latter is equivalent to the existence of integers X, Y and ℓ such that $\iota(u)X + \ell = i$ and $\iota(v)Y + \ell = j - \zeta(x)$. By subtracting these two equalities, we see that in this case the diophantine equation $\iota(u)X - \iota(v)Y = i - j + \zeta(x)$ has a solution, which is equivalent to requiring that $j - i \equiv \zeta(x) \pmod{\gcd(\iota(u), \iota(v))}$. Conversely, if this diophantine equation has a solution, then by letting $\ell := i - \iota(u)X$, which equals $j - \iota(v)Y - \zeta(x)$, we see that $i \equiv \ell (\operatorname{mod} \iota(u))$ and $j \equiv \ell + \zeta(x) (\operatorname{mod} \iota(v))$, and thus u_i is adjacent to v_j. □

Remark 5.1.4. Suppose $(\Delta, \lambda, \iota, \zeta)$ is a cyclic generalised voltage graph and let $\zeta' : \mathrm{D}(\Delta) \to \mathbb{Z}$ be the function obtained from ζ by reducing every $\zeta(x)$ modulo $\gcd(\iota(\operatorname{beg} x), \iota(\operatorname{end} x))$. That is, for each dart $x \in \mathrm{D}(\Delta)$, $\zeta'(x)$ is the unique integer such that $\zeta'(x) < d_x$ and $\zeta'(x) \equiv \zeta(x) \pmod{d_x}$ where $d_x = \gcd(\iota(\operatorname{beg} x), \iota(\operatorname{end} x))$. Then, from Lemma **??** we have $\operatorname{CycCov}(\Delta, \lambda, \iota, \zeta) = \operatorname{CycCov}(\Delta, \lambda, \iota, \zeta')$.

This construction is illustrated in Figure 5.1.1. Consider the cyclic generalised voltage graph $(\Delta, \lambda, \iota, \zeta)$ with darts a, b, x and y, and vertices u and v, shown in the bottom left of the figure. Its cyclic generalised cover, Γ, is shown immediately above it. Next to each vertex, in bold lettering, is its image under ι. That is, $\iota(u) = 3$ and $\iota(v) = 1$. Then, there are 3 distinct vertices in the fibre of u (labelled u_i with $i \in \mathbb{Z}_3$) and a single vertex in the fibre of v. Next to each dart is a number indicating its label λ. Since $\lambda(y) = 3$ and $\operatorname{beg}(y) = v$, there are 3 darts in the fibre of y beginning at each vertex in the fibre of v (which consists solely of v_0). Similarly, for each vertex u_i in the fibre of u there is a single dart in each of the fibres of x, a and b, that begins at u_i.

The voltages are given by $\zeta(a) = 1$, $\zeta(b) = 2$ and $\zeta(x) = \zeta(y) = 0$. Consider the dart a and see that, by Lemma 5.1.3 (and since $\zeta(a) = 1$, $\operatorname{beg}(a) = u$ and $\operatorname{end}(a) = u$) each vertex u_i is adjacent to all vertices u_j such that $j \equiv i + \zeta(a) = i + 1$ modulo $\iota(u) = 3$. That is, each u_i is adjacent to u_{i+1}. By the same token, but considering the dart b, we see that each u_i is adjacent to u_{i+2}. Finally, Lemma 5.1.3 also tells us that v_0 is adjacent to all u_0, u_1 and u_2.

5.1.2 Extendability of labelled graphs

We now turn our attention to the question of which labelled graphs (Δ, λ) are extendable (in the sense of Definition 5.1.1) and which functions ι then satisfy condition (5.1.1). This will then allow us to describe a cyclic generalised voltage graph in a more economical manner.

For a function $\lambda \colon \mathrm{D}(\Delta) \to \mathbb{N}$ and a walk $W = (x_1, \ldots, x_n)$ of Δ, let

$$\rho_\lambda(W) := \prod_{i=1}^{n} \frac{\lambda(x_i)}{\lambda(x_i^{-1})}. \tag{5.1.5}$$

Then clearly

$$\rho_\lambda(W^{-1}) = \rho_\lambda(W)^{-1} \text{ and } \rho_\lambda(W_1W_2) = \rho_\lambda(W_1)\rho_\lambda(W_2) \tag{5.1.6}$$

for every two walks for which the concatenation W_1W_2 is defined.

Now observe that if $\iota\colon \mathrm{V}(\Delta) \to \mathbb{N}$ is a function which together with λ satisfies the condition (5.1.1), then a repeated application of (5.1.1) to the darts x_i of the walk W implies that

$$\iota(\operatorname{end} x_n) = \iota(\operatorname{beg}(x_1))\rho_\lambda(W). \tag{5.1.7}$$

In particular, $\rho_\lambda(W) = 1$ for every closed walk W. This observations has the following useful consequence.

Lemma 5.1.5. *A labelled graph (Δ, λ) is extendable if and only if $\rho_\lambda(W) = 1$ for every closed walk W of Δ.*

Let $u \in \mathrm{V}(\Delta)$, $m \in \mathbb{N}$ and suppose that (Δ, λ) is extendable. Then a function $\iota\colon \mathrm{V}(\Delta) \to \mathbb{N}$ satisfying (5.1.1) and such that $\iota(u) = m$ exists if and only if m is divisible by $\operatorname{lcm}\{\alpha(v) : v \in \mathrm{V}(\Delta)\}$, where $\alpha(v)$ is the smallest integer for which $\alpha(v)\rho_\lambda(W_v) \in \mathbb{N}$ for one (and thus every) uv-walk W_v. Such a function ι is then unique.

Proof. Fix a vertex u of Δ. Suppose first that (Δ, λ) is extendable and that $\iota\colon \mathrm{V}(\Delta) \to \mathbb{N}$ is a function that together with λ satisfies (5.1.1). In view of (5.1.7), it follows that $\rho_\lambda(W) = 1$ for every closed walk W. Moreover, if $\iota(u) = m$, then $\iota(v) = m\rho_\lambda(W_v)$ where W_v is an arbitrary uv-walk. Since $\iota(v)$ is an integer, this implies that m is divisible by $\alpha(v)$. Since this is true for every vertex $v \in \mathrm{V}(\Delta)$, we see that m is divisible by $\operatorname{lcm}\{\alpha(v) : v \in \mathrm{V}(\Delta)\}$, as claimed.

Suppose now that $\rho_\lambda(W) = 1$ for every closed walk W. Let $v \in \mathrm{V}(\Delta)$ and suppose that W_1 and W_2 are two uv-walks. Then $W_1W_2^{-1}$ is a closed walk and hence $\rho_\lambda(W_1W_2^{-1}) = 1$. But then, in view of (5.1.6), $\rho_\lambda(W_1) = \rho_\lambda(W_2)$.

Now choose a uv-walk W_v, let $\alpha(v)$ be the smallest integer such that $\alpha(v)\rho_\lambda(W_v) \in \mathbb{N}$ and let

$$\iota(u) := \operatorname{lcm}\{\alpha(v) \mid v \in \mathrm{V}\}; \tag{5.1.8}$$

$$\iota(v) := \iota(u)\rho_\lambda(W_v) \text{ if } v \neq u. \tag{5.1.9}$$

In view of the the above argument, we see that this defines $\alpha(v)$ and $\iota(v)$ independently of the choice of the uv-walk W_v.

To show that ι together with λ satisfies (5.1.1), let $x \in \mathrm{D}(\Delta)$ and let X be the walk of length 1 consisting only of the dart x. Then $\iota(\operatorname{end} x) = \iota(u)\rho_\lambda(W_{\operatorname{end} x}) = \iota(u)\rho_\lambda(W_{\operatorname{beg} x})\rho_\lambda(X) = \iota(\operatorname{beg} x)\rho_\lambda(X) = \iota(\operatorname{beg} x)\frac{\lambda(x)}{\lambda(x^{-1})}$ and (5.1.1) thus holds. □

Remark 5.1.6. Note that the condition $\rho_\lambda(W) = 1$ in Lemma 5.1.5 only needs to be checked for all cycles W. To see that, assume that $\rho_\lambda(W) = 1$ for every cycle W of Δ and let $W = (x_1, \ldots, x_n)$ be a closed walk of shortest length such that $\rho_\lambda(W) \neq 1$. Then W is not a cycle. However, W is a reduced walk since otherwise, by removing a pair of consecutive mutually inverse darts, we obtain a shorter walk W' with $\rho_\lambda(W') = \rho_\lambda(W) \neq 1$, a contradiction. Hence there are i, j, $1 \leq i < j \leq n$ with $\operatorname{beg} x_i = \operatorname{beg} x_j$. Let $W_1 = (x_1, \ldots, x_{i-1}, x_j, \ldots, x_n)$ and $W_2 = (x_i, \ldots, x_{j-1})$. By the minimality of W, we see that $\rho_\lambda(W_i) = 1$ for all $i \in \{1, 2\}$. On the other hand, clearly $\rho_\lambda(W) = \rho_\lambda(W_1)\rho_\lambda(W_2)$, a contradiction.

Moreover, an easy argument (which we leave out) shows that is suffices to check the condition $\rho_\lambda(W) = 1$ on any complete set of fundamental cycles (recall that a complete set of *fundamental cycles relative to a fixed spanning tree* T is the set of all cycles that contain exactly one cotree dart).

Remark 5.1.7. The function ι that we define in the proof of Lemma 5.1.5 is such that $\iota(u)$ is the least positive integer m such that $m\rho_\lambda(W_v)$ is an integer for all $v \in \mathrm{V}(\Delta)$. It follows that $\gcd\{\iota(v) \mid v \in \mathrm{V}(\Delta)\} = 1$. This has two straightforward but important consequences. First, if $\iota' : \mathrm{D}(\Delta) \to \mathbb{N}$ is any function that together with λ satisfies (5.1.1), then $\iota' = c \cdot \iota$ for some positive integer c. Second, from Theorem 5.2.10, we see that $\operatorname{CycCov}(\Delta, \lambda, \iota, \zeta)$, where ζ is defined to be 0 on every dart, is a connected graph.

5.1.3 Depicting cyclic generalised voltage graphs

We will try to be as economical as possible when drawing a cyclic generalised voltage graph $(\Delta, \lambda, \iota, \zeta)$. We will draw every edge simply as a line segment connecting its endpoints, every loop as a closed curve beginning and ending at its unique endpoint, and every semi-edge as a pendant line segment. We will write the label $\lambda(x)$ near to each dart that belongs to an edge, so an edge will have two labels, one next to each of its endpoints. We will omit the label on a dart x whenever $\lambda(x) = \lambda(x^{-1}) = 1$. Moreover, since the index function ι is completely determined by its image on one vertex, it suffices to specify the image of ι on a distinguished vertex v. We will often use the letter m to denote $\iota(v)$. To indicate voltages, we will draw an arrowhead in the middle of every loop or edge $\{x, x^{-1}\}$. We will write $\zeta(x)$ next to the arrowhead if it is oriented from $\operatorname{beg} x$ to $\operatorname{end} x$ (otherwise, we will write $\zeta(x^{-1})$). We will omit the voltage on edges and loops having trivial voltage. A semi-edge x with no specified voltage in the picture will be assumed to have voltage $\iota(\operatorname{beg} x)/2$.

We now provide an example that illustrates the above described economical way of depicting the cyclic generalised voltage graphs and describe the cyclic generalised cover arising from it in as much an intuitive way as possible.

Consider the cyclic generalised voltage graph $(\Delta, \lambda, \iota, \zeta)$ in the bottom of Figure 5.1.2, consisting of three vertices u, v and w, and having one *loop* attached to the vertex u, two ordinary edges linking u with v and v with w, respectively, and a *semi-edge* attached to w. We will explain how it gives rise to its *cyclic generalised cover.*

The index function ι is given as follows. First, note that w is the distinguished vertex and has a positive integer m (in this case $m = 2$) assigned to it. This indicates that $\iota(w) = m$ or, simply put, that the vertex w gives rise to m distinct vertices in Γ (the fibre above w). The images of other vertices under ι (and thus also the size of the fibres above those vertices) are determined by $\iota(w)$. Recall that for all darts x we have $\lambda(x)\iota(\operatorname{beg} x) = \lambda(x^{-1})\iota(\operatorname{end} x)$. By letting x be the

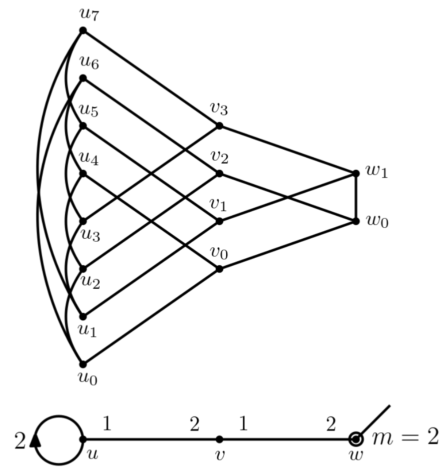


Figure 5.1.2: A cyclic generalised voltage graph and its covering graph

dart beginning at w and ending at v we have $\lambda(x)\iota(w) = \lambda(x^{-1})\iota(v)$, and so $\iota(v) = 2 \cdot \iota(w) = 4$. Similarly, $\iota(u) = 2\iota(v) = 8$.

Adjacency rules are given by Lemma 5.1.3. Recall that an edge lacking an arrowhead indicates that both darts comprising it have trivial voltage. For instance all darts in the edge between w and v, as well as those in the edge between v and u, have trivial voltage. We see that a vertex $w_i \in \mathrm{fib}(w)$ is adjacent to all vertices $v_j \in \mathrm{fib}(v)$ such that $j \equiv i \pmod 2$. That is each w_i is adjacent to v_i and v_{i+2}. Similarly each v_i is adjacent to u_i and u_{i+4}. Now, the loop at u has an arrowhead with the number 2 written next to it. That is, one dart in the loop at u has voltage 2 while the other has voltage -2 (or equivalently, 6, since the indices of the vertices in $\mathrm{fib}(u)$ are taken modulo $\iota(u) = 8$). Then every $u_i \in \mathrm{fib}(u)$ is adjacent to u_{i+2} and u_{i-2}. Finally, the semi-edge at w has no specified voltage in the picture. This means that it has voltage $\iota(w)/2 = 1$. Then every vertex $w_i \in \mathrm{fib}(w)$ is adjacent to w_{i+1}.

5.2 Cyclic generalised covers as a special case

Let $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ be a faithful generalised voltage graph. Since $\mathbb{Z}_n$ is cyclic, every subgroup $\omega(x) \leq \mathbb{Z}_n$, is determined uniquely by its index in $\mathbb{Z}_n$. This allows us to define two functions λ and ι such that $(\Delta, \lambda, \iota, \zeta)$ is a cyclic generalised voltage graph and $\mathrm{GenCov}(\Delta, \mathbb{Z}_n, \omega, \zeta) \cong \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$. Informally, we can think of a cyclic generalised voltage graph simply as a particular type of a generalised voltage graph. Let us make this more precise.

Let $\mathbb{Z}_n$ be the cyclic group of integers modulo n, and let $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ be a generalised voltage graph. Define $\iota : \mathrm{V}(\Delta) \to \mathbb{Z}$ and $\lambda : \mathrm{D}(\Delta) \to \mathbb{Z}$ by

$$\iota(v) = |\mathbb{Z}_n : \omega(v)|; \tag{5.2.1}$$

$$\lambda(x) = |\omega(\mathrm{beg}\, x) : \omega(x)|. \tag{5.2.2}$$

Now observe that $\omega(v)$ is the subgroup generated by $\iota(v)$, where we view the integer $\iota(v)$ as an element of $\mathbb{Z}_n$. Similarly, $\omega(x)$ is generated by its index in $\mathbb{Z}_n$, which equals $|\mathbb{Z}_n : \omega(x)| = |\mathbb{Z}_n : \omega(\operatorname{beg} x)| \cdot |\omega(\operatorname{beg} x) : \omega(x)| = \iota(\operatorname{beg} x)\lambda(x)$. Hence, the function ω can be reconstructed from the integer valued functions $\iota\colon \mathrm{V}(\Delta) \to \mathbb{Z}$ and $\lambda\colon \mathrm{D}(\Delta) \to \mathbb{Z}$ using the formulas

$$\omega(v) = \langle \iota(v) \rangle; \tag{5.2.3}$$

$$\omega(x) = \langle \iota(\operatorname{beg} x)\lambda(x) \rangle. \tag{5.2.4}$$

Furthermore, if the generalised voltage graph $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ is faithful, the order n of $\mathbb{Z}_n$ is determined by the functions ι and λ, as shown in the lemma below.

Lemma 5.2.1. *If $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ is a generalised voltage graph, then it is faithful if and only if*

$$n = \operatorname{lcm}\{\iota(\operatorname{beg} x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\}. \tag{5.2.5}$$

Proof. Let $G = \mathbb{Z}_n$. By definition, $(\Delta, G, \omega, \zeta)$ is faithful if and only if $\operatorname{core}_G(\cap_{x \in \mathrm{D}(\Delta)} \omega(x)) = 1$. Since G is abelian and since $\omega(x) = \langle \iota(\operatorname{beg} x)\lambda(x) \rangle$, this is equivalent to the condition that the group

$$H := \bigcap_{x \in \mathrm{D}(\Delta)} \langle \iota(\operatorname{beg} x)\lambda(x) \rangle$$

is trivial. Observe that H is precisely the subgroup of $\mathbb{Z}_n$ generated by $\operatorname{lcm}\{\iota(\operatorname{beg} x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\}$. Finally, $|\langle \operatorname{lcm}\{\iota(\operatorname{beg} x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\} \rangle| = 1$ if and only if $\operatorname{lcm}\{\iota(\operatorname{beg} x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\} = n$. $\square$

The discussion so far allows us to define a function Φ which assigns to each faithful generalised voltage graph $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ a cyclic generalised voltage graph $\Phi(\Delta, \mathbb{Z}_n, \omega, \zeta) := (\Delta, \lambda, \iota, \zeta)$ where ι and λ are given by (5.2.1) and (5.2.2). Conversely, one can define a mapping Ψ that assigns to each cyclic generalised voltage graph, $(\Delta, \lambda, \iota, \zeta)$ a faithful generalised voltage graph $\Psi(\Delta, \lambda, \iota, \zeta) := (\Delta, \mathbb{Z}_n, \omega, \zeta)$, where ω and n are given by (5.2.3), (5.2.4) and (5.2.5). The mappings Φ and Ψ are inverse to each other.

Theorem 5.2.2. *Let $(\Delta, \mathbb{Z}_n, \omega, \zeta)$ be a generalised voltage graph and let $(\Delta, \lambda, \iota, \zeta) = \Phi(\Delta, \mathbb{Z}_n, \omega, \zeta)$. Then* $\operatorname{CycCov}(\Delta, \lambda, \iota, \zeta) \cong \operatorname{GenCov}(\Delta, \mathbb{Z}_n, \omega, \zeta)$.

Proof. Let $\varphi\colon \mathrm{D}(\operatorname{GenCov}(\Delta, \mathbb{Z}_n, \omega, \zeta)) \to \mathrm{D}(\operatorname{CycCov}(\Delta, \lambda, \iota, \zeta))$ be given by

$$\varphi(x, \omega(x) + i) = x_i,$$

where the coset of $\omega(x) \leq \mathbb{Z}_n$ is written in additive notation and the subindex of x_i is taken modulo $\lambda(x)\iota(\operatorname{beg} x)$. To see that φ is well defined suppose $\omega(x) + i = \omega(x) + j$. Then $i - j \in \omega(x) = \langle \lambda(x)\iota(\operatorname{beg} x) \rangle$ which implies that $i \equiv j \pmod{\lambda(x)\iota(\operatorname{beg} x)}$. That is $x_i = x_j$. It is straightforward to verify that φ is indeed an isomorphism. $\square$

The following theorem is a straightforward but very important consequence of Theorems 4.2.1 and 5.2.2.

Theorem 5.2.3. *Let Γ be a graph and $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group. Then there exists functions $\lambda\colon \mathrm{D}(\Gamma/G) \to \mathbb{Z}$, $\iota\colon \mathrm{V}(\Gamma/G) \to \mathbb{Z}$ and $\zeta\colon \mathrm{D}(\Gamma/G) \to \mathbb{Z}$ such that $\Gamma \cong \mathrm{CycCov}(\Gamma/G, \lambda, \iota, \zeta)$.*

Proof. By Theorem 4.2.1, there exists a generalised voltage graph $(\Gamma/G, G, \omega, \zeta)$ such that $\Gamma \cong \mathrm{GenCov}(\Gamma/G, G, \omega, \zeta)$. By Theorem 5.2.2, $\mathrm{GenCov}(\Gamma/G, G, \omega, \zeta) \cong \mathrm{CycCov}(\Gamma/G, \lambda, \iota, \zeta)$ where λ and ι are given by formulas (5.2.2) and (5.2.1). □

Remark 5.2.4. We call the pair $(\Gamma/G, \lambda)$ in Theorem 5.2.3 above the *labelled quotient* of Γ by G.

We devote the remainder of this section to stating some fundamental properties of cyclic generalised covers. Each proposition hereafter is a special case, where the voltage group G is cyclic, of a more general proposition for a generalised voltage graph $(\Delta, G, \omega, \zeta)$ proved in Chapter 4. We state these results here in the language of cyclic generalised voltage graphs. Even though each of the following propositions is simply a translation of a (special case of a) proposition previously stated and proved, we provide a sketch of a proof for Theorems 5.2.10 and 5.2.11, as in these cases the translation might not be straightforward.

Lemmas 5.2.5, 5.2.6 and 5.2.7 are special cases of Lemma 4.3.1 and Corollaries 4.3.4 and 4.3.5, respectively

Lemma 5.2.5. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph, let $n = \mathrm{lcm}\{\iota(\mathrm{beg}\, x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\}$ and let $\Gamma = \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$. For an element $a \in \mathbb{Z}_n$ the mapping $f_a : \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma) \to \mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ given by $f_a(x_i) = x_{i+a}$ is an automorphism of Γ whose orbits (on vertices and darts) are precisely the fibres of Γ. Moreover, the group homomorphism $\varphi : \mathbb{Z}_n \to \mathrm{Aut}(\Gamma)$ mapping every $a \in \mathbb{Z}_n$ to f_a is an embedding.*

Lemma 5.2.6. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph and let $n = \mathrm{lcm}\{\iota(\mathrm{beg}\, x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\}$. For an integer a coprime to n let $\zeta_a(x) : \mathrm{D}(\Delta) \to \mathbb{Z}_n$ be given by the rule $\zeta_a(x) = \zeta(x)a$. Then $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta) \cong \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta_a)$.*

Lemma 5.2.7. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph, let $\Gamma = \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$, let $\varphi \in \mathrm{Aut}(\Delta)$ and let a be an integer coprime to n. If $\iota(v^\varphi) = \iota(v)$ for every $v \in \mathrm{V}(\Delta)$ and $\lambda(x^\varphi) = \lambda(x)$, $\zeta(x^\varphi) = a\zeta(x)$ for every $x \in \mathrm{D}(\Delta)$. Then the permutation mapping a dart x_i to the dart $(x^\varphi)_{ai}$ for every $x \in \mathrm{D}(\Delta)$ extends to an automorphism of Γ.*

Theorem 5.2.8 below is a special case of Theorem 4.4.3.

Theorem 5.2.8. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic voltage graph and let T be a spanning tree of Δ. Then there exists a voltage assignment ζ' such that $\zeta'(x) = 0$ for all $x \in \mathrm{D}(T)$ and $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta) \cong \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta')$.*

Definition 5.2.9. A voltage assignment in which $\zeta(x) = 0$ for all darts x belonging to a prescribed spanning tree T (as ζ' is in the lemma above) is said to be T-normalised.

Theorem 5.2.10. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph where ζ is T-normalised for some spanning tree T of Δ. Let $A = \{\zeta(x) | x \in \mathrm{D}(\Delta)\}$ and $B = \{\iota(v) | v \in \mathrm{V}(\Delta)\}$. Then,* $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$ *is connected if and only if* $\gcd(A \cup B) = 1$.

Proof. Let $(\Delta, \mathbb{Z}_n, \omega, \zeta) = \Psi(\Delta, \lambda, \iota, \zeta)$ and let $\bar{\Gamma} = \mathrm{GenCov}(\Delta, \mathbb{Z}_n, \omega, \zeta)$. Then Γ is connected if and only if $\bar{\Gamma}$ is connected. By Theorem 4.5.4, $\bar{\Gamma}$ is connected if and only if $\mathbb{Z}_n = \langle A, \bar{B} \rangle$ where $\bar{B} = \{\omega(v) | v \in \mathrm{V}(\Delta)\}$ and $n = \mathrm{lcm}\{\iota(\mathrm{beg}\, x)\lambda(x) \mid x \in \mathrm{D}(\Delta)\}$. However, $\omega(v)$ is precisely the subgroup of $\mathbb{Z}_n$ generated by $\iota(v)$. Hence $\bar{\Gamma}$ is connected if and only if $\mathbb{Z}_n = \langle A, B \rangle$. This equality, in turn, holds if and only if $\gcd(\{n\} \cup A \cup B) = 1$. Since n is a multiple of $\iota(v)$ for all $v \in \mathrm{V}(\Delta)$, we see that $\gcd(\{n\} \cup A \cup B) = \gcd(A \cup B)$. This completes the proof. □

Theorem 5.2.11. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph and let $\Gamma = \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$. Then Γ is a simple graph if and only if for all darts $x \in \mathrm{D}(\Delta)$ all of the following conditions hold:*

1. $\gcd(\lambda(x), \lambda(x^{-1})) = 1$ *for all $x \in \mathrm{D}(\Delta)$.*
2. $\zeta(x) \not\equiv \zeta(y) \pmod{\gcd(\iota(\mathrm{beg}\, x), \iota(\mathrm{end}\, x))}$ *for any two parallel darts $x, y \in \mathrm{D}(\Delta)$.*
3. $\zeta(x) \not\equiv 0 \pmod{\iota(\mathrm{beg}\, x)}$ *for all darts x in a semi-edge.*

Proof. Let $(\Delta, \mathbb{Z}_n, \omega, \zeta) = \Psi(\Delta, \lambda, \iota, \zeta)$ and let $\bar{\Gamma} = \mathrm{GenCov}(\Delta, \mathbb{Z}_n, \omega, \zeta)$. Then Γ is simple if and only if $\bar{\Gamma}$ is simple. Let $x \in \mathrm{D}(\Delta)$. Set $a = \iota(\mathrm{beg}\, x)$, $b = \iota(\mathrm{end}\, x)$ and $c = a\lambda(x)$. Note that a, b and c are the smallest integers such that $\omega(\mathrm{beg}\, x) = \langle a \rangle$, $\omega(\mathrm{end}\, x) = \langle b \rangle$ and $\omega(x) = \langle c \rangle$. By Theorem 4.6.3 (and by the fact that $\mathbb{Z}_n$ is abelian), $\bar{\Gamma}$ is simple if and only if the three following conditions hold:

(1') $h \notin \langle b \rangle$ for all $h \in \langle a \rangle \setminus \langle c \rangle$;

(2') $h + \zeta(y) - \zeta(x) \notin \langle b \rangle$ for all darts y, $y \neq x$, which are parallel to x in Δ and for all $h \in \langle a \rangle$;

(3') $\zeta(x) \notin \langle a \rangle$ if $x = x^{-1}$.

We will show that items (1), (2) and (3) of Theorem 5.2.11 are respectively equivalent to items (1'), (2') and (3') above.

Suppose (1') holds. Then, since $\mathbb{Z}_n$ is abelian, for all $h \in \langle a \rangle - \langle c \rangle$ we have $h \notin \langle b \rangle$. From formula (4.1.1) (and again from the fact that $\mathbb{Z}_n$ is abelian), $\langle c \rangle \leq \langle a \rangle \cap \langle b \rangle$ and so $\langle c \rangle = \langle a \rangle \cap \langle b \rangle = \langle \mathrm{lcm}(a, b) \rangle$. That is $c = \mathrm{lcm}(a, b)$. Now, $c = a \cdot \lambda(x)$ and $b = a \cdot \lambda(x)/\lambda(x^{-1})$ by equation (5.1.1). Hence $c = \mathrm{lcm}(a, b)$ if and only if $a \cdot \lambda(x) = \mathrm{lcm}(a, a \cdot \lambda(x)/\lambda(x^{-1}))$. The latter equality holds if and only if $\gcd(\lambda(x), \lambda(x^{-1})) = 1$. Therefore (1) holds if and only if (1').

Now, let $y \in \mathrm{D}(\Delta)$ be parallel to x. Suppose (2') holds. Then, for all $h \in \langle a \rangle$, $h + \zeta(y) - \zeta(x) \notin \langle b \rangle$. In particular, $ab \in \langle a \rangle$ and so b cannot divide $\zeta(y) - \zeta(x)$. It follows that $\zeta(x) \not\equiv \zeta(y)$ modulo $\gcd(a, b)$. Finally, recall that $\gcd(a,b) = \gcd(\iota(\operatorname{beg} x), \iota(\operatorname{end} x))$. Thus (2) holds. Conversely, suppose $\zeta(x) \not\equiv \zeta(y) \bmod (\gcd(a,b))$. If for some for some $h \in \langle a \rangle$ we have $h + \zeta(y) - \zeta(x) \in \langle b \rangle$, then, since $\gcd(a,b)$ divides both h and b, $\gcd(a,b)$ must also divide $\zeta(y) - \zeta(x)$, a contradiction. Thus (2') holds. We conclude that (2) is equivalent to (2').

Finally, it is a plain observation that (3) is equivalent to (3'). □

5.3 ccv-graphs

Since our main object of study are cubic graphs (that is, graphs that are 3-valent, connected and simple), we will be particularly interested in those cyclic generalised voltage graphs with covering graphs that are cubic. This motivates the following definition.

Definition 5.3.1. Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph. We say that $(\Delta, \lambda, \iota, \zeta)$ is a cyclic cubic voltage graph (or simply a ccv-graph) if $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$ is a cubic graph.

Throughout the remainder of this dissertation, *ccv*-graphs will become one of our most important tools. They will be used extensively in the analysis and classification of cubic graphs admitting cyclic group of automorphisms with particular properties.

Theorems 5.2.10 and 5.2.11, together with formula (5.1.3), give a characterization of cyclic generalised voltage graphs that are *ccv*-graphs. We state it in the following lemma for convenience.

Lemma 5.3.2. *Let $(\Delta, \lambda, \iota, \zeta)$ be a cyclic generalised voltage graph where ζ is $\mathcal{T}$-normalised for some spanning tree $\mathcal{T}$ of Δ. Let $A = \{\zeta(x) \mid x \in \mathrm{D}(\Delta)\}$ and $B = \{\iota(v) \mid x \in \mathrm{V}(\Delta)\}$. Then $(\Delta, \lambda, \iota, \zeta)$ is a ccv-graph if and only if the following conditions are satisfied:*

1. $\gcd(\lambda(x), \lambda(x^{-1})) = 1$ *for all* $x \in \mathrm{D}(\Delta)$;
2. $\zeta(x) \not\equiv \zeta(y) \pmod{\gcd(\iota(\operatorname{beg} x), \iota(\operatorname{end} x))}$ *for any two parallel darts* $x, y \in \mathrm{D}(\Delta)$;
3. $\zeta(x) \not\equiv 0 \pmod{\iota(\operatorname{beg} x)}$ *for all darts x in a semi-edge;*
4. $\gcd(A \cup B) = 1$;
5. $\mathit{val}_\lambda(v) = 3$ *for all* $v \in \mathrm{V}(\Delta)$.

In the lemma above, conditions (1)–(3) are there to guarantee the covering graph $\mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is a simple graph. Condition (4) guaranties that $\mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is connected; condition (5) that it is 3-valent.

Let (Δ, λ) be a dart-labelled graph and recall that (Δ, λ) can be extended to a cyclic generalised voltage graph if there are functions ι and ζ satisfying (5.1.1) and (5.1.2), respectively. Naturally, not every extension of a labelled graph (Δ, λ) is a *ccv*-graph. Some additional, but rather simple conditions on the labelling λ must be satisfied. If (Δ, λ) extends to a *ccv*-graph $(\Delta, \lambda, \iota, \zeta)$, we say $(\Delta, \lambda, \iota, \zeta)$ is a *ccv-extension* of (Δ, λ) and we call the covering graph $\mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ a *ccv-cover* of (Δ, λ).

Proposition 5.3.3. *Let (Δ, λ) be a connected dart-labelled graph and suppose it is extendable. Then (Δ, λ) can be extended to a ccv-graph $(\Delta, \lambda, \iota, \zeta)$ if and only if the following holds:*

1. *$val_\lambda(x) = 3$ for all $x \in \mathrm{D}(\Delta)$.*
2. *$\lambda(x) = \lambda(x^{-1})$ implies $\lambda(x) = 1$.*
3. *$\lambda(x) = \lambda(y) = 1$ for any two parallel darts x and y.*
4. *$\lambda(x) = 1$ for every dart x underlying a semi-edge.*

Proof. Suppose there are functions ι and ζ such that $(\Delta, \lambda, \iota, \zeta)$ is a ccv-graph. Observe that (1) holds by formula (**??**). Now suppose $\lambda(x) = \lambda(x^{-1})$ for some $x \in \mathrm{D}(\Delta)$. By item (1) of Theorem 5.2.11, we have $1 = \gcd(\lambda(x), \lambda(x^{-1})) = \lambda(x)$. Hence, (2) holds. Now suppose $x, y \in \mathrm{D}(\Delta)$ are two parallel darts. By (1), $\lambda(x) + \lambda(y) \leq 3$, so either $\lambda(x) = \lambda(y) = 1$ or $\lambda(x) = 1$ and $\lambda(y) = 2$. In the latter case, from (5.1.1) we have $\lambda(y^{-1}) = 2\lambda(x^{-1})$. Then $\lambda(y^{-1})$ is even and $\lambda(y^{-1}) \leq 3$. That is $\lambda(y^{-1}) = 2 = \lambda(y)$, contradicting item (2). Then (3) holds. Finally, item (4) follows at once from item (1) of Proposition 5.2.11.

For the converse suppose λ satisfies conditions (1)–(4). Suppose Δ is a dipole with vertices u and v. Let $\iota(v) = \iota(u) = 3$, and let $\{x_0, x_1, x_2\}$ be the set of darts incident to u. By letting $\zeta(x_i) = i$ and $\zeta(x_i^{-1}) = -\zeta(x_i)$ for all $x_i \in \{x_0, x_1, x_2\}$, we obtain a ccv-graph $(\Delta, \lambda, \iota, \zeta)$.

Now, suppose Δ is not a dipole. Let ι be the index function given by (5.1.8) and (5.1.9) in the proof of Lemma 5.1.5. Let c be the smallest positive integer such that $c \cdot \iota(v)$ is even if v is incident to a semi-edge or to a pair of parallel links, and $c \cdot \iota(v) \geq 3$ if v is incident to a loop. Define $\bar{\iota} : \mathrm{V}(\Delta) \to \mathbb{Z}$ by $\bar{\iota}(v) = c \cdot \iota(v)$ for all $v \in \mathrm{V}(\Delta)$.

To define an appropriate voltage assignment, let $n = \mathrm{lcm}\{\bar{\iota}(\mathrm{beg})\lambda(x) \mid x \in \mathrm{D}(\Delta)\}$ and let Δ' be a maximal simple subgraph of Δ (note that Δ' always exists and can be found by removing all darts, all loops and one edge from each pair of parallel edges in Δ). Let D^* be a subset of $\mathrm{D}(\Delta)$ containing exactly one dart for each edge of Δ. Define $\zeta^* : D^* \to \mathbb{Z}_n$ as follows:

$$\zeta^*(x) = \begin{cases} 0 & \text{if } x \in \mathrm{D}(\Delta'); \\ \iota(\mathrm{beg}\, x)/2 & \text{if } x \text{ underlies a semi-edge;} \\ 1 & \text{otherwise.} \end{cases}$$

We can now extend ζ^* to a voltage assignment ζ by letting $\zeta(x) = \zeta^*(x)$ if $x \in D^*$ and $\zeta(x) = \zeta^*(x^{-1})^{-1}$ if $x \notin D^*$. It is straightforward to see that ζ satisfies (5.1.2) and that it is T-normalised. Therefore, $(\Delta, \lambda, \bar{\iota}, \zeta)$ is a cyclic generalised voltage graph. We leave to the reader to verify that $(\Delta, \lambda, \bar{\iota}, \zeta)$ is indeed a ccv-graph. □

Let $(\Delta, \lambda, \iota, \zeta)$ be a ccv-graph, let $e := \{x, x^{-1}\}$ be an edge of Δ ans suppose $\lambda(x) \leq \lambda(x^{-1})$. Then we say e is an edge of type $[\lambda(x), \lambda(x^{-1})]$, or simply a $[\lambda(x), \lambda(x^{-1})]$-edge. If $\mathrm{beg}\, x = u$ and $\mathrm{beg}\, x^{-1} = v$ then we will also say, when there is no possibility of ambiguity, that uv is a $[\lambda(x), \lambda(x^{-1})]$-edge. The following is a direct consequence of Proposition 5.3.3,

Corollary 5.3.4. *A ccv-graph only has edges of type* $[1,1]$*,* $[1,2]$*,* $[1,3]$ *and* $[2,3]$*. In particular, parallel edges in a ccv-graph are necessarily* $[1,1]$*-edges.*

It should come as no surprise that given a labelled graph (Δ, λ), there exist different *ccv*-extension $(\Delta, \lambda, \iota, \zeta)$ and $(\Delta, \lambda, \iota, \zeta')$, where $\zeta \neq \zeta'$, such that $\mathrm{Cov}(\Delta, \lambda, \iota, \zeta) \cong \mathrm{Cov}(\Delta, \lambda, \iota, \zeta')$. Then, it would be convenient to take, among all the possible *ccv*-extensions giving out isomorphic covers, one with a voltage assignment that is as "nice" as possible.

Definition 5.3.5. Let $(\Delta, \lambda, \iota, \zeta)$ be a *ccv*-graph. The voltage ζ is a *simplified voltage* if for all $x \in \mathrm{D}(\Gamma)$ the following holds:

1. ζ is $\mathcal{T}$-normalised for a spanning tree $\mathcal{T}$ containing all $[i,j]$-edges with $i \neq j$;

2. $\zeta(x) < \iota(\operatorname{beg} x)$;

3. $\zeta(x) = \iota(\operatorname{beg} x)/2$ whenever x underlies a semi-edge;

4. $0 < \zeta(x)$ and $\zeta(x) \neq \iota(\operatorname{beg} x)/2$ whenever x underlies a loop;

5. $\zeta(x^{-1}) = \iota(\operatorname{beg} x) - \zeta(x)$ whenever $\zeta(x) \neq 0$.

Remark 5.3.6. One of the advantages of considering *ccv*-graphs with simplified voltage assignments is that the adjacency criteria (see Lemma 5.1.3) for the corresponding covering graphs becomes quite straightforward. Indeed, suppose $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-graph where ζ is simplified. Let $x \in \mathrm{D}(\Delta)$ and let $u = \operatorname{beg} x$ and $v = \operatorname{end} x$. Then for all $u_i \in \mathrm{fib}(u)$ we have:

1. If $u \neq v$ and uv is a $[1,1]$-edge, then $u_i \sim v_{i+\zeta(x)}$;

2. If $u \neq v$ and uv is a $[1,j]$-edge, $j \neq 1$, then $u_{i+k\cdot\iota(v)} \sim v_i$, with $0 \leq k < j$;

3. If $u = v$ and uv is a loop, then $u_i \sim u_{i+\zeta(x)}$ and $u_i \sim u_{i-\zeta(x)}$;

4. If x underlies a semi-edge, then $u_i \sim u_{i+(\iota(u)/2)}$.

By Lemma 4.4.3, we can always assume that the voltage in a cyclic generalised voltage graph $(\Delta, \lambda, \iota, \zeta)$ is T-normalised for some spanning tree T of Δ. In the particular case of *ccv*-graphs, one can assume the tree T to contain all $[i,j]$-edges with $i \neq j$.

For labelled graphs (Δ, λ) and (Δ', λ'), we say (Δ', λ') is a labelled subgraph of (Δ, λ) if Δ' is a subgraph of Δ and λ' is the restriction of λ to $\mathrm{D}(\Delta')$.

Proposition 5.3.7. *Let* $(\Delta, \lambda, \iota, \zeta)$ *be a ccv-graph. Then* $(\Delta, \lambda, \iota, \zeta)$ *has a spanning tree* $\mathcal{T}$ *that contains all the* $[i,j]$*-edges with* $i \neq j$*.*

Proof. Let (Δ', λ') be the subgraph of (Δ, λ) induced by all $[i,j]$-edges of $(\Delta, \lambda, \iota, \zeta)$ with $i \neq j$. We will show that Δ' is acyclic. Suppose to the contrary that Δ' contains a cycle C. Let u be the the vertex of C with the least index, that is $\iota(u) \leq \iota(v)$ for all $v \in \mathrm{V}(C)$. Let v and w be the two neighbours of u in C. Now, uv cannot be a $[1,1]$-edge as Δ' contains no such edges. If uv is a

$[1, j]$-edge with $j \in \{2, 3\}$, then $\iota(v) = \frac{1}{j} \cdot \iota(u) < \iota(u)$, which contradicts u having minimal index. Hence, vu must be a $[1, j]$-edge with $j \in \{2, 3\}$. Similarly, wu is a $[1, j]$-edge with $j \in \{2, 3\}$. This implies that the sum of the labels of darts in $\Delta(u)$ is at least 4, contradicting formula (**??**). Hence, Δ' is acyclic and thus can always be completed to a spanning tree $\mathcal{T}$ of Δ. □

Lemma 5.3.8. *Let $(\Delta, \lambda, \iota, \zeta)$ be a ccv-graph. Then there exists a simplified voltage assignment ζ' such that* $\mathrm{Cov}(\Delta, \lambda, \iota, \zeta) \cong \mathrm{Cov}(\Delta, \lambda, \iota, \zeta')$.

Proof. By Proposition 5.3.7, $(\Delta, \lambda, \iota, \zeta)$ admits a spanning tree T that contains all $[i, j]$-edges with $i \neq j$. By Theorem 5.2.8 we can assume without loss of generality that ζ is T-normalised. Let ζ' be the voltage obtained by reducing $\zeta(x)$ modulo $\gcd(\iota(\mathrm{beg}\, x), \iota(\mathrm{end}\, x))$ for all $x \in \mathrm{D}(\Delta)$. By Remark 5.1.4, $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta) = \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta')$. We must show that items (1)–(5) of Definition 5.3.5 hold. Observe that items (1) and (2) follow at once from our choice of ζ'.

Suppose a dart $x \in \mathrm{D}(\Delta)$ underlies a semi-edge. Then $\zeta'(x) \neq 0$ by Theorem 5.2.11. Moreover $\zeta'(x) = \zeta'(x^{-1})$ since $x = x^{-1}$. Then by equality (5.1.2) we have $\zeta'(x) \equiv -\zeta'(x) \pmod{\iota(\mathrm{beg}\, x)}$, or equivalently, $2 \cdot \zeta'(x) \equiv 0 \pmod{\iota(\mathrm{beg}\, x)}$. Since $0 < \zeta'(x) < \iota(\mathrm{beg}\, x)$, we see that $\iota(\mathrm{beg}\, x)$ is even and $\zeta'(x) = \iota(\mathrm{beg}\, x)/2$. Hence (3) holds.

Finally, if x underlies a loop, by Theorem 5.2.11, $\zeta'(x) \not\equiv \zeta'(x^{-1}) \pmod{\iota(\mathrm{beg}\, x)}$ and therefore $0 \neq \zeta'(x) \neq \iota(\mathrm{beg}\, x)/2$. Hence (4) holds. This completes the proof. □

Theorem 5.3.9. *Let Γ be a cubic graph, let $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group and let Γ/G be the quotient of Γ by G. Then there exists a labelling function $\lambda : \mathrm{D}(\Gamma/G) \to \mathbb{Z}$ such that the labelled graph $(\Gamma/G, \lambda)$ extends to a cvv-graph $(\Gamma/G, \lambda, \iota, \zeta)$ where ζ is simplified and $\Gamma \cong \mathrm{Cov}(\Gamma/G, \lambda, \iota, \zeta)$.*

Proof. By Theorem 5.2.3, Γ is the cyclic generalised cover of a cyclic generalised voltage graph $(\Gamma/G, \lambda, \iota, \zeta)$. Moreover, since Γ is cubic, $(\Gamma/G, \lambda, \iota, \zeta)$ is a *ccv*-graph, and thus the labelled graph $(\Gamma/G, \lambda)$ is extendable to a *ccv*-graph. By Lemma 5.3.8, we can assume ζ is simplified. □

5.4 Artefacts and exceptional graphs

In this section we will define a set of 'forbidden subgraphs' for a labelled graph, that we call artefacts. An artefact in a labelled graph (Δ, λ) is a labelled subgraph of (Δ, λ) that guaranties the existence of a particular subgraph (containing a short cycle) in any *ccv*-cover of (Δ, λ). Throughout this section, we will define 9 distinct artefacts, that we call A_i, $i \in \{1, \ldots, 9\}$. We will show that if a labelled graph contains a certain artefact then it either admits no vertex-transitive *ccv*-covers, or a vertex-transitive *ccv*-cover must be one of 9 exceptional graphs of small order (see Theorem 5.4.14 for a summary of the results of in this section).

5.4.1 The 9 exceptional graphs

We will now define the set of exceptional graphs and state a few uselful results involving them.

Definition 5.4.1. Let X be the set containing the following 9 cubic vertex-transitive graphs: K_4, $K_{3,3}$, the three-dimensional cube Q_3, the Petersen Graph, the Möbius-Kantor graph $\mathrm{GP}(8,3)$, the docecahedron $\mathrm{GP}(10,2)$, the Desargues graph $\mathrm{GP}(10,3)$, the Heawood graph and the Pappus graph.

Remark 5.4.2. With the exception of the Pappus graph, which is a tricirculant, all graphs in the set X are bicirculants.

Lemma 5.4.3. *[60, Theorem 1.5] Let Γ be a cubic girth-regular graph of girth $g \leq 5$. Then either the g-signature of Γ is $(0,1,1)$ or one of the following occurs:*

1. *$g = 3$ and $\Gamma \cong K_4$;*
2. *$g = 4$ and Γ has signature $(1,2,2)$ or is isomorphic to $K_{3,3}$ or the cube graph Q_3.*
3. *$g = 5$ and Γ is isomorphic to the Petersen graph or the dodecahedron $GP(10,2)$.*

The following result is almost folklore, it is mentioned in [11] and [20] but a direct proof is not provided. We state it here as a corollary of Lemma 5.4.3 (Theorem 5 of [60]).

Corollary 5.4.4. *If Γ is a cubic arc-transitive graph of girth smaller than 6, then Γ is isomorphic to one of the following: K_4, $K_{3,3}$, the three-dimensional cube Q_3, the Petersen Graph or the dodecahedron $GP(10,2)$.*

Lemma 5.4.5. *[20, Lemma 4.2] If Γ is a cubic arc-transitive graph of girth 6, then Γ has 6-signature $(2,2,2)$ or is isomorphic to one of the following: the Möbius-Cantor graph $GP(8,3)$, the Desargues graph $GP(10,3)$, the Heawood graph or the Pappus graph.*

5.4.2 Artefacts

Let A_1, A_2, A_3, A_4 and A_5 be the 5 labelled graphs depicted in the bottom row of Figure 5.4.1.

Lemma 5.4.6. *Let $(\Delta, \lambda, \iota, \zeta)$ be a ccv-graph and suppose $\Gamma := \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive. If for some $x \in \mathrm{D}(\Delta)$ we have $\lambda(x) = 3$, then Γ is arc-transitive.*

Proof. Let $u = \operatorname{beg} x$ and $u_0 \in \operatorname{fib}(u)$. Let a and b be two arbitrary darts of Γ. Since Γ is vertex-transitive, there exist automorphisms $\phi, \psi \in \mathrm{Aut}(\Gamma)$ such that $(\operatorname{beg} a)^\phi = u_0 = (\operatorname{beg} b)^\psi$. In particular, both a^ϕ and b^ψ begin at u_0. Furthermore, since $\lambda(x) = 3$, all three darts beginning at u_0 belong to the same orbit of $\langle \Phi \rangle$ (the cyclic group of automorphisms of Γ preserving the fibres) and so there exists $\gamma \in \mathrm{Aut}(\Gamma)$ such that $a^{\phi\gamma} = b^\psi$. Then $a^{\phi\gamma\psi^{-1}} = b$. We conclude Γ is arc-transitive. $\square$

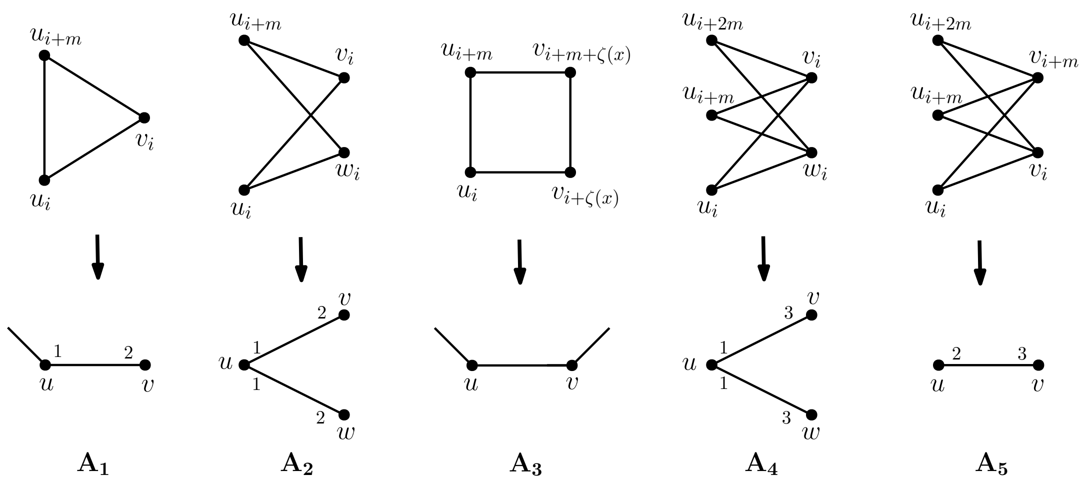


Figure 5.4.1: The five artefacts A_i with $i \in \{1, 2, 3, 4, 5\}$ (bottom row). Above each, a small subgraph of the cover of any *ccv*-graph containing A_i.

Lemma 5.4.7. *Let (Δ, λ) be a labelled graph and let Γ be a ccv-cover of (Δ, λ). Suppose (Δ, λ) contains an artefact A_j for some $j \in \{1, 2, 3, 4, 5\}$. Then:*

1. *if $j = 1$, Γ contains a 3-cycle;*
2. *if $j \in \{2, 3\}$, Γ contains a 4-cycle;*
3. *if $j \in \{4, 5\}$, Γ contains a copy of $K_{3,2}$.*

Proof. Suppose $\Gamma \cong \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of (Δ, λ). For the remainder of this proof, assume the notation of Figure 5.4.1.

First, suppose $j = 1$. Then $\iota(u) = 2 \cdot \iota(v) = 2m$ for some $m \in \mathbb{N}$. Since uv is a $[1, 2]$-edge, it follows from Remark 5.3.6 that each $v_i \in \mathrm{fib}(v)$ is adjacent to u_i and u_{i+m}. Furthermore, since u is incident to a semi-edge, u_i is incident to u_{i+m}. Then, (u_i, u_{i+m}, v_i) is a 3-cycle of Γ for all $i \in \{0, \ldots, m-1\}$.

Now, suppose $j = 2$. Then $\iota(u) = 2 \cdot \iota(v) = 2 \cdot \iota(w) = 2m$ for some $m \in \mathbb{N}$. By Remark 5.3.6, since uv and uw are $[1, 2]$-edges, then each of v_i and w_i is adjacent to u_i and u_{i+m}, for all $0 \leq i < m$. Then (v_i, u_i, w_i, u_{i+m}) is a 4-cycle of Γ.

If $j = 3$ then $\iota(u) = \iota(v) = 2m$ for some $m \in \mathbb{N}$ (recall that by Lemma 5.3.8 $\iota(u)$ is even whenever u is incident to a semi-edge. Let x be the dart of A_3 beginning at u and ending at v. Since both u and v are incident to a semi-edge, we have $u_i \sim u_{i+m}$ and $v_i \sim v_{i+m}$ for all $i \in \{0, \ldots, 2m-1\}$. Moreover, $u_i \sim v_{i+\zeta(x)}$. Then $(u_i, u_{i+m}, v_{i+m+\zeta(x)}, v_{i+\zeta})$ is a 4-cycle for all $i \in \{0, \ldots, 2m-1\}$.

If $j = 4$ then $\iota(u) = 3\iota(v) = 3\iota(w) = 3m$ for some $m \in \mathbb{N}$. By Remark 5.3.6, v_i is adjacent to u_i, u_{i+m} and u_{i+2m}. Similarly w_i is adjacent u_i, u_{i+m} and u_{i+2m}. That is, every vertex in $\{v_i, w_i\}$ is adjacent to every vertex in $\{u_i, u_{i+m}, u_{i+2m}\}$. It follows that Γ contains a copy of $K_{3,2}$.

Finally, if $j = 5$ then $2\iota(u) = 3\iota(v) = 6m$ for some $m \in \mathbb{N}$. Now $\gcd(\iota(u), \iota(v)) = m$ and $i \equiv i + km \pmod{m}$ for all $k \in \{0, 1, 2\}$. Then, from Remark 5.3.6 we see that each $v_i \in \mathrm{fib}(v)$ is adjacent to u_i, u_{i+m} and u_{i+2m}. Then every vertex in $\{u_i, u_{i+m}, u_{i+2m}\}$ is adjacent to every vertex in $\{v_i, v_{i+m}\}$. It follows that Γ contains a copy of $K_{3,2}$. □

Lemma 5.4.8. *Let (Δ, λ) be a labelled graph and suppose Γ is a vertex-transitive ccv-cover of (Γ, λ). If (Δ, λ) contains an artefact A_4 or A_5, then $\Gamma \cong K_{3,3}$.*

Proof. Since both A_4 and A_5 have a dart with label 3 and Γ is vertex-transitive, Γ is arc-transitive, by Lemma 5.4.6. Moreover, by Lemma 5.4.7, Γ contains a copy of $K_{3,2}$ and so, it contains a 4-cycle. By Corollary 5.4.4, Γ must be isomorphic to K_4, $K_{3,3}$ or Q_3. However, neither K_4 nor Q_3 contain a copy of $K_{3,2}$. The result follows. □

Let A_6, A_7, A_8 and A_9 be the graphs depicted in the bottom row of Figure 5.4.2.

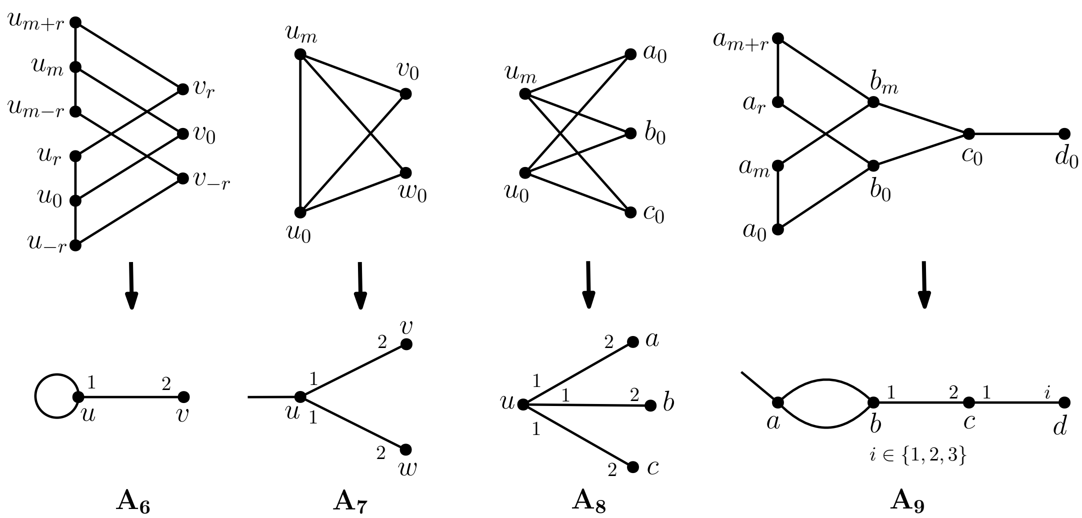


Figure 5.4.2: The four artefacts A_i with $i \in \{6, \dots, 9\}$ (bottom row). Above each, a small subgraph of the cover of any *ccv*-graph containing A_i.

Lemma 5.4.9. *Let (Δ, λ) be a labelled graph and suppose Γ is a vertex-transitive ccv-cover of (Γ, λ). If (Δ, λ) contains an artefact A_6, then Γ is isomorphic to $K_{3,3}$, the Petersen graph, the Pappus graph or the Heawood graph.*

Proof. First, suppose uv is a $[1, 2]$-edge consisting of two darts: x beginning at v, and x^{-1} beginning at u. Let $\{y, y^{-1}\}$ be a loop incident at u. Let m be the number of $\mathrm{Aut}(\Gamma)$-orbits on darts. Since $\lambda(x) = 2$, then for every vertex $u_i \in \mathrm{fib}(u)$ there are precisely two darts in $\mathrm{fib}(x)$ incident to u_i. These two darts must belong to the same $\mathrm{Aut}(\Gamma)$-orbit, since Γ has a cyclic subgroup of automorphism acting transitively in each fibre. Since Γ is vertex transitive, the orbits of $\mathrm{Aut}(\Gamma)$ on darts are precisely the orbits of $\mathrm{Aut}(\Gamma)_{u_i}$, the stabiliser of u_i in $\mathrm{Aut}(\Gamma)$, on its action on the darts beginning at u_i. Then, $\mathrm{Aut}(\Gamma)$ has at most 2 orbits on darts. Suppose $m = 2$. Then $\mathrm{Aut}(\Gamma)$ has two orbits on darts, say O_1

and O_2, and since Γ is vertex-transitive, every vertex of Γ is incident to precisely one dart in O_1 and two darts in O_2. Hence $|O_2| = 2 \cdot |O_1|$. Moreover, for every dart $z \in \mathrm{D}(\Gamma)$ both z and its inverse z^{-1} belong to the same $\mathrm{Aut}(\Gamma)$-orbit (for otherwise every edge of Γ has one dart in O_1 and the other in O_2, which implies that $|O_1| = |O_2|$, a contradiction). In particular, this implies that both $\mathrm{fib}(y)$ and $\mathrm{fib}(y^{-1})$ are subsets of O_2. On the other hand, $\mathrm{fib}(x) \subseteq O_2$ as $\lambda(x) = 2$, and so $\mathrm{fib}(x^{-1}) \subseteq O_2$. Then, all three darts incident to a vertex $\bar{u} \in \mathrm{fib}(u)$ lie in the same $\mathrm{Aut}(\Gamma)$-orbit, contradicting that $m = 2$. Therefore $m = 1$ and Γ is arc-transitive.

Let $k = \iota(v)$ (that is, $k = |\mathrm{fib}(v)|$) and recall that x is the dart beginning at v and ending at u. Since $\lambda(x) = 2$ there must be another dart, say z, incident to v. Since uv is a $[1,2]$-edge, we see that $\iota(u) = 2k$. Recall that $\zeta(x) = 0$, as uv is a $[1,2]$-edge (see Lemma 5.3.8), and let $\zeta(y) = r$. Notice that for all $i \in \mathbb{Z}_{2k}$ and all $j \in \{-1, 1\}$

$$C_{i,j} := (v_i, u_i, u_{i+jr}, v_{i+jr}, u_{i+jr+k}, u_{i+k}) \tag{5.4.1}$$

is a 6-cycle in Γ (see Figure 5.4.2). In particular v_0u_0 lies in 2 distinct 6-cycles, namely $C_{0,1}$ and $C_{0,-1}$. Furthermore, v_0u_k also lies on both $C_{0,1}$ and $C_{0,-1}$. Since Γ is arc-transitive, $z_0 \in \mathrm{fib}(z)$ must lie in a 6-cycle C. It is plain to see that C must visit either v_0u_0 or v_0u_m. That is, one of the edges incident to v_0 lies in at least 3 distinct 6-cycles and by Lemma 5.4.5, Γ is isomorphic to the Heawood graph, the Pappus graph or the generalised Petersen graph $\mathrm{GP}(\mathrm{i}, 3)$ with $i = 8, 10$. However the generalised Petersen graphs $\mathrm{GP}(\mathrm{i}, 3)$ with $i \in \{8, 10\}$ have no quotient by a cyclic group having a $[1,2]$-edge (this can be verified in the census [56]). Therefore Γ must be isomorphic to the Pappus graph or the Heawood graph.

Now, suppose uv is a $[1,3]$-edge. Then, Γ is arc-transitive. If $\iota(v)$ equals 1 or 2, Γ is isomorphic to the K_4 or the cube graph Q_3 respectively (see Figure **??**). If $\iota(v) \geq 3$, the edge u_0v_0 lies in 4 distinct 6-cycles and the result follows from Lemma 5.4.5. □

Lemma 5.4.10. *Let (Δ, λ) be a labelled graph and suppose Γ is a vertex-transitive ccv-cover of (Γ, λ). If (Δ, λ) contains an artefact A_7, then Γ is isomorphic to $K_{3,3}$.*

Proof. Assume the notation of Figure 5.4.2. By Lemma 5.4.7, both (u_0, u_m, v_0) and (u_0, u_m, w_0) are 3-cycles of Γ. This implies that every dart beginning at u_0 lies on a 3-cycle. Hence, the 3-signature of Γ is $(\epsilon_1, \epsilon_2, \epsilon_3)$ with $\epsilon_i > 0$. The result follows from Lemma 5.4.3. □

Lemma 5.4.11. *Let (Δ, λ) be a labelled graph and suppose Γ is a vertex-transitive ccv-cover of (Γ, λ). If (Δ, λ) contains an artefact A_8, then Γ is isomorphic to $K_{3,3}$.*

Proof. Suppose $\Gamma \cong \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ for some *ccv*-extension $(\Delta, \lambda, \iota, \zeta)$ of (Δ, λ). Assume the notation of Figure 5.4.2. Then $\iota(u) = 2\iota(a) = 2\iota(b) = 2\iota(c) = 2m$ for some $m \in \mathbb{N}$. It is not hard to see that each of a_0, b_0 and c_0 is adjacent to both u_0 and u_m. Then, every dart beginning at u_0 lies on 2 distinct 4-cycles. If the girth of Γ is 4, then the result follows from by Lemma 5.4.3. Suppose to the contrary, that the girth of Γ is 3. Since Γ is vertex-transitive, u_0 must lie on a 3-cycle. Then two neighbours of u_0, say a_0 and b_0, must be adjacent. Now, c_0 must also lie on a 3-cycle C. Note that C must contain exactly one of u_0 and u_m, since both are neighbours of c_0, but they are not adjacent to each other. Then, the third vertex in C must either be a_0 or b_0, but neither is adjacent to c_0; a contradiction. We conclude that the girth of Γ is 4 and by Lemma 5.4.3, Γ is isomorphic $K_{3,3}$ (since Q_3 does not contain a copy of $K_{2,3}$). □

Lemma 5.4.12. *If (Δ, λ) is a labelled graph containing A_9, then no ccv-cover of (Δ, λ) is vertex-transitive.*

Proof. Suppose $\Gamma \cong \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ for some *ccv*-extension $(\Delta, \lambda, \iota, \zeta)$ of (Δ, λ). Recall that we can assume ζ to be simplified. Let a, b, c and d be the vertices of A_9, as they are labelled in Figure 5.4.2. There are two edges connecting b to a. Without loss of generality, we may assume the darts on one of these edges have trivial voltage, as necessarily one of these edges must lie on a spanning tree of Δ and the voltage assignment ζ is simplified. As for the other edge, let $r \neq 0$ be the voltage of the dart underlying it and beginning at b (and so, the other dart, beginning at a has voltage $-r$). Let $\iota(c) = m$ so that $\iota(b) = \iota(a) = 2m$. It is not too hard to see that $(c_0, b_0, a_0, a_m, b_m)$ and $(c_0, b_0, a_r, a_{m+r}, b_m)$ are 5-cycles of Γ (note that this is true even if $r = m$). Then every dart beginning at b_0 lies on a 5-cycle. Since dc is a $[1, i]$-edge of (Δ, λ), we see that d_0 is adjacent to c_0 in Γ. Furthermore, since Γ is vertex-transitive, the dart beginning at d_0 and ending at c_0 lies on a 5-cycle $C := (d_0, c_0, u, v, w)$, for some $u, v, w \in \mathrm{V}(\Gamma)$. Clearly, $u \in \{b_0, b_m\}$ and $v \in \{a_0, a_r, a_m, a_{m+r}\}$, since both $\mathrm{fib}(c)$ and $\mathrm{fib}(b)$ are independent sets. Then $w \in \mathrm{fib}(a) \cup \mathrm{fib}(b)$ which leads us to a contradiction, since no vertex in $\mathrm{fib}(a) \cup \mathrm{fib}(b)$ is adjacent to a vertex in $\mathrm{fib}(d)$. Therefore, there is no 5-cycle tracing the edge d_0c_0 and thus Γ is not vertex-transitive. □

Lemma 5.4.13. *Let (Δ, λ) be a labelled graph and let Γ be a vertex-transitive ccv-cover of (Δ, λ) with more than 6 vertices. If A_1 is contained in (Δ, λ) then every vertex of (Δ, λ) is contained in either a 3-cycle or a copy of A_1, or is incident to a loop.*

Proof. Let $u \in \mathrm{V}(\Delta)$. By Lemma 5.4.7, Γ contains a triangle and since it is vertex-transitive, every vertex in $\mathrm{fib}(u)$ lies on a triangle. In particular, u_0 lies on a triangle T. The canonical projection $\pi : \Gamma \to \Delta$ maps T to a subgraph of Δ containing u. If all three vertices of T are mapped to u, then u is incident to a loop; if they are mapped to three distinct vertices, then clearly $\pi(T)$ is a triangle (that contains u). Suppose $\pi(T)$ has exactly two vertices. We can assume no dart in $\pi(T)$ has label 3, as this would make Γ a cubic arc-transitive graph of girth 3, and thus isomorphic to K_4 (contradicting that Γ has more than 6 vertices). Then, there are three possibilities for $\pi(T)$: it is either isomorphic to A_1, A_6 or it consists of a pair of parallel edges with one endpoint incident to a semi-edge. If $\pi(T)$ is isomorphic to A_1, we are done; if it isomorphic to A_6, then we are in contradiction with Lemma 5.4.9. Suppose u and v are joined by a pair of parallel edges e and e', and u (or v) is incident to a semi-edge s. Now, one edge in $\mathrm{fib}(e)$ lies on T, and thus, all edges in $\mathrm{fib}(e)$ must lie on a triangle. Similarly, every edge in $\mathrm{fib}(e')$ or $\mathrm{fib}(s)$ lies on a triangle. It follows that for any

$u_i \in \mathrm{fib}(u)$, all three edges incident to u_i lie on a triangle, and by Lemma 5.4.3 we have $\Gamma \cong K_{3,3}$, a contradiction. The result follows. □

The following theorem summarises the results of this section.

Theorem 5.4.14. *Let* $(\Delta, \lambda, \iota, \zeta)$ *be a ccv-graph and suppose* $\Gamma = \mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$ *is vertex-transitive. If any of the following conditions hold, then* Γ *belongs to the set* X *of exceptional graphs and is a* k*-multicirculant with* $k \in \{1, 2, 3\}$*:*

1. Δ *contains an artefact* A_i *with* $\{1, 2, 3\}$ *along with a* $[1, 3]$*-edge;*
2. Δ *contains an artefact* A_i *with* $\{4, ..., 9\}$*;*
3. Δ *agrees with the hypothesis of Lemma 5.4.13.*

Proof. First, suppose item (1) holds. By Lemma 5.4.7, Γ has girth 3 or 4 and by Lemma 5.4.6, Γ is arc-transitive. Then, by Lemma 5.4.3, Γ belongs to X. If (2) holds, then the result follows from Lemmas 5.4.8, 5.4.9, 5.4.10, 5.4.11 and 5.4.12. Now suppose (3) holds. Then by the proof of Lemma 5.4.13, Γ is isomoprhic to $K_{3,3}$, which is a circulant. □

Chapter 6

Cubic graphs admitting an automorphism with at most 3 orbits

In this Chapter we will use the language of cyclic generalised covers to present a classification of all cubic vertex-transitive graphs admitting an automorphism with at most three vertex-orbits. We do this in two steps. We first classify all cubic graphs with an automorphism with at most 3 orbits into 25 classes (Theorem 6.1.1). These classes correspond to the 25 possible dart-labelled quotients of a cubic graph by an automorphism with at most 3 orbits on vertices. We then determine which of these classes have vertex-transitive elements. As it transpires, every cubic vertex-transitive graph with an automorphism with 3 orbits admits a k-multicirculant automorphism with $k \in \{1, 2, 3\}$ (Theorem 6.2.1). Theorems 6.1.1 and 6.2.1 first appeared in [58].

6.1 Classification theorem

Let Γ be a graph and let $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group with at most three orbits on vertices. By Theorem 5.3.9, Γ is a *ccv*-cover an labelled graph on at most three vertices that can be extended to a *ccv*-graph. Let us turn our attention to Figure 6.1.1. The 25 graphs depicted here, which we denote by Δ_i for $i \in \{1, \ldots, 25\}$, are all the dart-labelled graphs on at most 3 vertices that can be extended to a ccv-graph. The proof that this family of 25 dart-labelled graphs is indeed complete is part of the proof of Theorem 6.1.1. It was obtained by checking all the graphs on at most 3 vertices and maximal valence at most 3, and then for each of these graphs finding all the labelings λ satisfying the conditions given in Proposition 5.3.3 (note that this is a finite problem for each graph). For each dart-labelled graph Δ_i we have chosen a distinguished vertex shown as a white vertex with a black circle around it and the letter m next to it. A voltage assignment, which may assign values r or s to some darts, has also been specified in the drawing.

Let $i \in \{1, \ldots, 25\}$ and let $m, r, s > 0$. We define $\Gamma_i(m; r, s)$ (alternatively

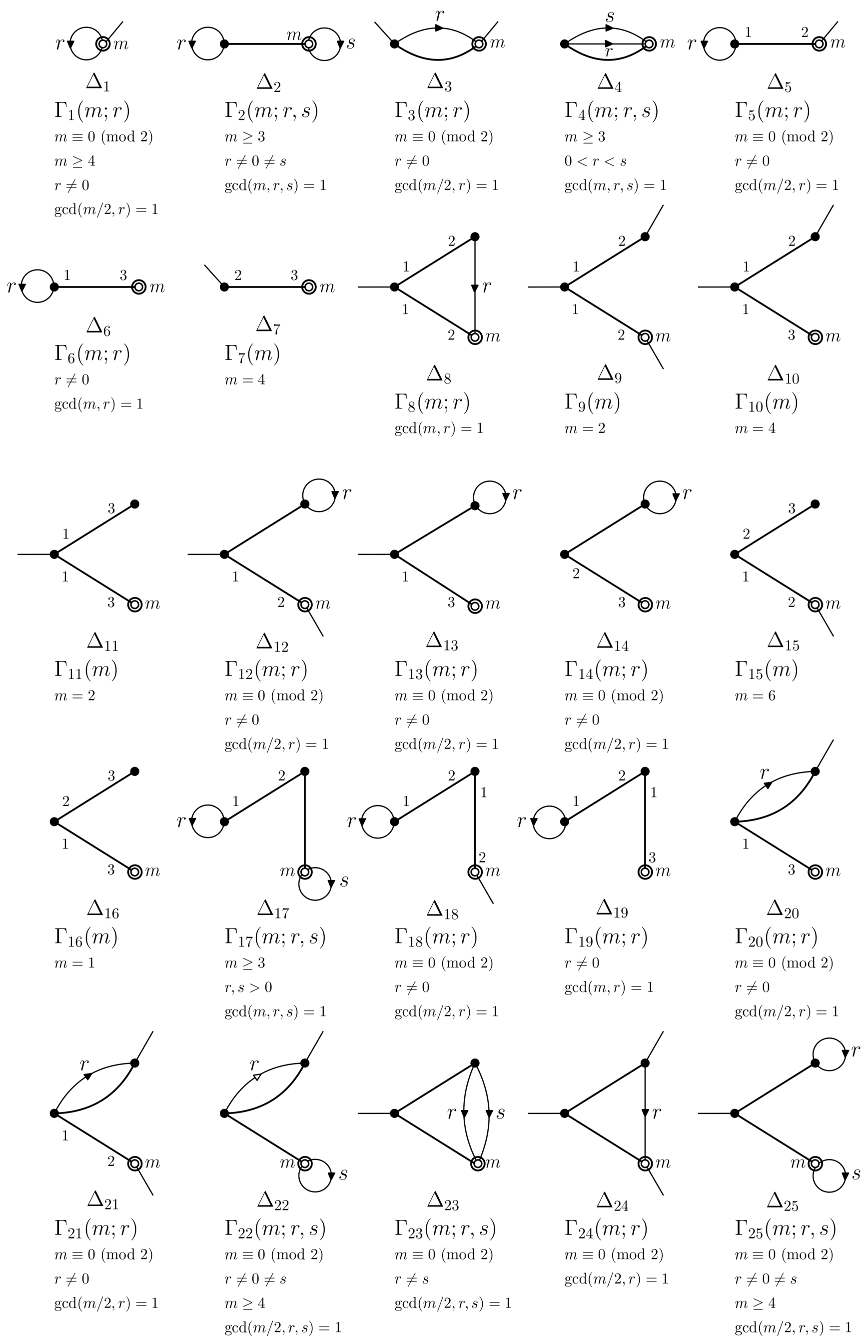


Figure 6.1.1: The 25 dart-labelled graphs on at most three vertices that can be extended to a ccv-graph.

$\Gamma_i(m;r)$ or $\Gamma_i(m)$) as the cyclic generalised cover of the cyclic generalised voltage graph $(\Delta, \lambda, \iota, \zeta)$ where $(\Delta, \lambda) = \Delta_i$, ι is the index function defined by $\iota(v_0) = m$, where v_0 is the distinguished vertex of Δ_i, and ζ is the voltage assignment shown in the picture, where we take the voltage of a semi-edge x to be $\iota(\operatorname{beg} x)/2$. We will also assume that the integers m, r and s satisfy the conditions listed underneath Δ_i in Figure 6.1.1. As we will see in Section 5.3, these conditions are sufficent and necessary for $(\Delta, \lambda, \iota, \zeta)$ to be a *ccv*-graph. Hence, the graph $\Gamma_i(m; r, s)$ (or $\Gamma_i(m; r)$ or $\Gamma_i(m)$, accordingly) is a connected, cubic, simple graph. We will say that a graph Γ is a ccv-cover of Δ_i for some $i \in \{1, \ldots, 25\}$ if Γ is isomorphic to $\Gamma_i(m; r, s)$ (or $\Gamma_i(m; r)$ or $\Gamma_i(m)$, accordingly).

Consider for instance the labelled graph Δ_6. The covering graph $\Gamma_6(1; 1)$ is the cyclic generalised cover of the ccv-graph obtained by assigning voltage $r = 1$ to (one dart underlying) the loop and letting $\iota(v_0) = 1$. Then, if u is the remaining vertex, we see that $\iota(u) = 3$. Observe that the ccv-graph thus obtained is precisely the generalised cyclic voltage graph shown in the bottom left of Figure 5.1.1. Then $\Gamma_6(1; 1)$ is isomorphic to K_4, as shown in the figure. Similarly, $\Gamma_6(2; 1)$ is isomorphic to the cube Q_3 (see Figure 5.1.1). The graph constructed in the example in the end of the preceding chapter and shown in Figure 5.1.2 is the graph $\Gamma_{18}(2; 2)$.

Theorem 6.1.1. *A graph Γ is a connected, cubic, simple graph with a cyclic group of automorphisms $G \leq \operatorname{Aut}(\Gamma)$ having at most 3 orbits on vertices if and only if it is isomorphic to one of the following:*

1. $\Gamma_7(2)$, $\Gamma_9(4)$, $\Gamma_{10}(2)$, $\Gamma_{11}(4)$, $\Gamma_{15}(6)$ *or* $\Gamma_{16}(1)$*;*
2. $\Gamma_1(m; r)$ *with m even,* $m \geq 4$, $r \neq 0$, $\gcd(\frac{m}{2}, r) = 1$*;*
3. $\Gamma_i(m; r)$ *with* $i \in \{3, 12, 13, 14, 18, 20, 21\}$*, m even,* $r \neq 0$, $\gcd(\frac{m}{2}, r) = 1$*;*
4. $\Gamma_i(m; r)$ *with* $i \in \{6, 19\}$, $r \neq 0$, $\gcd(m, r) = 1$*;*
5. $\Gamma_8(m; r)$ *with* $\gcd(m, r) = 1$*;*
6. $\Gamma_i(m; r, s)$ *with* $i \in \{2, 17\}$, $m \geq 3$, $r \neq 0 \neq s$, $\gcd(m, r, s) = 1$*;*
7. $\Gamma_4(m; r, s)$ *with* $m \geq 3$, $0 < r < s$, $\gcd(m, r, s) = 1$*;*
8. $\Gamma_5(m; r, s)$ *with m even,* $r \neq 0$, $\gcd(\frac{m}{2}, r) = 1$*;*
9. $\Gamma_{23}(m; r, s)$ *with m even,* $r \neq s$, $\gcd(\frac{m}{2}, r) = 1$*;*
10. $\Gamma_{24}(m; r, s)$ *with m even,* $\gcd(\frac{m}{2}, r) = 1$*;*
11. $\Gamma_i(m; r, s)$ *with* $i \in \{22, 25\}$*, m even,* $m \geq 4$, $r \neq 0 \neq s$, $\gcd(\frac{m}{2}, r, s) = 1$.

Proof. We would like to begin by making a few observations about the graphs Δ_i appearing in Figure 6.1.1. These 25 graphs comprise the complete list of graphs on at most 3 vertices with a dart-labelling that agrees with Proposition 5.3.3. Although it is a simple enough computation to be done by hand, in order to avoid human error, we used a computer programme written in SAGE [70] to construct, by brute force, all such graphs (up to label-preserving isomorphism). Then, any ccv-graph on at most 3 vertices can be obtained by extending some Δ_i to a cyclic generalised voltage graph. In light of Lemma 5.3.8, when extending a labelled graph Δ_i to a ccv-graph, we will only consider those voltage assignments that agree with the corresponding voltage shown in Figure 6.1.1 (that is, a voltage assignment that is trivial on every edge lacking an arrowhead, and that assigns voltage $\iota(\mathrm{beg}\,x)/2$ to every semi-edge x). The conditions under each Δ_i are derived from equalities (5.1.1) and (5.1.2), and from Theorems 5.2.10 and 5.2.11. Hence, an extension $(\Delta, \lambda, \iota, \zeta)$ of Δ_i, for some $i \in \{1, \dots, 25\}$, is a ccv-graph if and only if ι and ζ satisfy the corresponding conditions listed in Theorem 6.1.1 (and agree with the corresponding voltage assignment shown in Figure 6.1.1).

Let Γ be a cubic graph and suppose it admits a cyclic group of automorphism G having at most 3 vertex-orbits. By Theorem 5.3.9, Γ is a cyclic generalised cover of a *ccv*-graph $(\Gamma/G, \lambda, \iota, \zeta)$. Then $(\Gamma/G, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_i, for some $i \in \{1, \dots, 25\}$. That is Γ is isomorphic to $\Gamma_i(m; r, s)$ (or $\Gamma(m; r)$ or $\Gamma(m)$) for some m, r and s satisfying the corresponding conditions listed in Theorem 6.1.1.

For the converse, let $(\Delta, \lambda, \iota, \zeta)$ be a ccv-graph obtained by extending one dart-labelled graph Δ_i from Figure 6.1.1. Recall that the generalised cover of a ccv-graph is a cubic graph. Moreover, the group $\mathbb{Z}_n$, where $n = \mathrm{lcm}\{\lambda(x)\iota(\mathrm{beg}\,x) \mid x \in \mathrm{D}(\Delta_i)\}$, acts as a group of automorphism of $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$, and the orbits on vertices and darts under this action are precisely the fibres of $\mathrm{CycCov}(\Delta, \lambda, \iota, \zeta)$ (see Lemma 5.2.5). This completes the proof. □

6.2 Cubic vertex-transitive graphs admitting a cyclic group with at most 3 orbits

Theorem 6.2.1. *Let Γ be a connected, simple, cubic graph admitting a cyclic group of automorphisms having at most 3 orbits on the vertices of Γ. Then Γ is vertex-transitive if and only if it is isomorphic to the Tutte-Coxeter graph $\Gamma_{25}(10; 1, 3)$ or belongs to one of the following six infinite families:*

1. $\Gamma_1(m; r)$ *with m even,* $m \geq 4$, $\gcd(\frac{m}{2}, r) = 1$, $r \in \{1, 2\}$;
2. $\Gamma_2(m; r, 1)$ *with* $m \geq 3$, $r^2 \equiv \pm 1 \pmod{m}$, *or* $m = 10$ *and* $r = 2$;
3. $\Gamma_4(m; r, s)$ *with* $m \geq 3$, $r \neq s$, $\gcd(m, r, s) = 1$;
4. $\Gamma_{22}(m; 2, 1)$ *with* $m \geq 4$ *and* $\frac{m}{2}$ *odd;*
5. $\Gamma_{23}(m; r, 1)$ *with* $\frac{m}{2} \equiv 1 \pmod{4}$ *and* $r = (\frac{m}{2} + 3)/2$, *or* $\frac{m}{2} \equiv 3 \pmod{4}$ *and* $r = (\frac{3m}{2} + 3)/2$, *or* $m = 4$ *and* $r = 0$.

Remark 6.2.2. The class of graphs in item (1) of Theorem 6.2.1 is precisely the class of cubic vertex-transitive circulants. Meanwhile, the three classes of graphs in items (2), (3) and (4) correspond to the three families of cubic vertex-transitive bicirculants in Theorem 3.1.1. Finally, the graphs in items (5) and (6) are the graphs Y(k) or X(k) of Theorem 3.2.1, respectively. Thus, it follows from Theorem 6.2.1 that every cubic vertex-transitive graph admitting a cyclic group of automorphisms with at most three orbits on vertices is a k-multicirculant for some $k \in \{1, 2, 3\}$.

We will partition the set $\{1, \ldots, 25\}$ into three sets: $I_M = \{1, 2, 3, 4, 22, 23, 24, 25\}$, $I_X = \{5, 6, 8, 16, 18, 19, 21\}$ and $I_C = \{7, 9, 10, 11, 12, 13, 14, 15, 17, 20\}$. The reason for this is that a ccv-cover of Δ_i with $i \in I_M$ is necessarily a k-multicirculant for some $k \in \{1, 2, 3\}$. We can then resort to the classification of cubic vertex-transitive bicirculants (Theorem 3.1.1 and tricirculants (Theorem 3.2.1) to deal with ccv-covers of Δ_i when $i \in I_M$. For the remaining indices in $\{1, \ldots, 25\} \setminus I_M$, we distinguish those for which Δ_i admits no vertex-transitive ccv-cover (these conform the set I_C) and those who admit at least one vertex-transitive ccv-cover (these conform the set I_X). As it transpires, all vertex-transitive ccv-covers of Δ_i with $i \in I_X$ belong to the set X of exceptional graphs (see Definition 5.4.1). Theorem 6.2.1 will then follow from Claims 6.2.3 and 6.2.4 (which will be proved at the end of this section), as well as the classification of cubic vertex-transitive bicirculants and tricirculants (Theorems 3.1.1 and 3.2.1).

Claim 6.2.3. If $i \in I_X$, then a vertex-transitive ccv-cover of Δ_i must be one of the 9 exceptional graphs in E.

Claim 6.2.4. If $i \in I_C$ then Δ_i admits no vertex-transitive ccv-cover.

Before we prove the above claims, we need to look briefly into the set of exceptional graphs X and the possible dart-labelled graphs from which they arise. For each graph Γ in X, we have determined, with the aid of a computer programme written in SAGE [70], the values of $i \in \{1, \ldots, 25\}$ for which Γ is a cyclic generalised cover of Δ_i. For instance, K_4 is a cyclic generalised cover of Δ_1 and Δ_3 (and of no other Δ_i). Indeed, K_4 is isomorphic to $\Gamma_1(4; 1)$ and to $\Gamma_3(2; 1)$. The results are displayed in Table 6.1 below.

Graph	Cover of Δ_i	Graph	Cover of Δ_i
K_4	$i = 1, 3$	GP$(8, 3)$	$i = 2, 6$
$K_{3,3}$	$i = 1, 4, 5, 16$	Pappus	$i = 5, 22$
Q_3	$i = 2, 4, 6$	GP$(10, 2)$	$i = 2$
Petersen	$i = 2, 19, 21$	GP$(10, 3)$	$i = 2$
Heawood	$i = 4, 13, 18$		

Table 6.1: The exceptional graphs in E and the values of i for which they are a ccv-cover of Δ_i

Proof of Claim 6.2.3. Let $i \in I_X := \{5, 6, 8, 16, 18, 19, 21\}$ and suppose Γ is a vertex-transitive ccv-cover of Δ_i. If $i \in \{5, 6, 18, 19\}$ then by Lemma 5.4.9 we have $\Gamma \in X$. If $i = 16$ then by Lemma 5.4.7 the girth of Γ is at most 4 and by Lemma 5.4.6 Γ is arc-transitive. Then, by Lemma 5.4.3, $\Gamma \in X$. If $i = 8$, then by Lemma 5.4.7 Γ has 3-signature $(\epsilon_1, \epsilon_2, \epsilon_3)$, where $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, and by Lemma 5.4.3 Γ is isomorphic to K_4, which is in X. It remains to see what happens when $i = 21$. Suppose Γ is a vertex-transitive ccv-cover of Δ_{21}. Then, by Theorem 6.1.1, Γ is isomorphic to $\Gamma_{21}(m; r)$ for two integer $m, r > 0$ such that m is even and $\gcd(\frac{m}{2}, r) = 1$. With the notation of Figure 6.2.1 (right), let $\iota(w) = m$ and let g be the girth of $\Gamma_{21}(m;r)$. Observe that the vertex u_0 lies on two distinct 5-cycles, namely $(w_0, u_0, v_0, v_m, u_m)$ and $(w_0, u_0, v_r, v_{r+m}, u_m)$. That is, every edge incident to u_0 lies on a 5-cycle. If $g = 5$ then by Lemma 5.4.3, $\Gamma_{21}(m;r)$ is isomorphic to the Petersen graph or the dodecahedron, and we are done since both graphs are in X. Clearly, $g \neq 3$ as no vertex in $\mathrm{fib}(w)$ lies on a 3-cycle (vertices in $\mathrm{fib}(w)$ are only adjacent to vertices in $\mathrm{fib}(u)$, which is an independent set). If $g = 4$ then w must lie on a 4-cycle. It is not too difficult to see that this can only happen if $v_m = v_r$ and $v_0 = v_{r+m}$. However, (u_0, v_m, v_0) is then a 3-cycle, contradicting $g \neq 3$. We conclude that if $i = 21$, then $\Gamma \in X$. We have shown that if $i \in I_X$, then a vertex-transitive ccv-cover of Δ_i must be an exceptional graph belonging to X.

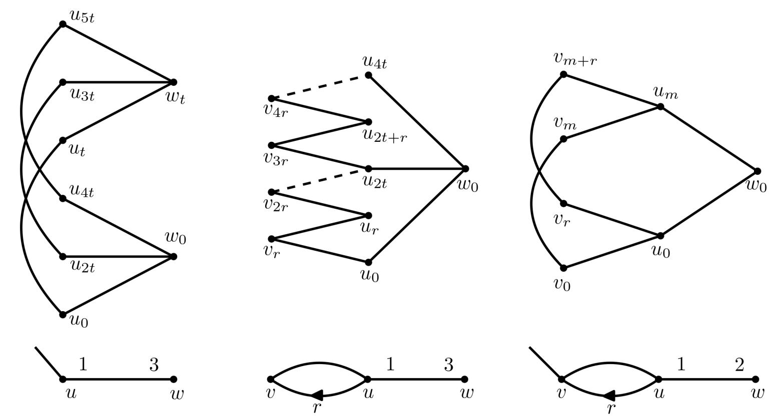


Figure 6.2.1: On the top row, from left to write, a subgraph of $\Gamma_{13}(m;r)$, $\Gamma_{20}(m;r)$ and $\Gamma_{21}(m;r)$, respectively. Underneath each, the subgraph of Δ_i to which it projects.

Proof of Claim 6.2.4. Let $i \in I_C := \{7, 9, 10, 11, 12, 13, 14, 15, 17, 20\}$ and suppose Γ is a vertex-transitive ccv-cover of Δ_i. Since no element of X is a ccv-cover of Δ_i for $i \in I_C$ (see Table 6.1) we see that $\Gamma \notin X$. Suppose $i \in \{7, 14, 15\}$. Then by Lemma 5.4.7, the girth of Γ is 3 or 4 and by Lemma 5.4.6, Γ is arc-transitive. It then follows from Lemma 5.4.3 that $\Gamma \in X$, a contradiction. If $i = 9$ then by Lemma 5.4.7, Γ has 3-signature $(\epsilon_1, \epsilon_2, \epsilon_3)$, with $\epsilon_1, \epsilon_2, \epsilon_3, > 0$, and by Lemma 5.4.3 we see that Γ is isomorphic to K_4, which is a element of X, leading us again to a contradiction. If $i \in \{10, 11\}$ then by Lemma 5.4.7, Lemma 5.4.6 and Lemma 5.4.3 we see, once more, that $\Gamma \in X$. If $i = 17$, $\Gamma \in X$ by Lemma 5.4.9. The cases when $i \in \{12, 13, 20\}$ require slightly deeper analysis. In the subsequent paragraphs, we will assume the vertices of the graphs Δ_i, with $i \in \{12, 13, 20\}$, are named as follows: in Figure 6.1.1, let u be the leftmost vertex; v the top vertex; w the bottom vertex, so that w is in fact the distinguished vertex of Δ_i.

Suppose $i = 12$. Then $\Gamma \cong \Gamma_{12}(m;r)$ for some integers $m, r > 0$ such that m is even and $\gcd(m/2, r) = 1$. Let $m = \iota(w)$. By Lemma 5.4.7 every vertex in $\mathrm{fib}(u)$ lies on a 3-cycle. Then v_0 must also lie on a 3-cycle C. Note that since a vertex u_0 is the only neighbour of v_0 in $\mathrm{fib}(u)$, all three vertices in C must belong to $\mathrm{fib}(v)$. That is, $C = (v_0, v_r, v_{2r})$ or $C = (v_0, v_{-r}, v_{-2r})$. Then $3r \equiv 0 \pmod{2m}$. However, by item (1) of Lemma 5.3.8, $r < \iota(\mathrm{beg}\, v) = 2m$ which implies that either $3r = 2m$ or $3r = 4m$. Moreover, since $\gcd(m/2, r) = 1$, then necessarily $r = 4$ and $m = 6$. The reader can verify that the graph $\Gamma_{12}(6, 4)$ is not vertex-transitive.

Now, suppose $i = 13$. Then $\Gamma \cong \Gamma_{13}(m; r)$ for some integers $m, r > 0$ such that m is even and $\gcd(m/2, r) = 1$. Let $\iota(w) = m = 2t$ and let g be the girth of Γ. Note that Γ must be arc-transitive as Δ_{13} has a $[1, 3]$-edge. If $g < 6$ then by Lemma 5.4.3, $\Gamma \in X$, a contradiction. Suppose $g \geq 6$. Let $\Gamma' \leq \Gamma$ be the subgraph induced by $\mathrm{fib}(u) \cup \mathrm{fib}(w)$. It is plain to see that Γ' contains a 6-cycle (see Figure 6.2.1, left) and thus $g = 6$. Moreover, every edge incident to w_i, $i \in \mathbb{Z}_{\iota(w)}$, lies on two distinct 6-cycles of Γ'. The edge u_0u_{3t} also lies on two distinct 6-cycles, both contained in Γ'. Since $\Gamma_{13}(m; r)$ is arc-transitive, the edge u_0v_0 must lie on a 6-cycle C. However, C must necessarily visit either u_0w_0 or u_0u_{3t}. That is, one of u_0w_0 or u_0u_{3t} lies on three distinct 6-cycles. It follows that the 6-signature of Γ is not $(2, 2, 2)$ and by Lemma 5.4.5, $\Gamma \in X$, a contradiction.

Finally, suppose $i = 20$ and Γ is isomorphic to $\Gamma_{20}(m; r)$ where m is even, $r \neq 0$ and $\gcd(m/2, r) = 1$. Let $\iota(w) = m = 2t$ and let g be the girth of Γ. Observe that Γ must be arc-transitive since Δ_{20} has a $[1, 3]$-edge. If $g < 6$, then by Lemma 5.4.5, $\Gamma \in X$, a contradiction. Suppose $g \geq 6$ and see that if $r \equiv 0 \pmod{2t}$ then (w_0, u_0, v_r, u_r) is a 4-cycle, contradicting our assumption that $g \geq 6$. Hence we may assume $r \not\equiv 0 \pmod{2t}$. Now, note that $\iota(v) = \iota(u) = 3 \cdot \iota(w) = 6t$. Then the voltage of the semi-edge at v must be $\iota(v)/2 = 3t$. This implies that each v_i is adjacent to v_{i+3t}, and in particular that $(u_0, v_0, v_{3t}, u_{3t}, v_{3t+r}, v_r)$ is a 6-cycle of $\Gamma_{20}(m; r)$. Then w_0 must also lie on a 6-cycle C. Since both $\mathrm{fib}(w)$ and $\mathrm{fib}(u)$ are independent sets, if a cycle visits $\mathrm{fib}(w)$ twice, it must have length at least 8. It follows that w_0 is the only vertex in C belonging to $\mathrm{fib}(w)$. It is straightforward to see that a 6-cycle through w_0 must necessarily be $C_i = (w_0, u_{it}, v_{it+r}, u_{it+r}, v_{it+2r}, u_{it+2r})$, for some $i \in \{0, 2, 4\}$. That is, in Figure 6.2.1 (middle), the dotted lines must be edges. This implies that $2r \equiv 0 \pmod{2t}$. Moreover, since $0 < r < 3t$ we see that $2r = 2t$. Then the edge u_0v_0 lies on both C_0 and C_4, which are distinct cycles since $r \not\equiv 0 \pmod{2t}$. Since u_0v_0 also lies in C, it follows from Lemma 5.4.5 that $\Gamma \in X$, once more a contradiction. This completes the proof of Claim 6.2.4.

Proof of Theorem 6.2.1. Let Γ be a simple, connected, cubic graph admitting a cyclic group of automorphism G with at most 3 orbits on vertices. Suppose Γ is vertex-transitive. Then, Γ is a ccv-cover of Δ_i for some $i \in \{1, \ldots, 25\}$. By Claim 6.2.4, $i \notin I_C$. If $i \in I_X$, then by Claim 6.2.3, Γ must be one of the exceptional graphs in X. However, every graph in X is either isomorphic to the Tutte-Coxeter graph or belongs to one of the six infinite families described in Theorem 6.2.1. Indeed, $K_4 \cong \Gamma_i(4; 1)$, $K_{3,3} \cong \Gamma_4(3; 1, 2)$, $Q_3 \cong \Gamma_2(4; 1, 1)$, $\mathrm{GP}(5, 2) \cong \Gamma_2(5; 2, 1)$, the Heawood graph is isomorphic to $\Gamma_4(7; 1, 3)$, $\mathrm{GP}(8, 3) \cong \Gamma_2(8; 3, 1)$, the Pappus graph is isomorphic to $\Gamma_{22}(6; 2, 1)$ and $\mathrm{GP}(10, 3) \cong \Gamma_2(10; 3, 1)$. It remains to see what happens when $i \in I_M = \{1, 2, 3, 4, 22, 23, 24, 25\}$.

If $i = 1$, then $\Gamma \cong \Gamma_1(m; r)$ for some integers m and r where $m \geq 4$, m is even, $r \neq 0$ and $\gcd(\frac{m}{2}, r)$ (see Theorem 6.1.1). Observe that $\gcd(m, r) \in \{1, 2\}$.

If $\gcd(m, r) = 1$ then r is a unit in the ring $\mathbb{Z}_m$. That is, r has a multiplicative inverse r^{-1} in $\mathbb{Z}_m$, and $\gcd(m, r^{-1}) = 1$. Then by Lemma 5.2.7, $\Gamma_1(m; r) \cong \Gamma_1(m; 1)$. On the other hand, if $\gcd(m, r) = 2$, then $\frac{r}{2}$ is a unit in $\mathbb{Z}_m$ and, once again by Lemma 5.2.7, $\Gamma_1(m; r) \cong \Gamma_1(m; 2)$.

Suppose $i \in \{2, 3, 4\}$. Note that for integers m, r and s, the graphs $\Gamma_2(m; r, s)$ and $\Gamma_3(m; r, s)$ correspond, respectively, to the graphs $I(m, r, s)$ and $H(m, r, s)$ of Theorem 3.1.1. It then follows that if $i = 2$ then $\Gamma \cong \Gamma_2(m; r, 1)$, for some $m \geq 3$ and $r^2 \equiv \pm 1 \pmod{m}$, or $m = 10$ and $r = 2$. By the same token, if $i = 4$ then $\Gamma \cong \Gamma_4(m; r, s)$ where $m \geq 3$, $r \neq s$ and $\gcd(m, r, s) = 1$. If $i = 3$ then by Theorem 3.1.1, Γ is also a *ccv*-cover of either Δ_1 or Δ_2

Now, suppose $i \in \{22, 23, 24, 25\}$. Observe that the graphs $\Gamma_{22}(m; r, s)$, $\Gamma_{23}(m; r, s)$, $\Gamma_{24}(m; r, s)$ and $\Gamma_{25}(m; r, s)$ correspond to the graphs $T_1(\frac{m}{2}, r, s)$, $T_2(\frac{m}{2}, r, s)$, $T_3(\frac{m}{2}, r)$ and $T_4(\frac{m}{2}, r, s)$ of Theorem 3.2.1. Thus, if $i = 22$ then Γ is isomorphic to $\Gamma_{22}(m; 2, 1)$ with $\frac{m}{2}$ odd. By the same theorem if $i = 23$ then $\Gamma \cong \Gamma_{23}(m; r, 1)$ with $\frac{m}{2} \equiv 1 \pmod{4}$ and $r = (\frac{m}{2} + 3)/2$, or $\frac{m}{2} \equiv 3 \pmod{4}$ and $r = (\frac{3m}{2} + 3)/2$, or $m = 2$ and $r = 0$. If $i = 24$ then by Theorem 3.2.45, Γ is a circulant and thus also a cover of Δ_1. Finally if $i = 25$ then by Theorem 3.2.1 Γ is isomorphic to Tutte-Coxeter graph.

For the converse, suppose Γ is the Tutte-Coxeter graph or belongs to one of the 6 infinite families described in Theorem 6.2.1. It is well known that the Tutte-Coxeter graph is arc-transitive and thus also vertex-transitive. A cover of Δ_1 must be vertex-transitive as it has a cyclic group of automorphism with a single orbit on vertices. It is shown in Theorem 3.1.1, that the graphs belonging to the families of items (2) and (3) and (4) of Theorem 6.2.1 are vertex-transitive. Finally, the graphs in items (5) and (6) of Theorem 6.2.1 are vertex-transitive by Theorem 3.2.1. This completes the proof.

Chapter 7

Cubic-vertex transitive graph with long regular orbits

In this chapter we will study cubic vertex-transitive graphs admitting a cyclic group of automorphisms with a long orbit. Given a permutation g on a set Ω, let c_g be a longest cycle appearing in the decomposition of g into a product of disjoint cycles, and let $o(g)$ and $\ell(g)$ denote the order of g and the length of c_g, respectively. If a permutation g is such that $o(g) = \ell(g)$, then c_g is called a *regular cycle of* g and the points of Ω moved by c_g is called a *regular orbit* of the cyclic group $\langle g \rangle$. We say that a graph Γ *admits a regular orbit* of length ℓ if it admits an automorphism with a regular cycle of length ℓ (where the automorphism is considered as a permutation on the vertex-set). We define $o(\Gamma) := \max\{o(g) \mid g \in \mathrm{Aut}(\Gamma)\}$ and $\ell(\Gamma) := \max\{\ell(g) \mid g \in \mathrm{Aut}(\Gamma)\}$. We will be interested in the following general questions: What can be said about a cubic vertex-transitive graph Γ for which the value $\ell(\Gamma)$ is large? Under what conditions does the equality $o(\Gamma) = \ell(\Gamma)$ hold?

We show in Section 7.1 that every automorphism of a cubic vertex-transitive graph Γ not isomorphic to $K_{3,3}$ admits a regular cycle, and thus $\ell(\Gamma) = o(\Gamma)$ (Theorem 7.1.1). We also give a bound on the number of vertex-orbits of an automorphism of Γ in function of the size of its largest orbit (Thorem 7.1.4). In Section 7.2, we obtain a complete classification of cubic vertex-transitive graphs Γ admitting an automorphism g with $\ell(g) \geq |\mathrm{V}(\Gamma)/3|$, thus extending the results on cubic vertex-transitive k-multicirculants (for instance, Theorems 3.1.1 and 3.2.1) and further generalising Theorem 6.2.1. In particular, each such graph is a *ccv*-cover of one of the seven labelled graphs shown in Figure 7.2.7 (see Theorem 7.2.22 for details).

7.1 Vertex orbits in cubic vertex-transitive graphs

7.1.1 Regular orbits

Theorem 7.1.1. *If Γ is a cubic vertex-transitive graph not isomorphic to $K_{3,3}$, then every automorphism of Γ has a regular cycle and $o(\Gamma) = \ell(\Gamma)$.*

Proof. Let $g \in \mathrm{Aut}(\Gamma)$ have order n, let $G = \langle g \rangle$ and $\Delta = \Gamma/G$. By Theorem **??**, Γ is the cover of some *ccv*-graph $(\Delta, \lambda, \iota, \zeta)$. Recall that the orbits of G on the vertices of Γ are identified with the fibres of vertices of Δ, and that for each $v \in \mathrm{V}(\Delta)$ we have $\iota(v) = |fib(v)|$. Then, since $G = \langle g \rangle$, we have $n = \mathrm{lcm}\{\iota(v) \mid v \in \mathrm{V}(\Delta)\}$. Let $u \in \mathrm{V}(\Delta)$ be such that $\iota(u) \geq \iota(v)$ for all $v \in \mathrm{V}(\Delta)$. We will show that $\iota(v) \mid \iota(u)$ for all $v \in \mathrm{V}(\Delta)$, and so $\iota(u) = n$. Let $w \in \mathrm{V}(\Delta) \setminus \{u\}$ and let $W = (x_1, \ldots, x_k)$ be a uw-path. Since $\Gamma \ncong K_{3,3}$, we see from Lemma 5.4.8 that (Δ, λ) contains no edges of type $[2, 3]$. That is, if for any $y \in \mathrm{D}(\Delta)$ we have $\lambda(y) = 3$, then $\lambda(y^{-1}) = 1$. Now, let x be a dart traced by W. If $\lambda(x) = 3$ then $\lambda(x^{-1}) = 1$ and $\mathrm{beg}\, x$ has only one neighbour, namely, $\mathrm{end}\, x$. Then necessarily $\mathrm{beg}\, x = u$, which contradicts $\iota(u)$ being maximal (as $\iota(\mathrm{end}\, x)$ would equal $3 \cdot \iota(u)$ by formula (5.1.1)). If $\lambda(x^{-1}) = 3$ then $\lambda(x) = 1$ and x is the last dart traced by W. Thus $\lambda(x_i), \lambda(x_i^{-1}) \in \{1, 2\}$ for all $i < k$. Let $W^- = (x_1, \ldots, x_{k-1})$ and let $X = (x_k)$ be the walk consisting only of the dart x_k. Now, consider the function ρ_λ defined given in formula (5.1.5) and observe that

$$\rho_\lambda(W^-) = \prod_{i=1}^{k-1} \frac{\lambda(x_i)}{\lambda(x_i^{-1})} = \frac{1}{2^a},$$

for some $a \in \{0, 1, 2\}$. And so

$$\rho_\lambda(W) = \rho_\lambda(W^-)\rho_\lambda(X) = \frac{1}{2^a}\rho_\lambda(X) = \frac{1}{2^b 3^c},$$

for some $b \in \{0, 1, 2\}$ and $c \in \{0, 1\}$. Then by equality (5.1.7) we have

$$\iota(w) = \frac{\iota(u)}{2^b 3^c}.$$

That is, $\iota(w)$ is a divisor of $\iota(u)$ and since our choice of w was arbitrary, we have $\iota(v) \mid \iota(u)$ for all $v \in \mathrm{V}(\Delta)$. The result follows. □

7.1.2 Bounding orbits

The main result of this subsection is Theorem 7.1.4. It will be essential in the classification of cubic vertex-transitive graphs with a long regular orbit (Theorem 7.2.22).

Proposition 7.1.2. *Let Γ be a cubic G-vertex-transitive graph, let $v \in V(\Gamma)$ and denote by G_v the stabilizer of v in G. If $g \in G_v$, then the order of g is at most* 6.

Proof. Suppose that G has m orbits on its action on the darts of Γ. Then, since γ is vertex-transitive, we have $m \in \{1, 2, 3\}$.

If $m = 3$, then the connectivity of Γ implies that the stabilizer G_v in G of any vertex $v \in \mathrm{V}(\Gamma)$ is trivial, and the result follows.

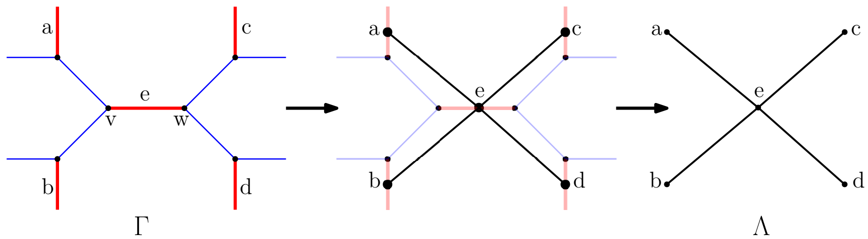


Figure 7.1.1: The neighbourhood of an edge with endpoints v and w in Γ, and the neighbourhood of the corresponding vertex in Λ.

If $m = 1$, then Γ is arc-transitive. By a celebrated result by Tutte [72], Γ is s-arc-transitive, $s \leq 5$, and G_v is isomorphic to either $\mathbb{Z}_3$, S_3, $S_3 \times S_2$, S_4 or $S_4 \times S_2$, depending on whether s equals 1, 2, 3, 4 or 5, respectively. In any of the five possible cases, no element of G_v has order greater than 6.

If $m = 2$, then G_v must fix a dart x beginning at v. Let $\mathcal{T}$ be the orbit of x under the action of G. Note that for every $u \in \mathrm{V}(\Gamma)$, there exists exactly one dart in $\mathcal{T}$ beginning at u. Denote by $\bar{\mathcal{T}}$ the set of edges containing the darts in $\mathcal{T}$. That is, $\bar{\mathcal{T}} = \{\{x, x^{-1}\} \mid x \in \mathcal{T}\}$.

We can thus construct a connected tetravalent graph Λ as follows. Let $\bar{\mathcal{T}}$ be the vertex-set of Λ and let $\mathrm{D}(\Gamma) \setminus \mathcal{T}$ be the dart-set of Λ. For $y \in \mathrm{D}(\Lambda) = \mathrm{D}(\Gamma) \setminus \mathcal{T}$, we let $\mathrm{beg}_\Lambda\, y$ be the unique $x \in \mathrm{V}(\Lambda) = \bar{\mathcal{T}}$ with $\mathrm{beg}_\Gamma\, x = \mathrm{beg}_\Gamma\, y$, and let $\mathrm{inv}_\Lambda\, y = \mathrm{inv}_\Gamma\, y$. Informally, Λ is constructed from Γ by contracting every edge in $\bar{\mathcal{T}}$ (see Construction 7 in Section 4 of [56] for details about this construction and the graph Λ).

It is not difficult to see that Λ is arc-transitive and that G acts on the vertices of Λ. In particular if $e \in \bar{\mathcal{T}}$ then G_e acts on the four edges incident to e (in Γ) as it acts on the four vertices adjacent to e (in Λ). Further this action must be isomorphic to the natural action of D_4, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$ on a set of 4 elements.

Let $e \in \bar{\mathcal{T}}$ have endpoints v and w and assume the notation in Figure 7.1.2. Then $N := \{a, b, c, d\}$ is the set of neighbours of e in Λ. Notice that G_v is isomorphic to the subgroup of G_e that fixes $\{a, b\}$ and $\{c, d\}$ set-wise.

If the permutation group G_e^N induced by the action of G_e on N is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$, then it follows from the connectivity of Λ that $G_e \cong \mathbb{Z}_4$ or $G_e \cong \mathbb{Z}_2^2$, respectively.

Suppose that $G_e^N \cong D_4$. Then, $(G_e, G_x, G_{\{x,x^{-1}\}})$, where x is the dart starting at e and ending in a (in Λ), is a dihedral amalgam of type $(4, 2)$ (see [15]). If follows from the main theorem of [15] that the subgroup H of G_e that fixes both $\{a, b\}$ and $\{c, d\}$ set-wise satisfies the following:

$$\begin{aligned} H &= \langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle, \\ a_i^2 &= 1, \qquad 0 \leq i \leq n-1, \\ [a_i, a_j] &= r_{i,j} \qquad 0 \leq i, j \leq n-1, \end{aligned}$$

where $r_{i,j}$ is in the center $\mathrm{Z}(H)$ of H and has order at most 2. In particular $N/\mathrm{Z}(H)$ is an elementary abelian 2-group implying that $g^2 \in \mathrm{Z}(H)$ for every

$g \in H$. However since N is generated by involutions, the exponent of $\mathrm{Z}(H)$ is at most 2, implying that the exponent of N is at most 4. Then, since $G_v \cong H$, we see that G_v has exponent at most 4.

We have shown that in all possible cases, the order of an element $g \in G_v$ has order at most 6.First, observe that if Γ is isomorphic to a prism or a Möbius ladder, then the result follows, as a vertex stabilizer is isomorphic to either S_3, if Γ is arc-transitive, or $\mathbb{Z}_2$ otherwise. We may thus assume without loss of generality that Γ is not isomorphic to a prism or a Möbius ladder.

Suppose that G has m orbits in its action on the darts of Γ. Then, since Γ is vertex-transitive, we have $m \in \{1, 2, 3\}$.

If $m = 3$, then the connectivity of Γ implies that the stabilizer G_v in G of any vertex $v \in \mathrm{V}(\Gamma)$ is trivial, and the result follows.

If $m = 1$, then Γ is arc-transitive. By a result of Djoković and Miller [16] (which is based on the celebrated work of Tutte [72] on cubic arc-transitive graphs), G_v is isomorphic to either $\mathbb{Z}_3$, S_3, $S_3 \times S_2$, S_4 or $S_4 \times S_2$. In none of the five possible cases, an element of G_v has order greater than 6.

If $m = 2$, then G_v must fix a dart x beginning at v. Let $\mathcal{T}$ be the orbit of x under the action of G. Note that $x \in \mathcal{T}$ if and only if $x^{-1} \in \mathcal{T}$ and that for every $u \in \mathrm{V}(\Gamma)$, there exists exactly one dart in $\mathcal{T}$ beginning at u. Denote by $\bar{\mathcal{T}}$ the set of edges containing the darts in $\mathcal{T}$; that is, $\bar{\mathcal{T}} = \{\{x, x^{-1}\} \mid x \in \mathcal{T}\}$. Then $\bar{\mathcal{T}}$ is a perfect matching in Γ.

We can thus construct a connected tetravalent graph Λ as follows. Let $\bar{\mathcal{T}}$ be the vertex-set of Λ and let $\mathrm{D}(\Gamma) \setminus \mathcal{T}$ be the dart-set of Λ. For $y \in \mathrm{D}(\Lambda)$, we let $\mathrm{beg}_\Lambda\, y$ be the unique edge $\{x, x^{-1}\} \in \bar{\mathcal{T}}$ with $\mathrm{beg}_\Gamma\, y \in \{\mathrm{beg}_\Gamma\, x, \mathrm{beg}_\Gamma\, x^{-1}\}$, and let $\mathrm{inv}_\Lambda\, y = \mathrm{inv}_\Gamma\, y$. Informally, Λ is constructed from Γ by contracting every edge in $\bar{\mathcal{T}}$ (see Construction 7 in Section 4 of [56] for details about this construction and the graph Λ). As was proved in [56, Lemma 9], since Γ is neither a prism or a Möbius ladder, Λ is a simple graph. Clearly every $g \in G$ induces an automorphism of Λ. Since G is transitive on $\mathrm{D}(\Gamma) \setminus \mathcal{T}$, we see that G acts arc-transitively on Λ.

Let $e \in \bar{\mathcal{T}}$ have endpoints v and w and assume the notation in Figure 7.1.2. Then $N := \{a, b, c, d\}$ is the set of neighbours of e in Λ. Since G acts transitively in the arcs of Λ, it follows that the permutation group G_e^N induced by the action of G_e on the set N is transitive. Observe also that $\{\{a, b\}, \{c, d\}\}$ is a G_e-invariant partition of N and that G_v is isomorphic to the subgroup of G_e that fixes $\{a, b\}$ and $\{c, d\}$ set-wise. Hence, G_e^N is permutation isomorphic to one of the three imprimitive transitive groups of degree four: D_4, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$.

If $G_e^N \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$, then it follows from the connectivity of Λ that $G_e \cong \mathbb{Z}_4$ or $G_e \cong \mathbb{Z}_2^2$, respectively. Suppose now that $G_e^N \cong D_4$. Then, $(G_e, G_x, G_{\{x,x^{-1}\}})$, where x is the dart starting at e and ending in a (in Λ), is a dihedral amalgam of type $(4, 2)$ (see [15]). If follows from the main theorem of [15] that the subgroup H of G_e that fixes both $\{a, b\}$ and $\{c, d\}$ set-wise satisfies the following:

$$\begin{aligned} H &= \langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle, \\ a_i^2 &= 1, \qquad 0 \le i \le n-1, \\ [a_i, a_j] &= r_{i,j} \qquad 0 \le i, j \le n-1, \end{aligned}$$

where each $r_{i,j}$ is in the center $\mathrm{Z}(H)$ of H and has order at most 2. In particular $N/\mathrm{Z}(H)$ is an elementary abelian 2-group implying that $g^2 \in \mathrm{Z}(H)$ for every $g \in H$. However since N is generated by involutions, the exponent of $\mathrm{Z}(H)$ is at most 2, implying that the exponent of N is at most 4. Then, since $G_v \cong H$, we see that G_v has exponent at most 4.

We have shown that in all possible cases, the order of an element $g \in G_v$ has order at most 6. □

Corollary 7.1.3. *Let Γ be a cubic vertex-transitive graph and let $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group with generator g. If u^G is the largest orbit of G on $\mathrm{V}(\Gamma)$, then $|v^G| \geq \frac{1}{6}|u^G|$ for all $v \in \mathrm{V}(\Gamma)$.*

Proof. Let u^G be the largest orbit of $G = \langle \varphi \rangle$ and let v^G be any other orbit. Suppose $|u^G| = p$ and $|v^G| = q$ for some $p, q \in \mathbb{Z}$. Set $u = u_0$ and define $u_{i+1} = u_i^g$ for all $i \in \mathbb{Z}_p$. Similarly, let $v = v_0$ and define $v_{i+1} = v_i^g$ for all $i \in \mathbb{Z}_q$. Then $g^q \in \mathrm{Aut}(\Gamma)_v$ and by Proposition 7.1.2 the order of g^q is r for some $r \leq 6$. Hence $g^{qr} = 1$. Since $u = u^{g^p} = u^{g^{qr}}$ we see that $p \leq qr \leq 6q$. Therefore $\frac{1}{6}|u^g| \leq |v^g|$. □

Theorem 7.1.4. *Let k be a positive integer, let Γ be a cubic vertex-transitive graph of order n, and let $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group of order at least n/k. Then the number of G-orbits on $\mathrm{V}(\Gamma)$ is strictly less than $6k - 5$.*

Proof. By Corollary 7.1.3, every G-orbit on vertices has size at least $\frac{n}{6k}$. Then, the total number of G-orbits is bounded above by $6k$, with equality holding only if every orbit has size exactly $\frac{n}{6k}$. However, at least one orbit has size $\frac{n}{k}$. The result follows. □

We would like to point out that the bound given in Theorem 7.1.4 can be improved. The split Praeger-Xu graphs (see [51] for the definition), which are cubic and vertex-transitive of order $4m$, admit a cyclic group of automorphisms of order 2 with a regular orbit of size $2 = \frac{4m}{2m}$ that partitions the vertex-set into $2(2m) - 2$ distinct orbits. Clearly, a circulant graph admits an automorphism

We suspect this is the most extreme case and pose the following conjecture.

Conjecture 7.1.5. Let k be a positive integer, let Γ be a cubic vertex-transitive graph of order n, and let $G \leq \mathrm{Aut}(\Gamma)$ be a cyclic group of order at least n/k. Then the number of G-orbits on $\mathrm{V}(\Gamma)$ is smaller or equal than $2k - 2$.

7.2 Cubic vertex-transitive graphs with long regular orbits

Let $\mathcal{G}$ be the set of cubic vertex-transitive graphs of order n, admitting a cyclic group $G \leq \mathrm{Aut}(\Gamma)$ with an orbit on vertices of size $\frac{n}{3}$ or greater. Let $\mathcal{Q}$ be the set of labelled quotients $(\Gamma/G, \lambda_G)$ where $\Gamma \in \mathcal{G}$ and $G \leq \mathrm{Aut}(\Gamma)$ is cyclic with an orbit of size $\frac{|\mathrm{V}(\Gamma)|}{3}$ or greater. Observe that if $(\Gamma/G, \lambda) \in \mathcal{Q}$, then (since Γ is cubic) the graph Γ/G is connected and the labelling λ satisfies the conditions stated in Proposition 5.3.3; that is, $\lambda(x) = 1$ for every dart x underlying a loop, a semi-edge or a link that is parallel to another link, and $\mathrm{val}_\lambda = 3$ for all darts $x \in \mathrm{D}(\Delta)$. Moreover, since Γ is vertex-transitive and G has an orbit containing at least one third of the total number of vertices of Γ, then further restrictions are set on the labelling λ. In Theorem 7.2.1 we state necessary conditions for a labelled graph (Δ, λ) to admit a vertex-transitive *ccv*-cover Γ having a group of automorphisms with an orbit on vertices of size at least $|\mathrm{V}(\Gamma)|/3$. Then Theorem 7.2.1, allow us to determine a set $\mathcal{Q}^*$, containing $\mathcal{Q}$, consisting of 363 labelled-graph such that every cubic vertex-transitive graph Γ admitting a group of automorphisms with an orbit on vertices of size at least $|\mathrm{V}(\Gamma)|/3$, is the *ccv*-cover of an element of $\mathcal{Q}^*$. The algorithm used to construct $\mathcal{Q}^*$ is explained in Subsection 7.2.1. As it transpires, the immense majority of the elements of $\mathcal{Q}^*$ contain an artefact, and so we can restrict our analysis to only a handful of labelled graphs.

Let Γ be a cubic graph admitting a cyclic subgroup of automorphisms G. Then Γ is isomorphic to the cover of some *ccv*-graph $(\Delta, \lambda, \iota, \zeta)$ with a voltage group isomorphic to G. We can slightly abuse language and identify Γ with $\mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$. Suppose Γ is vertex-transitive. Then, by Proposition 7.1.3, for any two vertices $\bar{u}$ and $\bar{v}$ of Γ, we have $\frac{1}{6}|\bar{u}^G| \leq |\bar{v}^G| \leq 6|\bar{u}^G|$. Since the orbits of G on $\mathrm{V}(\Gamma)$ are identified with the fibres of $\mathrm{V}(\Delta)$ we see that

$$\frac{1}{6}\iota(u) \leq \iota(v) \leq 6 \cdot \iota(u) \tag{7.2.1}$$

for any two $u, v \in \mathrm{V}(\Delta)$ (recall that $\iota(u) = |\mathrm{fib}(u)|$ for all $u \in \mathrm{V}(\Delta)$). If in addition we suppose G has an orbit containing at least one third of the total number of vertices of Γ, then there must exists a vertex $\hat{u} \in \mathrm{V}(\Delta)$ such that

$$3 \cdot \iota(\hat{u}) \geq \sum_{v \in \mathrm{V}(\Delta)} \iota(v) = |\mathrm{V}(\Gamma)|. \tag{7.2.2}$$

From the discussion in the preceding paragraphs (and Proposition 5.3.3) with obtain the following Theorem.

Theorem 7.2.1. *Let (Δ, λ) be a labelled graph, let Γ be a ccv-cover of (Δ, λ) and let T be a spanning tree of Δ. If Γ is vertex-transitive and admits an orbit of size at least $\frac{|\mathrm{V}(\Gamma)|}{3}$ then the following holds:*

1. *(Δ, λ) is extendable;*
2. *$val_\lambda(v) = 3$ for all vertices $v \in \mathrm{V}(\Delta)$;*
3. *$\lambda(x) = \lambda(x^{-1})$ implies $\lambda(x) = 1$;*
4. *$\lambda(x) = \lambda(y) = 1$ for any two parallel darts x and y;*
5. *$\lambda(x) = 1$ for every dart x underlying a semi-edge,*

moreover, there exists a vertex $\hat{u} \in \mathrm{V}(\Delta)$ such that:

(6) $\frac{1}{6} \leq \rho_\lambda(W_v)$,

(7) $\sum_{v \in \mathrm{V}(\Delta) \setminus \{\hat{u}\}} \rho_\lambda(W_v) \leq 2$;

for every $v \in \mathrm{V}(\Delta) \setminus \{\hat{u}\}$, *where* W_v *denotes the unique* $\hat{u}v$*-path in* T.

Proof. That items (1)–(5) hold follows at once from Proposition 5.3.3. Let $u \in \mathrm{V}(\Gamma)$ be such that $|u^G|$ has maximum cardinality amongst all the vertex orbits of G. Then $|u^G| \geq \frac{n}{3}$. Let $\hat{u} = \pi(u)$ and let $v \in \mathrm{V}(\Delta) \setminus \{\hat{u}\}$ (recall that the natural projection π maps every vertex $v_i \in \mathrm{fib}(v)$ to v). Then by (7.2.1) we have $\frac{1}{6}\iota(\hat{u}) \leq \iota(v)$, but by (5.1.7)) we can replace $\iota(v)$ by $\rho_\lambda(W_v)\iota(\hat{u})$. Thus (1) holds. To see that (2) holds, subtract $\iota(\hat{u})$ on both sides of inequality (7.2.2) and replace $\iota(v)$ by $\rho_\lambda(W_v)\iota(\hat{u})$. □

7.2.1 Construction of $\mathcal{Q}^*$

Let $\bar{\mathcal{Q}}$ be the set of all labelled graphs that can be extended to a *ccv*-graph. If we denote by $\mathcal{Q}^*$ the set of all labelled graphs $(\Delta, \lambda) \in \bar{\mathcal{Q}}$ agreeing with conditions (1)–(7) of Theorem 7.2.1, then Γ must be the cover of some element of $\mathcal{Q}^*$. Note that if $(\Delta, \lambda) \in \mathcal{Q}^*$, then by Theorem 7.1.4, Δ has at most 13 vertices, and by Theorem 7.2.1, $\lambda(x) \leq 3$ for all $x \in \mathrm{D}(\Delta)$. It follows that $\mathcal{Q}^*$ is a finite set. Then we can make a computer assisted enumeration of the elements of $\mathcal{Q}^*$. First construct the set $\mathcal{T}$ of all labelled trees (T, λ) rooted at a vertex $\hat{u}$ (which plays the role of the vertex $\hat{u}$ of Theorem 7.2.1) that satisfy conditions (3)–(7) of Theorem 7.2.1 and such that $\mathrm{val}_\lambda(v) \leq 3$ for all $v \in \mathrm{V}(T)$. There are 229 such trees. Recall that, by Proposition 5.3.7, every *ccv*-graph $(\Delta, \lambda_\Delta, \iota, \zeta)$ admits a spanning tree T' containing all $[i, j]$-edges with $i \neq j$. Moreover, the labelling $\lambda_{T'}$ of T' induced by λ_Δ must agree with conditions (3)–(7) of Theorem 7.2.1. Hence $(T', \lambda_{T'}) \in \mathcal{T}$ and thus every labelled graph in $\mathcal{Q}^*$ can be obtained from an appropriate tree $(T, \lambda) \in \mathcal{T}$ by repeatedly adding $[1, 1]$-edges, loops or semi-edges to (T, λ) until the graph obtained satisfies condition (2) of Theorem 7.2.1. To be precise, we can obtain $\mathcal{Q}^*$ by completing every tree $(T, \lambda) \in \mathcal{T}$ to a graph (in all possible ways) that satisfies condition (2) of Theorem 7.2.1, by any combination of the following operations:

1. Drawing an edge of type $[1, 1]$ between two vertices v and w provided that $\mathrm{val}_\lambda(v), \mathrm{val}_\lambda(w) \leq 2$ and $\rho_\lambda(W_v) = \rho_\lambda(W_w)$, where W_v and W_w denote the unique walk joining $\hat{u}$ with v and w, respectively, in T.

2. Attaching a semi-edge (whose only dart has label 1) to a vertex v provided that $\mathrm{val}_\lambda(v) = 2$.

3. Attaching a loop (whose darts have label 1) to a vertex v provided that $\mathrm{val}_\lambda(v) = 1$.

We claim that the set of labelled graphs thus obtained is precisely the set $\mathcal{Q}^*$. It is straightforward to see that any graph obtained from an element of $\mathcal{T}$ by means of the above operations satisfies conditions (2)–(7) of Theorem 7.2.1. It remains to see that such a graph is extendable.

Suppose (Δ, λ) is a graph obtained by completing a tree $(T, \lambda') \in \mathcal{T}$. By Lemma 5.1.5 we need to verify that $\rho_\lambda(C) = 1$ for every cycle C in (Δ, λ). It suffices to show this for fundamental cycles (that is, cycles containing exactly one cotree dart). Suppose C is a cycle of (Δ, λ) containing exactly one cotree (relative to the spanning tree T) dart x and let $X = (x)$ be the walk consisting only of the dart x. Since x is a cotree dart, then $\lambda(x) = \lambda(x^{-1}) = 1$ and so $\rho_\lambda(X) = 1$. Now, let $\operatorname{beg} x = v$ and $\operatorname{end} x = w$ and let u be any vertex of Δ. Let W_v and W_w denote, respectively, the unique uv-walk and the unique uw-walk in T. Then $C = W_v X (W_w)^{-1}$. However, by the equalities in (5.1.6) we have

$$\rho_\lambda(W_v X W_w^{-1}) = \rho_\lambda(W_v)\rho_\lambda(X)\rho_\lambda(W_w)^{-1} = \rho_\lambda(W_v)\rho_\lambda(W_w)^{-1} = 1,$$

where the third equality follow from the fact that $\rho_\lambda(W_v) = \rho_\lambda(W_w)$.

The task of computing $\mathcal{Q}^*$ is simple enough to be brute-forced with a computer programme written in SAGE [70]. The complete list of graphs in $\mathcal{Q}^*$ is presented in the Appendix. There are 363 non-isomorphic labelled graphs in $\mathcal{Q}^*$ (two labelled graphs are said to be isomorphic if there is a label-preserving graph isomorphism). Recall that of the goal of this section is to prove Theorem 7.2.22. In particular, since cubic vertex-transitive k-multicirculants with $k \in \{1, 2, 3\}$ were characterized in Chapter 3, our immediate goal is to determine which elements of $\mathcal{Q}^*$ admit a vertex-transitive *ccv*-cover that is not a k-multicirculant with $k \in \{1, 2, 3\}$. In light of Theorem 5.4.14, we can exclude the elements of $\mathcal{Q}^*$ that either contain an artefact A_i with $i \in \{1, 2, 3\}$ along with a $[1, 3]$-edge, an artefact A_i with $i \in \{4, \dots, 9\}$ or agree with Lemma 5.4.13. Moreover, in light of Theorem 6.1.1, we need not concern ourselves with the elements of $\mathcal{Q}^*$ having less than 4 vertices. This leaves us with the 11 labelled graphs Δ_i, $i \in \{1, \dots, 11\}$, shown in Figure 7.2.1 (along with a voltage assignment, where voltages are denoted by r or s), to be analysed. In the remainder of this section we will show that for $i \in \{1, \dots, 10\}$, Δ_i can only admit a vertex-transitive *ccv*-cover if it is a k-multicirculant. The remaining graph, Δ_{11}, will be shown in Section 7.2.3 to admit infinitely many vertex-transitive *ccv*-covers that are not k-multicirculants with $k \in \{1, 2, 3\}$.

7.2.2 The graphs Δ_i

We present in Figure 7.2.1 the 11 dart labelled graphs Δ_i, $i \in \{26, \dots, 36\}$. Consider a labelled graph Δ_i and suppose $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_i. By Lemma 5.3.8, we may assume that ζ is a simplified voltage assignment and agrees with Figure 7.2.1. Recall that for a symbol $\alpha \in \{r, s\}$, an edge with an arrow oriented from, say, u to v, with the letter α next to it, indicates that the dart x underlying this edge and beginning at u has voltage α, for some $0 \le \alpha < \iota(v)$. A loop with the letter α next to it, indicates that one of the underlying darts, say x, has voltage α for some integer $0 < \alpha < \iota(\operatorname{beg} x)$. For a semi-edge x, $\zeta(x) = \iota(\operatorname{beg} x)/2$. All other darts belong to a spanning $\mathcal{T}$ and have trivial voltage. The vertices of each Δ_i are named in Figure 7.2.1, but we refrain from naming the darts in the figure so as not to overburden it.

Let (Δ, λ) be a labelled graph, let Γ be a *ccv*-cover of (Δ, λ) and let $\pi\colon \Gamma \to \Delta$ be the corresponding projection. If W is a uv-walk in Δ, then a *lift* of W based at a vertex $u_i \in \operatorname{fib}(u)$ is a walk $\overline{W} = (x_1, x_2, \dots, x_n)$ beginning at u_i such that

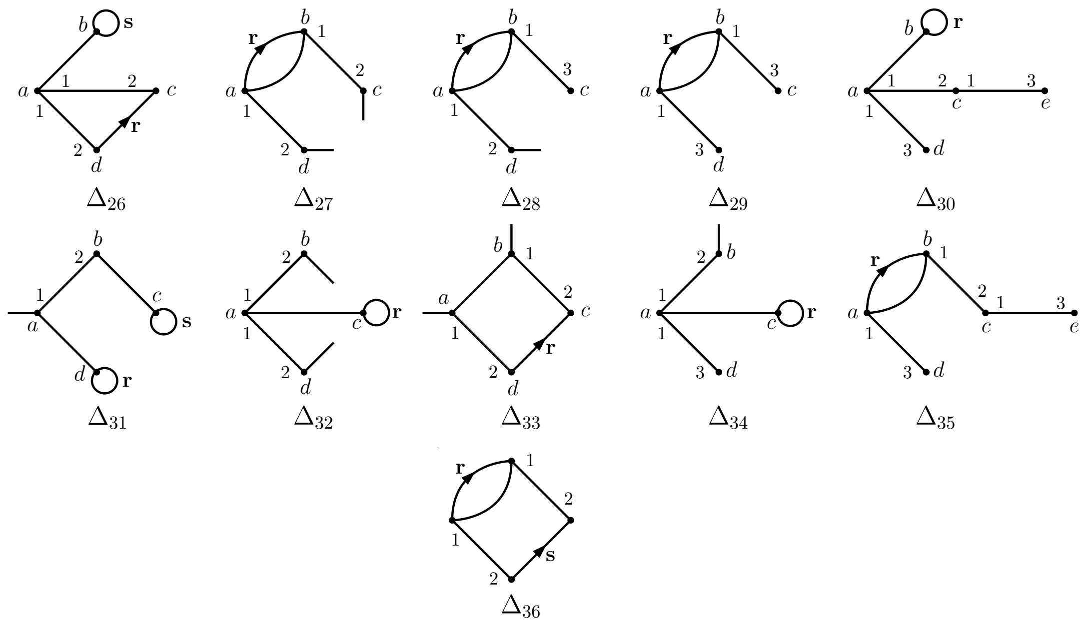


Figure 7.2.1: Elements of $\mathcal{Q}^*$ with more than 3 vertices that do not satisfy the hypothesis of Theorem 5.4.14.

the projection $\pi(\overline{W}) := (\pi(x_1), \pi(x_2), \dots, \pi(x_n))$ is equal to W. We denote by $\mathcal{L}(W)$ the set of all lifts of W based at u_0.

We say W is λ-reduced if $x_{i+1} \neq x_i^{-1}$ whenever $\lambda(x_i^{-1}) = 1$, and $x_n \neq x_1^{-1}$ whenever $\lambda(x_1) = 1$. Clearly, every reduced walk is λ-reduced.

Let $(\Delta, \lambda, \iota, \zeta)$ be a *ccv*-graph and set $n = \operatorname{lcm}\{\lambda(x)\iota(\operatorname{beg} x) \mid x \in \mathrm{D}(\Delta)\}$. Let $W = (x_1, x_2, \dots, x_k)$ be a walk in Δ and let $d = \gcd\{\iota(\operatorname{beg} x_i) \mid x_i \in W\}$. We define the *endset* of W as

$$\operatorname{end}(W) = \sum_{i=0}^{k} \zeta(x_i) + \langle d \rangle,$$

where $\langle d \rangle$ denotes the subgroup of $\mathbb{Z}$ generated by d, and where the addition is computed modulo $\iota(\operatorname{end} x_k)$. The following lemma is a special case of Lemma 4.5.2.

Lemma 7.2.2. *Let $(\Delta, \lambda, \iota, \zeta)$ be a ccv-graph and $\Gamma = \operatorname{Cov}(\Delta, \lambda, \iota, \zeta)$. Let W be a uv-walk for some $u, v \in \mathrm{V}(\Delta)$. If $\overline{W} \in \mathcal{L}(W)$, then the final vertex of $\overline{W}$ is v_j for some $j \in \operatorname{end}(W)$. Conversely, for every $j \in \operatorname{end}(W)$ there exists a lift of W beginning at u_0 and ending at v_j.*

Lemma 7.2.3. *Let $(\Delta, \lambda, \iota, \zeta)$ be a ccv-graph and $\Gamma = \operatorname{Cov}(\Delta, \lambda, \iota, \zeta)$. If C is a cycle in Γ, then $\pi(C)$ is a λ-reduced closed walk in Δ and $0 \in \operatorname{end}(\pi(C))$.*

Proof. Let C be a cycle in Γ. Then C is a $u_a u_a$-walk for some vertex $u_a \in \operatorname{fib}(u)$ and some $u \in \mathrm{V}(\Delta)$. We must show that if x and x^{-1} are two consecutive darts of $\pi(C)$, or if x and x^{-1} are the first and last darts traced by $\pi(C)$, then $\lambda(x) \neq 1$. Suppose that for a dart x, $\pi(C)$ traces x^{-1} immediately after x. Then there exist two darts $x_i \in \pi^{-1}(x) = \mathrm{fib}(x)$ and $(x^{-1})_j \in \pi^{-1}(x^{-1}) = \mathrm{fib}(x^{-1})$ such that C traces x_i and $(x^{-1})_j$ consecutively. Since C is reduced by definition, we have $(x^{-1})_j \neq (x_i)^{-1}$. However both $(x^{-1})_j$ and $(x_i)^{-1}$ belong to $\mathrm{fib}(x^{-1})$ and clearly $\mathrm{beg}(x^{-1})_j = \mathrm{beg}(x_i)^{-1}$. That is, there are two distinct darts in $\mathrm{fib}(x^{-1})$ beginning at the same vertex. This implies that $\lambda(x^{-1}) \geq 2$.

If x and x^{-1} are the first and the last darts traced by $\pi(C)$, then an analogous argument shows that $\lambda(x) \neq 1$. This implies that $\pi(C)$ is λ-reduced.

Now, recall that Γ admits an automorphism Φ that maps every dart x_i to x_{i+1}. Then Φ^{-a} maps the vertex u_a to u_0, and so $\Phi^{-a}(C)$ is a reduced closed walk beginning and ending at u_0. Moreover, $\Phi^{-a}(C) \in \mathcal{L}(\pi(C))$. Since the final vertex of $\Phi^{-a}(C)$ is u_0, it follows from Lemma 7.2.2 that $0 \in \mathrm{end}(\pi(W))$. $\square$

Throughout the rest of this section we will assume $(\Delta, \lambda) := \Delta_i$ for some $i \in \{1, \ldots, 11\}$, that Γ is the cover of a *ccv*-extension $(\Delta, \lambda, \iota, \zeta)$ of Δ_i, and that $\pi\colon \Gamma \to \Delta$ is the associated projection. Note that Γ is completely determined by the values of the voltages r and s, and $m := \gcd\{\iota(u) \mid u \in \mathrm{V}(\Delta)\}$. Indeed, Γ is uniquely determined by the quadruple $(\Delta, \lambda, \iota, \zeta)$, and Δ and λ are given. Recall that the index function ι is determined by its value on a single vertex along with the labelling λ, which is given. Let $u \in \mathrm{V}(\Delta)$ be any vertex and for each $v \in \mathrm{V}(\Delta) \setminus \{u\}$ chose (arbitrarily) a uv-walk W_v. By equality (5.1.7), we have $\iota(v) = \rho_\lambda(W_v)\iota(u)$ for all $v \in \mathrm{V}(\Delta) \setminus \{u\}$. Let c be the smallest positive integer such that $c \cdot \rho_\lambda(W_v)$ is an integer for all $v \in \mathrm{V}(\Delta) \setminus \{u\}$. Then $\iota(u) = c \cdot m$. Note that c depends only on λ and our choice of u. Then, ι is completely determined by λ and m. Finally, since we can assume ζ to be simplified, we know that every dart x underlying a semi-edge has voltage $\iota(\mathrm{beg}\, x)/2$, and any other dart not labelled r or s has voltage 0. The values of r and s, along with the function ι thus completely determine ζ.

We are now ready to analyse the labelled graphs Δ_i. The technique employed in the following pages relies mainly in finding a closed walk W of length n in Δ_i such that $\mathcal{L}(W)$ contains a cycle of length n, regardless of the specific values of the voltages r and s. Such a walk can often be found by finding an artefact A_j in Δ_i. If we suppose that Γ is vertex-transitive, then for every vertex v_i of Γ, at least once dart incident to v_i must lie on an n-cycle. Then by Lemma 7.2.3, this will imply that for every vertex $v \in \mathrm{V}(\Delta)$ a specific dart incident to v lies on a closed walk W' of length n such that $0 \in \mathrm{end}(W')$. Since every element of $\mathrm{end}(W')$ can be seen as a linear combination of m, r and s, $0 \in \mathrm{end}(W')$ implies a relation between m, r and s, which along with the fact that $\gcd(m, r, s) = 1$ (see item (4) of Lemma 5.3.2), is often enough to completely determine their values (up to a few options). This, in turn, determines the graph Γ.

Lemma 7.2.4. *If Γ is a vertex-transitive ccv-cover of Δ_{26}, then Γ is a bicirculant graph of order* 12.

Proof. Let $(\Delta, \lambda, \iota, \zeta)$ be a *ccv*-extension of Δ_{26} such that $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$. Let $\iota(c) = m$ and note that $\iota(d) = m$ as c and d are connected through a $[1,1]$-edge. Similarly $\iota(a) = \iota(b) = 2m$. Let $W = ((ac)_0, (ca)_0, (ad)_0, (da)_0)$ and note that every reduced walk in $\mathcal{L}(W)$ is a cycle of length 4 (see Lemma

5.4.7). Then, every dart in the fibre of $(ac)_0$ or $(ad)_0$ lies on a 4-cycle. Since Γ is vertex-transitive, every dart in the fibre of $(bb)_s$ (or of $(bb)_{-s}$) must lie on a 4-cycle. Suppose C is a 4-cycle (in Γ) through b_0b_s. Then $\pi(C)$ is a λ-reduced walk of length 4 through $(bb)_s$. Clearly, $\pi(C) = ((bb)_s, (bb)_s, (bb)_s, (bb)_s)$ and $\mathrm{end}(\pi(C)) = \{4s\}$. By Lemma 7.2.3, $0 \in \mathrm{end}(\pi(C))$ and so we see that $4s \equiv 0 \pmod{2m}$. Since $0 < s < m$, we have $2s = m$. This shows that $(b_0, b_s, b_{2s}, a_{2s}, c_{2s}, a_{2s+m})$ is a 6-cycle in Γ since $a_{2s+m} = a_0$ and $a_0 \sim b_0$. Then the 6-signature of Γ is $(\epsilon_1, \epsilon_2, \epsilon_3)$ where $\epsilon_i > 0$. That is, every dart of Γ lies in at least one 6-cycle. In particular, there is a 6-cycle C' through c_0d_{-r}. Then $\pi(C')$ is a λ-reduced walk of length 6 through $(cd)_{-r}$. By inspecting Figure 7.2.1, it is not difficult to see that $\pi(C')$ must be one of the following:

$$\begin{aligned}
W_1 &= ((cd)_{-r}, (da)_0, (ac)_0, (cd)_{-r}, (da)_0, (ac)_0),\\
W_2 &= ((cd)_{-r}, (da)_0, (ab)_0, (bb)_s, (ba)_0, (ac)_0),\\
W_3 &= ((cd)_{-r}, (da)_0, (ab)_0, (bb)_{-s}, (ba)_0, (ac)_0).
\end{aligned}$$

Now, $\mathrm{end}(W_1) = \{-2r\}$, $\mathrm{end}(W_2) = \{s - r\}$ and $\mathrm{end}(W_3) = \{-s - r\}$. Since $0 \in \mathrm{end}(W_i)$ for some $i \in \{1, 2, 3\}$, we see that one of the following holds modulo m,

$$\begin{aligned}
-2r &\equiv 0,\\
s - r &\equiv 0,\\
-s - r &\equiv 0.
\end{aligned}$$

Since $0 \le r < m$ and $2s = m$, we see that either $r = 0$ or $s = r$. However, $\gcd(m, r, s) = 1$ by Lemma 5.3.2. Therefore, $r = s = 1$ and $m = 2$. That is, the functions ζ and ι are completely determined and so is Γ. It can be verified that Γ is a bicirculant isomorphic to the Franklin graph (see [4] for definition and properties). □

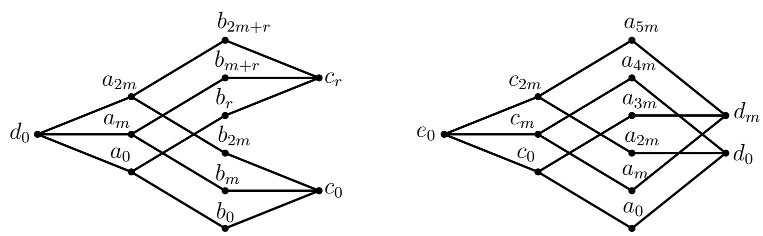


Figure 7.2.2: For each of the two cases in the proof of Lemma 7.2.6, a subgraph of Γ containing 4 distinct 6-cycles through the dart d_0a_0 (left) and through the dart e_0c_0 (right), respectively

Lemma 7.2.5. *If Γ is a ccv-cover of Δ_i, $i \in \{27, 28\}$, then Γ is not vertex-transitive.*

Proof. Let $(\Delta, \lambda, \iota, \zeta)$ be a *ccv*-extension of Δ_{27} and suppose $\Gamma := \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive. Let $\iota(d) = 2m$ and for $j \in \{0, 1\}$ let $W_j =$

$((da)_0, (ab)_{jr}, (bc)_0, (cb)_0, (ba)_{-jr}, (ad)_0)$. Observe that every edge incident to a_0 lies on a 6-cycle belonging to $\mathcal{L}(W_0) \cup \mathcal{L}(W_1)$. Then d_0d_m must lie on a 6-cycle C of Γ and $\pi(C)$ is a λ-reduced closed walk of length 6. It is straightforward to see that no λ-reduced closed walk of length 6 in Δ_{27} traces the dart $(dd)_m$, a contradiction. Therefore Γ is not vertex-transitive.

Now suppose $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{28} and $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive. Let $\iota(d) = 2m$ and $W = ((da)_0, (ab)_r, (ba)_0, (ad)_0, (da)_0, (ab)_0, (ba)_{-r}, (ad)_0)$. Observe that every edge incident to a_0 lies on an 8-cycle of $\mathcal{L}(W)$, which implies the existence of an 8-cycle C through d_0d_m, since Γ is vertex-transitive. Then $\pi(C)$ is a λ-reduced closed walk of length 8 through $(dd)_m$. Once more, one can verify that no such walk exists in Δ_{28}. We conclude Γ is not vertex-transitive. □

Lemma 7.2.6. *If Γ is a vertex-transitive ccv-cover of Δ_j, with $j \in \{29, 30\}$, then $\Gamma \in X$ (where X is the ste of exceptional graphs defined in 5.4.1).*

Proof. Suppose $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_4 and let $\iota(a) = m$. Since Δ_{29} has a $[1, 3]$-edge, it must be arc-transitive by Lemma 5.4.6. Now, if $r \equiv 0 \pmod m$, then (d_0, a_0, c_m, a_m) is a 4-cycle of Γ and by Lemma 5.4.5, $\Gamma \in X$. Suppose that $r \not\equiv 0 \pmod m$. For $i \in \{0, 1\}$, let

$$W_i = ((da)_0, (ab)_{ir}, (bc)_0, (cb)_0, (ba)_{-ir}, (ad)_0).$$

Observe that every dart in the fiber of $(da)_0$ lies on 4 distinct 6-cycle in $\mathcal{L}(W_0) \cup \mathcal{L}(W_1)$ (see Figure 7.2.2, left). Then by Lemma 5.4.5 we have $\Gamma \in X$.

Now, suppose $\Gamma := \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{30} and consider the walk

$$W = ((ec)_0, (ca)_0, (ad)_0, (da)_0, (ac)_0, (ce)_0).$$

Observe that every dart in e_0c_0 lies on 4 distinct 6-cycle in $\mathcal{L}(W)$ (see Figure 7.2.2, right) and by Lemma 5.4.5, $\Gamma \in X$. □

Lemma 7.2.7. *If Γ is a vertex-transitive ccv-cover of Δ_{31}, then Γ is a tricirculant of order* 18.

Proof. Suppose $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{31}. Then, for some $m \in \mathbb{Z}$ we have $\iota(c) = \iota(d) = m$ and $\iota(a) = \iota(b) = 2m$. Let $W = ((aa)_m, (ad)_0, (da)_0)$ and observe that every reduced walk in $\mathcal{L}(W)$ is a 3-cycle (see Lemma 5.4.7). Then every dart in the fibre of $(aa)_m$ or $(ad)_0$ lies on a 3-cycle. Since Γ is vertex-transitive, it must be 3-cycle-regular. In particular, every dart in the fibre of $(bb)_r$ lies on a 3-cycle and so, in Γ, there is a 3-cycle C through the edge b_0b_r. Then $\pi(C)$ is a λ-reduced closed walk of length 3 that traces $(bb)_r$. It is straightforward to see that necessarily $\pi(C) = ((bb)_r, (bb)_r, (bb)_r)$. Moreover, by Lemma 7.2.3, $0 \in \mathrm{end}(\pi(C)) = \{3r\}$ and so

$$3r \equiv 0 \pmod{2m}. \tag{7.2.3}$$

Now, every dart in the fibre of $(cc)_s$ must also lie on a 3-cycle, and by an analogous argument,

$$3s \equiv 0 \pmod{m}. \tag{7.2.4}$$

Since Γ is connected, by Lemma 5.3.2 we see that $\gcd(m, r, s) = 1$ and by (7.2.3) and (7.2.4), we see that the only possibility is that $m = 3$, $r = 2$ and $s = 1$. One can readily verify that Γ is isomorphic to the truncation of $K_{3,3}$, and thus is a vertex-transitive tricirculant of order 18. □

Lemma 7.2.8. *If Γ is a vertex-transitive ccv-cover of Δ_{32}, then Γ is a bicirculant of order* 12.

Proof. Suppose $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ is vertex-transitive where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{32}. Then $\iota(c) = \iota(a) = 2m$ and $\iota(b) = \iota(d) = m$ for some $m \in \mathbb{Z}$. Let $W = ((ad)_0, (da)_0, (ab)_0, (ba)_0)$ and observe that every reduced walk in $\mathcal{L}(W)$ is a 4-cycle. In particular, a_0b_0 and a_0d_0 lie on a 4-cycle, and thus, the vertex-transitivity of Γ implies that there is a 4-cycle C through c_0c_r. It follows that $\pi(C)$ is a λ-reduced closed walk of length 4 through the dart $(cc)_r$. It is plain to see that $\pi(C) = ((cc)_r, (cc)_r, (cc)_r, (cc)_r)$ and thus $\mathrm{end}(\pi(C)) = \{4r\}$. Then $4r \equiv 0 \pmod{2m}$ and $\gcd(m, r) = 1$. Since $0 < r < m$, we see that $r = 1$ and $m = 2$. Then Γ is a cubic bicirculant of order 12 and is in fact isomorphic to the Franklin graph. □

Lemma 7.2.9. *If Γ is a vertex-transitive ccv-cover of Δ_{33}, then Γ is the triangular prism $GP(3,1)$.*

Proof. Let $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ be vertex-transitive where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{33}. Let $\iota(c) = m$ so that $\iota(a) = \iota(b) = 2m$. Consider the walk $W = ((aa)_m, (ab)_0, (bb)_m, (ba)_0)$ and see that both a_0a_m and a_0b_0 lie on a 4-cycle in $\mathcal{L}(W)$. Since Γ is vertex-transitive, then one of d_0a_0 or d_0a_m must lie on a 4-cycle C. Then $\pi(C)$ is a λ-reduced closed walk of length 4 through the dart $(da)_0$. Clearly $\pi(C) = ((da)_0, (ab)_0, (bc)_0, (cd)_{-r})$ and $\mathrm{end}(\pi(C)) = \{-r\}$. Then, by Lemma 7.2.3 we have $r \equiv 0 \pmod{m}$. Since $\gcd(m, r) = 1$ and $r < m$, we see that $m = 1$ and $r = 0$. Then Γ can be seen to be isomorphic to the triangular prism. □

Lemma 7.2.10. *If Γ is a ccv-cover of Δ_{34}, then Γ is not vertex-transitive.*

Proof. Let $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{34}. Then, for some $m \in \mathbb{Z}$ we have $\iota(a) = \iota(c) = 12$, $\iota(b) = 6m$ and $\iota(d) = 4m$. Observe that the order of Γ is $34m$. Suppose Γ is vertex-transitive and consider the walk

$$W = ((da)_0, (ab)_0, (ba)_0, (ad)_0, (da)_0, (ab)_0, (ba)_0, (ad)_0).$$

Observe that every dart beginning at d_0 lies on an 8-cycle belonging to $\mathcal{L}(W)$. This implies the existence of a λ-reduced walk W' of length 8 through the dart $(bb)_{3m}$. Observe that then W' must be one (or the inverse) of the following 6 walks, where $i \in \{-1, 1\}$:

$$W_{1,i} = ((bb)_{3m}, (ba)_0, (ac)_0, (cc)_{ir}, (cc)_{ir}, (cc)_{ir}, (ca)_0, (ab)_0),$$

$$W_{2,i} = ((bb)_{3m}, (ba)_0, (ac)_0, (cc)_{ir}, (ca)_0, (ad)_0, (da)_0, (ab)_0),$$
$$W_{3,i} = ((bb)_{3m}, (ba)_0, (ad)_0, (da)_0, (ac)_0, (cc)_{ir}, (ca)_0, (ab)_0).$$

Let $\mathcal{W}$ be the set containing the six walks $W_{j,i}$, $j \in \{1,2,3\}$ and $i \in \{-1,1\}$, along with their inverses. Denote by $\mathrm{end}(\mathcal{W})$ the union of endsets over the elements of $\mathcal{W}$. A tedious but straightforward computation shows that

$$\mathrm{end}(\mathcal{W}) = \{\pm 2m, \pm(m+r), \pm(m-r), \pm(3m+r), \pm(3m+3r)\}.$$

Then $z \equiv 0 \pmod{6m}$ for some $z \in \mathrm{end}(\mathcal{W})$. This implies that $m = 1$ and $r = km$ for some $k \in \{1,3,5\}$. Then Γ is one of three possible graphs of order 34. One can check that in neither one of the three possible cases is Γ vertex-transitive. $\square$

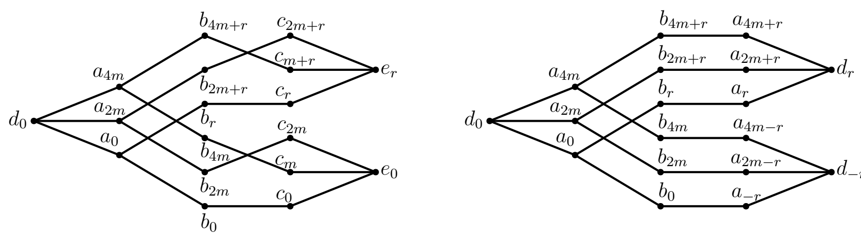


Figure 7.2.3: The 8 distinct 8-cycles through the dart d_0a_0 in the proof of Lemma 7.2.11

Lemma 7.2.11. *If Γ is a ccv-cover of Δ_{35}, then Γ is not vertex-transitive.*

Proof. Let $\Gamma = \mathrm{Cov}(\Delta, \lambda, \iota, \zeta)$ where $(\Delta, \lambda, \iota, \zeta)$ is a *ccv*-extension of Δ_{35}. Suppose Γ is vertex-transitive. For some $m \in \mathbb{Z}$ we have $\iota(e) = m$, $\iota(c) = 3m$, $\iota(a) = \iota(b) = 6m$ and $\iota(d) = 2m$. Observe that the order of Γ is $18m$. As one can verify with the census of cubic vertex-transitive graphs [56], no *ccv*-cover of Δ_{10} is vertex-transitive if $m \in \{1,2\}$. Then assume $m > 2$. Furthermore, $r \not\equiv 0 \pmod{m}$, for otherwise $m = 1$ (since $\gcd(m,r) = 1$). Furthermore, since Δ_{10} has a $[1,3]$-edge, Γ is arc-transitive. Now, for $i, j \in \{0,1\}$ and $k \in \{2,4\}$ consider the walks in Γ:

$$\begin{aligned} W_{i,j} &= (d_0, a_0, b_{ir}, c_{ir}, e_{ir}, c_{(1+j)m+ir}, b_{(4-2j)m+ir}, a_{(4-2j)m}, d_0), \\ W'_{i,k} &= (d_0, a_0, b_{(1-i)r}, a_{(1-2i)r}, d_{(1-2i)r}, a_{km+(1-2i)r}, b_{km+(1-i)r}, a_{km}). \end{aligned}$$

Note that each of these 8 walks is an 8-cycle through a_0d_0 (see Figure 7.2.3). Since Γ is arc-transitive, then there must be 8 distinct 8-cycles through c_0b_0. Now, consider the walks

$$W_1 = ((cb)_0, (ba)_0, (ad)_0, (da)_0, (ab)_0, (bc)_0, (cd)_0, (dc)_0)$$
$$W_2 = ((cb)_0, (ba)_{-r}, (ad)_0, (da)_0, (ab)_r, (bc)_0, (cd)_0, (dc)_0)$$
$$W_3 = ((cd)_0, (ba)_0, (ab)_r, (bc)_0, (cb)_0, (ba)_{-r}, (ab)_0, (bc)_0)$$
$$W_4 = ((cd)_0, (ba)_{-r}, (ab)_0, (bc)_0, (cb)_0, (ba)_0, (ab)_r, (bc)_0)$$

Observe that for all $i \in \{1,2,3,4\}$, all lifts of W_i that trace the dart c_0b_0 are 8-cycles. There are exactly 6 such cycles. It follows that there are an additional two 8-cycles through c_0b_0 that do not project to any of the four walks W_i, $i \in \{1,2,3,4\}$. Each of these two cycles projects to a λ-reduced closed walk of length 8 based at c and visiting the dart $(cb)_0$. Moreover, such projections must be different than W_i with $i \in \{1,2,3,4\}$. Let $\mathcal{W}$ be the set of all λ-reduced closed walks of length 8 based at c and visiting the dart $(cb)_0$, that are distinct from W_i with $i \in \{1,2,3,4\}$. Let $\mathrm{end}(\mathcal{W})$ be the union of the endsets of all the elements of $\mathcal{W}$. A computer assisted calculation shows that $\mathrm{end}(\mathcal{W}) = \{\pm(2m+r), \pm(m+2r), \pm(m+r), \pm(r), \pm(2r), \pm(3r), \pm(2m+2r)\}$. Then $z \equiv 0 \pmod{3m}$ for some $z \in \mathrm{end}\,\mathcal{W}$. This implies that $m = 1$ and $r \in \{1,2,3,4,5\}$ or $m = 2$ and $r \in \{5,7\}$, a contradiction. We conclude that Γ is not vertex-transitive. $\square$

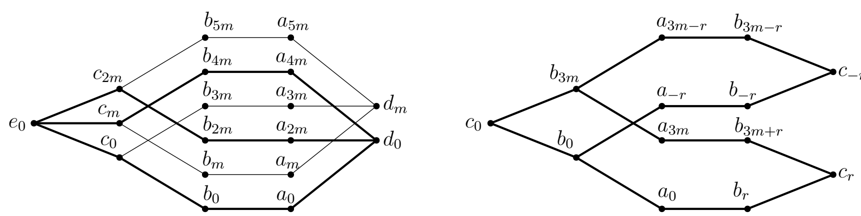


Figure 7.2.4: Two subgraphs of Γ in the proof of Lemma 7.2.11. On the left, a subgraph containing all lifts of W_1 that visit c_0 (8-cycles through c_0b_0 shown in bold edges). On the right, a subgraph containing all lifts of W_3 and W_4 that visit c_0.

Since every exceptional graph of X, along with the Franklin graph and the truncation of $K_{3,3}$ has at most 20 vertices, we have the following Proposition, which sums up the contents of this section.

Proposition 7.2.12. *Let Γ be a cubic vertex-transitive graph admitting a cyclic group $G \leq \mathrm{Aut}(\Gamma)$ with an orbit on vertices of size $\mathrm{V}(\Gamma)/3$ or greater. If Γ has more than 20 vertices, then Γ is either a k-multicirculant for some $k \in \{1,2,3\}$ or is a ccv-cover of Δ_{36}.*

7.2.3 The graph Δ_{36}

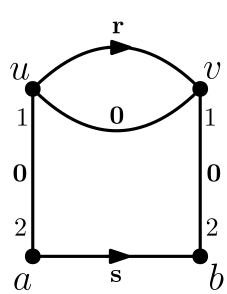


Figure 7.2.5: The voltage assignment giving rise to the graph $\Gamma_{36}(m,r,s)$.

Let m be a positive integer, and let r and s be two distinct elements of $\mathbb{Z}_{2m}$. Let $\Delta_{36}(m,r,s)$ be the the *ccv*-extension of Δ_{36} shown in Figure 7.2.5, where voltages are shown in bold characters next to each edge and $\iota(u) = \iota(v) = 2m$ and $\iota(a) = \iota(b) = m$. Let $\Gamma_{36}(m,r,s)$ be the cover of $\Delta_{36}(m,r,s)$. We may assume that $\gcd(m,r,s) = 1$, $0 < r < 2m$ and $0 \le s < m$ by Lemma 5.3.8. Recall that $\Gamma_{36}(m,r,s)$ admits an automorphism Φ of order $2m$ whose orbits on vertices and darts are precisely the fibres of vertices and darts. Consulting the census of cubic vertex-transitive graphs [56], one can verify that the only vertex-transitive *ccv*-covers of Δ_{36} of order 6 or 18 are $K_{3,3} \cong \Gamma_{36}(1,1,0)$ and the Pappus Graph (isomorphic to $\Gamma_{36}(3,1,2)$), respectively. Furthermore, Δ_{36} does not admit vertex-transitive *ccv*-covers of order 12. We will henceforth assume $m > 3$. The goal of this section is to prove Theorem 7.2.13 below.

Theorem 7.2.13. *For positive integers $m, r, s \in \mathbb{Z}$, the graph $\Gamma_{36}(m,r,s)$ is vertex-transitive if and only if m is odd and $\Gamma_{36}(m,r,s) \cong \Gamma_{36}(m,1,2)$.*

Lemma 7.2.14. *For $m > 3$, if $\Gamma_{36}(m,r,s)$ is vertex-transitive, then one of the following holds modulo m:*

i) $s - 2r \equiv 0$ ii) $s + r \equiv 0$

Proof. First, consider the following walks in Δ_{36}:

$$\begin{aligned} W_1 &= ((au)_0, (uv)_0, (vb)_0, (bv)_0, (vu)_0, (ua)_0); \\ W_2 &= ((au)_0, (uv)_r, (vb)_0, (bv)_0, (vu)_{-r}, (ua)_0). \end{aligned}$$

Observe that every reduced lift of W_1 or W_2 is a 6-cycle in $\Gamma_{36}(m,r,s)$. Indeed, a reduced lift of W_1 or W_2 must be, respectively, $(a_i, u_i, v_i, b_i, v_{i+m}, u_{i+m}, a_{i+m})$ or $(a_i, u_i, v_{i+r}, b_{i+r}, v_{i+r+m}, u_{i+m}, a_{i+m})$, with $i \in \{0, .., m-1\}$ (since the indices of vertices in fib(a) are computed modulo $m = \iota(a)$, we see that $a_{i+m} = a_i$ and so both these walks are cycles). This shows that every dart incident to u_0 or v_0 lies on at least one 6-cycle. Since $\Delta_{36}(m,r,s)$ is vertex-transitive, every dart incident to a_0 must also lie in at least one 6-cycle. In particular, the dart a_0b_s lies in a 6-cycle C. Such a cycle is a lift of some λ-reduced walk W of length 6 in Δ_{36} that traces $(ab)_s$. By inspecting Figure 7.2.5, we see that W must be one of the 2 following walks:

$$\begin{aligned} W_3 &= ((ab)_s, (bv)_0, (vu)_{-r}, (uv)_0, (vu)_{-r}, (ua)_0); \\ W_4 &= ((ab)_s, (bv)_0, (vu)_0, (uv)_r, (vu)_0, (ua)_0). \end{aligned}$$

One can see that end$(W_3) = \{s - 2r\}$ and end$(W_4) = \{s + r\}$, and thus $s - 2r \equiv 0 \pmod m$ when $C \in \mathcal{L}(W_3)$, and $s + r \equiv 0 \pmod m$ when $C \in \mathcal{L}(W_4)$, by Lemma 7.2.3. □

Corollary 7.2.15. *If $m > 3$ and $\Gamma_{36}(m,r,s)$ is vertex-transitive, then* $\gcd(m,r) = 1$.

Proof. This follows from the fact that one of the congruences of Lemma 7.2.14 holds and $\gcd(m,r,s) = 1$. □

Lemma 7.2.16. *For $m > 3$, if $\Gamma_{36}(m, r, s)$ is vertex-transitive then it has girth 6.*

Proof. Note that $\Delta_{36}(m, r, s)$ has no closed walks of odd length. Therefore, $\Gamma_{36}(m, r, s)$ only has cycles of even length. Observe that if W is a λ-reduced closed walk of length 4 based at a, then $\mathrm{end}(W) = \{\pm(r - s), \pm r, \pm s\}$. If $\Gamma_{36}(m, r, s)$ has a 4-cycle visiting $a_0 \in \mathrm{fib}(a)$, then $z \equiv 0 \pmod m$ for some $z \in \mathrm{end}(W)$. Additionally, by Lemma 7.2.14, either $s - 2r \equiv 0$ or $s + r \equiv 0$ modulo m. If $r - s \equiv 0 \pmod m$ and $s - 2r \equiv 0$ then $m = 1$ (since $\gcd(m, r, s) = 1$). If on the other hand $r - s \equiv 0 \pmod m$ and $s + r \equiv 0$, then $m = 2$. Similarly, if one the two congruences of Lemma 7.2.14 hold and either s or r is congruent to 0 modulo m then either $m = 1$ or $m = 2$. In all cases we contradict $m > 3$. We conclude that $\Gamma_{36}(m, r, s)$ has no cycles of length 4 and thus has girth 6. □

For a vertex u in $\Gamma_{36}(m, r, s)$ we can now easily count how many 6-cycles visit each of the darts incident to x and compute the 6-signature of $\Gamma_{36}(m, r, s)$.

Lemma 7.2.17. *For $m > 3$, if $\Gamma_{36}(m, r, s)$ is vertex-transitive, then it has 6-signature $(2, 3, 3)$*

Proof. We will show that a 6-cycle in Γ through the vertex a_0 must be a lift of W_i for some $i \in \{1, 2, 3, 4\}$, where W_i is as in the proof of Lemma 7.2.14. Suppose to the contrary that there is an additional 6-cycle C visiting a_0. Then, $\pi(C)$ is a λ-reduced closed walk of length 6 based at a, and $\pi(C) \neq W_i$ for all $i \in \{1, 2, 3, 4\}$. However, as one can confirm by inspecting Figure 7.2.5, the only two such walks in Δ_{36} are $((au)_0, (uv)_0, (vu)_{-r}, (uv)_0, (vu)_{-r}, (ua)_0)$ and $((au)_0, (uv)_r, (vu)_0, (uv)_r, (vu)_0, (ua)_0)$. Hence $\mathrm{end}(\pi(C)) \subseteq \{2r, -2r\}$, and so $2r \equiv 0 \pmod m$. In addition, one of the two congruences of Lemma 7.2.14 must hold. From this, we can see that $m \in \{1, 2\}$, a contradiction. Thus the only 6-cycles through a_0 are the cycles W_i with $i \in \{1, 2, 3, 4\}$. It follows that the darts a_0u_0 and a_0u_m each lie on exactly three distinct 6-cycles (one lift of each W_i, $i \in \{1, 2\}$, and one lift of either W_3 or W_4), while a_0b_s lies on exactly two distinct 6-cycles (both lifts of W_i for exactly one value of $i \in \{3, 4\}$). We conclude that the 6-signature of $\Gamma_{36}(m, r, s)$ is $(2, 3, 3)$. □

From Section 2.4 of [61], we see that a cubic vertex-transitive graph of order $6m$, girth 6 and with 6-signature $(2, 3, 3)$ is isomorphic to one of the Cayley graphs Θ_m or Σ_m defined below (the graph Θ_m is called Δ_m in [61]; we change the notation here to avoid confusion). In what follows, D_m denotes the dihedral group of order $2m$ with presentation $D_m = \langle \rho, \tau \mid \rho^m, \tau^2, (\rho\tau)^2 \rangle$.

For a positive integer m, let $\Theta_m = \mathrm{Cay}(D_{3m}, \{\tau, \rho^k\tau, \rho^m\tau\})$ of order $6m$, where $k = 3/\gcd(3, m)$. The full automorphism group of Θ_m is isomorphic to D_{3m}, making it a regular graphical representation of D_{3m}. However, the automorphism Φ^m of Γ is non-trivial and fixes every vertex in $\mathrm{fib}(a) \cup \mathrm{fib}(b)$. Then Γ cannot be isomorphic to Θ_i. Therefore it must be isomorphic to the graph Σ_m defined below.

For a positive integer m, let $\Sigma_m = \mathrm{Cay}(D_m \times \mathbb{Z}_3, \{(\rho\tau, 0), (\tau, 1), (\tau, 2)\})$ of order $6m$. The full automorphism group of Σ_m has order $12m$ and is isomorphic to the direct product $D_m \times S_3$. Observe that if m is even, then the group $D_m \times S_3$ has no element of order $2m$. However, $\Phi \in \mathrm{Aut}(\Gamma)$ has order $2m$, from which we see that if m is even, Γ is not isomorphic to Σ_m. That is, for m even and $r, s \in \mathbb{Z}_{2m}$, $\Gamma_{36}(m, r, s)$ is not vertex-transitive. The following lemma sums up the preceding paragraphs

Lemma 7.2.18. *If $m > 3$ and $\Gamma_{36}(m, r, s)$ is vertex-transitive, then m is odd and $\Gamma_{36}(m, r, s) \cong \Sigma_m$.*

Lemma 7.2.19. *For an odd positive integer m, $\Gamma_{36}(m, 1, 2) \cong \Sigma_m$.*

Proof. We will give an explicit isomorphism between Σ_m and $\Gamma_{36}(m, 1, 2)$. First, for $i \in \mathbb{Z}_{2m}$ let $\alpha(i) = 0$ if i is even and $\alpha(i) = 1$ otherwise. Recall that the vertices of Σ_m are of the form $(\rho^i\tau^j, k)$ with $i \in \mathbb{Z}_m$, $j \in \{0, 1\}$ and $k \in \mathbb{Z}_3$. Now the mapping $f : \Sigma_m \to \Gamma_{36}(m, 1, 2)$ given by

$$\begin{array}{lll} (\rho^i\tau, 0) \mapsto u_{\alpha(i)m+i}, & (\rho^i\tau, 1) \mapsto u_{(1-\alpha(i))m+i}, & (\rho^i\tau, 2) \mapsto b_{i+1} \\ (\rho^i, 0) \mapsto v_{(1-\alpha(i))m+i+1}, & (\rho^i, 1) \mapsto v_{\alpha(i)m+i}, & (\rho^i, 2) \mapsto a_i \end{array}$$

is a graph isomorphism. □

From Lemma 7.2.19 we see that for a fixed odd integer m, a vertex-transitive $\Gamma_{36}(m, r, s)$ must be isomorphic to $\Gamma_{36}(m, 1, 2)$. This, together with the fact that every Cayley graph is vertex-transitive, completes the proof of Theorem 7.2.13.

Remark 7.2.20. For an odd integer m, the graph $\Delta_{36}(m, 1, 2)$ can be constructed from the depleted wreath graph $\mathrm{DW}(m, 3)$ (defined in [62]) by means of a split operation. The graph $\mathrm{DW}(m, 3)$ is the tetravalent graph with vertex-set $\mathbb{Z}_m \times \mathbb{Z}_3$ where every vertex (i, j) is adjacent to $(i \pm 1, j \pm 1)$. The split of $\mathrm{DW}(m, 3)$ is obtained by replacing every vertex (i, j) by two adjacent vertices $(i, j)^-$ and $(i, j)^+$ and making each $(i, j)^+$ adjacent to $(i + 1, j \pm 1)^-$. The resulting graph is isomorphic to $\Delta_{36}(m, 1, 2)$.

Lemma 7.2.21. *Let $m > 3$ be an odd integer. If $3 \mid m$ then $\Delta_{36}(m, 1, 2)$ is not a k-multicirculant for any $k \in \{1, 2, 3\}$.*

Proof. First, note that $\Gamma_{36}(m, 1, 2)$ cannot be a circulant, as cubic circulants have girth 4. From Theorem 3.2.30 and Proposition 3.2.41 we know a vertex-transitive tricirculant of girth 6 with more than 18 vertices has 6-signature $(2, 2, 2)$. Hence $\Gamma_{36}(m, 1, 2)$ is not a tricirculant. Finally, if $3 \mid m$ then the group $D_m \times S_3$ has no elements of order $3m$, so $\Gamma_{36}(m, 1, 2)$ is not a bicirculant. The result follows. □

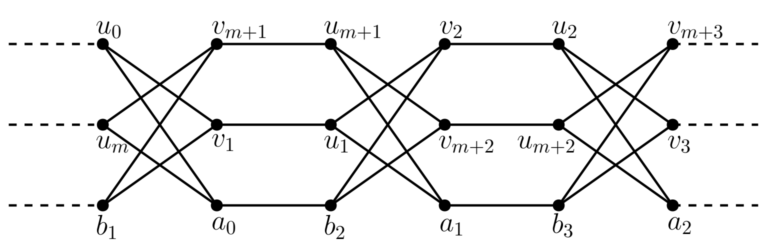


Figure 7.2.6: A section of the graph $\Gamma_{36}(m, 1, 2)$ with $m \geq 3$.

7.2.4 Classification theorem

We are now ready to state and prove the classification theorem for cubic vertex-transitive graphs Γ admitting a regular orbit of size $|\mathrm{V}(\Gamma)|/3$. The graphs in items (1)-(6) of Theorem 7.2.22 below correspond to the *ccv*-covers of the respective *ccv*-graphs in Figure 7.2.7, where we take $\iota(v) = m$ for all vertices (since all edges are $[1,1]$ edges, all vertices have the same ι-value) and the voltage is given in the figure. The graphs in item (7) are *ccv*-covers of Δ_{36} in Figure 7.2.7, and are defined at the beginning of Section 7.2.3.

Theorem 7.2.22. *Let Γ be a cubic vertex-transitive graph of order n. Then $\ell(\Gamma) \geq n/3$ if and only if Γ is the ccv-cover of one the seven labelled graphs of Figure 7.2.7. That is, Γ is isomorphic to one of the following:*

1. *$C(m;r)$ with m even, $m \geq 4$, $\gcd(\frac{m}{2}, r) = 1$, $r \in \{1,2\}$;*
2. *$I(m;r,1)$ with $m \geq 3$, $r^2 \equiv \pm 1 \pmod m$, or $m = 10$ and $r = 2$;*
3. *$H(m;r,s)$ with $m \geq 3$, $r \neq s$, $\gcd(m,r,s) = 1$;*
4. *$T_1(m;r,1)$ with $\frac{m}{2} \equiv 1 \pmod 4$ and $r = (\frac{m}{2}+3)/2$, or $\frac{m}{2} \equiv 3 \pmod 4$ and $r = (\frac{3m}{2}+3)/2$ or $m = 4$ and $r = 0$;*
5. *$T_2(m;2,1)$ with $m \geq 4$ and $\frac{m}{2}$ odd;*
6. *$T_4(5,1,3)$;*
7. *$\Gamma_{36}(m,1,2)$ with m odd.*

Proof. Let Γ be a cubic vertex-transitive graph of order n such that $\ell(\Gamma) \geq n/3$. From Proposition 7.2.12, we see that if Γ has more than 20 vertices, then it is either a k-multicirculant graph for some $k \in \{1,2,3\}$, or Γ is a *ccv*-cover of $\Delta_1 1$. Meanwhile, all cubic vertex-transitive grahs on at most 20 vertices are a k-multicirculant for some $k \in \{1,2,3\}$ (as can be verified in the census [56]). If Γ is a circulant, then it follows from the paragraph below Lemma 3.0.1 that Γ is isomorphic to one of the graphs in item (1). If Γ is a bicirculant, then by Theorem 3.1.1 and the proof of Theorem 6.2.1, Γ belongs to one of the classes in items (2) or (4).

If Γ is a tricirculant, then it belongs to one of four classes of tricirculants of Theorem 3.2.1 (in particular, Γ is a tricirculant of type i for some $i \in \{1,2,3,4\}$). However, if Γ is of type 3, then Γ is a circulant by Theorem 3.2.45, and thus is one of the graphs in item (1). If Γ is of type 1, 2 or 4, then Γ belongs to the class of graphs in item (5), (6) or (7) respectively. Finally, if Γ is a *ccv*-cover of Δ_{36}, then from Lemma 7.2.19, we see that Γ is isomorphic to $\Gamma_{36}(m,1,2)$ with m odd.

Now, suppose Γ belongs to one of the six families corresponding to items (1)–(6) of Theorem 7.2.22. Then it follows from Theorems 3.1.1 and 3.2.1 and the discussion below Lemma 3.0.1 that Γ is vertex-transitive If Γ is isomorphic to $\Gamma_{36}(m,1,2)$ with m odd, then Γ is vertex-transitive by Lemma 7.2.19. By construction, all graphs in items (1)–(7) admit an orbit containing at least one third of the total number of vertices of the graph. $\square$

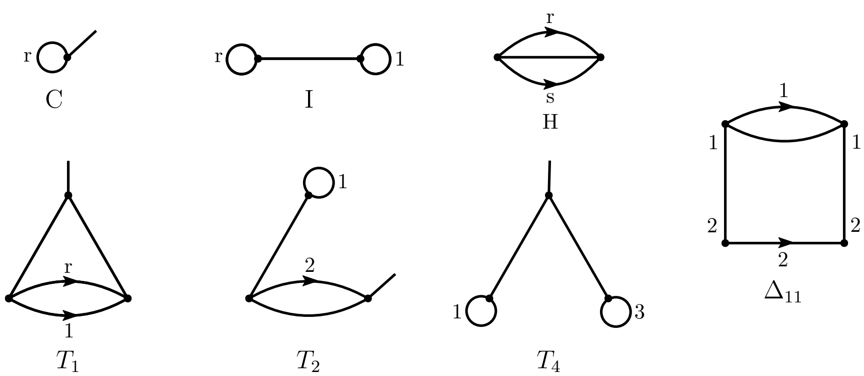


Figure 7.2.7: The seven labelled graphs whose combined vertex-transitive *ccv*-covers conform the class of cubic vertex-transitive graphs Γ admitting a regular orbit of length at least $\frac{|V(\Gamma)|}{3}$.

Remark 7.2.23. The graph $\mathrm{I}(10, 2)$ of item (2) is isomorphic to the dodecahedron $\mathrm{GP}(10, 2)$, the graph $\mathrm{T}_1(4, 0, 1)$ of item (4) is isomorphic to the truncated tetrahedron and the graph $\mathrm{T}_4(5, 1, 3)$ of item (6) is isomorphic to the Tutte-Coxeter graph. The graphs in items (1)–(5) correspond to the graphs in items (1)–(5) of Theorem 6.2.1, respectively.

Remark 7.2.24. The set of labelled graphs shown in Figure 7.2.7 is of minimal size in the sense that no smaller subset $X \subseteq \mathcal{Q}^*$ has the property that any cubic vertex-transitive graph Γ with a regular orbit of size $\frac{|V(\Gamma)|}{3}$ is a *ccv*-cover of some graph in X.

Observe that the graph $K_{3,3}$ is a circulant, and thus, it admits an automorphism φ such that the cyclic group $\langle\varphi\rangle$ is transitive on the vertices of $K_{3,3}$. Then $\ell(K_{3,3}) = o(K_{3,3}) = 6$, and the only orbit of $\langle\varphi\rangle$ is a regular orbit. It follows from Theorem 7.1.1 that the set $\mathcal{G}$ is precisely the set of cubic vertex-transitive graphs Γ admitting a regular orbit of size $|V(\Gamma)|/3$.

Corollary 7.2.25. *A cubic vertex-transitive graph Γ of order n admits a regular orbit of size $n/3$ or greater if and only if Γ belongs to one of the eight families of Theorem 7.2.22.*

Let $\Gamma \in \mathcal{G}$ have order n, let $g \leq \mathrm{Aut}(\Gamma)$ be an automorphism with $\ell(g) = n/3$ and let $G = \langle g\rangle$. If g is a k-multicirculant automorphism, then the labelled quotient $(\Gamma/G, \lambda)$ must be the graph Δ_i of Figure 6.1.1 for some $i \in \{1, 2, 3, 4, 22, 23, 24, 25\}$. Otherwise, from combination of Theorems 5.4.14 and 6.2.1, and the results in Section 7.2 we see that Γ must either belong to the set X of exceptional graphs, or be isomorphic to the Franklin graph (see the proof of Lemmas 7.2.4 and 7.2.8), the truncation of $K_{3,3}$ (see Lemma 7.2.7), the prism P_3 (see Lemma 7.2.9) or the graph $\Gamma_{36}(\frac{n}{6}, 1, 2)$. It follows that the set $\mathcal{Q}$ is comprised of the labelled graphs Δ_i with $\{1, 2, 3, 4, 22, 23, 24, 25, 36\}$ of Figures 6.1.1 and 7.2.1 along with all possible labelled quotient $(\Gamma/G, \lambda)$ where Γ is isomorphic to the Franklin graph, the truncation of $K_{3,3}$, the prism P_3 or an element of X, and G is a cyclic group with an orbit of size $|V(\Gamma)|/3$. We present the 20 labelled graphs conform the set $\mathcal{Q}$ in Figure 7.2.8. The labelled graph within a dashed box are the only elements of $\mathcal{Q}$ admitting infinitely many vertex-transitive *ccv*-covers. A vertex-transitive *ccv*-cover of a graph not inside a dashed box has order at most 30.

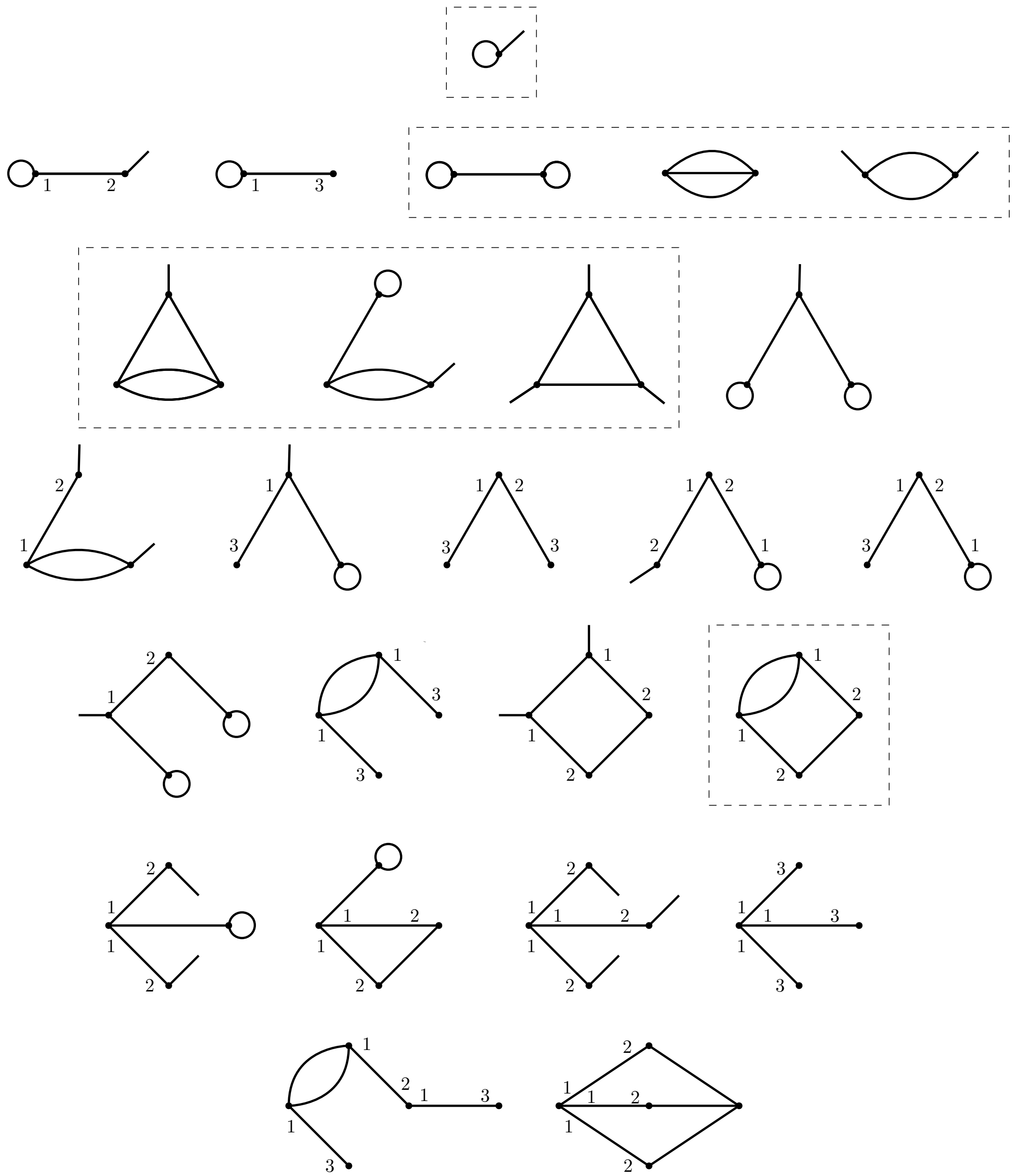


Figure 7.2.8: The twenty possible labelled quotient $(\Gamma/G, \lambda)$ where $G \leq \mathrm{Aut}(\Gamma)$ is a cyclic group admitting an orbit of size $\frac{|V(\Gamma)|}{3}$ or greater.

Chapter 8

Conclusions

The contributions to algebraic graph theory within this dissertation can be roughly divided in two different but interrelated categories: classification results of cubic-vertex transitive graphs and the study of automorphisms of large order in such graphs, and the development of new tools for constructing graphs admitting a group of automorphism with a given structure or action type. Let us briefly discuss these contributions and suggest possible paths for future research.

With regard to classification results, we first provide a full classification of cubic vertex-transitive tricirculants, thus extending the results of [54] and [35]. We then extend this to encompass all cubic vertex-transitive admitting a cyclic group of automorphism with at most three orbits on vertices. In particular we show that all such graph admit a k-multicirculant automorphism for some $k \in \{1, 2, 3\}$. As a final step in this direction, we provide the complete classification of cubic vertex-transitive graphs Γ admitting a cyclic group with an orbit on vertices of size $|\mathrm{V}(\Gamma)|/3$. All such graphs can be constructed as a *ccv*-cover of eight labelled graph on at most four vertices. We would like to point out that infinitely many such graphs do not admit a k-multicirculant automorphism with $k \in \{1, 2, 3\}$. Indeed, the class of graphs denoted $\Gamma_{36}(m, r, s)$ admit an infinite subclass of graphs such that $k = 6$ is the smallest integer for which they admit a k-multicirculant automorphism.

It is then natural to consider the following problem, which is related to the problem of finding multicirculant automorphisms of large order in vertex-transitive graphs. For a positive integer n, let $\mu(n)$ denote the smallest integer for which (with finitely many exceptions) every vertex-transitive cubic graph admitting a regular orbit of size at least $\frac{|\mathrm{V}(\Gamma)|}{n}$ is a k-multicirculant for some $k \leq \mu(n)$. Clearly $\mu(1) = 1$, and from the results in this dissertation, we have $\mu(2) = 2$ and $\mu(3) = 6$. What can be said about the function μ?

We believe that for every odd integer $n > 1$, there is a family of cubic vertex-transitive graphs (consisting of *ccv*-covers of a labelled graph on $n + 1$ vertices) such that $k = 2n$ is the smallest integer k for which a graph belonging to this class is a k-multicirculant. This would suggest that $\mu(n) \geq 2n$ for an odd $n > 1$.

As for results regarding the action of automorphism of large order in vertex-transitive graphs, we show that, with the exception of $K_{3,3}$, every automorphism of a cubic vertex-transitive graph Γ admits a regular cycle. That is, the order of an automorphism g matches the size of the largest orbit of $\langle g \rangle$. Moreover, we show that if an automorphism g of a vertex-transitive cubic graph Γ has an orbit of size $|\mathrm{V}(\Gamma)|/k$ for some integer $k > 1$, then the number of $\langle g \rangle$-orbits on vertices is bounded from above by $6k - 5$. In the particular case when $k = 3$, the number of orbits turns out to be much smaller (4 instead of 13). In Chapter 7 we point out that this bound might be as small as $2k - 2$. The main result of [55] states that a non-trivial automorphism of a cubic vertex-transitive graph on n vertices cannot fix more than $n/3$ vertices. If this result is confirmed, it would undoubtedly be a useful tool for improving the bound on the number of orbits of an automorphism with a 'long' orbit in a cubic vertex-transitive graph. It would also be of interested to investigate if a similar bound exists for r-valent vertex-transitive graphs with $r > 3$.

Finally, we introduce the notion of a generalised voltage graph and the associated generalised cover construction which allow us to construct graphs admitting a group of automorphism isomorphic to a given group, and to encode the structure of large graphs in a compact format. It generalises some well-known constructions such as the Sabidussi double coset graph, the Cayley graph or the derived graph constructions. The usefulness of this new tool is made evident in Chapters 5, 6 and 7, where we use it to study and classify cubic graphs admitting a cyclic group of automorphisms with orbit-size and orbit-number constrains.

The theory of generalised voltage graphs presented here serves as a general framework from which new graph constructing methods can be derived by restricting its scope to graphs admitting groups of automorphisms with a particular type of action or group structure. The voltage graph and derived cover construction, where we consider only groups with a semiregular action on vertices, is a prime example of the former, while the cyclic generalised cover construction presented in Chapter 5, where we consider only cyclic groups, is a example of the latter. The question arises whether other useful or otherwise nice constructions can be derived from the theory of generalised voltage graphs when we consider other types of group-actions (other than semiregular, regular or arc-transitive) or groups (other than cyclic groups).

Bibliography

[1] A. Ádám, Research problem 2–10, *J. Combin. Theory*, **2** (1967), p. 393.

[2] B. Alspach, T. D. Parsons, A construction for vertex-transitive graphs, *Canadian J. Math.* **34** (1982), 307–318.

[3] S. Bhoumik, E. Dobson, J. Morris, On the automorphism groups of almost all circulant graphs and digraphs., *Ars Math. Contemp.* **7** (2014), 487–506.

[4] Bondy, J. A. and Murty, U. S. R. Graph Theory with Applications. New York: North Holland, p. 244, 1976.

[5] I. Z. Bouwer, W. W. Chernoff, B. Monson, Z. Star, The Foster census. Charles Babbage Research Centre (1988).

[6] P. Cameron, M. Giudici, G. A. Jones, W. M. Kantor, M. H. Klin, D. Marušič, L. A. Nowitz, Transitive permutation groups without semiregular subgroups, J. Lond. Math. Soc. **66** (2002) 325–333.

[7] P. Cameron, J. Sheehan, P. Spiga, Semiregular automorphisms of vertex-transitive cubic graphs, European J. Combin. **27** (2006) 924–930.

[8] M. Conder, All trivalent (cubic) symmetric graphs on up to 10000 vertices, available online at https://www.math.auckland.ac.nz/ conder/

[9] M. Conder, Graph Symmetries; in *PhD Summer School in Discrete Mathematics*, Famnit Lectures 2, University of Primorska Press, Koper, 2013; available online at http://www.hippocampus.si/ISBN/978-961-6832-62-5.pdf

[10] M. Conder, R. Grande, On embeddings of circulant graphs, *Electron. J. Combin.* **22** (2015), no. 2, Paper 2.28, 27 pp.

[11] M. Conder, R. Nedela, Symmetric cubic graphs of small girth, *J. Combin. Theory Ser. B* **97** (2007), 757–768.

[12] M. Conder, T. Pisanski, A. Žitnik, Vertex-transitive graphs and their arc-types, Ars Math. Contemporanea **12** (2017), 383–413.

[13] H. S. M. Coxeter, W. O. J. Moser, Generators and Relations for Discrete Groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, 14 (4th ed.), Springer Verlag (1980).

[14] A. Devillers, M. Giudici, C.-H. Li, C. Praeger, An infinite family of biquasiprimitive 2-arc transitive cubic graphs, *J. Algebraic Combin.* **35** (2012), 173–192.

[15] D. Djoković, A class of finite group-amalgams, *Proc. Amer. Math. Soc.* **80** (1980), 22–26.

[16] D. Djoković and G.L. Miller, Regular Groups of Automorphisms of Cubic Graphs, *J. Combin. Theory, Ser. B* **29** (1980) 195–230

[17] S.-F. Du, M.-Y. Xu, A classification of semisymmetric graphs of order $2pq$, *Comm. in Algebra* **28** (2000), 2685–2715.

[18] X.-G. Fang, C. Praeger, Finite two-arc transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27** (1999), 3727–3754.

[19] Y.-Q. Feng, D.-W. Yang, J.-X. Zhou Arc-transitive cyclic and dihedral covers of pentavalent symmetric graphs of order twice a prime *Ars Math. Contemp.* **15** (2018), 499–522.

[20] Y.-Q. Feng, R. Nedela, Symmetric cubic graphs of girth at most 7, Acta Univ. M. Belii Ser. Math. **13** (2006), 33–35.

[21] J. Folkman, Regular line-symmetric graphs, J. Combin. Theory **3**(3) (1967), 215-232.

[22] B. Frelih, K. Kutnar, Classification of cubic symmetric tetracirculants and pentacirculants, European J. Combin. **34** (2013), 169–194.

[23] R. Frucht, J.E. Graver, M.E. Watkins, The groups of the generalized Petersen graphs, Proc. Camb. Phil. Soc. **70** (1971) 211–218.

[24] M. Giudici, C.-H. Li, C. Praeger, Analysing finite locally s-arc transitive graphs, *Trans. Amer. Math. Soc.* **356** (2004), 291–317.

[25] M. Giudici, C. H. Li, B. Xia, An infinite family of vertex-primitive 2-arc-transitive digraphs, *J. Combin. Theory Ser. B* **127** (2017), 1–13.

[26] M. Giudici, I. Kovács, C.-H. Li, G. Verret, Cubic arc-transitive k-multicirculants, J. Combin. Theory Ser. B **125** (2017), 80–94.

[27] M. Giudici, C. Praeger, P. Spiga, Finite primitive permutation groups and regular cycles of their elements, *J. Algebra*, **421**, (2015), 27–55.

[28] S. Guest P. Spiga, Finite primitive groups and regular orbits of group elements *Trans. Amer. Math. Soc.* **369** (2017), 997–1024.

[29] D. Goldschmidt, Automorphisms of trivalent graphs, *Ann. Math.* **111** (1980), 377–406.

[30] J. Gross, Voltage Graphs, *Discrete Math.* **9** (1974), 3, 239–246.

[31] J. Gross, T. W. Tucker, Generating all graph coverings by permutation voltage assignment, Discrete Math. **18** (1977), 273-283.

[32] J. Gross , T. W. Tucker, *Topological graph theory*, Wiley-Interscience, New York, 1987.

[33] M. Hladnik, D. Marušič, T. Pisanski, Cyclic Haar graphs, *Discrete Math.* **244** (2002), 137–152.

[34] R. Jajcay, Š. Miklavič, P. Šparl, G. Vasiljević, On certain edge-transitive bicirculants, *Electron. J. Combin.* **26** (2019), no. 2, Paper 2.6, 28 pp.

[35] I. Kovács, K. Kutnar, D. Marušič, S. Wilson, Classification of cubic symmetric tricirculants, Electronic J. Combin. 19(2) (2012), P24, 14 pages.

[36] I. Kovács, B. Kuzman, A. Malnič, S. Wilson, *Characterization of edge-transitive 4-valent bicirculants*, *J.' Graph Theory* **69** (2012), 441–463.

[37] B. Kuzman, A. Malnič, P. Potočnik, Tetravalent Vertex- and Edge-Transitive Graphs Over Doubled Cycles, *J. Combin. Theory Ser. B.* **131** (2018), 109–137.

[38] B. Ling, B. G. Lou, C. X. Wu Pentavalent symmetric graphs of order four times an odd square-free integer, *Ars. Math. Contemp.* **16** (2019), 81–95.

[39] M. Mačaj, Strongly regular graphs from coverings?, a talk delivered at the conference *ATCAGC 2010, Auckland, 15-19 February*, https://www.math.auckland.ac.nz/ conder/ATCAGC2010-Abstracts.pdf

[40] A. Malnič, R. Nedela, M. Škoviera, Lifting graph automorphisms by voltage assignments, Europ. J. Combin. **21** (2000), 927-947.

[41] A. Malnič, D. Marušič, P. Potočnik, Elementary abelian covers of graphs, J. Alg. Combin. **20** (2004), 71–97.

[42] D. Marušič, On vertex symmetric digraphs, Discrete Math. **36** (1981) 69–81.

[43] D. Marušič, Semiregular automorphisms in vertex-transitive graphs of order $3p^2$, Electronic J Combin. **25** (2018) P2.25.

[44] D. Marušič, Semiregular automorphisms in vertex-transitive graphs with a solvable group of automorphisms, Ars Math. Comptemp. **13** (2017) 461–468.

[45] D. Marušič, Strongly regular bicirculants and tricirculants *Ars Combin.* **25** (1988), 11–15.

[46] D. Marušič, R. Scapellato, Permutation groups, vertex-transitive digraphs and semiregular automorphism, European J. Combin. **19** (1998), 707–712.

[47] A. D. Mednykh, R. Nedela, Harmonic morphisms of graphs and the Riemann-Hurwitz theorem, *Dokl. Math.* **93**, 23–26 (2016).

[48] A. D. Mednykh, I. A. Mednykh, R. Nedela, On the Oikawa and Arakawa theorems for graphs, *Tr. Inst. Mat. Mekh.* **23** (2017), 243–252; translation in *Proc. Steklov Inst. Math.* **304** (2019), 133–140.

[49] A. D. Mednykh, I. A. Mednykh, The number of spanning trees in circulant graphs, its arithmetic properties and asymptotic *Discrete Math.* **342** (2019), 1772–1781.

[50] R.C. Miller, The trivalent symmetric graphs of girth at most 6, *J. Combin. Theory*, **20** (1971), 163–192.

[51] J. Morris, P. Spiga, G. Verret, Semiregular automorphisms of cubic vertex-transitive graphs and the abelian normal quotient method; *Electro. J. Combin.* **20** (2015), P3.32.

[52] J. Pan, B. Ling, S. Ding, On prime-valent symmetric graphs of square-free order, *Ars. Math. Contemp.* **15** (2018), 53–65.

[53] P. Potočnik, Edge-colourings of cubic graphs admitting a solvable vertex-transitive group of automorphisms, *J. Combin. Theory Ser. B*, **91** (2004), 289–300.

[54] T. Pisanski, A classification of cubic bicirculants, Discrete Math. **307** (2007), 567-578.

[55] P. Potočnik, P. Spiga, On the number of fixed points of automorphisms of vertex-transitive graphs of bounded valency, arXiv:1909.05456

[56] P. Potočnik, P. Spiga, G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, Journal of Symbolic Computation **50** (2013), 465-477

[57] P. Potočnik, M. Toledo, Classification of cubic vertex-transitive tricirculants, *Ars Math. Contemp.* **18** (2020) 1–31.

[58] P. Potočnik, M. Toledo, Finite cubic graphs admitting a cyclic group of automorphisms with at most three orbits on vertices. Discrete Math. Accepted. *arXiv:2003.05186*

[59] P. Potočnik, M. Toledo, Generalised voltage graphs, *arXiv:1910.08421*.

[60] P. Potočnik, J. Vidali, Girth-regular graphs, arXiv:1802.01881.

[61] P. Potočnik, J. Vidali, Cubic vertex-transitive graphs of girth 6, arXiv:2005.01635

[62] P. Potočnik, S. Wilson, Recipes for edge-transitive tetravalent graphs, *The art of discrete and applied mathematics* (2020)

[63] Y.-L. Qin, B. Xia, S. Zhou, Stability of circulant graphs, *J. Combin. Theory Ser. B* **136** (2019), 154–169.

[64] G. Sabidussi, On a Class of Fixed-Point-Free Graphs, *Proceedings of the American Mathematical Society*, **9**(5) (1958), 800-804.

[65] J. Siemons, A. Zalesskii Intersections of matrix algebras and permutation representations of $PSL(n, q)$ *J. Algebra*, **226** (2000), 451–478.

[66] J. Siemons, A. Zalesskii Regular orbits of cyclic subgroups in permutation representations of certain simple groups *J. Algebra*, **256** (2002), pp. 611-625.

[67] J. Širáň, Coverings of graphs and maps, ortogonality, and eigenvectors, *J. Alg. Combin.* **14** (2001), 57–72.

[68] M. Škoviera, A contribution to the theory of voltage graphs, *Discrete Math.* **61**(1986), 281–292.

[69] P. Spiga, Semiregular elements in cubic vertex-transitive graphs and the restricted Burnside problem, Math. Proc. Cambridge Phil. Soc. **157** (2014), 45–61.

[70] W. A. Stein et al., *SageMath, the Sage Mathematics Software System (Version 8.3)*, The Sage Developers, 2018, [`https://www.sagemath.org`].

[71] W.T. Tutte, 1947. A family of cubical graphs. Proc. Cambridge Philos. Soc. **43**, 459-474.

[72] W.T. Tutte, Connectivity in graphs, Univ. of Toronto Press, Toronto, 1966.

[73] W.T. Tutte, A family of cubical graphs. *Proc. Cambridge Philos. Soc.*, **43** (1947), 459–474.

[74] W.T. Tutte, On the symmetry of cubic graphs, *Canad. J. Math.*, **11** (1959), 621–624.

[75] G.-X. Wang, Z.-P. Lu, The two-arc-transitive graphs of square-free order admitting alternating or symmetric groups, *J. Aust. Math. Soc.* **104** (2018), 127–144.

Povzetek v slovenskem jeziku

Grafi, ki premorejo visoko stopnjo simetrije, so deležni velike pozornosti, saj je njihova struktura velikokrat določena že s samim delovanjem podgrupe avtomorfizmov. Za graf pravimo, da je vozliščno-tranzitiven, če za poljubni vozlišči obstaja avtomorfizem, ki slika eno vozlišče v drugo. Hitro opazimo, da so vsa vozlišča vozliščno-tranzitivnega grafa iste stopnje. V primeru, ko je stopnja enaka 1 oz. 2, graf predstavlja unijo disjunktnih povezav oz. disjunktnih ciklov. V primeru, ko je stopnja enaka 3, pa je struktura grafov kompleksnejša. Take grafe imenujemo kubični grafi. Le-ti sestavljajo bogato in raznoliko družino, zato ne preseneča dejstvo, da so vozliščno-tranzitivni kubični grafi najbolj raziskani med grafi z visoko stopnjo simetrije. Veliko lastnosti grafov z visoko stopnjo simetrije je bilo najprej raziskanih na primeru kubičnih grafov (glej na primer [73, 74]). Določene podrazrede vozliščno-tranzitivnih grafov Γ velikokrat raziskujemo s pomočjo delovanja specifične vozliščno-tranzitivne grupe avtomorfizmov G, saj je s tem njihova struktura natančno določena. Zanimivi so tudi grafi, ki niso vozliščno-tranzitivni, vendar kljub temu premorejo visoko stopnjo simetrije.

Grafi Γ, z grupo avtomorfizmov $\mathrm{Aut}(\Gamma)$, ki premore ciklično podgrupo G z majhnim številom orbit na množici vozlišč $\mathrm{V}(\Gamma)$, imajo veliko zanimivih lastnosti in so bili vključeni v veliko raziskav. V primeru, ko ima ciklična grupa G eno samo orbito na $\mathrm{V}(\Gamma)$, graf pripada družini cirkulantov, ki je bila deležna številnih raziskav (v [1] zasledimo eno izmed zgodnjih omemb cirkulantov, nekateri novejši rezultati iz tega področja pa so predstavljeni v člankih [49, 3, 10, 63]). V primeru, ko G razdeli $\mathrm{V}(\Gamma)$ na k enako velikih orbit, pravimo grafu k-multicirkulant. Za 1-, 2- in 3-multicirkulante pogosto uporabljamo kar izraze cirkulanti, bicirkulanti oz. tricirkulanti. Ko študiramo k-multicirkulante, pogosto predpostavimo tudi dodatne pogoje, kot so vozliščna tranzitivnost, povezavna tranzitivnost ali ločna tranzitivnost grafa (glej npr. [2, 22, 26, 34, 36, 45, 54]). Multicirkulanti so tudi glavna tema znane *domneve o multicirkulantih* [6, 42, 43, 44, 69], ki pravi, da so končni vozliščno-tranzitivni grafi k-multicirkulanti za nek k, ki je manjši od števila vozlišč v grafu.

Pri študiju k-multicirkulantov pogosto uporabljamo teorijo napetostnih grafov in regularnih krovov (glej npr. [32, 40, 67]). če je Γ graf na n vozliščih in premore ciklično grupo avtomorfizmov G, ki razbije množico vozlišč na k-orbit enake velikosti, potem lahko graf Γ skonstruiramo iz grafa Δ na k vozliščih z uporabo posebne funkcije $\zeta\colon \mathrm{D}(\Delta) \to G$. Funkciji ζ pravimo *dodelitev napetosti*, trojici (Δ, G, ζ) pa *napetostni graf.* V tem primeru je graf Γ *regularni krov* grafa Δ. Področje raziskovanja napetostnih grafov ima dolgo zgodovino, ki se je začela z delom Grossa in Tuckerja [30, 32]. Zaradi dela Malniča, Nedele in Škoviere [40] so napetostni grafi postali eno izmed pomembnejših orodij pri študiju simetrije grafov (glej npr. [19, 22, 37, 35, 40, 41, 53]).

Primer, ko so orbite grupe G različnih velikosti, je bil deležen manj pozornosti. Razlog za to lahko pripišemo pomanjkanju primernih orodij za študij teh grafov. Dasiravno za študij k-multicirkulantov uporabljamo teorijo napetostnih grafov, za grafe z G orbitami različnih velikosti v literaturi ni zaslediti primerljive teorije. To kaže na potrebo po razvoju novih algebrajskih in kombinatoričnih orodij za preučevanje grafov z grupo avtomorfizmov G, ki ne zadošča tako strogim simetričnim pogojem.

čeprav so glavni predmet tega raziskovalnega dela enostavni povezani grafi, začnemo z vpeljavo splošnejših grafov, kjer dovolimo vzporedne povezave, zanke in štrclje. To nam bo omogočilo točne definicije in dobro definirana teoretična orodja. Graf je urejena četverica $(V, D; \mathrm{beg}, \mathrm{inv})$, kjer sta D in $V \neq \emptyset$ disjunktni končni množici *polpovezav* in *vozlišč*. Preslikava $\mathrm{beg}\colon D \to V$ vsaki polpovezavi x določi njeno *začetno vozlišče* $\mathrm{beg}\, x$, preslikava $\mathrm{inv}\colon D \to D$ pa je involucija, ki izmenja vsako polpovezavo x z njeno *nasprotno polpovezavo*, ki jo često označujemo z x^{-1}. *Končno vozlišče* polpovezave x, $\mathrm{end}\, x$, je začetno vozlišče polpovezave x^{-1}. Začetno in končno vozlišče polpovezave x sta *krajišči* polpovezave x. če sta dve različni vozlišči u in v krajišči polpovezave x, pravimo, da sta sosednji, in pišemo $u \sim v$. Različni polpovezavi x in y sta vzporedni, če imata enaki začetni in končni vozlišči. *Soseščina* vozlišča v je definirana kot množica polpovezav, ki imajo v za začetno vozlišče. *Stopnja* vozlišča v je kardinalnost soseščine vozlišča v. Graf, v katerem je vsako vozlišče stopnje k, $k \in \mathbb{N}$, je *regularen graf stopnje k*.

Orbitam preslikave inv pravimo *povezave*. Povezava, ki vsebuje polpovezavo x, je *štrcelj*, če velja $\mathrm{inv}\, x = x$. če pa velja $\mathrm{inv}\, x \neq x$ in hkrati $\mathrm{beg}\,(x^{-1}) = \mathrm{beg}\ x$, jo imenujemo *zanka*. Pojem "povezava" bomo rajši rezervirali za povezave, ki niso niti zanke niti štrclji. Povezavi e in e' sta vzporedni, če e vsebuje polpovezavo, ki je vzporedna neki polpovezavi v e'. V kontekstu te definicije je enostaven graf (v klasičnem smislu) tisti graf, ki ne vsebuje zank, štrcljev ali vzporednih povezav. Enostaven graf Γ je mogoče s to splošnejšo definicijo predstaviti kot graf z množico vozlišč $V = \mathrm{V}(\Gamma)$ in množico lokov $D = \{(u, v) \mid u, v \in \mathrm{V}(\Gamma), u \sim v\}$, kjer je $\mathrm{beg}(u, v) = u$ in $\mathrm{inv}(u, v) = (v, u)$. Polpovezavam v enostavnem grafu ponavadi pravimo *loki*. Enostavni, povezani regularni grafi stopnje 3 se imenujejo *kubični grafi*.

Pojmi morfizmov, izomorfizmov in avtomorfizmov so naravne posplošitve tistih za enostavne grafe. Natančnejše definicije so opisane v [41]. Grupo avtomorfizmov grafa Γ označimo z $\mathrm{Aut}(\Gamma)$.

če je G grupa, ki deluje na množici X, potem za vsak $x \in X$ označimo z x^G orbito G, ki vsebuje x. Stabilizator x v G označimo z G_x. Orbitam G na X po navadi pravimo G-orbite. če je Γ graf in $G \leq \mathrm{Aut}(\Gamma)$, je kvocient Γ po G graf $\Gamma/G := (V', D', \mathrm{beg}', \mathrm{inv}')$, katerega vozlišča in polpovezave so orbite G na $\mathrm{V}(\Gamma)$ in $\mathrm{D}(\Gamma)$, tako da $\mathrm{beg}'\, x^G = (\mathrm{beg}\, x)^G$ in $\mathrm{inv}\, x^G = (\mathrm{inv}\, x)^G$ (tj. sosednost je naravno podedovana iz Γ). če je $\phi \in \mathrm{Aut}(\Gamma)$, je kvocient Γ vzdolž ϕ kvocient Γ po ciklični grupi $\langle\phi\rangle$. če je Γ enostaven in ima $\langle\phi\rangle$ k različnih orbit enake velikosti na $\mathrm{V}(\Gamma)$, potem je Γ k-multicirkulant in ϕ k-multicirkulantni avtomorfizem.

Naj bo Γ graf in $G \leq \mathrm{Aut}(\Gamma)$. Pravimo, da je G vozliščno-, povezavno- oz. ločno-tranzitivna, če je G tranzitivna na vozliščih, povezavah oz. lokih grafa Γ. Pravimo, da je G semiregularna (na vozliščih grafa Γ), če je stabilizator $G_v \leq G$ vsakega vozlišča $v \in \mathrm{V}(\Gamma)$ trivialen; tj. če noben $g \in G$ ne pribije vozlišča $v \in \mathrm{V}(\Gamma)$. Pravimo, da je G regularna (na vozliščih), če je semiregularna in tranzitivna na $\mathrm{V}(\Gamma)$.

Struktura grafa, ki dopušča razmeroma visoko stopnjo simetrije, je pogosto povsem določena z delovanjem podgrupe grupe avtomorfizmov. Kar nekaj konstrukcij nam omogoča, da izrazimo strukturo grafa z visoko stopnjo simetrije z delovanjem grupe in tako rekonstruiramo graf iz majhne količine podatkov. V nadaljevanju bomo opisali tri zglede.

Naj bo G grupa, naj bo $H \leq G$ in naj bo $a \in G \setminus H$ tak element, da je $a^2 \in H$. *Odsekovni graf* $\mathrm{Cos}(G, H, a)$ je graf, katerega vozlišča so desni odseki H v G, kar označimo z G/H, povezave pa so oblike (Hg, Hsg), kjer sta $g \in G$ in $s \in HaH$. Dobro je znano, da je G-ločno-tranzitiven graf Γ izomorfen $\mathrm{Cos}(G, G_v, a)$, kjer je G_v stabilizator vozlišča $v \in \mathrm{V}(\Gamma)$ in $a \in G$ avtomorfizem, ki izmenja v z enim od sosedov; več podrobnosti lahko najdete v [9, razdelek 1.2] ali [50].

Naj bo zdaj G grupa in naj bo $S \subset G$ taka podmnožica G, da $1_G \notin S$. *Cayleyjev graf* $\mathrm{Cay}(G, S)$ je enostaven graf, katerega vozlišča so elementi G, tako da je $g \sim gs$ za vse $g \in G$ in $s \in S$. Slavni Sabidussijev rezultat [64] pravi, da je enostaven graf Cayleyjev natanko tedaj, ko dopušča regularno grupo avtomorfizmov. Natančneje, struktura G-regularnega grafa Γ je povsem določena z G in množico $S \subset G$, ki vsebuje vse avtomorfizme G, ki preslikajo izbrano vozlišče v v enega od sosedov v (ker je G regularna, je $|S| = \mathrm{val}(v)$). V tem primeru je Γ izomorfen $\mathrm{Cay}(G, S)$.

Splošneje, če grupa avtomorfizmov G deluje semiregularno na vozliščih Γ, potem lahko s konstrukcijo regularnega krova graf Γ rekonstruiramo iz manjšega, t. i. napetostnega grafa, ki ga zdaj podrobneje definiramo. Naj bo Δ graf in naj bo G grupa. Dodelitev napetosti je funkcija $\zeta\colon \mathrm{D}(\Delta) \to G$, ki dodeli *napetost* vsaki polpovezavi grafa Δ, tako da velja $\zeta(x^{-1}) = \zeta(x)^{-1}$ za vsako polpovezavo x. Trojici (Δ, G, ζ) pravimo *napetostni graf*. *Regularni krov* napetostnega grafa (Δ, G, ζ), zapisano $\mathrm{Cov}(\Delta, G, \zeta)$, je graf z množico vozlišč $\mathrm{V}(\Delta) \times G$ in povezavami $(u, g) \sim (v, h)$, pri čemer je $u \sim v$ v Δ in $h = \zeta(x)g$ za neko polpovezavo $x \in \mathrm{D}(\Delta)$, tako da je $\mathrm{beg}\, x = u$ in $\mathrm{end}\, x = v$. Množici $\{(x, g) \mid g \in G\}$ za $x \in \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta)$ pravimo *vlakno* nad x in jo označimo s $\mathrm{fib}(x)$. Grupa G deluje kot grupa avtomorfizmov z desnim množenjem na drugi komponenti vozlišč in polpovezav $\mathrm{Cov}(\Delta, G, \zeta)$. Orbite G na vozliščih in polpovezavah so natanko vlakna $\mathrm{Cov}(\Delta, G, \zeta)$. še več, delovanje G je regularno na vlaknu vsakega vozlišča in polpovezave ter semiregularno na vozliščih in polpovezavah $\mathrm{Cov}(\Delta, G, \zeta)$.

Ključno dejstvo v teoriji napetostnih grafov je, da je vsak graf Γ s semiregularno grupo avtomorfizmov G izomorfen grafu $\mathrm{Cov}(\Gamma/G, G, \zeta)$, kjer je $\zeta\colon \mathrm{D}(\Gamma/G) \to G$ primerna dodelitev napetosti, pri čemer Γ/G označuje kvocient Γ po G.

še posebej uporabna je na področju multicirkulantov. Dolgoletna domneva o policirkulantih trdi, da je vsak vozliščno-tranzitiven graf k-multicirkulant za nek k. Posebej zanimiv je študij pogojev, pod katerimi vozliščno-tranzitivni grafi dopuščajo k-multicirkulantni avtomorfizem visokega reda (posledično je k razmeroma majhen); glej na primer [7, 69]. Po drugi strani obstoj k-multicirkulantnega avtomorfizma z majhnim k pogosto omeji strukturo vozliščno-tranzitivnega grafa do te mere, da je mogoča klasifikacija. Tako imamo serijo klasifikacij kubičnih ločno-tranzitivnih k-multicirkulantov za $k \leq 5$ (glej [22, 35]). Konkretneje, kubični ločno-tranzitivni tricirkulanti so bili povsem klasificirani v [35], kjer so pokazali, da obstajajo samo štirje takši grafi: $K_{3,3}$, Pappusov graf, Tutte-Coxeterjev graf in graf na 54 vozliščih. Popolna klasifikacija kubičnih ločno-tranzitivnih 4- in 5-multicirkulantov je opisana v [22]. Obstaja neskončno mnogo povezanih kubičnih ločno-tranzitivnih 4-multicirkulantov, medtem ko sta povezana kubična ločno-tranzitivna 5-multicirkulanta le dva. To delo je doseglo vrhunec v lepem članku [26], kjer so pokazali, da za vsa števila k, ki so deljiva brez kvadrata in tuja številu 6, obstaja le končno mnogo kubičnih ločno-tranzitivnih k-multicirkulantov.

Za širši in strukturno bogatejši razred kubičnih vozliščno-tranzitivnih grafov, ki niso nujno ločno-tranzitivni, raziskovanje ni obrodilo tako bogatih rezultatov, vsaj delno zaradi pomanjkanja močnih algebrajskih orodij, ki so na voljo v ločno-tranzitivnem primeru.

Znano je, da ima vsak kubičen vozliščno-tranzitiven graf k-multicirkulantni avtomorfizem [46] in da ne obstaja tak k, da bi bil vsak kubičen vozliščno-tranzitiven graf k-multicirkulant [69]. Iz klasifikacij je razvidno, da je kubičen graf vozliščno-tranzitiven 1-cirkulant natanko tedaj, ko je kubičen Cayleyjev graf ciklične grupe. še več, vozliščno-tranzitivni kubični bicirkulanti so bili klasificirani v [54], kjer so pokazali tudi, da obstajajo tri neskončne družine vozliščno-tranzitivnih bicirkulantov, od katerih vsaka ustreza enemu od treh povezanih kubičnih grafov na dveh vozliščih brez vzporednih polpovezav.

Pričujoča doktorska disertacija ima dva glavna cilja. Prvi je, grobo rečeno, študij in klasifikacija kubičnih vozliščno-tranzitivnih grafov, ki dopuščajo ciklično grupo avtomorfizmov z orbito na vozliščih dane relativne velikosti (glede na red grafa). Najprej želimo posplošiti rezultate iz [35] in [54] ter podati popolno klasifikacijo kubičnih tricirkulantov, v smislu njihovih možnih kvocientov s tricirkulantnimi avtomorfizmi in karakterizacije vozliščne-tranzitivnosti za vsak možen razred. Določili bomo tudi nekatere strukturne lastnosti teh grafov, če so vozliščno-tranzitivni (kot npr. ožino, signaturo ožine, dvodelnost, povezavno-tranzitivnost itd.). Nadalje posplošimo ta rezultat z razširitvijo klasifikacije na veliko širšo družino kubičnih vozliščno-tranzitivnih grafov, ki imajo ciklično grupo avtomorfizmov G z orbito vozlišč, ki vsebuje vsaj $\frac{1}{3}$ skupnega števila vozlišč v grafu. Klasificiramo grafe v tej družini v smislu njihovih možnih kvocientov z G. V ta namen potrebujemo nove metode za konstruiranje grafov iz kvocientov, kar nas vodi k našemu drugemu cilju.

Drugi cilj je razvoj novih orodij za konstrukcijo grafov, ki dopuščajo podgrupo grupe avtomorfizmov izomorfno dani grupi. želimo posplošiti teorijo napetostnih grafov in študirati grafe brez kakršnihkoli simetrijskih omejitev. V ta namen bomo vpeljali koncept *posplošenega napetostnega grafa* $(\Delta, G, \omega, \zeta)$, kjer je Δ graf (imenovan *temeljni graf*), G poljubna grupa, $\omega\colon \mathrm{D}(\Delta) \cup \mathrm{V}(\Delta) \to S(G)$, kjer $S(G)$ označuje množico vseh podgrup grupe G, in $\zeta\colon \mathrm{D}(\Delta) \to G$. Vpeljali bomo tudi graf $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$, imenovan *posplošeni krov* grafa $(\Delta, G, \omega, \zeta)$. Pokazali bomo, da je graf Γ z grupo avtomorfizmov $G \leq \mathrm{Aut}(\Gamma)$

izomorfen posplošenemu krovu $\mathrm{GenCov}(\Delta/G, G, \omega, \zeta)$, kjer je Γ/G kvocient grafa Γ z grupo G, ω in ζ pa sta primerno izbrani preslikavi (ne zahtevamo, da G deluje na kakršenkoli poseben način). Karakterizirali bomo strukturne lastnosti grafa $\mathrm{GenCov}(\Delta/G, G, \omega, \zeta)$ v smislu preslikav ω in ζ, raziskali pa bomo tudi lastnosti njegove grupe avtomorfizmov. To orodje bomo uporabili v prej omenjeni klasifikaciji kubičnih vozliščno-tranzitivnih grafov Γ, ki dopuščajo grupo avtomorfizmov G z orbito velikosti vsaj $\frac{|\mathrm{V}(\Gamma)|}{3}$, v smislu njihovega kvocienta, kadar delovanje grupe G ni semiregularno.

Naj bo dano pozitivno celo število k, graf Γ in grupa $G \leq \mathrm{Aut}(\Gamma)$. G-orbita na $\mathrm{V}(\Gamma)$ je $[k]$-orbita, če ima orbito velikosti vsaj $\frac{n}{k}$, kjer je n red grafa Γ. Označimo z $\mathcal{G}$ razred vseh kubičnih grafov, ki imajo ciklično grupo avtomorfizmov s [3]-orbito. Pri naši analizi razreda grafov $\mathcal{G}$ bomo lahko raziskovali multicirculantne grafe (tiste, ki dopuščajo ciklično grupo avtomorfizmov, katerih orbite so vse iste velikosti) v tej družini s pomočjo teorije napetostnih grafov, podobno kot je to storjeno v [22, 35, 54]. Kubični k-multicirkulant Γ reda n mora biti regularni krov napetostnega grafa (Δ, G, ζ), kjer je Δ povezan kubični graf na k vozliščih in brez vzporednih polpovezav, G pa je ciklična grupa reda $\frac{n}{k}$. Kot je bilo že omenjeno, so kubični k-multicirkulanti za $k \in \{1, 2\}$ dobro znani in klasificirani. Posebej, kubični vozliščno-tranzitivni cirkulant je izomorfen prizmi ali Möbiusovi lestvi. Pisanski [54] je pokazal, da je kubični vozliščno-tranzitivni bicirkulant bodisi cirkulant bodisi je izomorfen cikličnemu Haarovemu grafu ali pa posplošenemu Petersenovemu grafu $\mathrm{GP}(n, m)$, kjer je $m^2 \equiv \pm 1 \pmod{n}$.

Za kubične tricirkulante ni težko videti, da obstajajo 4 neizomorfni povezani kubični grafi na 3 vozliščih brez vzporednih polpovezav, ki jih bomo označili z Δ_i, kjer je $i \in \{1, 2, 3, 4\}$. Kubični tricirkulant Γ je tedaj regularen krov napetostnega grafa (Δ_i, G, ζ) za neki $i \in \{1, 2, 3, 4\}$. Razred kubičnih tricirkulantov se tedaj deli na štiri neskončne razrede, vsak izmed njih pa ustreza množici regularnih krovov grafa Δ_i (graf Γ je graf *Tipa i*, kadar je regularni krov grafa Δ_i); glej Teorem REF. Nato je treba karakterizirati vozliščno-tranzitivnost za vsak razred. Ker ni močnih algebrajskih orodij, kot so tista, ki so na voljo za kubične ločno-tranzitivne grafe (glej [35, 22]), je potreben bolj kombinatorični pristop.

Izrek 8.0.1. Kubični graf Γ je vozliščno-tranzitiven tricirkulant, če in samo če je red grafa Γ 6k za neko pozitivno število k in hkrati velja eno od naslednjega:

(1) Γ je tipa 1, $\mathrm{k} > 1$, k je liho število in Γ je izomorfen grafu $\mathrm{X(k)}$ (glej Definicijo 3.2.5), ali pa je Γ izomorfen prisekanemu tetraedru $\mathrm{Tr(K4)}$.

(2) Γ je tipa 2, $\mathrm{k} > 1$, k je liho število in Γ je izomorfen grafu $\mathrm{Y(k)}$ (glej Definicijo 3.2.29).

(3) Γ je tipa 3 in je izomorfen prizmi $\mathrm{P_{3k}}$ kjer je k liho število, ali pa je Γ izomorfen Mobiusovi lestvi $\mathrm{M_{3k}}$.

(4) Γ je tipa 4, $\mathrm{k} = 5$ in Γ je izomorfen Tutte-Coxeterjevemu grafu.

Pri razširitvi obsega te raziskave na vozliščno-tranzitivne grafe iz $\mathcal{G}$ bi se bilo naravno lotiti klasifikacije te družine grafov v smislu njihovih možnih kvocientov s ciklično grupo avtomorfizmov s [3]-orbito. Pri tem naletimo na dve glavni oviri.

Prva je ta, da če je $\Gamma \in \mathcal{G}$ in je $G \leq \mathrm{Aut}(\Gamma)$ ciklična grupa s $[k]$-orbito, potem ima kvocientni graf Γ/G lahko, vsaj načeloma, poljubno mnogo vozlišč. Iz tega sledi, da lahko imamo neskončno mnogo kvocientnih grafov Γ/G, kjer je $\Gamma \in \mathcal{G}$ in je $G \leq \mathrm{Aut}(\Gamma)$ ciklična s [3]-orbito, zaradi česar je klasifikacija (s končnim številom razredov) preko kvocientov nemogoča. Druga ovira pa je, da je teorija regularnih krovov dobljenih iz napetostnih grafov, ki je naše močno orodje, kadar obravnavamo kubične k-cirkulantne grafe, v tem kontekstu nezadostna, saj ni vsak graf iz $\mathcal{G}$ regularni krov nad svojim kvocentom Γ/G po ciklični grupi automorfizmov s [3]-orbito. Zato moramo za reševanje tega problema razviti nova orodja.

Oglejmo si podrobneje prvo oviro. Vprašanje, ali je klasifikacija vozliščno-tranzitivnih grafov iz $\mathcal{G}$ s pomočjo njihovih kvocientov po ciklični grupi s [3]-orbito možna, se skrči na naslednje vprašanje.

Vprašanje 8.0.2. Naj bo Γ kubičen vozliščno-tranzitiven graf in naj bo $G \in \mathrm{Aut}(\Gamma)$ ciklična grupa, ki ima [3]-orbito na vozliščih. Ali je skupno število orbit grupe G na $\mathrm{V}(\Gamma)$ omejeno?

če je število G-orbit omejeno, potem je red kvocientnega grafa Γ/G prav tako omejen. Tedaj je skupno število možnih kvocientnih grafov Γ/G, kjer ima $G \leq \mathrm{Aut}(\Gamma)$ [3]-orbito, končno. Pokažemo, da je odgovor na vprašanje 8.0.2 pritrdilen tudi v bolj splošnem kontekstu.

Izrek 8.0.3. Naj bo k pozitivno celo število in naj bo Γ kubičen vozliščno-tranzitiven graf, ki dopušča ciklično grupo $G \leq \mathrm{Aut}(\Gamma)$ s $[k]$-orbito na vozliščih. Potem je število orbit grupe G na vozliščih omejeno s $6k - 5$.

O klasifikaciji vozliščno-tranzitivnih grafov iz razreda $\mathcal{G}$ laho razmislimo v smislu njihovih kvocientov s ciklično grupo avtomorfizmov G s [3]-orbito. To nas vodi k drugi oviri: za graf $\Gamma \in \mathcal{G}$ ciklična grupa $G \leq \mathrm{Aut}(G)$ s [3]-orbito morda ne deluje semiregularno na vozliščih grafa Γ. Tedaj Γ ni regularni krov nobenega napetostnega grafa $(\Gamma/G, G, \zeta)$. Možen način reševanja te težave je, da priskrbimo splošnejšo konstrukcijo (kot je konstrukcija regularnega krova), ki nam bo dovoljevala rekonstruirati katerikoli graf Γ z grupo avtomorfizmov G (ki morda ni semiregularna) iz kvocientnega grafa Γ/G.

Vprašanje 8.0.4. če je Γ graf in $G \leq \mathrm{Aut}(\Gamma)$, kakšno informacijo o delovanju G na polpovezavah in vozliščih grafa Γ je treba zabeležiti, da se bo dalo rekonstruirati Γ iz kvocientnega grafa Γ/G?

V primeru, ko je G semiregularna, je dovolj zabeležiti množico posebnih avtomorfizmov grafa Γ, ki imajo en element za vsako G-orbito na polpovezavah. Bodimo natančnejši. Naj bo $\mathcal{T}$ transverzala za delovanje G na $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Delta)$ (tj. na množici, ki vsebuje natančno en element vsake orbite na polpovezavah in vozliščih). Predpostavimo, da za vsako polpovezavo $x \in \mathcal{T}$ velja $\mathrm{beg}\, x \in \mathcal{T}$. Za vsako polpovezavo $x \in \mathcal{T}$ naj bo ζ_x avtomorfizem grafa Γ, ki preslika x^{-1} v transverzalo $\mathcal{T}$. Potem je struktura Γ povsem določena z grupo G, kvocientom Γ/G in množico $\{\zeta_x \mid x \in \mathrm{D}(\Gamma) \cap \mathcal{T}\}$. Res, Γ je izomorfen regularnemu krovu $\mathrm{Cov}(\Gamma/G, G, \zeta)$, kjer je dodelitev napetosti $\zeta\colon \mathrm{D}(\Gamma/G) \to G$ dana z $\zeta(x^G) = \zeta_x$ za vse $x \in \mathrm{D}(\Gamma/G)$ (spomnimo se, da so polpovezave Γ/G orbite grupe G na $\mathrm{D}(\Gamma)$). V primeru, ko G ni semiregularna, potrebujemo dodatno informacijo o stabilizatorjih elementov $\mathcal{T}$. V ta namen definiramo *posplošeni napetostni graf* in njegov *posplošeni krov*, s čimer posplošimo konstrukcijo regularnega krova iz napetostnega grafa, ki je bila vpeljana v klasičnem delu Grossa in Tuckerja [30, razdelek 2.1.1].

Definicija 8.0.5. Naj bo Δ povezan graf in G grupa. Naj bosta $\omega\colon \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta) \to S(G)$ in $\zeta\colon \mathrm{D}(\Delta) \to G$ dve preslikavi, tako da za vse $x \in \mathrm{D}(\Delta)$ velja naslednje:

$$\omega(x) \leq \omega(\mathrm{beg}_\Delta x); \tag{8.0.1}$$

$$\omega(x) = \omega(x^{-1})^{\zeta(x)}; \tag{8.0.2}$$

$$\zeta(x^{-1})\zeta(x) \in \omega(x). \tag{8.0.3}$$

Potem pravimo, da je četverica $(\Delta, G, \omega, \zeta)$ *posplošeni napetostni graf*, preslikavama ω in ζ pa pravimo *utežna funkcija* oz. *dodelitev napetosti.*

Definicija 8.0.6. Naj bo $(\Delta, G, \omega, \zeta)$ posplošeni napetostni graf in naj bo Γ graf, ki je definiran z:

- $\mathrm{V}(\Gamma) = \{(v, \omega(v)g) \mid g \in G, v \in \mathrm{V}(\Delta)\}$;
- $\mathrm{D}(\Gamma) = \{(x, \omega(x)g) \mid g \in G, x \in \mathrm{D}(\Delta)\}$;
- $\mathrm{beg}_\Gamma(x, \omega(x)g) = (\mathrm{beg}_\Delta(x), \omega(\mathrm{beg}_\Delta x)g)$;
- $\mathrm{inv}_\Gamma(x, \omega(x)g) = (\mathrm{inv}_\Delta x, \omega(\mathrm{inv}_\Delta x)\zeta(x)g)$.

Potem Γ imenujemo *posplošeni krov*, ki je porojen iz $(\Delta, G, \omega, \zeta)$ in ga označimo z $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$.

Izrek 8.0.7. če je $(\Delta, G, \omega, \zeta)$ posplošeni napetostni graf, potem graf $\mathrm{GenCov}(\Delta, G, \omega, \zeta)$ dopušča grupo avtomorfizmov, ki je izomorfna grupi G.

Izrek 8.0.8. Naj bo Γ graf, $G \leq \mathrm{Aut}(\Gamma)$ in naj bo $S(G)$ množica vseh podgrup grupe G. Naj bo $\mathcal{T}$ transverzala delovanja grupe G na $\mathrm{V}(\Gamma) \cup \mathrm{D}(\Gamma)$ in naj bo $\mathrm{beg}\, x \in \mathcal{T}$ za vsako polpovezavo $x \in \mathcal{T}$.

Naj bo $\zeta\colon \mathrm{D}(\Gamma/G) \to G$ poljubna preslikava z lastnostjo, da za vsako polpovezavo x v $\mathcal{T}$ element $\zeta(x^G)$ preslika x^{-1} v polpovezavo iz $\mathcal{T}$.

Naj bo $\omega\colon \mathrm{V}(\Delta) \cup \mathrm{D}(\Delta) \to S(G)$ preslikava, dana z $\omega(x^G) = G_x$ za vse $x \in \mathcal{T}$, kjer G_x označuje stabilizator polpovezave x v grupi G.

Potem je $(\Gamma/G, G, \omega, \zeta)$ posplošeni napetostni graf in velja

$$\Gamma \cong \mathrm{GenCov}(\Gamma/G, G, \omega, \zeta).$$

V primeru (regularnega) napetostnega grafa (Γ, G, ζ) je mogoče strukturne lastnosti krova $\mathrm{Cov}(\Gamma, G)$, kot so npr. povezanost ali ožina, pogosto izpeljati iz dodelitve napetosti ζ. Podobne rezultate pokažemo za posplošene napetostne grafe v Poglavju 4 kjer karakteriziramo lastnosti posplošenega krova v smislu funkcij ω in ζ.

Naj bo Γ graf in $G \leq \mathrm{Aut}(\Gamma)$. Nadalje, naj bo $x \in \mathrm{D}(\Gamma)$ polpovezava in $u = \mathrm{beg}\, x$. Potem je x^G polpovezava v Γ/G z začetnim vozliščem u^G. Z $\lambda_G(x^G)$ označimo število polpovezav Γ v orbiti x^G, ki se začnejo v poljubnem vozlišču v u^G (to je neodvisno od tega, katero vozlišče u^G izberemo).

Tedaj je $\lambda_G\colon \mathrm{D}(\Gamma/G) \to \mathbb{N}$ dobro definirana funkcija in paru $(\Gamma/G, \lambda_G)$ rečemo označen kvocient. Za klasifikacijo elementov $\mathcal{G}$ uporabimo pojem *cikličnega posplošenega napetostnega grafa* in z njim povezano konstrukcijo posplošenega krovnega grafa, ki ga lahko smatramo za poenostavljen poseben primer konstrukcije posplošenega krova, kjer je napetostna grupa ciklična. V grobem pri tej konstrukciji začnemo z označenim grafom (Δ, λ) (skupaj z dodatnima celoštevilskima funkcijama *indeks* ι in *dodelitev napetosti* ζ) in konstruiramo povezan graf Γ (*ccv-krov, kadar je* Γ *kubičen*), katerega označen kvocient je (Δ, λ).

Izrek 8.0.9. Naj bo Γ kubičen graf in $G \leq \mathrm{Aut}(\Gamma)$ ciklična grupa. Potem obstajata funkciji $\iota\colon \mathrm{V}(\Gamma/G) \to \mathbb{Z}$ in $\zeta\colon \mathrm{D}(\Gamma/G) \to \mathbb{Z}$, tako da je Γ *ccv*-krov označenih kvocientov $(\Gamma/G, \lambda_G)$.

Naj bo $\mathcal{Q}$ množica označenih kvocientnih grafov $(\Gamma/G, \lambda_G)$, kjer je $\Gamma \in \mathcal{G}$ in $G \leq \mathrm{Aut}(\Gamma)$ ciklična s [3]-orbito. Po izrekih 8.0.3, je $\mathcal{Q}$ končna in vsak graf $\mathcal{G}$ je *ccv*-krov nekega označenega grafa $(\Delta, \lambda) \in \mathcal{Q}$. Klasifikacijo kubičnih vozliščno transitivnih tricirkulantov lahko razširimo, tako da zaobjame tudi vse vožliščno tranzitivne grafe v $\mathcal{G}$, tako da za vsak $(\Delta, \lambda) \in \mathcal{Q}$ določimo nujne in zadostne pogoje, da je njihov *ccv*-krov element $\mathcal{G}$.

Izrek 8.0.10. Naj bo Γ kubičen vozliščno tranzitiven graf reda n. Potem Γ dopušča regularno orbito dolžine vsaj $\frac{n}{3}$ natanko tedaj, ko je *ccv*-krov enega od sedmih označenih grafov (Slika 7.2.7). T.j., Γ je izomorfen enemu od naslednjih grafov:

1. $C(m; r)$, za sod m, $m \geq 4$, $\gcd(\frac{m}{2}, r) = 1$ in $r \in \{1, 2\}$;
2. $I(m; r, 1)$ za $m \geq 3$ in $r^2 \equiv \pm 1 \pmod{m}$, ali $m = 10$ in $r = 2$;
3. $H(m; r, s)$ za $m \geq 3$, $r \neq s$ in $\gcd(m, r, s) = 1$;
4. $T_1(m; r, 1)$ za $\frac{m}{2} \equiv 1 \pmod 4$ in $r = (\frac{m}{2} + 3)/2$, ali $\frac{m}{2} \equiv 3 \pmod 4$ in $r = (\frac{3m}{2} + 3)/2$, ali $m = 4$ in $r = 0$;
5. $T_2(m; 2, 1)$ za $m \geq 4$ in liho $\frac{m}{2}$;
6. $T_4(5, 1, 3)$;
7. $\Gamma_{11}(m, 1, 2)$ za lih m.

Appendix

The following is a complete list of all labelled graphs in $\mathcal{Q}^*$. Each item, labelled Gi for some $0 \leq i \leq 362$, is the (python-readable) list of the links of a labelled graph in $\mathcal{Q}^*$. Each link is represented by a triple of the form $(u, v, (l_1, l_2))$ where u is the initial vertex, v is the final vertex and the label (l_1, l_2) indicates that this is an edge of type $[l_1, l_2]$. Equivalently, we may think of $(u, v, (l_1, l_2))$ as a pair of mutually inverse darts $\{x, x^{-1}\}$ where $\operatorname{beg} x = u$, $\operatorname{beg} x^{-1} = v$, $\lambda(x) = l_1$ and $\lambda(x^{-1}) = l_2$. Note that loops and semi-edges are not included on this list, as an element of $\mathcal{Q}^*$ is completely determined by its links. The list is divided in 10 sublists as follows:

- Graphs $G0$–$G165$ contain a copy of A_6.
- Graphs $G166$–$G227$ contain a copy of A_7.
- Graphs $G228$–$G269$ contain a copy of A_1 and vertex v that is not contain in either a copy of A_1, a triangle or a loop.
- Graphs $G270$–$G307$ contain a copy of A_i with $i \in \{2, 3, 4\}$ and a $[1, 3]$-edge
- Graphs $G308$–$G321$ contain a copy of A_5.
- Graphs $G322$–$G333$ contain a copy of A_8.
- Graphs $G334$–$G346$ have order 3 or less.
- Graph $G347$ has no cycles (except for semi-edges) and by Lemma 5.2.10, only admits a single connected cover, which is not vertex-transitive.
- Graphs $G348$–$G351$ contain a copy of A_9.
- Graphs $G352$–$G362$ are the eleven graphs Δ_i with $i \in \{1, \ldots, 11\}$.

List of elements of $\mathcal{Q}^*$:

```
G0= [(0, 1, (1, 2))]
G1= [(0, 1, (1, 2)), (1, 2, (1, 1))]
G2= [(0, 1, (1, 2)), (1, 2, (1, 2))]
G3= [(0, 1, (1, 2)), (1, 2, (1, 3))]
G4= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1))]
G5= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 3, (1, 1))]
G6= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2))]
G7= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 3))]
G8= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (2, 1))]
```

G9= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (2, 3))]
G10= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 2))]
G11= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 1))]
G12= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 1)), (3, 4, (1, 1))]
G13= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1))]
G14= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 3))]
G15= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 3))]
G16= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2))]
G17= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 4, (1, 1))]
G18= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 3)), (2, 4, (1, 3))]
G19= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1))]
G20= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 1))]
G21= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 1))]
G22= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1))]
G23= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2))]
G24= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2))]
G25= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 3))]
G26= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 3))]
G27= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (2, 3))]
G28= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 3))]
G29= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 2))]
G30= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 2)), (4, 5, (1, 1))]
G31= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 3)), (3, 5, (1, 3))]
G32= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1))]
G33= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 2)), (3, 5, (1, 2))]
G34= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 2)), (3, 5, (1, 2)), (4, 5, (1, 1))]
G35= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 2)), (3, 5, (1, 3))]
G36= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 2)), (4, 5, (1, 1))]
G37= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (1, 2))]
G38= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (1, 3))]
G39= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (2, 3))]
G40= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (1, 3)), (3, 6, (1, 3))]
G41= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1))]
G42= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1))]
G43= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 5, (1, 1))]
G44= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1))]
G45= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1))]
G46= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1))]
G47= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 3))]
G48= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1))]
G49= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1))]
G50= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G51= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2)), (4, 5, (1, 1))]
G52= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2)), (4, 5, (1, 1))]
G53= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1))]
G54= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G55= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 1))]
G56= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 3))]
G57= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1))]
G58= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G59= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G60= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1)), (5, 6, (1, 1))]
G61= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G62= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1)), (5, 6, (2, 3))]
G63= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 6, (1, 1))]
G64= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G65= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G66= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (5, 6, (1, 1))]
G67= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]

G68= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G69= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (5, 6, (2, 3))]
G70= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1))]
G71= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1))]
G72= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G73= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (2, 1))]
G74= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (2, 3))]
G75= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1))]
G76= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1))]
G77= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G78= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G79= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G80= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 1))]
G81= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G82= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 2))]
G83= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 3))]
G84= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (2, 3))]
G85= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 2)), (4, 6, (1, 3))]
G86= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 2)), (4, 6, (1, 2))]
G87= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 2)), (4, 6, (1, 2)), (5, 6, (1, 1))]
G88= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 3)), (4, 6, (1, 3))]
G89= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1))]
G90= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1))]
G91= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G92= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G93= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (2, 1))]
G94= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (2, 3))]
G95= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1))]
G96= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (5, 6, (1, 1))]
G97= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (6, 7, (1, 1))]
G98= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G99= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1))]
G100= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1))]
G101= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G102= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 7, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G103= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G104= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G105= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G106= [(0, 1, (1, 2)), (1, 2, (1, 1)), (2, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 3))]
G107= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G108= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G109= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G110= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G111= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G112= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 1))]
G113= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 1))]
G114= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 3))]
G115= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 3))]
G116= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1))]
G117= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7,

(1, 1))]
G118= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 6, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1))]
G119= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1))]
G120= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G121= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G122= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 6, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1))]
G123= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G124= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G125= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G126= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (2, 1)), (5, 6, (1, 2))]
G127= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 6, (1, 1)), (4, 5, (2, 1)), (5, 6, (1, 2))]
G128= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (2, 1)), (5, 6, (1, 3))]
G129= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1))]
G130= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1))]
G131= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 6, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1))]
G132= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (5, 7, (1, 1))]
G133= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G134= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G135= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 6, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (5, 7, (1, 1))]
G136= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G137= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G138= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (2, 3))]
G139= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 6, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (2, 3))]
G140= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (2, 1)), (4, 5, (1, 2)), (5, 6, (1, 1))]
G141= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1))]
G142= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1))]
G143= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G144= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (5, 7, (1, 1))]
G145= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G146= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G147= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (5, 7, (1, 1)), (5, 7, (1, 1)), (6, 7, (1, 1))]
G148= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (2, 3))]
G149= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (5, 7, (2, 3))]
G150= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (6, 7, (1, 1))]
G151= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (4, 6, (1, 1)), (6, 7, (1, 1))]
G152= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (6, 7, (1, 1)), (6, 7,

(1, 1))]
G153= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (1, 1)), (6, 7, (2, 3))]
G154= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 1)), (4, 6, (2, 3)), (5, 7, (2, 3))]
G155= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1))]
G156= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1))]
G157= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1))]
G158= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1))]
G159= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G160= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1))]
G161= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G162= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (1, 1)), (6, 7, (1, 1))]
G163= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (2, 3))]
G164= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (2, 3))]
G165= [(0, 1, (1, 2)), (1, 2, (1, 2)), (2, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (2, 3))]

G166= [(0, 1, (1, 2)), (0, 2, (1, 2))]
G167= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 2, (1, 1))]
G168= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1))]
G169= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 3, (1, 1))]
G170= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2))]
G171= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 3))]
G172= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 1))]
G173= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 1)), (3, 4, (1, 1))]
G174= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1))]
G175= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 2))]
G176= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 3))]
G177= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 1))]
G178= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 1))]
G179= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1))]
G180= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 2))]
G181= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2))]
G182= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 3))]
G183= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 3))]
G184= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (2, 3))]
G185= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 3))]
G186= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 2))]
G187= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 2)), (4, 5, (1, 1))]
G188= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 3)), (3, 5, (1, 3))]
G189= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2))]
G190= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 4, (1, 1))]
G191= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 3))]
G192= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1))]
G193= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 3)), (2, 4, (1, 3))]
G194= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 2)), (3, 5, (1, 2))]
G195= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 2)), (3, 5, (1, 2)), (4, 5, (1, 1))]
G196= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 2)), (3, 5, (1, 3))]
G197= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 2)), (4, 5, (1, 1))]
G198= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (1, 2))]
G199= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (1, 3))]
G200= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (2, 3))]
G201= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 4, (1, 3)), (3, 5, (1, 3)), (3, 6, (1, 3))]
G202= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (3, 4, (1, 2)), (4, 5, (1, 1))]

G203= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 1)), (2, 3, (1, 1)), (3, 4, (1, 2)), (4, 5, (1, 1))]
G204= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1))]
G205= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 5, (1, 1))]
G206= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1))]
G207= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1))]
G208= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G209= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 1))]
G210= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 3))]
G211= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1))]
G212= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G213= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G214= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 6, (1, 1))]
G215= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G216= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G217= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (5, 6, (1, 1))]
G218= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G219= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G220= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 2)), (3, 5, (1, 1)), (5, 6, (2, 3))]
G221= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1)), (5, 6, (1, 1))]
G222= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G223= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 4, (1, 3)), (3, 5, (1, 1)), (5, 6, (2, 3))]
G224= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G225= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G226= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G227= [(0, 1, (1, 2)), (0, 2, (1, 2)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 3))]

G228= [(0, 1, (1, 2)), (0, 2, (1, 3))]
G229= [(0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 3))]
G230= [(0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 2))]
G231= [(0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 3))]
G232= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1))]
G233= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2))]
G234= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 3))]
G235= [(0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 3)), (2, 4, (1, 3))]
G236= [(0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 2)), (3, 4, (1, 1))]
G237= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 1))]
G238= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1))]
G239= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2))]
G240= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 3))]
G241= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (2, 3))]
G242= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 1)), (3, 5, (1, 3))]
G243= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 3))]
G244= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 2))]
G245= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 2)), (4, 5, (1, 1))]
G246= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 3)), (3, 5, (1, 3))]
G247= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1))]
G248= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 1)), (3, 4, (1, 2))]
G249= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 1)), (4, 5, (1, 2))]
G250= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 3))]
G251= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 1)), (3, 4, (1, 1)), (4, 5, (1, 3))]
G252= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2)), (4, 5, (1, 1))]
G253= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2)), (3, 5, (1, 3)), (4, 6, (1, 1))]
G254= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1))]
G255= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G256= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 1))]
G257= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 3))]
G258= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1))]
G259= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1))]
G260= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 6, (1, 1)), (5, 6,

(1, 1))]
G261= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 1)), (3, 4, (1, 2)), (4, 5, (1, 1)), (5, 6, (2, 3))]
G262= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G263= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G264= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G265= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (2, 3))]
G266= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 1)), (5, 6, (1, 3))]
G267= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 6, (1, 1)), (5, 7, (2, 3))]
G268= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (2, 3))]
G269= [(0, 1, (1, 2)), (0, 2, (1, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1)), (5, 6, (1, 1)), (6, 7, (2, 3))]

G270= [(0, 1, (1, 3)), (0, 2, (1, 3))]
G271= [(0, 1, (1, 1)), (0, 2, (1, 3)), (0, 3, (1, 3))]
G272= [(0, 1, (1, 3)), (0, 2, (1, 3)), (0, 3, (1, 3))]
G273= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3))]
G274= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3))]
G275= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 2, (1, 1))]
G276= [(0, 1, (1, 1)), (0, 2, (1, 3)), (1, 3, (1, 3))]
G277= [(0, 1, (1, 1)), (0, 2, (1, 3)), (1, 3, (1, 3)), (1, 4, (1, 3))]
G278= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 3))]
G279= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 3)), (2, 3, (1, 1))]
G280= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 1))]
G281= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 2))]
G282= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 3))]
G283= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 1))]
G284= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 1)), (2, 4, (1, 1))]
G285= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2))]
G286= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 3))]
G287= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 1)), (3, 4, (1, 3)), (3, 5, (1, 3))]
G288= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2)), (2, 5, (1, 3))]
G289= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 3)), (2, 5, (1, 3))]
G290= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 1)), (4, 5, (1, 3))]
G291= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 1)), (4, 5, (2, 3))]
G292= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 1)), (4, 5, (1, 3)), (4, 6, (1, 3))]
G293= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 2)), (4, 5, (1, 1))]
G294= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 1)), (2, 5, (1, 3))]
G295= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 1)), (4, 5, (1, 3))]
G296= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 1)), (2, 4, (1, 1)), (4, 5, (1, 3))]
G297= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2)), (2, 5, (1, 2))]
G298= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2)), (2, 5, (1, 2)), (4, 5, (1, 1))]
G299= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2)), (2, 5, (1, 3))]
G300= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2)), (4, 5, (1, 1))]
G301= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 3)), (2, 5, (1, 3))]
G302= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 3)), (2, 5, (1, 3)), (3, 6, (1, 3))]
G303= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 2)), (4, 5, (1, 1)), (5, 6, (1, 1))]
G304= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 2)), (4, 5, (1, 1)), (5, 6, (1, 1)), (5, 6, (1, 1))]
G305= [(0, 1, (1, 2)), (0, 2, (1, 3)), (0, 3, (1, 3)), (1, 4, (1, 2)), (4, 5, (1, 1)), (5, 6, (2, 3))]
G306= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2)), (2, 5, (1, 3)), (4, 6, (1, 1))]
G307= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 3)), (1, 4, (1, 2)), (4, 5, (1, 1)), (5, 6, (2, 3))]

G308= [(0, 1, (2, 3))]
G309= [(0, 1, (1, 1)), (0, 2, (2, 3))]
G310= [(0, 1, (1, 2)), (0, 2, (2, 3))]
G311= [(0, 1, (1, 3)), (0, 2, (2, 3))]
G312= [(0, 1, (1, 1)), (0, 2, (1, 3)), (1, 3, (2, 3))]
G313= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 1))]
G314= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 2))]
G315= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 3))]
G316= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 1)), (3, 4, (1, 3))]
G317= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 1)), (3, 4, (2, 3))]

G318= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 2)), (3, 4, (1, 1))]
G319= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1))]
G320= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (1, 1)), (4, 5, (1, 1))]
G321= [(0, 1, (1, 2)), (0, 2, (2, 3)), (1, 3, (1, 2)), (3, 4, (1, 1)), (4, 5, (2, 3))]

G322= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2))]
G323= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 2, (1, 1))]
G324= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 1))]
G325= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 1)), (2, 3, (1, 1))]
G326= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 1)), (2, 4, (1, 1))]
G327= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 1)), (2, 4, (1, 1)), (3, 4, (1, 1))]
G328= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2))]
G329= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2)), (2, 3, (1, 1))]
G330= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2)), (2, 5, (1, 2))]
G331= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2)), (2, 5, (1, 2)), (4, 5, (1, 1))]
G332= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2)), (4, 5, (1, 1))]
G333= [(0, 1, (1, 2)), (0, 2, (1, 2)), (0, 3, (1, 2)), (1, 4, (1, 2)), (2, 3, (1, 1)), (4, 5, (1, 1))]

G334= []
G335= [(0, 1, (1, 1))]
G336= [(0, 1, (1, 1)), (0, 1, (1, 1))]
G337= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 1, (1, 1))]
G338= [(0, 1, (1, 3))]
G339= [(0, 1, (1, 1)), (0, 2, (1, 1))]
G340= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 1))]
G341= [(0, 1, (1, 1)), (0, 2, (1, 1)), (1, 2, (1, 1))]
G342= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 1)), (1, 2, (1, 1))]
G343= [(0, 1, (1, 1)), (0, 2, (1, 2))]
G344= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2))]
G345= [(0, 1, (1, 1)), (0, 2, (1, 3))]
G346= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 3))]

G347= [(0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 2))]

G348= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 1))]
G349= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 2))]
G350= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 3))]
G351= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 2)), (3, 4, (1, 1))]]

G352= [(0, 1, (1, 1)), (0, 2, (1, 2)), (0, 3, (1, 2))]
G353= [(0, 1, (1, 1)), (0, 2, (1, 2)), (0, 3, (1, 2)), (2, 3, (1, 1))]
G354= [(0, 1, (1, 1)), (0, 2, (1, 2)), (0, 3, (1, 3))]
G355= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 2))]
G356= [(0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 3, (1, 1))]
G357= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 2)), (2, 3, (1, 1))]
G358= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 3))]
G359= [(0, 1, (1, 1)), (0, 2, (1, 2)), (2, 3, (1, 1))]
G360= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 3)), (1, 3, (1, 3))]
G361= [(0, 1, (1, 1)), (0, 2, (1, 2)), (0, 3, (1, 3)), (2, 4, (1, 3))]
G362= [(0, 1, (1, 1)), (0, 1, (1, 1)), (0, 2, (1, 2)), (1, 3, (1, 3)), (2, 4, (1, 3))]