UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

Master's thesis
(Magistrsko delo)

Matching extensions in regular graphs with small diameter

(Razširljivost prirejanj v regularnih grafih z majhnim premerom)

Ime in priimek: Ivana Koroman
Študijski program: Matematične znanosti, 2. stopnja
Mentor: prof. dr. Štefko Miklavič

Koper, julij 2019

Ključna dokumentacijska informacija

Ime in PRIIMEK: Ivana KOROMAN

Naslov zaključne naloge: Razširljivost prirejanj v regularnih grafih z majhnim premerom

Kraj: Koper

Leto: 2019

Število listov: 59 Število slik: 15 Število referenc: 23

Mentor: prof. dr. Štefko Miklavič

UDK: 519.17(043.2)

Ključne besede: l-razširljiv, $l\frac{1}{2}$-razširljiv, regularen graph, povezavno-regularen graf, krepko regularen graf, premer 2, premer 3

Math. Subj. Class. (2010): 05C70

Izvleček:
Glavni predmet magistrskega dela je problem razširljivosti prirejanj v regularnih grafih z majhnim premerom. V magistrskem delu je ta problem formalno definiran. Nadalje je v magistrskem delu motivirano raziskovanje tega problema v posebnih grafovskih razredih. Povzet je dokaz izreka, ki pravi, da so vsi regularni grafi z lihim številom vozlišč in premerom 2 $0\frac{1}{2}$-razširljivi. Podana je kalsifikacija $1\frac{1}{2}$-razširljivih povezavno-regularnih grafov z lihim številom vozlišč in premerom 2, ter $2\frac{1}{2}$-razširljivih krepko regularnih grafov z lihim številom vozlišč. Za grafe s premerom 3 je dokazano, da so vsi regularni grafi s sodim številom vozlišč 0-razširljivi, ter vsi regularni grafi z lihim številom vozlišč $0\frac{1}{2}$-razširljivi. Konstruirani so tudi primeri regularnih in povezavno-regularnih grafov premera 3, ki niso 2-razširljivi oziroma niso $1\frac{1}{2}$-razširljivi.

Key words documentation

Name and SURNAME: Ivana KOROMAN

Title of final project paper: Matching extensions in regular graphs with small diameter

Place: Koper

Year: 2019

Number of pages: 59 Number of figures: 15 Number of references: 23

Mentor: Prof. Štefko Miklavič, PhD

UDK: 519.17(043.2)

Keywords: l-extendable, $l\frac{1}{2}$-extendable, regular graph, edge-regular graph, strongly regular graph, diameter 2, diameter 3

Math. Subj. Class. (2010): $05C70$

Abstract: The main topic of the Master thesis is extendability of matchings in regular graphs with small diameter. In the Master thesis we formally define this problem and motivate its research on restricted graph classes. We prove that all regular graphs with odd order and diameter 2 are $0\frac{1}{2}$-extendable. We classify $1\frac{1}{2}$-extendable edge-regular graphs with odd order and diameter 2. We provide several examples of non-$2\frac{1}{2}$-extendable edge-regular graphs with diameter 2 and odd order and classify all $2\frac{1}{2}$-extendable strongly regular graphs with an odd number of vertices. For graphs with diameter 3, we prove that all regular graphs with an even number of vertices have a perfect matching and that all regular graphs with odd order are $0\frac{1}{2}$-extendable. Also we construct several examples of non-2-extendable and non-$1\frac{1}{2}$-extendable regular and edge-regular graphs with diameter 3.

Contents

List of Figures

List of Abbreviations

WLOG	without losing of generality
i.e.	that is

Acknowledgement

I would like thank to my mentor and professor Štefko Miklavič for his time, patience, help and all reviews of my thesis.
I express my gratitude to UP FAMNIT for caring for its students.
Special gratitude to my dearest for unconditional love and support.

1 Introduction

Matching theory is one of the fundamental areas in graph theory. It studies the structures and properties of matchings. Over the last hundred years, matching theory has played a catalytic role in developing a number of more general combinatorial methods. Since its appearance, much new and interesting progress has been made. In last decades, matching extension theory is of particular interest.

In 1979, an interesting problem was posed by Sumner. The problem was to characterize the graphs with the property that every matching of a graph can be extended to a perfect matching. It turns out that the complete graphs and the complete bipartite graphs are the only two classes of graphs which satisfy this property. Obviously, the above condition is very strong since the matchings of arbitrary size have to be extended to a perfect matching. However, relaxing this property by requiring only the matching with the same size to be extended to a perfect matching, leads to an interesting refinement.

Let l be a non-negative integer. A connected graph Γ with vertex set $V(\Gamma)$ and an even number of vertices is l-extendable if it contains a perfect matching, $l \leq \frac{|V(\Gamma)|}{2}$, and any matching of l edges is contained in a perfect matching. The extendability of Γ is defined as maximum l such that Γ is l-extendable.

The definition of l-extendability was first introduced by Plummer in the 1980s (see [19]) for graphs with even order. Since then, this combinatorial parameter was studied from the various points of view and for many classes of interesting graphs.
For example, Chan et al. characterized all 2-extendable Cayley graphs on dihedral groups and on abelian groups. Holton and Lou first studied l-extendability of strongly regular graphs (see [13]), and conjectured that all but a few strongly regular graphs are 2-extendable. This conjecture was proved by Lou and Zhu (in [16]), since they proved that the Petersen graph, the complete multipartite graph $K_{3\times2}$ and K_4 are the only non-2-extendable strongly regular graphs with $k \geq 3$.
In 2005, Brouwer and Haemers proved that all distance-regular graphs with an even number of vertices are 1-extendable and in 2014, Cioabă and Li (see [9]) proved that any connected strongly regular graph with an even number of vertices is 3-extendable except for a small number of exceptions. These results were generalized by Cioabă

et al. in 2017 (see [10]). They showed that all distance-regular graphs with diameter greater than 2 are 2-extendable and obtain several lower bounds for the extendability of distance-regular graphs of valency $k \geq 3$.
Miklavič and Šparl (see [18]) proved that there are only few non-2-extendable Deza graphs with diameter 2. Alajbegović et al. (in [1]) gave the classification of 2-extendable quasistrongly regular graphs with diameter 2. Recently, Kutnar et al. (in [14]) gave a classification of 2-extendable edge-regular graphs with diameter 2.
From the algorithmic point of view there are not many results. Zhang and Zhang (see [23]) obtained an $O(mn)$ algorithm for determining the extendability of a bipartite graph with n vertices and m edges. The complexity of determining the extendability of a non-bipartite graph is still an open problem.

The notion of extendability was extended to graphs of odd order by Yu (in [21]), but until now there are not too many results in this area. Some results can be found in [15] and [9]. One of the well known results is that it is possible to characterize all $l\frac{1}{2}$-extendable graphs in terms of l-extendable and $(l+1)$-extendable graphs. Such a characterization was done by Yu and it can be found in [21].

Motivated by above results for graphs with even order, we started studying graphs with an odd number of vertices and small diameter.
In this Master's thesis we will give some new results about the extendability of matchings in regular graphs with an odd number of vertices and diameter 2. Also, we will obtain some results about regular graphs with diameter 3. For graphs with diameter 3 we will also pay attention on regular graphs with even order.

Structure of the thesis. The thesis is divided into 4 main parts. Since graphs with diameter 1 are complete graphs and matching extensions are trivial, those graphs will be skipped. Therefore, we will start our work with graphs with diameter 2.
First part (Chapter 2) will be devoted to the extendability of matchings in the family of regular graphs with an odd number of vertices and diameter 2. We will prove that all such graphs are $0\frac{1}{2}$-extendable and construct an infinite family of such graphs, which are not $1\frac{1}{2}$-extendable graphs.
After this construction, we will restrict our attention to the family of edge-regular graphs. In Chapter 3 we will work with edge-regular graphs with an odd number of vertices and diameter 2. The main result of this part will be the proof that cycle on 5 vertices is the only non-$1\frac{1}{2}$-extendable graph satisfying all above conditions.
Chapter 4 will be about $2\frac{1}{2}$-extendability. We will give some basic results about edge-regular graphs with diameter 2 and some examples of non-$2\frac{1}{2}$-extendable graphs. The

$2\frac{1}{2}$-extendability of edge-regular graphs with diameter 2 and an odd number of vertices seems to be pretty hard problem, so restricting to the family of strongly regular graphs is natural step. In this part we will consider primitive and imprimitive strongly regular graphs and obtain some important results. Namely, we will prove that there is only 1 non-$2\frac{1}{2}$-extendable primitive strongly regular graph and that is Paley graph on 9 vertices. Also, we will prove that the complete multipartite graph $K_{3\times 3}$ is the only non-$2\frac{1}{2}$-extendable connected imprimitive strongly regular graph with an odd number of vertices.
The last part of the thesis will be devoted to the extedability of regular graphs with diameter 3. Since there are not too many results in this area, in this part we will also consider graphs with an even number of vertices. We will prove that all such graphs have a perfect matching. We will give some conclusions about 1-extendability of regular graphs with an even number of vertices and diameter 3 and construct several examples of non-2-extendable graphs satisfying previous conditions. We will prove the result that all regular graphs with an odd number of vertices and diameter 3 are $0\frac{1}{2}$-extendable. This chapter will be concluded with some examples of non-$1\frac{1}{2}$-extendable regular graphs with an odd number of vertices and diameter 3 and presentation of possible directions for further research.

Since Graph Theory is a young branch of mathematics, there are many new terminologies and knowledge accumulated in its development. Therefore, there are often many names or notions defined for a same entity. In the next section we will give the terms and notions used in this thesis.

1.1 Preliminaries

We will start this section with some basic definitions from graph theory and then we will devote to more specific definitions and theorems directly related to the topics discussed below.

Let Γ be a finite, simple, undirected graph with vertex set $V(\Gamma)$ of order n. Whenever there exists an edge between vertices x and y from $V(\Gamma)$ we say that x and y are adjacent and we denote that by $x \sim y$.
The complement of a graph Γ is the graph $\overline{\Gamma}$ with the same vertex set as Γ, where two distinct vertices are adjacent if and only if they are not adjacent in Γ.
With $deg_\Gamma(x)$ we will usually denote the number of neighbours of x in Γ.
We use the notation $d(x, y)$ for the distance between vertices x and y. The set of all vertices of Γ which are at distance i from vertex x, where i is a non-negative integer,

will be denoted by $N_i(x) = \{y \in V(\Gamma) : d(x,y) = i\}$. For $i = 1$ that will be the set of neighbours of x and we will simply write $N(x)$. The diameter of a graph Γ is $max_{u,v \in V(\Gamma)} d(u,v)$.
We say that a graph Γ is regular with valency k (or k-regular) if $|N(x)| = k$ for every $x \in V(\Gamma)$.
Note that the connectivity of a graph is the minimum number of vertices one has to remove in order to make it disconnected (or empty).
A set of vertices that induces an empty subgraph is called an independent set. Independence number of a graph Γ (denoted by $\alpha(\Gamma)$) is the size of the largest independent set in Γ.
An isomorphism from a graph Γ to a graph Γ^* is a bijection $f : V(\Gamma) \to V(\Gamma^*)$ such that uv is an edge in Γ if and only if $f(u)f(v)$ is an edge in Γ^*. We say that two simple graphs Γ and Γ^* are isomorphic if there is an isomorphism between them.

All graphs considered in this thesis will be finite and simple (no loops and no multiple edges).

Definition 1.1. A matching (or independent edge set) of a graph Γ is a set of edges such that no two of them share a common vertex.
A perfect matching (or a 1-factor) of a graph Γ is a matching of Γ such that every vertex of the graph is incident with exactly one edge of the matching.

From the definition of a perfect matching is clear that only graphs with even order may have a perfect matching.
The notion of an l-extendable graph was introduced at the beginning of the thesis, but because of its importance now we will formally state it.

Definition 1.2. Let Γ be a connected graph of even order at least $2l + 2$, where l is a non-negative integer. Graph Γ is *l-extendable* if it contains a matching of size l and if every such matching is contained in a perfect matching of Γ. Otherwise, Γ is said to be *non-l-extendable.*

Observe that Γ is 0-extendable if it contains a perfect matching.

For example, Petersen graph is 1-extendable (and also 0-extendable). This can be easily seen from Figure 1. Blue edges represent one perfect matching of Petersen graph. Since this graph has a nice symmetry, by rotating the graph we can easily see that an arbitrary edge of Petersen graph is contained in some matching.
It is clear that matching containing edges $\{(0,1),(1,1)\}$ and $\{(1,0),(1,2)\}$ cannot be

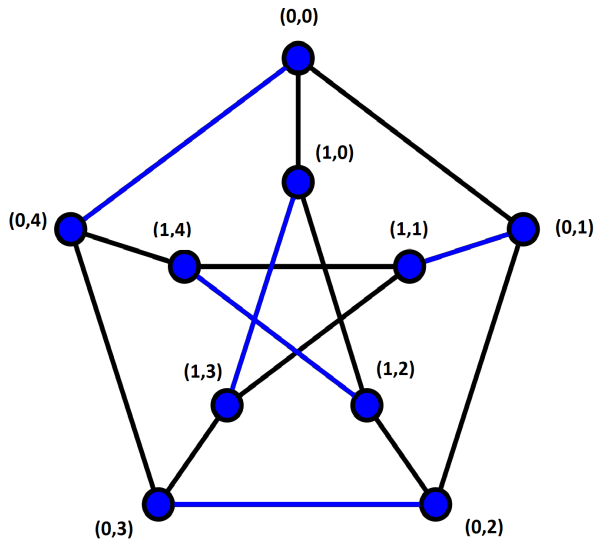

Figure 1: Petersen graph

extended to a perfect matching. Therefore, Petersen graph is not 2-extendable.

Note that if S is a subset of $V(\Gamma)$ then $\Gamma - S$ will denote the subgraph of Γ induced on the set $V(\Gamma)\backslash S$. If $S = \{x\}$ we abbreviate $\Gamma - \{x\}$ with $\Gamma - x$.

For graphs with an odd number of vertices, $l\frac{1}{2}$-extendability can be defined.

Definition 1.3. Let l denote a non-negative integer and let Γ be a connected graph of odd order at least $2l+3$. Graph Γ is $l\frac{1}{2}$-*extendable* if for any vertex $x \in V(\Gamma)$ graph $\Gamma - x$ is l-extendable. Otherwise, Γ is said to be *non-*$l\frac{1}{2}$*-extendable*.

Definition of l-extendable graphs was first introduced by Plummer in 1980 and this definition requirement for an l-extendable graph to have order at least $2l+2$ was not stated. This requirement is important because in this case if $l \geq 1$, l-extendability of Γ implies $(l-1)$-extendability of Γ, otherwise it could happen that l-extendable graph is not $(l-1)$-extendable.
To demonstrate this fact, let us observe a graph Γ from Figure 2. This graph has order 4 and it can be easily seen that a matching containing 2 arbitrary independent edges is a perfect matching of Γ. On the other hand, it is obvious that a matching containing an edge $\{v_2, v_4\}$ cannot be extended to a perfect matching of Γ. Hence Γ is not 1-extendable.

Since we use l-extendability to define $l\frac{1}{2}$-extendable graphs, the requirement for graphs with an odd number of vertices to be of order at least $2l+3$ seems to be natural.

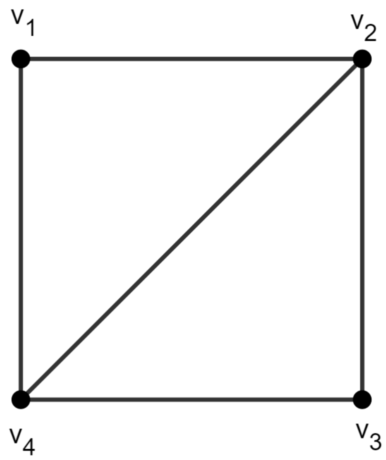

Figure 2: A non-1-extendable graph Γ

Note that in the rest of this thesis we will use term components instead of term connected components, where the term component of a graph Γ represents its maximal connected subgraph of Γ. We will say that component is even if it has an even number of vertices, and odd if it has an odd number of vertices. We will denote by $\sigma(\Gamma)$ the number of odd components of Γ. Components with cardinality 1 will be called singletons.

The following theorem is a famous result of Tutte from 1947 and it will be used several times in this work. Tutte's theorem is the most important result in non-bipartite matching theory and implies many other important results. The proof can be found in [11, Theorem 2.2.1] or [17, Theorem 3.1.1].

Theorem 1.4. *(Tutte) A graph Γ has a perfect matching if and only if for every subset $S \subseteq V(\Gamma)$ we have $\sigma(\Gamma - S) \leq |S|$.*

This theorem gives a good characterization of the existence of a perfect matching in a graph. It is clear that for proving that a graph has a perfect matching is enough to list one. If we want to exibit the fact that observed graph does not contain a perfect matching, it will be enough to find a set S which does not satisfy Tutte's condition.

Of particular use in the rest of this work will also be the following Mantel's theorem which gives us the upper bound on the number of edges for graphs without triangles:

Theorem 1.5. *(Mantel) The maximum number of edges in a triangle-free graph of order n is $\lfloor \frac{n^2}{4} \rfloor$.*

The proof of Mantel's theorem can be found in [20, Theorem 1.3.23].

Further new terminology and notation will be defined when it is required. For some possible undefined (basic) terms we refer interested reader to [3] or [11]. For further reading about graph factors and matching extensions we refer to [22].

1.1.1 Relationships between l-extendability and $l\frac{1}{2}$-extendability

As we have already seen, $l\frac{1}{2}$-extendability is defined using l-extendability. Hence, we can say that $l\frac{1}{2}$-extendability is, in some sense, stronger than l-extendability.
But what about relation between $l\frac{1}{2}$-extendability and $(l+1)$-extendability?
It was proved in [21, Theorem 2.3] that if we join a new vertex to all vertices of $l\frac{1}{2}$-extendable graph Γ, the resulting graph will be $(l+1)$-extendable. Hence, we can say that $l\frac{1}{2}$-extendability is weaker than $(l+1)$-extendability.
On the other hand, not all $(l+1)$-extendable graphs have a property that deletion of an arbitrary vertex will result in a $l\frac{1}{2}$-extendable graph. An example of such a graph is the 6-cycle C_6.
After all, it seems natural to think about $l\frac{1}{2}$-extendability as lying between l-extendability and $(l+1)$-extendability. Also, it is possible to characterize all $l\frac{1}{2}$-extendable graphs in terms of l-extendable and $(l+1)$-extendable graphs. Such a characterization can be found in [21].

We end this introductory chapter by considering extendability of matchings in cycles.

1.1.2 Extendability of cycles

Let us observe a cycle C_n with an odd number of vertices. Let u be an arbitrary vertex in C_n. Graph $C_n - u$ is a path P_{n-1} with odd number of edges. Obviously such graph contains a perfect matching and so C_n is $0\frac{1}{2}$-extendable.
But is it $1\frac{1}{2}$-extendable? Note that C_3 does not satisfy the condition on the number of vertices from the definition of $l\frac{1}{2}$-extendability, so there is no sense to speak about $1\frac{1}{2}$-extendability of this graph. Let $n \geq 5$ and let us choose the middle edge e in P_{n-1}. It can be easily seen that the matching containing this edge cannot be extended to a perfect matching of P_{n-1}. Therefore, C_n is not $1\frac{1}{2}$-extendable for any odd n.

What about the extendability of cycles with an even number of vertices? Obviously each even cycle contains a perfect matching. Also, it can be easily seen that a matching containing an arbitrary edge of an even cycle can be extended to a perfect matching. Therefore, even cycles are 1-extendable. Since there is no sense to speak about 2-extendability of C_4 let us suppose that $n \geq 6$. Let us choose edges e_1 and e_2 such that there is a vertex v which is adjacent to one endpoint of both edges. It is clear that the matching containing edges e_1 and e_2 cannot be extended to a perfect matching of C_n. Therefore, the even cycles are not 2-extendable.

Since we completely determined l-extendability and $l\frac{1}{2}$-extendability of cycles, from now on we will assume that graph Γ is regular with valency $k \geq 3$.

2 Extendability of regular graphs

In this chapter we will focus on the matching extensions of regular graphs with an odd number of vertices and diameter 2. First section will be about $0\frac{1}{2}$-extendability and in the second section we will work on $1\frac{1}{2}$-extendability of regular graphs with an odd number of vertices and diameter 2.

2.1 On $0\frac{1}{2}$-extendability of regular graphs with diameter 2

Lemma 2.1. *Let Γ be a regular graph of valency k, odd order and diameter 2. Let $S \subseteq V(\Gamma)$ be such that $\Gamma - S$ is not connected and let C be a component of $\Gamma - S$. Then there are at least k edges between C and S in Γ.*

The proof of this lemma is the same as for graphs with an even number of vertices and is therefore omitted. It can be found in [18, Lemma 2.1].

Lemma 2.2. *Let Γ be a regular graph of valency k, an odd number of vertices and diameter 2. Let $S \subseteq V(\Gamma)$ be such that $\Gamma - S$ is not connected. Let C be a component of $\Gamma - S$ such that there are exactly k edges between S and C. Then C is either singleton component or a complete graph with k vertices.*

Proof. Assume that C is not a singleton. Since Γ is of diameter 2, each vertex from C must have at least one neighbour in S. Hence, if C has more than k vertices, then there are more than k edges from C to S, which is impossible.
Suppose that the cardinality of component C is less than k. Because of regularity of Γ, there are $k - deg_C(v)$ edges from each vertex $v \in C$ to S. Therefore, total number of edges between C and S is $\sum_{v\in C}\big(k - deg_C(v)\big)$.
Let us denote $M = \max\{deg_C(v);\ v \in C\}$ and note that $M \leq |C| - 1$.
As $|C| < k$, we have

$$\sum_{v\in C} deg_C(v) \leq\ |C|M <\ kM, \tag{2.1}$$

Therefore,

$$\begin{aligned}\sum_{v\in C}\big(k-deg_C(v)\big) &= |C|k-\sum_{v\in C}\big(deg_C(v)\big)\\ &> |C|k-kM\\ &= k\left(|C|-M\right)\\ &\geq k\,,\end{aligned} \tag{2.2}$$

Hence, there are more than k edges from S to C, which is impossible.
Therefore, $|C| = k$. If a component C is not a complete graph, then, because of regularity, there are at least $k+2$ edges from C to S. Contradiction. □

Remark 2.3. *For k-regular graphs with an odd number of vertices, k must be even. This can be easily seen from the fact that k-regular graph of order n has* $\frac{nk}{2}$ *edges.*

Theorem 2.4. *Let* Γ *be a regular graph with an odd number of vertices and diameter* 2*. Then* Γ *is* $0\frac{1}{2}$*-extendable.*

Proof. Suppose that Γ is not $0\frac{1}{2}$-extendable. Then there exists $x \in V(\Gamma)$ such that $\Gamma - x$ is not 0-extendable (that is, it does not contain a perfect matching).
By Tutte's result there exists $S \subseteq V(\Gamma - x)$ such that

$$\sigma\big(V(\Gamma - x) - S\big) > |S|. \tag{2.3}$$

Let us denote $S' = S \cup \{x\}$. Let l denote the number of odd components of $\Gamma - S'$. Note that connected components of $\Gamma - S'$ are the same as connected components of $\Gamma - x - S$, so (by inequality (2.3)) we have that $l > |S|$. Let t denote the number of edges between S' and odd components of $\Gamma - S'$. From Lemma 2.1 it follows that there are at least k edges between each (odd) component and S'. On the other hand, each vertex from S' has at most k neighbours in odd components of $\Gamma - S'$ (because each vertex in Γ has degree k). Therefore, we have the following:

$$lk \leq t \leq k|S'| = k(|S|+1). \tag{2.4}$$

From the previous inequality it follows that $l \leq |S| + 1$ which, together with the fact that $l > |S|$, gives us $l = |S| + 1$. Using this fact and inequality (2.4) we get that $t = k(|S| + 1)$. It follows that equality holds in (2.4), and so there are exactly k edges from each odd component to S'. Hence, by Lemma 2.2, all odd components are either singletons or complete graphs with k vertices. The second possibility is not possible since components are odd and k is even. Therefore, all odd components are singletons, so there are exactly $|S| + 1$ vertices contained in odd components of graph $\Gamma - S'$.

Let X denote the union of all even components of $\Gamma - S'$. Let us calculate the total number of vertices in graph Γ:

$$|V(\Gamma)| = |S| + 1 + |S| + 1 + |X| = 2(|S| + 1) + |X|. \tag{2.5}$$

Since $|X|$ is an even number, from equation (2.5) it follows that $|V(\Gamma)|$ is an even number. Contradiction. □

Since we proved that all regular graphs with an odd number of vertices and diameter 2 are $0\frac{1}{2}$-extendable, now we wonder what is happening with $1\frac{1}{2}$-extendability of regular graphs with an odd number of vertices and diameter 2. Are they all $1\frac{1}{2}$-extendable?

2.2 On $1\frac{1}{2}$-extendability of regular graphs with diameter 2

As Γ is a regular graph with valency k and an odd number of vertices, k must be an even number. Since the only graph with diameter 2 and valency $k = 2$ is the cycle on 5 vertices and as it was proved before it is not $1\frac{1}{2}$-extendable, from now on we will observe graphs with $k \geq 4$.

In order to prove that not all regular graphs with an odd number of vertices, diameter 2 and $k \geq 4$ are $1\frac{1}{2}$-extendable we give the following construction of an infinite family of regular graphs which are not $1\frac{1}{2}$-extendable.

Construction 2.5. *Let $k \geq 4$ be an even integer and let Γ be a graph with vertex set $\{u\} \cup V \cup W$ where $V = \{v_i \mid i \in \mathbb{Z}_k\}$ and $W = \{w_i \mid i \in \mathbb{Z}_k\}$, and edge set $E_1 \cup E_2 \cup E_3$ where $E_1 = \{uv_i \mid i \in \mathbb{Z}_k\}$, $E_2 = \{v_iw_j \mid i \in \mathbb{Z}_k,\ j \in \mathbb{Z}_k,\ i \leq j \leq i + k - 2\}$ and $E_3 = \{w_iw_{i+1} \mid i \in \mathbb{Z}_k,\ i = 2s + 1,\ 0 \leq s \leq \frac{k}{2} - 1\}$.*

For example, for $k = 4$ we have a graph from Figure 3.

Note that $\mathbb{Z}_k$, in the previous construction, denotes the additive group of integers modulo k.

Proposition 2.6. *Let Γ be as in Construction 2.5. Then graph Γ is k-regular graph with an odd number of vertices and diameter 2 which is not $1\frac{1}{2}$-extendable.*

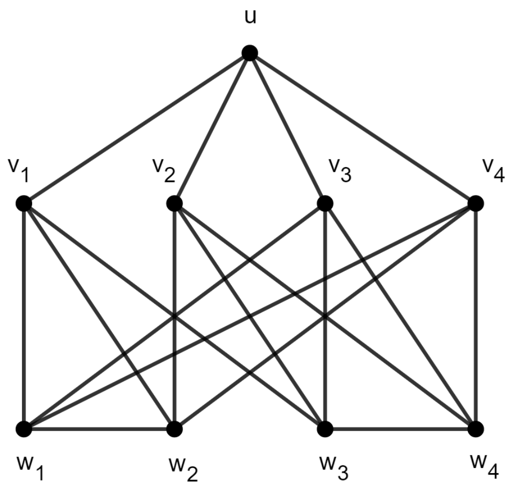

Figure 3: $(9, 4)$-regular graph

Proof. Note that number of vertices in graph Γ is

$$|V(\Gamma)| = 1 + |V| + |W| = 1 + k + k = 1 + 2k, \tag{2.6}$$

which is obviously an odd number.

Regularity: Observing the edge set E_1 it can be easily seen that $deg_\Gamma(u) = k$. By construction, each vertex v_i is adjacent to vertex u, has no neighbours in V and has exactly $k-1$ neighbours in W. Hence $deg_\Gamma(v_i) = k$. Note that none of vertices from W is adjacent to u, each of them is adjacent to exactly $k-1$ vertices from V and to exactly one vertex from vertex set W. Therefore, $deg_\Gamma(w_i) = k$. We proved that each vertex of Γ has k neighbours, so Γ is k-regular graph.

Diameter: It is obvious that each vertex from V is at distance 1 from vertex u and that each vertex from W is at distance 2 from vertex u. Also, each two vertices from V are at distance 2 because vertex u is their common neighbour. Let us observe vertices from W. Each w_i is adjacent to exactly $k-1$ vertices from V, so it is at distance 1 from them. The only vertex from V which, by construction, is not adjacent to w_i is vertex v_{i+1}, but that vertex is adjacent to both vertices w_{i-1} and w_{i+1}. Since exactly one of them is adjacent to w_i, it follows that each vertex from W is at distance at most 2 from any vertex from V. Note that each two vertices of W, by construction, have $k-2$ common vertices in V, so they are at distance at most 2. Therefore, diameter of Γ is 2.

Non-extendability: Let us observe the graph $\Gamma - u$ and a matching containing an edge from the edge set E_3. In order to cover vertices from V by a matching, we have to choose edges from the edge set E_2, but with each new edge in the matching we will cover one vertex from W. As at the beginning there were k uncovered vertices from

V and $k-2$ uncovered vertices from W, no matter how we choose edges from E_2 to include them in matching there will always stay two uncovered vertices from V. Since a matching of graph $\Gamma - u$ containing an arbitrary edge from E_3 cannot be extended to a perfect matching of $\Gamma - u$, it follows that Γ is not $1\frac{1}{2}$-extendable. □

Now when we know that there exists an infinite family of non-$1\frac{1}{2}$-extendable graphs, it will be useful to find some properties of such graphs. First, let us fix some notation.

Notation 2.7. *Let Γ be a k-regular graph with an odd number of vertices and diameter 2, which is not $1\frac{1}{2}$-extendable. Let us denote by x the vertex of $V(\Gamma)$ such that the graph $\Gamma - x$ is not 1-extendable. Denote by $e = \{y, z\}$ an edge such that the matching containing this edge cannot be extended to a perfect matching of $\Gamma - x$. This means that the graph $\Gamma' = \Gamma - \{x, y, z\}$ is not 0-extendable, so by Tutte's result it follows that there exists $S' \subseteq V(\Gamma')$ such that $\sigma(\Gamma' - S') > |S'|$. Denote $S = S' \cup \{x, y, z\}$.*

Note that the connected components of $\Gamma' - S'$ are the same as the connected components of $\Gamma - S$.

Remark 2.8. *With reference to Notation 2.7 every vertex in an odd component of $\Gamma - S$ has at least one neighbour in S (because the diameter of Γ is 2).*

Lemma 2.9. *With reference to Notation 2.7 we have that $\sigma(\Gamma - S) = |S| - 1$.*

Proof. Let us denote

$$t = \sigma(\Gamma - S) = \sigma(\Gamma' - S') > |S'|, \tag{2.7}$$

and let $C_1, C_2, ..., C_t$ be odd components of $\Gamma - S$.
$|V(\Gamma)|$ is an odd number, so the cardinality of $S' \cup \{x, y, z\} \cup C_1 \cup C_2 \cup ... \cup C_t$ is odd, and so the cardinality of $S' \cup C_1 \cup C_2 \cup ... \cup C_t$ is even. Therefore, $|S'|$ and $|C_1 \cup C_2 \cup ... \cup C_t|$ have the same parity. The numbers $|C_1|, |C_2|, ..., |C_t|$ are odd, so the parity of t is the same as the parity of $|C_1 \cup C_2 \cup ... \cup C_t|$. Therefore, t and $|S'|$ have the same parity. This conclusion together with equation (2.7) gives us

$$t \geq |S'| + 2. \tag{2.8}$$

Let us denote by $S'' = S' \cup \{y, z\}$. Obviously, $S'' \subseteq V(\Gamma - x)$. As Γ is regular graph with an odd number of vertices, by Theorem 2.4, Γ is $0\frac{1}{2}$-extendable. Therefore, $\Gamma - x$ is 0-extendable. So, by Tutte's result it follows that $\sigma(\Gamma - x - S'') \leq |S''| = |S'| + 2$. Note that the connected components of $\Gamma - x - S''$ are the same as the connected components of $\Gamma - S$. Therefore, we have:

$$t = \sigma(\Gamma - S) \leq |S'| + 2. \tag{2.9}$$

Using (2.8) and (2.9) we get $t = |S'| + 2 = |S| - 1$. □

Proposition 2.10. *With reference to Notation 2.7 we can assume that all components of* $\Gamma - S$ *are odd.*

Proof. Suppose that $\Gamma - S$ has an even component C. Pick an arbitrary vertex $c \in C$ and set $S^* = S \cup \{c\}$. Note that each component of $\Gamma - S$ different from component C is also a component of $\Gamma - S^*$. The remaining components of $\Gamma - S^*$ are the components of the subgraph of Γ induced on vertex set $C - c$. This set is of odd size, so at least one of the new components must be odd. Therefore, we have:

$$\sigma(\Gamma - S^*) \geq \sigma(\Gamma - S) + 1 = |S| - 1 + 1 = |S| = |S^*| - 1. \tag{2.10}$$

Graph Γ is regular graph with an odd number of vertices, so by Theorem 2.4, it is $0\frac{1}{2}$-extendable. Therefore, for each vertex $w \in V(\Gamma)$, graph $\Gamma - w$ contains a perfect matching. Applying Tutte's result on S^* and some vertex w such that $S^* \subseteq V(\Gamma - w)$ we get $\sigma(\Gamma - w - S^*) \leq |S^*|$. Combining this result with the obvious fact that $\sigma(\Gamma - w - S^*) \geq \sigma(\Gamma - S^*)$ we get that

$$\sigma(\Gamma - S^*) \leq |S^*|. \tag{2.11}$$

Using (2.10) and (2.11) we get

$$|S^*| - 1 \leq \sigma(\Gamma - S^*) \leq |S^*|. \tag{2.12}$$

Let us denote $t^* = \sigma(\Gamma - S^*)$ and let $C_1, C_2, ..., C_{t^*}$ be odd components of $V(\Gamma)$. Cardinality of $S^* \cup C_1 \cup C_2 \cup ... \cup C_{t^*}$ is odd (because $|V(\Gamma)|$ is odd). Therefore, $|S^*|$ and $|C_1 \cup C_2 \cup ... \cup C_{t^*}|$ have different parity. Since numbers $|C_1|$, $|C_2|$,..., $|C_{t^*}|$ are odd, the parity of t^* is the same as the parity of $|C_1 \cup C_2 \cup ... \cup C_{t^*}|$. Therefore, t^* and $|S^*|$ have different parities. This fact, together with inequality (2.12) gives us $\sigma(\Gamma - S^*) = |S^*| - 1$.
Repeating this process of enlarging S until no even component of $\Gamma - S$ exists we can eliminate all even components. □

For the rest of this work we may assume that $\Gamma - S$ has no even components.

Since the existence of an infinite family of regular graphs with an odd number of vertices and diameter 2, which are not $1\frac{1}{2}$-extendable, have been shown, it makes sense to observe edge-regular graphs with an odd number of vertices and diameter 2.

3 On $1\frac{1}{2}$-extendability of edge-regular graphs with diameter 2

In this chapter we will recall the definition of an edge-regular graph and a famous Moore's bound on the number of vertices of an edge-regular graph of diameter 2. In the first section we will consider edge-regular graphs with diameter 2, an odd number of vertices and valency 4, while the second section is devoted to edge-regular graphs with diameter 2, an odd number of vertices and valency greater than 4.

Definition 3.1. A connected k-regular graph Γ with n vertices is *edge-regular*, if there exists an integer λ such that each two adjacent vertices of Γ have exactly λ common neighbours.

Numbers n, k and λ are called parameters of graph Γ, and we say that Γ is an (n, k, λ) edge-regular graph. It is clear that $\lambda \leq k - 1$. In fact, $\lambda = k - 1$ if and only if Γ is the complete graph with $k + 1$ vertices.
It is a well known fact that edge-regular graph with parameters (n, k, λ) has $\frac{nk\lambda}{6}$ triangles. Also, there exists a famous Moore's upper bound on the number of vertices of an edge-regular graph.

Proposition 3.2. *(Moore's bound) Let Γ be an (n, k, λ) edge-regular graph with diameter 2. Then $7 \leq n \leq 1 + k + k(k - \lambda - 1)$.*

Remark 3.3. *Original Moore's bound gives just upper bound on the number of vertices. But as all graphs considered in this chapter are with an odd number of vertices and diameter 2, which exclude complete graphs (since they have diameter 1), we may add lower bound on the number of vertices.*

In this chapter we will prove that all edge-regular graphs with an odd number of vertices, diameter 2 and $k \geq 4$ are $1\frac{1}{2}$-extendable. We will consider cases $k = 4$ and $k > 4$ separately.

3.1 Graphs with valency $k = 4$

In this section we will consider edge-regular graphs with an odd number of vertices, diameter 2 and valency 4. Of our particular interest will be the Paley graph on 9 vertices, which is, as we will show, $1\frac{1}{2}$-extendable.
Also, we will show that all triangle-free edge-regular graphs with an odd number of vertices, diameter 2 and valency 4, must have 11, 13 or 15 vertices and that all of them are $1\frac{1}{2}$-extendable.

First, let us define Paley graphs. To do this we recall that if q is a prime power, then $GF(q)$ denotes finite (Galois) field with q elements.

Definition 3.4. Let q be a prime power, $q \equiv 1\ (mod\, 4)$. The Paley graph PG_q is the graph whose vertex set is the set of elements of the field $GF(q)$, two vertices being adjacent if their difference is a non-zero square in $GF(q)$.

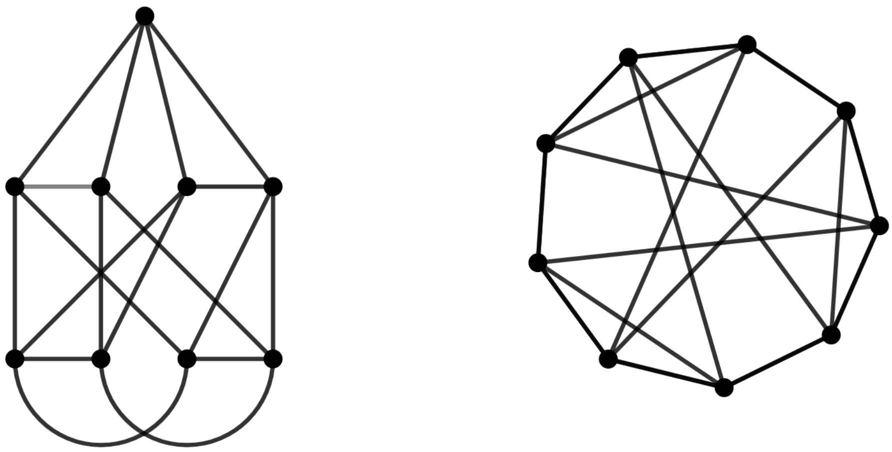

Figure 4: Paley graph on 9 vertices.

Proposition 3.5. *Let Γ be an $(n, 4, \lambda)$ edge-regular graph with odd order and diameter 2. Then Γ is either the Paley graph on 9 vertices, or $\lambda = 0$. Moreover, if $\lambda = 0$ then $n \in \{11, 13, 15\}$.*

Proof. Since $k = 4$ and $\lambda < k - 1$ it follows $\lambda \in \{0, 1, 2\}$. Graph Γ is edge-regular, so by Proposition 3.2, we have that $7 \leq n \leq 1 + k + k(k - \lambda - 1)$ which implies $7 \leq n \leq 17 - 4\lambda$.

Case 1: $\lambda = 2$. Since $7 \leq n \leq 9$ and n is odd, we have $n \in \{7, 9\}$. The number of triangles in Γ is equal to $\frac{nk\lambda}{6} = \frac{4n}{3}$. Therefore, n is divisible by 3. So $n = 9$. Let us suppose that such a graph exist and try to construct it. Let u denote our starting vertex and let $N(u) = V = \{v_1, v_2, v_3, v_4\}$. Let $W = \{w_1, w_2, w_3, w_4\}$ denote the other vertices. Since diameter of Γ is 2, each vertex from W must have at least one neighbour in V. Since $\lambda = 2$ each vertex from V must have exactly 2 common neighbours with u. Therefore, each vertex in V has exactly 2 neighbours in V. Let us observe vertex v_1. It must have one neighbour in W. WLOG suppose that w_1 is its neighbour. Since $\lambda = 2$ they must have 2 common neighbours and they must be in V. WLOG suppose that these vertices are v_2 and v_3. Now we have that $v_3 \sim v_4$, because v_3 must have two neighbours in V and it cannot be adjacent to v_2 (since in that case v_2 and v_3 would have 3 common neighbours). Using similar argument we also find that $v_2 \sim v_4$. Vertex u is a common neighbour of vertices v_3 and v_4. Therefore, vertex w_1 is the only possibility for their second common neighbour. Now, each vertex from V has 4 neighbours, so vertices w_2, w_3, w_4 cannot have any neighbours in V. Contradiction.

Case 2: $\lambda = 1$. Since in this case $7 \leq n \leq 13$, we have that $n \in \{7, 9, 11, 13\}$. The number of triangles in Γ is $\frac{nk\lambda}{6} = \frac{2n}{3}$, so n is divisible by 3. Therefore, $n = 9$. It can be easily seen that such a graph is unique, up to isomorphism. This graph is the well known Paley graph on 9 vertices.

Case 3: $\lambda = 0$, $7 \leq n \leq 17$, so $n \in \{7, 9, 11, 13, 15, 17\}$. Let Γ be a $(n, 4, 0)$ edge-regular graph with vertex set $\{u\} \cup V \cup W$ where $V = N(u) = \{v_1, v_2, v_3, v_4\}$ and W are vertices on distance 2 from vertex u. Now we split our analysis into the following subcases depending on the value of n:

Subcase 1: $n = 7$. Since $\lambda = 0$, there are no triangles in Γ, so by Mantel's result it follows that the number of edges in Γ is at most $\lfloor \frac{n^2}{4} \rfloor = \lfloor \frac{49}{4} \rfloor = 12$. On the other hand, regular graph with 7 vertices and valency 4 has $\frac{7 \cdot 4}{2} = 14$ edges. Contradiction.

Subcase 2: $n = 9$. In this case $W = \{w_1, w_2, w_3, w_4\}$ and Γ has 18 edges. Vertex u has 4 neighbours in V and each vertex from V has 3 neighbours in W. Therefore, there are 2 edges in a subgraph of Γ induced by W. WLOG we may assume that vertex v_1 is adjacent to vertices w_1, w_2, w_3, so these two edges must have vertex w_4 as an endpoint (otherwise we would obtain a triangle). Also WLOG we may assume that w_4 is adjacent to vertex v_2 (since w_4 is at distance 2 from u it must have at least one neighbour in V). Since there are no triangles in Γ, vertex v_2 cannot be adjacent to any vertex from V or to any neighbours of vertex w_4 in W. Therefore, vertex v_2 can have at most 2 neighbours in W. Contradiction.

Subcase 4: $n = 11$. Graph has parameters $(11, 4, 0)$.

Subcase 5: $n = 13$. Graph has parameters $(13, 4, 0)$.

Subcase 6: $n = 15$. Graph has parameters $(15, 4, 0)$.

Subcase 7: $n = 17$. In this case $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}\}$. As the diameter of Γ is 2, each vertex from W must have at least one neighbour in V. There are 12 vertices in W and each vertex from V has exactly 3 neighbours in W. Hence it follows that each vertex from W has exactly one neighbour in V and 3 more neighbours in W. Therefore, the girth[1] of this graph is 5. Note that edge-regular graph $(17, 4, 0)$ meets the Moore upper bound, and is therefore the so-called Moore graph. But it is known (Hoffman-Singleton theorem, [2, Chapter 23]) that the Moore graph with girth 5 could only exist if $k = 2, 3, 7$ or 57. Therefore, no $(17, 4, 0)$ edge-regular graph exist. □

Using the previous proposition we will show that all edge-regular graphs with an odd number of vertices, diameter 2 and valency 4 are $1\frac{1}{2}$-extendable.

Lemma 3.6. *Let Γ be an edge-regular graph with parameters $(n, 4, 1)$, an odd number of vertices and diameter 2. Then Γ is a Paley graph on 9 vertices which is $1\frac{1}{2}$-extendable.*

Proof. From Proposition 3.5 it follows that Γ is a Paley graph. Suppose to the contrary that Γ is not $1\frac{1}{2}$-extendable. That means that there is a vertex $x \in V(\Gamma)$ such that $\Gamma - x$ is not 1-extendable. Therefore, there exists an edge $e = \{y, z\}$ such that $\Gamma - \{x, y, z\}$ does not have a perfect matching. Let S be as in Notation 2.7. As Γ is regular graph with an odd number of vertices, it follows from Lemma 2.9 that $\sigma(\Gamma - S) = |S| - 1$. Let $C_1, C_2, ..., C_{|S|-1}$ denote the odd components of $\Gamma - S$. Vertices y and z are adjacent and all other vertices from S may have at most $k = 4$ neighbours in $C_1 \cup C_2 \cup ... \cup C_{|S|-1}$. Therefore, there are at most $4|S| - 2$ edges between S and $C_1 \cup C_2 \cup ... \cup C_{|S|-1}$. There are at least $k = 4$ edges between S and each odd component, so there are at least $4(|S| - 1)$ edges between S and odd components. On the other hand, Γ is 4-regular graph with 9 vertices, so it has 18 edges and one of them is edge e in S. Therefore, $4|S| - 4 < 18$, which implies that $|S| \leq 5$. As S contains vertices x, y and z, we have that $|S| \geq 3$. Hence $|S| = \{3, 4, 5\}$. Now we can split our analysis into the following three cases depending on the cardinality of S.

Case 1: $|S| = 3$. Γ has two odd components and since we assumed that there are no even components in $\Gamma - S$ it follows that there are 6 vertices in these two components. If one of the components is singleton, vertex from that component has 4 neighbours in S which is impossible. Therefore, $|C_1| = |C_2| = 3$. This shows that there are at least 6 edges between each of C_i, $i \in \{1, 2\}$ and S, so at least 12 edges between $C_1 \cup C_2$ and S. Therefore, we get $12 \leq 4|S| - 2 = 10$. Contradiction.

Case 2: $|S| = 4$ and there are 3 odd components which altogether contain 5 vertices. So, there are two singleton components. Vertices from singleton components have 4

[1] the lenght of a shortest circuit in the graph

neighbours in S, so y and z have at least two common neighbours. Contradiction.
Case 3: $|S| = 5$. There are 4 components and all of them are singletons. There are 18 edges in Γ, and there are exactly 16 edges between S and odd components (because each vertex from each odd component has 4 neighbours in S). So there are 2 edges in S, one of which is edge $e = \{y, z\}$. Let us denote the other one by e_1. As $\lambda = 1$, each 2 adjacent vertices must have a common neighbour. If one of the two adjacent vertices is in $\Gamma - S$, their common neighbour must be in S. Therefore, each vertex from each odd component must be adjacent to those vertices of S which are endpoints of e and e_1. But since $k = 4$ each of these endpoints have at most 3 neighbours in odd components. Contradiction. □

Lemma 3.7. *Let* Γ *be a triangle-free edge-regular graph with an odd number of vertices, valency* 4 *and diameter* 2*. Then* Γ *is* $1\frac{1}{2}$*-extendable.*

Proof. From Proposition 3.5 it follows that $n = |V(\Gamma)| \in \{11, 13, 15\}$. Suppose that graph Γ is not $1\frac{1}{2}$-extendable, so there exists a vertex $x \in V(\Gamma)$ such that $\Gamma - x$ is not 1-extendable and there exists an edge $e = \{y, z\}$ such that $\Gamma - \{x, y, z\}$ does not have a perfect matching. Let S be as in Notation 2.7. Note that there are at most $4|S| - 2$ edges between S and odd components. As Γ is regular graph with an odd number of vertices, it follows from Lemma 2.9 that $\sigma(\Gamma - S) = |S| - 1$. Let $C_1, C_2, ..., C_{|S|-1}$ denote the odd components of $\Gamma - S$. Since there are at least k edges between each odd component and S, it follows that there are at least $4(|S| - 1)$ edges between S and odd components. This number, of course, must be smaller than the total number of edges in Γ (since there is at least one edge in S). Now we can split our analysis into the following 3 cases.

Case 1: $n = 11$. In this case graph Γ has 22 edges, so $4(|S| - 1) < 22$ which implies $|S| < \frac{26}{4}$. As $|S| \geq 3$ is an integer, it follows that $|S| = \{3, 4, 5, 6\}$.
Subcase 1.1: $|S| = 3$. In this case there are two components of $\Gamma - S$ which together contain 8 vertices. Since $k = 4$ there is no singleton component (vertex from singleton component would have 4 neighbours in S which is not possible). So, the only remaining possibility is that one component has cardinality 3 and the other one has cardinality 5. Let $|C_1| = 3$. As Γ is triangle-free and components are connected it follows that C_1 is the 3-vertex path P_3. End vertices of this path are adjacent to all vertices in S (because $k = 4$ and $|S| = 3$), but that is impossible since y and z do not have common neighbours.
Subcase 1.2: $|S| = 4$. In this case there are three components which altogether contain 7 vertices. There are two options: there are two singleton components and one component with 5 vertices, or there is one singleton component and 2 components with

3 vertices. Vertices from singleton components must be adjacent to all vertices in S. Contradiction (with the fact that y and z do not have common neighbours).
Subcase 1.3: $|S| = 5$. There are 4 components containing altogether 6 vertices, so there must be 3 singletons and one component with 3 vertices. As there are 12 edges between singleton components and S, there are at most $4|S|-2-12 = 6$ edges between S and the fourth component. As this component must be P_3 it follows that it sends 8 edges to S. Contradiction.
Subcase 1.4: $|S| = 6$. In this case there are 5 singleton components. Therefore, there are 20 edges between S and odd components. Since graph Γ has 22 edges, it follows that there are exactly two edges in S (one of them is edge e). Let $u \in S$ be a vertex which is not an endpoint of any of the edges in S (such a vertex exists since $|S| = 6$ and two edges can cover at most 4 vertices). Vertex u has 4 neighbours in odd components (because $k = 4$ and it does not have any neighbours in S). Since there are 5 odd components, there exists a vertex in an odd component, say v, which is not adjacent to u. Vertices u and v are not adjacent and do not have any common neighbours, so they are at distance greater than 2. Contradiction.

Case 2: $n = 13$. Graph Γ has 26 edges and $4(|S| - 1) < 26$ implies that $|S| \in \{3, 4, 5, 6, 7\}$.
Subcase 2.1: $|S| = 3$, so there are two components containing 10 vertices. Using similar approach like in the previous case, we can conclude that there are no singleton components neither components with 3 vertices. So, the only remaining case is to have two components both of cardinality 5. As there are at most $4|S| - 2 = 10$ between S and these odd components, it follows from Lemma 2.1 that there are exactly 5 edges between each of these components and S. Therefore, by Lemma 2.2, the subgraphs induced on these components are the complete graphs on 5 vertices. But this is a contradiction with the fact that Γ is triangle-free.
Subcase 2.2: $|S| = 4$, so there are at most 14 edges from S to odd components. In this case we have 9 vertices in 3 odd components. Note that there are no singleton components (otherwise y and z would have a common neighbour which is impossible since $\lambda = 0$). Now we are left with the case of 3 components with cardinality 3. Each of them is the 3-vertex path P_3, and so there are 8 edges from each odd component to S (each end vertex of P_3 has 3 neighbours in S and middle vertex has 2 neighbours in S). Therefore, there are $3 \cdot 8 = 24$ edges between odd components and S which is not possible.
Subcase 2.3: $|S| = 5$. There are 8 vertices in 4 odd components, and there are at most 18 edges between S and odd components. If there are 3 singleton components and one component with 5 vertices, then there are at least $4 + 4 + 4 + 6 = 18$ edges

between S and odd components. By Mantel's result, there are at most 6 edges in 5-vertex component, and so there are at least 8 edges between 5-vertex component to S. This gives us in total $4 + 4 + 4 + 8 = 20$ edges between odd components and S, a contradiction. The remaining possibility is that there are 2 singleton components and two components with cardinality 3. In this case there are $4 + 4 + 8 + 8 = 24$ edges between odd components and S. Contradiction.
Subcase 2.4: $|S| = 6$. Maximal number of edges between S and odd components is 22. Here we have 7 vertices contained in 5 odd components, so the only possibility is that there are 4 singletons and one component with 3 vertices. Therefore, there are $4 + 4 + 4 + 4 + 8 = 24$ edges between odd components and S. Contradiction.
Subcase 2.5: $|S| = 7$. All 6 components are singletons. There are 24 edges between S and odd components and therefore there are exactly two edges in S. Let us observe an arbitrary vertex v in S which is not an endpoint of any of the edges from S (such a vertex exists because two edges can cover at most 4 vertices and $|S| = 7$). As $k = 4$, v has 4 neighbours in odd components. As there are 6 odd components it follows that there are 2 vertices in odd components which are not adjacent to v. As v does not have any neighbours in S and there are no edges between odd components it follows that distance between this vertices and v is greater than 2. Contradiction (with the fact that diameter of Γ is 2).

Case 3*:* $n = 15$. There are 30 edges in Γ. As $4(|S| - 1) < 30$ it follows that $|S| \in \{3, 4, 5, 6, 7, 8\}$. Depending on the cardinality of S, we have the following subcases.
Subcase 3.1: $|S| = 3$. In this case, there are at most $4|S| - 2 = 10$ edges between S and components, and there are 12 vertices in two components. As each vertex in odd component has at least one neighbour in S, it follows that there are at least 12 edges from odd components to S. Contradiction.
Subcase 3.2: $|S| = 4$. There are 3 components which altogether contain 11 vertices. Since there are no singleton components (otherwise y and z would have a common neighbour), there are two components with 3 vertices and one component with 5 vertices. As components with 3 vertices are 3-vertex paths, there are 8 edges from each of them to S. Therefore, there are at least $8 + 8 + 5 = 21$ edges from odd components to S. This is impossible, since there are at most 14 edges between S and odd components.
Subcase 3.3: $|S| = 5$. In this case there are 4 components and at most 18 edges between S and $\Gamma - S$. The case when there are 3 singletons and one component with 7 vertices is impossible, since there would be at least $4+4+4+7 = 19$ edges between S and odd components. If there are 2 singletons, then the other two components must be with cardinalities 3 and 5. So, there are at least $4 + 4 + 8 + 5 = 21$ edges between S and

$\Gamma - S$. Impossible. The only remaining possibility is that there are 3 components with cardinality 3 and one singleton. But, in this case there are at least 28 edges from odd components to S. Contradiction.
Subcase 3.4: $|S| = 6$. There are 9 vertices in 5 odd components and there are at most 22 edges between S and them. If there are 3 singletons and two components with cardinality 3, there are at least 28 edges between them and S. Impossible. So, there must be 4 singletons and one component with 5 vertices. Similarly as above, it follows from Mantel's result that there are at least 8 edges between the 5-vertex component and S. Hence, there are at least 24 edges between odd components and S. Contradiction.
Subcase 3.5: $|S| = 7$. There are 8 vertices in 6 components. In this case there must be 5 singletons and a 3-vertex component. Hence, there are at least 28 edges between S and $\Gamma - S$. Contradiction (because there are at most 26 such edges).
Subcase 3.6: $|S| = 8$. There are 7 singletons and exactly 28 edges from S to them. Since there are 30 edges in Γ, S contains exactly 2 edges. Let us observe an arbitrary vertex v in S which is not an endpoint of any of the edges from S. Since $k = 4$, v has 4 neighbours in the odd components. As there are 7 components, it follows that there are 2 vertices in the odd components which are not adjacent to v. As v does not have any neighbours in S and there are no edges between components, it follows that the distance between these vertices and v is greater than 2. Contradiction. □

In the following theorem we will summarize the results obtained in this section.

Theorem 3.8. *Let Γ be an edge-regular graph with an odd number of vertices, diameter 2 and valency 4. Then Γ is $1\frac{1}{2}$-extendable.*

3.2 Graphs with valency $k \geq 6$

In this section we will consider edge-regular graphs with an odd number of vertices, diameter 2 and $k > 4$. First, let us fix some notation.

Notation 3.9. *Let Γ be a non-$1\frac{1}{2}$-extendable edge-regular graph with parameters (n, k, λ), n odd, $k \geq 6$ and diameter 2. Let $S \subseteq V(\Gamma)$ be such that $|S| \geq 3$, S contains at least one edge, $\sigma(\Gamma - S) = |S| - 1$ and $\Gamma - S$ has no even components. Let the subgraph induced by S contains a vertex x such that $\Gamma - x$ is not 1-extendable, and there is an edge $e = \{y, z\}$ in S such that the matching containing this edge cannot be extended to a perfect matching of $\Gamma - x$.*

Remark 3.10. *Since $|S| \geq 3$ it follows that $\sigma(\Gamma - S) \geq 2$. As Γ is of diameter 2 it follows that each vertex from $V(\Gamma) - S$ has at least one neighbour in S.*

Lemma 3.11. *With reference to Notation 3.9 each vertex in* $\Gamma - S$ *has at least* 2 *neighbours in* S.

Proof. Suppose on the contrary that there is a vertex v in an odd component C which has just one neighbour in S. Let us denote this neighbour by u and let $C_1, C_2, ..., C_{|S|-2}, C$ be odd components of $\Gamma - S$. Denote $m_i = |C_i|$, $1 \leq i \leq |S| - 2$ and WLOG assume that $m_1 \leq m_2 \leq ... \leq m_{|S|-2}$. As the diameter of Γ is 2, the unique neighbour u of v must be adjacent to all vertices in $C_1 \cup C_2 \cup ... \cup C_{|S|-2}$. Since u an v have λ common neighbours and all of them are in C (because u is the only neighbour of v which is in S), we have:

$$k \geq 1 + m_1 + m_2 + ... + m_{|S|-2} + \lambda. \tag{3.1}$$

On the other hand, for each vertex $w \in C_1$ it holds that $N(w_1) \subseteq C_1 \cup S$. As vertex u is adjacent to each vertex of component C_1, w has at most λ neighbours in C_1, so

$$k \leq \lambda + |S|. \tag{3.2}$$

Now we can split our analysis into two cases:
Case 1: $m_1 \geq 3$. In this case, we have:

$$k \geq 1 + m_1 + m_2 + ... + m_{|S|-2} + \lambda \geq 1 + 3(|S| - 2) + \lambda, \tag{3.3}$$

which together with (3.2) gives us

$$1 + 3(|S| - 2) + \lambda \leq \ k \ \leq \lambda + |S|, \tag{3.4}$$

which implies $|S| \leq \frac{5}{2}$, and therefore $|S| \leq 2$. This is a contradiction with the fact that $|S| \geq 3$.
Case 2: $m_1 = 1$. Let w_1 be the unique vertex in C_1. As all neighbours of w_1 are in S it follows that $k \leq |S|$. On the other hand, as u and w_1 have λ common neighbours and all of them are contained in S, vertex u must have at least λ neighbours in S, so $k \geq 1 + 1 + m_2 + ... + m_{|S|-2} + 2\lambda$. Therefore,

$$|S| - 2 + m_{|S|-2} + 2\lambda \ \leq 1 + 1 + m_2 + ... + m_{|S|-2} + 2\lambda \ \leq \ k \ \leq |S|, \tag{3.5}$$

which implies

$$2\lambda \leq 2 - m_{|S|-2}. \tag{3.6}$$

Since λ is non-negative integer and $m_{|S|-2}$ is odd we have $m_{|S|-2} = 1$ (and therefore all odd components, except possibly C, are singletons) and $\lambda = 0$. As Γ is triangle free it follows that at least one of y and z is not adjacent to w, and so $k \leq |S| - 1$. Now, using (3.1) we get $k = |S| - 1$ and because of that we can conclude that u does not have any neighbours in S and there is exactly one vertex in S which is not adjacent to w_1. The same conclusion holds for each $w_i \in C_i$, $1 \leq i \leq |S|-2$ which together with the fact that $\lambda = 0$ imply that each w_i $(1 \leq i \leq |S|-2)$ is adjacent to exactly one of vertices y and z and to all other vertices from S. Let us observe vertex x different from vertices u, y and z (such a vertex exists since otherwise $|S| = 3$ would imply $k = 2$ which is not possible). Since diameter of Γ is 2 and vertices from $S - u$ are not adjacent to u and v, each vertex from $S - u$ must be adjacent to at least one neighbour of v. As x is adjacent to all vertices in $C_1 \cup C_2 \cup ... \cup C_{|S|-2}$ and $k = |S| - 1$, it is adjacent to exactly one neighbour of v, say v_1. Therefore, x is not adjacent to any of the vertices from S. As vertex x must be at distance 2 from other neighbours of v it follows that all of them must be adjacent to v_1. Since $\lambda = 0$ it follows that v has only one neighbour in C. Therefore, $k = 2$. Contradiction. □

The proof of the following lemma is the same as for graphs with an even number of vertices, so it will be omitted. It can be found in [1, Lemma 4.3].

Lemma 3.12. *With reference to Notation 3.9, suppose C is a component of $\Gamma - S$ which is not a singleton. Then there are at least $\frac{3k}{2}$ edges between C and S.*

Using this result we will prove that there is at most one component of $\Gamma - S$ which is not a singleton.

Lemma 3.13. *With reference to Notation 3.9, $\Gamma - S$ has at most one component with cardinality at least 3.*

Proof. Suppose to the contrary that $\Gamma - S$ has two odd components C_1 and C_2 such that their cardinalities are at least 3. Let us denote $m_1 = |C_1|$ and $m_2 = |C_2|$. WLOG we may assume that $m_1 \leq m_2$. Let f denote the number of edges between S and $\Gamma - S$. Since Γ contains at least one edge we have that

$$f \leq k|S| - 2. \tag{3.7}$$

By Lemma 3.12 there are at least $\frac{3k}{2} + \frac{3k}{2} = 3k$ edges between $C_1 \cup C_2$ and S. By Lemma 2.1, there are at least $k(|S| - 3)$ edges between the remaining $|S| - 3$ odd components and S. Therefore, there are at least $3k + k(|S| - 3) = k|S|$ edges between the odd components and S. Together with (3.7) this shows that $k|S| \leq k|S| - 2$, a contradiction. □

Using the previous results, we can prove that all components of $\Gamma - S$ are singletons.

Proposition 3.14. *With reference to Notation 3.9 all components of $\Gamma - S$ are singletons.*

Proof. Let us suppose that $\Gamma - S$ has a component, say C, with cardinality at least 3. By Lemma 3.13, $\Gamma - S$ has exactly one such component. Let $|C| = m = 2l + 1 \geq 3$ and let $w_1, w_2, ..., w_{|S|-2}$ denote the vertices of the singleton components. Let s denote the number of edges contained in S and f denote the number of edges between S and C. Counting the number of edges between S and $\Gamma - S$ in two ways we get:

$$k|S| - 2s = k(|S| - 2) + f, \tag{3.8}$$

and so

$$f = 2k - 2s. \tag{3.9}$$

As, by Lemma 3.12 $f \geq \frac{3k}{2}$ holds, we get that

$$2k - 2s \geq \frac{3k}{2}, \tag{3.10}$$

which implies $s \leq \frac{k}{4}$.
Observe that $N(w_i) \subseteq S$ for $i \in \{1, 2, ..., |S| - 2\}$ and note that there are $\frac{k\lambda}{2}$ edges contained in $N(w_i)$ (because there are k vertices in $N(w_i)$ and each of them has λ neighbours in $N(w_i)$). This implies $\frac{k\lambda}{2} \leq s \leq \frac{k}{4}$. This shows that $\lambda = 0$, so Γ is triangle-free, and therefore the subgraph of Γ induced by C is triangle-free. Now we can use the result of Mantel on C, so we get that the number of edges contained in C is at most

$$\left\lfloor \frac{m^2}{4} \right\rfloor = \left\lfloor \frac{(2l+1)^2}{4} \right\rfloor = \left\lfloor \frac{4l^2 + 4l + 1}{4} \right\rfloor = \left\lfloor l^2 + l + \frac{1}{4} \right\rfloor = l(l+1). \tag{3.11}$$

Therefore, $f \geq km - 2l(l + 1)$.
Now, using (3.8) we get

$$\begin{aligned} k|S| - 2s =& k(|S| - 2) + f \\ \geq& k(|S| - 2) + km - 2l(l+1) \\ =& k(|S| - 2) + k(2l+1) - 2l(l+1), \end{aligned} \tag{3.12}$$

which implies

$$2l(l + 1) \geq 2s + k(2l - 1) \geq k(2l - 1) + 2. \tag{3.13}$$

Therefore,

$$k \leq \left\lfloor \frac{2l^2 + 2l - 2}{2l - 1} \right\rfloor = \left\lfloor l + \frac{3}{2} - \frac{1}{2(2l-1)} \right\rfloor \leq l + 1. \tag{3.14}$$

As each vertex from C sends at least two edges to S, equation (3.9) gives us $2k - 2s \geq 2m$, and so

$$s \leq k - m \leq l + 1 - 2l - 1 \leq -1. \tag{3.15}$$

This is a contradiction with the definition of s. □

Proposition 3.15. *With reference to Notation 3.9 there are exactly $\frac{k}{2}$ edges contained in S and graph Γ is triangle-free.*

Proof. Let the number of edges in S be denoted by s. As all components are singletons, by Proposition 3.14, counting the number of edges between S and $\Gamma - S$ in two different ways we get the following equation:

$$k|S| - 2s = k(|S| - 1) = k|S| - k, \tag{3.16}$$

which obviously implies that $s = \frac{k}{2}$, as claimed.
Now, let us prove that Γ is triangle-free, that is, that $\lambda = 0$. Observe, like in the proof of the previous proposition, that $N(w_i) \subseteq S$ for all $i \in \{1, 2, ..., |S| - 1\}$ and that the number of edges in the subgraph of Γ induced by $N(w_i)$ is $\frac{k\lambda}{2}$. Therefore, $\frac{k\lambda}{2} \leq \frac{k}{2}$ which implies that $\lambda \leq 1$. As λ is non-negative integer we have $\lambda \in \{0, 1\}$.
Suppose that $\lambda = 1$ and let $C_1, C_2, ..., C_{|S|-1}$ be components of $\Gamma - S$. First, let us prove that each vertex of S has at least two neighbours in S. Let v be an arbitrary vertex from S which does not have any neighbours in S. Since $k \geq 6$ v has at least 6 neighbours in $\Gamma - S$. Then as $\lambda = 1$, v must have one common neighbour with each of them. Since all components are singletons, those neighbours must be in S. Contradiction. Now, let us suppose that v has exactly one neighbour in S, say u. Then vertex v is adjacent to $k - 1$ vertices in $C_1 \cup C_2 \cup ... \cup C_{|S|-1}$. However, in that case (since $\lambda = 1$) v and each w_i, $i \in \{1, 2, ..., |S| - 1\}$ must have a common neighbour, and that neighbour must be u (because all components are singletons and all neighbours of vertices from components are in S). So, u and v are adjacent vertices and they have $k - 1$ common neighbours, which implies $1 = \lambda = k - 1$ and $k = 2$. Contradiction. Therefore, each vertex in S has at least two neighbours in S. Since $|S| \geq k$ there are at least k edges in S. Contradiction (with the fact that there are exactly $\frac{k}{2}$ edges in S). □

Using the previous Proposition we will show that set S from Notation 3.9 has exactly $\frac{3k}{2}$ vertices.

Proposition 3.16. *With reference to Notation 3.9 we have that $|S| = \frac{3k}{2}$.*

Proof. Let us observe a vertex w_j $(1 \leq j \leq |S| - 1)$ from an arbitrary odd component. Vertex w_j has k neighbours in S. By Proposition 3.15, there are exactly $\frac{k}{2}$ edges in S. Hence there are at most $\frac{k}{2}$ vertices in S which are at distance 2 from w_j (note that there are no edges in the subgraph induced by $N(w_j)$ since Γ is triangle-free, by Proposition 3.15). Since there are no vertices in Γ at distance greater than 2 from vertex w_j, it follows that $|S| \leq k + \frac{k}{2} = \frac{3k}{2}$.
Let us prove the other inequality. Suppose to the contrary that $|S| < \frac{3k}{2}$. Let w_i $(1 \leq i \leq |S| - 1)$ be a vertex from an odd component of $\Gamma - S$. Let V denote the set

of vertices from $S\backslash N(w_i)$ and let us observe a vertex v from V. Since the diameter of Γ is 2, v must have at least one neighbour in $N(w_i)$. Let us denote that neighbour by u. By Proposition 3.15, we have that $\lambda = 0$, so vertices u and v do not have common neighbours. Since there are exactly $\frac{k}{2}$ edges in S, vertices u and v (together) may have at most $\frac{k}{2} - 1$ neighbours in S. Therefore, they have at least $2(k-1) - (\frac{k}{2} - 1) = \frac{3k}{2} - 1$ neighbours in odd components. That is a contradiction with the fact that there are $|S| - 1 < \frac{3k}{2} - 1$ singleton components of $\Gamma - S$.
Therefore, $|S| \geq \frac{3k}{2}$, which together with $|S| \leq \frac{3k}{2}$ gives us the desired result. □

Lemma 3.17. *With reference to Notation 3.9, let w_i be a vertex in an odd component C_i of $\Gamma - S$ and let $V = S - N(w_i)$ $(1 \leq i \leq |S| - 1)$. Then there are no edges in the subgraph induced by V and any vertex from V is adjacent to exactly one vertex from $N(w_i)$. Moreover, all vertices from V are adjacent to the same vertex from $N(w_i)$.*

Proof. Suppose that there is an edge in the subgraph induced by V, or there is a vertex in V adjacent to more than one vertex from $N(w_i)$. In both cases, there is a vertex in V which is an endpoint of at least two edges in S. There are exactly $\frac{k}{2}$ edges in S and $\frac{k}{2}$ vertices in V. Hence there is a vertex in V which does not have any neighbours in $N(w_i)$, and such a vertex is at distance more than 2 from w_i. Contradiction.
In order to prove the second part of the claim, it will be enough to prove that there is a vertex in $N(w_i)$ which is adjacent to all vertices from V. Let us suppose to the contrary, that there is no such a vertex in $N(w_i)$. Let us observe a vertex $u \in N(w_i)$ which has at least one neighbour in V (such a vertex exists since there are $\frac{k}{2}$ edges in S and Γ is triangle-free). Vertex u is adjacent to at most $\frac{k}{2} - 1$ vertices in S, and therefore to at least $k - (\frac{k}{2} - 1) = \frac{k}{2} + 1$ vertices in odd components. Let $v \in V$ be an arbitrary neighbour of u. Vertex v is adjacent to u and to $k - 1$ vertices in odd components (because by the first part of this Lemma, it does not have any neighbours in V or in $N(w_i) - u$). As Γ is, by Proposition 3.15, triangle-free, we have that there are at least $(\frac{k}{2} + 1) + (k - 1) = \frac{3k}{2}$ vertices in odd components. Therefore, $\frac{3k}{2} \leq |S| - 1$ which, since by Proposition 3.16 $|S| = \frac{3k}{2}$, implies $\frac{3k}{2} \leq \frac{3k}{2} - 1$. Contradiction. □

Now we can finally prove the main result of this chapter.

Theorem 3.18. *Let Γ be an edge-regular graph with parameters (n, k, λ), n odd, $k \geq 6$ and diameter 2. Then Γ is $1\frac{1}{2}$-extendable.*

Proof. Let us suppose that an edge-regular graph Γ with an odd number of vertices, diameter 2 and $k \geq 6$ is not $1\frac{1}{2}$-extendable. Then, with reference to Notation 3.9, Proposition 3.14 holds, all components of $\Gamma - S$ are singletons and (by Lemma 2.9) there are $|S| - 1$ of them.
Let w_i, $1 \leq i \leq |S| - 1$, be a vertex in an odd component. It has k neighbours in S,

so $|N(w_i)| = k$. By Lemma 3.17, there is a vertex $u \in N(w_i)$ which is adjacent to $\frac{k}{2}$ vertices in $V = S - N(w_i)$. Vertex u has $\frac{k}{2}$ neighbours in odd components. Each vertex v from V has $k-1$ neighbours in odd components. The total number of vertices in odd components is $\frac{3k}{2} - 1$ (because, by Proposition 3.16, $|S| = \frac{3k}{2}$) and u and v do not have common neighbours (since Γ is, by Proposition 3.15, triangle-free). So it follows that all vertices from V are adjacent to the same $k-1$ vertices in odd components.
Let us observe a vertex w_j, $1 \leq j \leq |S|-1$, from some odd component which is adjacent to all vertices from V. Vertex w_j has $\frac{k}{2}$ neighbours in V and $\frac{k}{2}$ neighbours in $N(w_i)$. Therefore, there are $|N(w_i)| - \frac{k}{2} = k - \frac{k}{2} = \frac{k}{2}$ vertices in $N(w_i)$ which are not adjacent to w_j. There are no edges in the subgraph induced by $N(w_i)$, and all vertices from V are adjacent to the same vertex in $N(w_i)$. Hence there are $\frac{k}{2}$ vertices in S which are not adjacent to w_j. Also, $\frac{k}{2} - 1$ of them do not have any common neighbours with w_j (w_j and u have $|V|$ common neighbours). Therefore, they are at distance greater than 2 from w_j. Contradiction. □

In the following theorem we will summarize the results obtained in this chapter.

Theorem 3.19. *Let Γ be an edge-regular graph with an odd number of vertices and diameter 2. If Γ is non-$1\frac{1}{2}$-extendable graph, then it is isomorphic to the graph C_5.*

4 On $2\frac{1}{2}$-extendability of edge-regular graphs with diameter 2

This chapter is devoted to the $2\frac{1}{2}$-extendability of edge-regular graphs and strongly regular graphs with an odd number of vertices and diameter 2.

In the first section we represent some basic properties for non-$2\frac{1}{2}$-extendable edge-regular graphs with an odd number of vertices and diameter 2 which will be used in the rest of this chapter. The second section will be about examples of non-$2\frac{1}{2}$-extendable edge-regular graphs of diameter 2 and an odd number of vertices.

Since there are some edge-regular graphs of diameter 2 which are not $2\frac{1}{2}$-extendable, we will narrow our attention to strongly regular graphs. In the last section we will prove that there are only 2 non-$2\frac{1}{2}$-extendable strongly regular graphs with an odd number of vertices.

4.1 Basic results about edge-regular graphs with diameter 2

In the previous chapter we proved that C_5 is the only edge-regular graph with an odd number of vertices and diameter 2 which is not $1\frac{1}{2}$-extendable. It was already mentioned that C_5 does not satisfy the lower bound on the number of vertices given in the definition of the $2\frac{1}{2}$-extendable graph, so there is no sense to talk about the $2\frac{1}{2}$-extendability of C_5. But what about the $2\frac{1}{2}$-extendability of other edge-regular graphs with an odd number of vertices and diameter 2?

Before we start proving the properties of edge-regular graphs with an odd number of vertices and diameter 2, which are not $2\frac{1}{2}$-extendable, let us fix some notation.

Notation 4.1. *Let Γ be a non-$2\frac{1}{2}$-extendable edge-regular graph with parameters (n, k, λ), an odd number of vertices and diameter 2, which is not C_5. Let x denote the vertex of Γ such that $\Gamma - x$ is not 2-extendable. Let $e_1 = \{y_1, z_1\}$ and $e_2 = \{y_2, z_2\}$ be two edges of Γ such that $\Gamma' = \Gamma - \{x, y_1, y_2, z_1, z_2\}$ does not contain a perfect matching.*

By Tutte's result, there exists a subset S' *of* $V(\Gamma)$ *such that* $\sigma(\Gamma' - S') > |S'|$. *Let* $S = S' \cup \{x, y_1, y_2, z_1, z_2\}$. *Note that the components of* $\Gamma' - S'$ *are the same as the components of* $\Gamma - S$, *and therefore* $\sigma(\Gamma' - S') = \sigma(\Gamma - S)$.
Note that since $|\Gamma|$ *is odd,* $|S|$ *and* $\sigma(\Gamma - S)$ *have different parity.*

Now we can find the number of components of $\Gamma - S$.

Lemma 4.2. *With reference to Notation 4.1 we have* $\sigma(\Gamma - S) = |S| - 3$.

Proof. By Theorem 3.19, we have that Γ is $1\frac{1}{2}$-extendable. So, $\Gamma - x$ is 1-exendable. Therefore, $\Gamma'' = \Gamma - \{x, y_1, z_1\}$ contains a perfect matching. Let us denote $S'' = S' \cup \{x, y_1, z_1\}$. By Tutte's result, we have $\sigma(\Gamma'' - S'') \leq |S''|$. As the components of $\Gamma'' - S''$ are the same as the components of $\Gamma - S$ we have the following: $\sigma(\Gamma - S) \leq |S''| = |S'| + 3 = |S| - 2$. As $\sigma(\Gamma - S)$ and $|S|$ have different parity, it follows that $\sigma(\Gamma - S) \leq |S| - 3$. On the other hand, since Γ' does not contain a perfect matching, $\sigma(\Gamma - S) = \sigma(\Gamma' - S') > |S'| = |S| - 5$. Therefore, again because of different parity of $|S|$ and $\sigma(\Gamma - S)$, we get $\sigma(\Gamma - S) \geq |S| - 3$. Therefore, $\sigma(\Gamma - S) = |S| - 3$. □

Similarly as in the proof of Proposition 2.10 we can show, that we could assume, that all components of $\Gamma - S$ are odd.

4.2 Some examples of non-$2\frac{1}{2}$-edge-regular graphs

Since we proved that all (n, k, λ)-edge-regular graphs with $n > 5$ odd and diameter 2 are $1\frac{1}{2}$-extendable, it is natural to ask if all such graphs are $2\frac{1}{2}$-extendable.
In this section we will give two examples of non-$2\frac{1}{2}$-extendable edge-regular graphs with an odd number of vertices and diameter 2.

First, let us prove the following proposition.

Proposition 4.3. *Let* Γ *be an* $(n, k, 0)$*-edge-regular graph with an odd number of vertices and diameter* 2 *for which there exists a vertex* u *of* Γ, *such that* $N_2(u)$ *contains two independent edges and* $|N_2(u)| < k + 4$. *Then* Γ *is not* $2\frac{1}{2}$*-extendable.*

Proof. Let Γ be a graph satisfying all given conditions. Suppose that Γ is $2\frac{1}{2}$-extendable. Let us observe $\Gamma - u$ and let us choose any two independent edges from the subgraph induced on $N_2(u)$. The matching containing these two edges will cover four vertices in $N_2(u)$. Therefore, as $|N_2(u)| < k + 4$ there will be less than k uncovered vertices in $N_2(u)$. Since Γ is triangle-free, there are no edges in the subgraph induced on $N(u)$. Since $|N(u)| = k$, there will always stay at least one uncovered vertex in $N(u)$. Hence our starting matching cannot be extended to a perfect matching of $\Gamma - u$. Contradiction. □

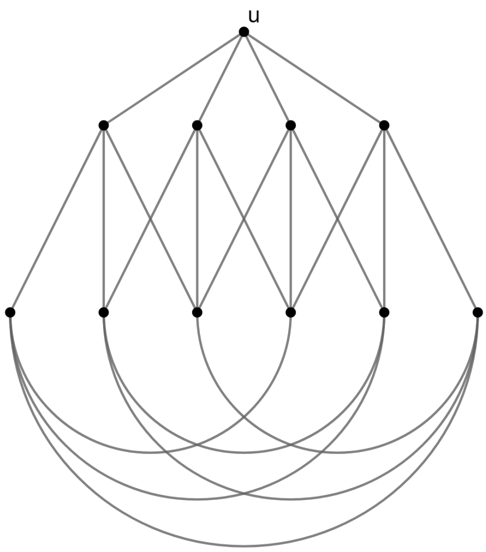

Figure 5: $(11, 4, 0)$-edge-regular graph

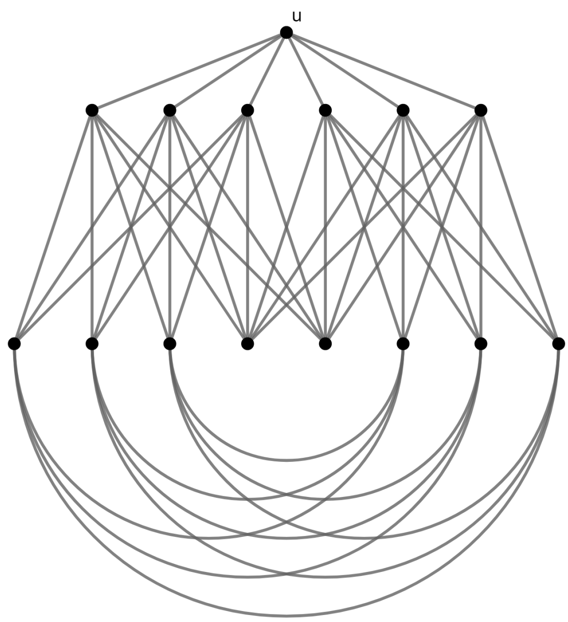

Figure 6: $(15, 6, 0)$-edge-regular graph

Let us consider an edge-regular graph from Figure 5.
It is easy to see that this graph is an $(11, 4, 0)$-edge-regular graph. This graph has diameter 2 since obviously each two non-adjacent vertices have at least one common neighbour. Let us show that it is not $2\frac{1}{2}$-extendable. In this case $k = 4$ and $|N_2(u)| = 6$, so it follows from Proposition 4.3 that this graph is not $2\frac{1}{2}$-extendable.

Let us consider an edge-regular graph from Figure 6.
It is easy to see that this graph is an $(15, 6, 0)$-edge-regular graph. Obviously each two non-adjacent vertices have at least one common neighbour, so this graph has diameter 2. In this case $k = 6$ and $|N_2(u)| = 8$. Hence, by Proposition 4.3 this graph is not $2\frac{1}{2}$-extendable.

Since there are some edge-regular graphs with an odd number of vertices and diameter 2 which are not $2\frac{1}{2}$-extendable, we will restrict our attention to the family of strongly regular graphs.

4.3 Extendability of strongly regular graphs

At the beginning of this section we will recall a definition of a strongly regular graph and some fundamental results about these graphs. Later on we will prove that there are only two non-$2\frac{1}{2}$-extendable strongly regular graphs with an odd number of vertices and diameter 2.

Definition 4.4. Let Γ be a regular graph that is neither complete nor empty. Then Γ is said to be strongly regular with parameters (n, k, λ, μ) if it has n vertices, it is k-regular, every pair of adjacent vertices have λ common neighbours and every pair of distinct non-adjacent vertices have μ common neighbours.

Remark 4.5. *If μ is non-zero, strongly regular graph is a distance-regular graph with diameter 2. If $\mu = 0$, then Γ is a disjoint union of complete graphs of the same size. Since by Definition 1.2 and Definition 1.3 we are only interested in connected graphs, we will assume that $\mu \geq 1$.*

The parameters of strongly regular graphs are not independent and they must obey the following relation:

$$k(k - \lambda - 1) = \mu(n - k - 1). \tag{4.1}$$

We can obtain this relation by counting the number of edges between the neighbours and non-neighbours of an arbitrary vertex u from $V(\Gamma)$ in two ways. Let t denote the number of edges between the neighbours and non-neighbours of u. Each vertex in Γ

has k neighbours and $n-k-1$ non-neighbours. Each non-neighbour of u is adjacent to μ neighbours of u, so $t = \mu(n-k-1)$. On the other hand, each of the k neighbours of u is adjacent to u itself and to λ neighbours of u, so it is adjacent to $k-\lambda-1$ non-neighbours of u. Thus, $t = k(k-\lambda-1)$.

It is well known that the complement of a strongly regular graph with parameters (n, k, λ, μ) is again a strongly regular graph, and it has parameters

$$(n,\ n-k-1,\ n-2k+\mu-2,\ n-2k+\lambda). \tag{4.2}$$

If Γ is a strongly regular graph with parameters (n, k, λ, μ) we will often write just Γ is (n, k, λ, μ) strongly regular graph.

Strongly regular graphs are special case of edge-regular graphs, so (as we already proved) C_5 is the only strongly regular graph which is not $1\frac{1}{2}$-extendable. But, what about the $2\frac{1}{2}$-extendability of other strongly regular graphs?

Note that C_5 is the only strongly regular graph with $k = 2$ and an odd number of vertices. So, we can start our analysis observing strongly regular graphs with $k = 4$.

Proposition 4.6. *The only strongly regular graph with an odd number of vertices and $k = 4$ is the Paley graph on 9 vertices, which is not $2\frac{1}{2}$-extendable.*

Proof. From the relation (4.1) it can be easily seen that $n \leq 17$ for any possible combination of λ and μ. From the Brouwer's list of strongly regular graphs (see [4]) we see that the Paley graph on 9 vertices is indeed the only strongly regular graph with $k = 4$.

Let us observe the Paley graph on 9 vertices labelled as in the following figure.

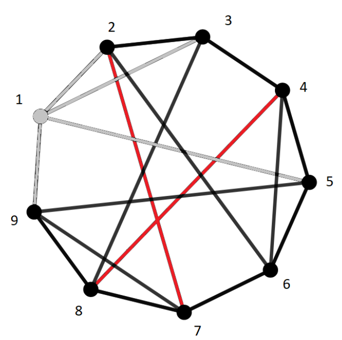

Figure 7: Paley graph on 9 vertices

Let us observe $\Gamma - \{1\}$ and the matching containing edges $\{2,7\}$ and $\{8,4\}$. The only way to cover vertex 9 is to add edge $\{9,5\}$ in a matching. But, in this case, there is no possibility to cover vertices 3 and 6 by a matching. So, our starting matching cannot be extended to a perfect matching of $\Gamma - \{1\}$. Therefore, the Paley graph on 9 vertices is not $2\frac{1}{2}$-extendable. □

Now we can observe strongly regular graphs with $k \geq 6$. First, let us fix some additional notation.

Notation 4.7. *Let Γ be a (n,k,λ,μ) strongly regular graph, with an odd number of vertices and $k \geq 6$ which is not $2\frac{1}{2}$-extendable. Since the class of strongly regular graphs is a subclass of edge-regular graphs, we will use Notation 4.1 with the additional assumption that Γ is strongly regular with $k \geq 6$.*

Lemma 4.8. *With reference to Notation 4.7, each vertex of $\Gamma - S$ sends at least μ edges to S.*

Proof. As by Lemma 4.2, $\sigma(\Gamma - S) = |S| - 3$ and $|S| \geq 5$ it follows that $\Gamma - S$ has at least 2 odd components. Since vertices from different components are not adjacent, the result follows from the definition of the strongly regular graphs. □

Let us recall that by Lemma 2.1 there are at least k edges from each component of $\Gamma - S$ to S. This result will be of particular use in the rest of this chapter.

Using the previous results we can prove that each vertex of $\Gamma - S$ sends at least 2 edges to S.

Lemma 4.9. *With reference to Notation 4.7 each vertex of each component of $\Gamma - S$ has at least $max\{2,\mu\}$ neighbours in S.*

Proof. If $\mu \geq 2$ the claim follows from Lemma 4.8.
Let $\mu = 1$. As Γ is strongly regular, we have that $n = k(k-\lambda)+1$ (by (4.1)). Suppose to the contrary that there is a vertex v in an odd component C of $\Gamma - S$ which has only one neighbour in S. Let u be that neighbour. Let $C_1, C_2, ..., C_{|S|-4}$ be the components of $\Gamma - S$ different from C. Let $|C_i| = m_i$ $(1 \leq i \leq |S|-4)$ and WLOG let us assume that $m_1 \leq m_2 \leq ... \leq m_{|S|-4}$. Let $w_1 \in C_1$ be an arbitrary vertex. Then

$$deg(w_1) = k \leq m_1 - 1 + |S|. \tag{4.3}$$

Since $\mu = 1$, each vertex in each odd component of $\Gamma - S$ different from component C is adjacent to u. As u is adjacent to all vertices in C_1 and has λ common neighbours with w_1, it follows that

$$deg(w_1) = k \leq |S| + \lambda. \tag{4.4}$$

Since vertex u is adjacent to vertex v and all vertices in $C_1 \cup C_2 \cup ... \cup C_{|S|-4}$ and it has λ common neighbours with v (and all of them are in C) we have that

$$deg(u) = k \geq 1 + m_1 + m_2 + ... + m_{|S|-4} + \lambda. \tag{4.5}$$

Let us consider the following two cases depending on the value of m_1.

Case 1: $m_1 \geq 3$. Using (4.4) and (4.5) we get $1 + 3(|S| - 4) + \lambda \leq |S| + \lambda$, which implies that $|S| \leq 5$. Since $|S| \geq 5$, we get $|S| = 5$. Using this fact together with (4.3) we have that $k \leq m_1 + 4$. But as k is even and m_1 is odd it follows that $k \leq m_1 + 3$. Equation (4.5) together with the fact that $|S| = 5$ gives us $k \geq m_1 + 1 + \lambda$. Therefore, $\lambda \leq 2$. Let us consider the following 3 subcases depending on the value of λ.
Subcase 1.1 $\lambda = 2$. Then by (4.4) we have that $k \leq 7$. Since k is even we get $k = 6$. Therefore, we get that Γ has parameters $(25, 6, 2, 1)$, but there is no such strongly regular graph (by [4]).
Subcase 1.2 $\lambda = 1$. Since k is even and $k \geq m_1 + 2$ we get that $k \geq m_1 + 3$. By (4.4) and the assumption that $k \geq 6$ it follows that $k = 6$. Therefore, we get that Γ has parameters $(31, 6, 1, 1)$, but there is no such strongly regular graph (by [4]).
Subcase 1.3 $\lambda = 0$. In this case, by (4.4) it follows that $k \leq 5$. Contradiction.

Case 2: $m_1 = 1$. Let w_1 denote the unique vertex in C_1. It has k neighbours in S, so $k \leq |S|$. As u and w_1 have λ common neighbours and all of them are in S, vertex u must have at least λ neighbours in S. So, $deg(u) = k \geq 1 + 1 + m_2 + ... + m_{|S|-4} + 2\lambda$. Therefore,

$$|S| - 4 + m_{|S|-4} + 2\lambda \leq 1 + 1 + m_2 + ... + m_{|S|-4} + 2\lambda \leq k \leq |S|, \tag{4.6}$$

which implies

$$m_{|S|-4} \leq 4 - 2\lambda. \tag{4.7}$$

As $m_{|S|-4}$ is odd, non-negative integer, it follows that $\lambda \in \{0, 1\}$ and $m_{|S|-4} \in \{1, 3\}$. Now we can split our analysis into the following two subcases:
Subcase 2.1: $m_{|S|-4} = 1$. All components of $\Gamma - S$ different from C are singletons. Each vertex from each singleton has k neighbours in S. As $\mu = 1$ any two vertices from $C_1 \cup ... \cup C_{|S|-4}$ have exactly one common neighbour in S and that neighbour must be vertex u. So,

$$|S| \geq 1 + (k - 1)(|S| - 4) \geq 1 + 5(|S| - 4), \tag{4.8}$$

which implies $|S| < 5$. Contradiction.
Subcase 2.2: $m_{|S|-4} = 3$. Then $\lambda = 0$, and

$$deg(u) = k \geq 1 + 1 + m_2 + ... + m_{|S|-5} + 3 \geq |S| - 1. \tag{4.9}$$

As $k \leq |S|$, it follows that $k \in \{|S|-1, |S|\}$. But, in both cases w_1 must be adjacent to both endpoints of at least one of the edges in S (there are at least two independent edges in S). Contradiction (with the fact that $\lambda = 0$). □

Lemma 3.12 from the previous chapter holds also with reference to Notation 4.7. In this chapter it will be again of particular use, so we will recall it.

Lemma 4.10. *With reference to Notation 4.7, let C be a component of $\Gamma - S$ which is not a singleton. Then the number of edges between S and C is at least $\frac{3k}{2}$.*

From the definition of strongly regular graphs it is obvious that $\mu \leq k$. In this chapter, we will consider cases $\mu = k$ and $\mu < k$ separately.

4.3.1 Imprimitive strongly regular graphs with diameter 2

Before we start our analysis of strongly regular graphs with $\mu = k$, let us recall some important definitions and facts.

Definition 4.11. A strongly regular graph Γ is imprimitive if Γ, or its complement is disconnected. Otherwise, Γ is primitive.

Since we are interested only in connected graphs, we assume that the complement of Γ is disconnected. It follows from equation (4.2) that this is the case if and only if $n - 2k + \lambda = 0$. By (4.1) this happens if and only if $k = \mu$. In this case Γ is a complete multipartite graph $K_{a \times m}$, with parameters $n = am$, $k = \mu = (a-1)m$, $\lambda = (a-2)m$.

In this subsection we will show that there is exactly one imprimitive strongly regular graph with an odd number of vertices and diameter 2 which is not $2\frac{1}{2}$-extendable. Before we do that, we will state one important result given by Yu [21, Theorem 2.5].

Theorem 4.12. *A graph Γ is $1\frac{1}{2}$-extendable if and only if for any $S \subseteq V(\Gamma)$, $S \neq \emptyset$,*

1. *$\sigma(\Gamma - S) \leq |S| - 1$ and*
2. *if both $\sigma(\Gamma - S) = |S| - 1$ and $|S| \geq 3$, then S is independent.*

Now we can prove the main result of this subsection.

Proposition 4.13. *Let Γ be a connected imprimitive strongly regular graph with an odd number of vertices. If Γ is not $2\frac{1}{2}$-extendable, then it is isomorphic to $K_{3\times 3}$.*

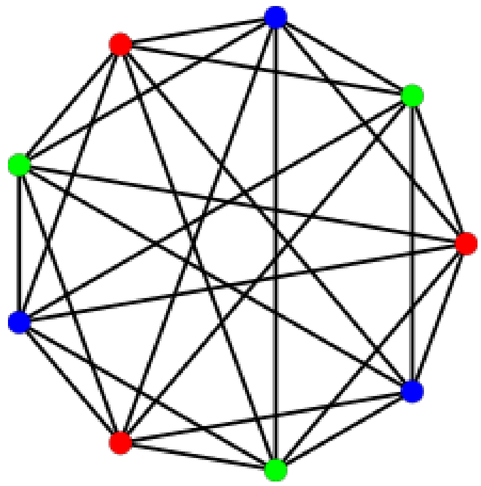

Figure 8: Complete multipartite graph $K_{3\times3}$.

Proof. Let us suppose that Γ is not $2\frac{1}{2}$-extendable. Then Γ is a graph from Notation 4.7 and $\sigma(\Gamma - S) = |S| - 3$. Since each vertex from each component sends $k = \mu$ edges to S and all components of $\Gamma - S$ are connected, it follows that all components of $\Gamma - S$ are singletons. As each vertex from an odd component has k neighbours in S it follows that $k \leq |S|$.

Let us denote by w_i the unique vertex from component C_i, $1 \leq i \leq |S|-3$. Since $k = \mu$ it follows that $N(w_i) = N(w_j)$ for $1 \leq i, j \leq |S| - 3$. Let us suppose that $k < |S|$. Let us pick a vertex $y \in S - N(w_1)$. As w_1 is not adjacent to y and $\mu = k$ it follows that $N(y) = N(w_1)$. Let us consider set $S' = S - y$. We have that $|S'| = |S| - 1$ and $\sigma(\Gamma - S') = |S| - 3 + 1 = |S'| - 1$. Since S contains at least 2 independent edges, S' contains at least one edge. Hence by Theorem 4.12 it follows that Γ is not $1\frac{1}{2}$-extendable. This is in contradiction with Theorem 3.18. Therefore $k = |S|$. Moreover, as each two adjacent vertices have $\lambda = (a-2)m$ common neighbours, it follows that a vertex from $N(w_1)$ must have $(a-2)m$ neighbours in $N(w_1) = S$. Hence,

$$|S| = k = |S| - 3 + (a-2)m, \tag{4.10}$$

which implies

$$3 = (a-2)m. \tag{4.11}$$

Since Γ is not complete and has an odd number of vertices it follows that $m \geq 3$. This fact together with (4.11) gives us $m = 3$ and $a = 3$. Therefore, $\Gamma = K_{3\times3}$.

Let us show that Γ is not $2\frac{1}{2}$-extendable. Since $m = a = 3$ it follows that Γ contains three independent sets of size 3, say I_1, I_2 and I_3. Let $u \in I_1$ be an arbitrary vertex and let us observe $\Gamma - u$. Let us choose any two edges with endpoints in $I_1 - u$ and I_2. Then there is one uncovered vertex in I_2, 3 uncovered vertices in I_3 and no uncovered vertices in $I_1 - u$. No matter how we choose the next edge in the matching, there will always remain 2 uncovered vertices in I_3. Therefore, the starting matching cannot be extended to a perfect matching of Γ. This shows that $K_{3\times 3}$ is not $2\frac{1}{2}$-extendable. □

From now on, we will consider primitive strongly regular graphs with an odd number of vertices.

4.3.2 Primitive strongly regular graphs

Cioabă and Li in [9] mentioned that it can be proved that the Paley graph on 9 vertices is the only primitive strongly regular graph with an odd number of vertices which is not $2\frac{1}{2}$-extendable. We already proved that the Paley graph on 9 vertices is not $2\frac{1}{2}$-extendable (Proposition 4.6), so in this subsection we will prove that all primitive strongly regular graphs with an odd number of vertices and $k \geq 6$ are $2\frac{1}{2}$-extendable.

The following lemma is a generalization of [9, Lemma 16] and it will be of particular use in the rest of this chapter.

Lemma 4.14. *Let Γ be a primitive (n, k, λ, μ) strongly regular graph and S be a disconnecting set of vertices. If $\Gamma - S$ contains at least two singleton components, then S contains at least $\mu(k-\mu) + \frac{k\lambda}{2}$ edges.*

Proof. Let C_1 and C_2 be two singleton components of $\Gamma - S$ and let $w_i \in C_i$, $i = 1, 2$. All neighbours of w_i are in S. As Γ is (n, k, λ, μ) strongly regular graph, it follows that $|N(w_1)| = |N(w_2)| = k$ and $|N(w_1) \cap N(w_2)| = \mu$. Let u be an arbitrary vertex from $N(w_1)\backslash N(w_2)$. Vertex u has λ common neighbours with w_1. Therefore, there are at least $\frac{k\lambda}{2}$ edges in the subgraph induced by $N(w_1)$. Now, let v be an arbitrary vertex from $N(w_2)\backslash N(w_1)$. Since vertices v and w_1 are not adjacent, they have μ common neighbours. As $|N(w_2)\backslash N(w_1)| = k-\mu$, it follows that there are at least $\mu(k-\mu)$ edges from $N(w_2)\backslash N(w_1)$ to $N(w_1)$. Obviously, none of this edges is contained in subgraph of Γ induced by $N(w_1)$. Therefore, there are at least $\mu(k-\mu) + \frac{k\lambda}{2}$ edges in S. □

Of great importance in the rest of this chapter will be the following lemma. Proof is given in [9, Lemma 11], so we will omit it.

Lemma 4.15. *Let Γ be a primitive (n, k, λ, μ) strongly regular graph. If A is a subset of vertices with $3 \leq |A| \leq \frac{n}{2}$ and A^c denotes its complement, then $e(A, A^c) \geq 3k - 6$.*

In the previous lemma, $e(A, A^c)$ denotes the number of edges with one endpoint in A and the other in A^c.

Note that in the rest of this chapter all claims will be with reference to Notation 4.7 with additional assumption that Γ is primitive.

Since there are no edges between the components of $\Gamma - S$, the number of edges between the component C of $\Gamma - S$ and its complement will be the same as the number of edges between C and S. From now on, let f denote the number of edges between S and $\Gamma - S$.

Lemma 4.16. *There are at most* 2 *non-singleton components in* $\Gamma - S$.

Proof. Let us suppose that there are at least 3 components of $\Gamma - S$ with cardinality at least 3, say C_1, C_2 and C_3.
It is obvious that at most one of these components can have cardinality greater than $\frac{n}{2}$. Therefore, we can apply Lemma 4.15 to at least two non-singleton components. Using this together with Lemma 4.10 we have

$$f \geq k(|S| - 6) + \frac{3k}{2} + 2(3k - 6) = k|S| + \frac{3k}{2} - 12. \tag{4.12}$$

On the other hand, as there are at least 2 edges in S, it holds that

$$f \leq k|S| - 4. \tag{4.13}$$

Combining the inequalities (4.12) and (4.13) we get that $k \leq \frac{16}{3}$. This is a contradiction with the assumption that $k \geq 6$. □

Brouwer and Mesner (see [8]) proved the following result which we will use to prove that there is at least one singleton component in $\Gamma - S$.

Proposition 4.17. *If* Γ *is a primitive strongly regular graph of valency* k*, then* Γ *is* k*-connected. Any disconnecting set of size* k *must be the neighbourhood of some vertex.*

Remark 4.18. *The set* S *used in the Notation 4.7 is a disconnecting set of* Γ*. So, using the Proposition 4.17 we may conclude that* $|S| \geq k \geq 6$.

Using the following two lemmas we will prove that there is at most one non-singleton component in $\Gamma - S$.

Lemma 4.19. *If there are exactly two components of* $\Gamma - S$ *with cardinality at least* 3*, then there is at least one singleton component.*

Proof. Suppose to the contrary that there are no singleton components in $\Gamma - S$. Then there are exactly two components in $\Gamma - S$ (both with cardinality at least 3). Therefore, $\sigma(\Gamma - S) = |S| - 3 = 2$ which implies $|S| = 5$. But this is a contradiction with Proposition 4.17 and Remark following it. □

Lemma 4.20. *If there are exactly two components of* $\Gamma - S$ *with cardinality at least* 3*, then* $\lambda \in \{0, 1\}$*.*

Proof. Let us denote by e the number of edges in S. As by the previous lemma there is at least one singleton component in $\Gamma - S$, it follows that $e \geq \frac{k\lambda}{2}$ (because there are $\frac{k\lambda}{2}$ edges in the subgraph induced on the neighbourhood of a singleton component). By Lemma 4.15 it follows that $f \geq k(|S| - 5) + \frac{3k}{2} + 3k - 6$. If $\lambda \geq 2$ we get $f = k|S| - 2e \leq k|S| - k$. Hence, $k \leq 4$. Contradiction. Therefore, $\lambda \in \{0, 1\}$. □

Lemma 4.21. *There is at most one component in* $\Gamma - S$ *with cardinality at least* 3*.*

Proof. Suppose that there are exactly two non-singletons in $\Gamma - S$, say C_1 and C_2. At least one of them has cardinality at most $\frac{n}{2}$.
In the previous lemma we proved that $\lambda \in \{0, 1\}$, so now we can split our analysis into the following two cases.
Case 1: $\lambda = 1$. Let us split our analysis into the two cases depending on the number of singleton components.
Subcase 1.1: There are at least 2 singletons. Using Lemma 4.14 and the fact that $1 \leq \mu \leq k - 1$ we have that there are at least $\mu(k - \mu) \geq k - 1$ edges in S. Therefore, $f \leq k|S| - 2(k - 1)$. On the other hand, $f \geq k(|S| - 5) + \frac{3k}{2} + 3k - 6$. Hence, $k \leq \frac{16}{3} < 6$. Contradiction.
Subcase 1.2: There is at most one singleton component. Since $|S| \geq 6$ and $\sigma(\Gamma - S) = |S| - 3$, there is exactly one singleton. Therefore, $|S| = k = 6$. Since $k(k - \lambda - 1) = (n - k - 1)\mu$, we get that the possible parameter sets are: $(31, 6, 1, 1)$, $(19, 6, 1, 2)$, $(15, 6, 1, 3)$ and $(13, 6, 1, 4)$. Regarding [4], among these parameter sets, strongly regular graph $(15, 6, 1, 3)$ is the only one which exists. But, for this graph there is no disconnecting set S such that $|S| = 6$ and $\Gamma - S$ contains one singleton and 2 components with cardinality at least 3. Suppose to the contrary. Vertex from singleton component is adjacent to all vertices in S. Since $\mu = 3$, each vertex from $C_1 \cup C_2$ has exactly 3 neighbours in S. On the other hand, at least one of non-singleton components must have cardinality 3 and each vertex in it must have at least 4 neighbours in S. Contradiction.
Case 2: $\lambda = 0$. Similarly as in the previous case we can prove that there is exactly one singleton component in $\Gamma - S$ and $|S| = k = 6$. Therefore, vertex from the singleton component is adjacent to all vertices in S, but that is impossible since $\lambda = 0$ and there are at least 2 edges in S. □

Using the previous result we can prove that all components of $\Gamma - S$ are singletons.

Lemma 4.22. *All components of* $\Gamma - S$ *are singletons.*

Proof. Suppose that there is exactly one non-singleton component of $\Gamma - S$, say C. Let $C_1, C_2, ..., C_{|S|-4}$ denote the singleton components of $\Gamma - S$ and let $w_i \in C_i$, $1 \leq i \leq$

$|S|-4$.
Denote by e the number of edges in S. Obviously, $f = k|S| - 2e$. Since there are k edges from each singleton component to S and at least $\frac{3k}{2}$ edges from a non-singleton component to S we have that $f \geq k(|S|-4) + \frac{3k}{2}$. Therefore, $e \leq \frac{5k}{4}$.
First we will show that $\mu \in \{1, k-1\}$ and $\lambda = 0$.
By Lemma 4.14 we get

$$\mu(k-\mu) + \frac{k\lambda}{2} \leq \frac{5k}{4}. \tag{4.14}$$

If $2 \leq \mu \leq k-2$ then $\mu(k-\mu) \geq 2(k-2)$. But $2(k-2) \leq \frac{5k}{4}$ gives us $3k \leq 16$. Contradiction. Therefore, $\mu \in \{1, k-1\}$ and so $\frac{k\lambda}{2} + k - 1 \leq \frac{5k}{4}$. This implies $\lambda = 0$.
In this case there are at least $k-1$ edges in S. Depending on the value of $|C|$ we have the following two cases:
Case 1: $|C| > \frac{n}{2}$. Since $|S| > k$ we have that $n = |S| + |S| - 4 + |C| > 2k - 4 + \frac{n}{2}$. This implies $n > 4k-8$. On the other hand, inequality $f \geq k|S| - 4k + 2 \cdot \frac{n}{2}$ together with $f \leq k|S| - 2(k-1)$ gives us $n \leq 2k+2$. Therefore, we get $k < 5$. Contradiction.
Case 2: $|C| \leq \frac{n}{2}$. Using Lemma 4.15 we get $f \geq k(|S|-4) + 3k - 6$ which together with $f \leq k|S| - 2(k-1)$ gives us $k \leq 8$. As k is even and $k \geq 6$ we have $k \in \{6, 8\}$. Using relation (4.1) together with the facts that n is odd and $\mu \in \{1, k-1\}$ we can obtain the possible parameter sets. If $k = 6$ we get $(37, 6, 0, 1)$ and $(13, 6, 0, 5)$. If $k = 8$ we get $(65, 8, 0, 1)$ and $(17, 8, 0, 7)$. By [4] there is no strongly regular graph with any of the above parameter sets. □

Using the fact that all components of $\Gamma - S$ are singletons, we can obtain several observations.

Lemma 4.23. *With reference to Notation 4.7 we have*

- *(i) there are at least* 4 *singletons in* $\Gamma - S$*,*
- *(ii) there are exactly* $\frac{3k}{2}$ *edges in* S*,*
- *(iii)* $\lambda \leq 1$*, and*
- *(iv) if* $\lambda = 1$ *then* $\mu \in \{1, k-1\}$*.*

Proof. (i) If there are at most 3 singleton components in $\Gamma - S$, then $|S| \leq 6$. This fact together with $|S| \geq k \geq 6$ gives us $k = |S| = 6$. Each singleton component is adjacent to all vertices from S, so $\mu = k$. But, this is a contradiction with the fact that Γ is primitive.
(ii) As there are $|S|-3$ singleton components in $\Gamma - S$, there are exactly $k(|S|-3)$ edges between S and odd components. Γ is strongly regular with valency k, so it has $\frac{nk}{2}$ edges. As $n = 2|S| - 3$ we get that the total number of edges in S is $\frac{(2|S|-3)k}{2} - k(|S|-3) = \frac{3k}{2}$.

(iii) Suppose that $\lambda \geq 2$. By Lemma 4.14 there are at least $k+\mu(k-\mu)$ edges in S. On the other hand, by (ii) there are exactly $\frac{3k}{2}$ edges in S, so we have that $k+\mu(k-\mu) \leq \frac{3k}{2}$. This implies $\mu(k-\mu) \leq \frac{k}{2}$. Contradiction.
(iv) Let $\lambda = 1$. Suppose to the contrary that $2 \leq \mu \leq k-2$. Then there are $\frac{k}{2}+\mu(k-\mu) \geq \frac{k}{2}+2(k-2)$ edges in S. Using (ii) we get $\frac{k}{2}+2(k-2) \leq \frac{3k}{2}$. Therefore, $k \leq 4$. Contradiction. □

We proved that all components of a non-$2\frac{1}{2}$-extendable graph must be singletons and that $\lambda \in \{0, 1\}$. We can prove the main result of this chapter by obtaining the contradictions in each case depending on the value of λ. But, before that we will need some theoretical background from the spectral graph theory.

Definition 4.24. Let Γ be a simple graph. The adjacency matrix $A(\Gamma)$ is the integer matrix with rows and columns indexed by the vertices of Γ, such that the uv-entry is 1 if vertices u and v are adjacent, and 0 otherwise.

The spectrum of a matrix is the list of its eigenvalues together with their multiplicities. The spectrum of a graph is the spectrum of its adjacency matrix.
Let us denote by $k = \theta_1 \geq \theta_2 \geq ... \geq \theta_n$ the eigenvalues of Γ. It is known that if Γ is a strongly regular graph that it has exactly three distinct eigenvalues k, θ_2 and θ_n (see [12, Section 10.2]). Let r be the multiplicity of θ_2 and g be the multiplicity of θ_n. The eigenvalue k has multiplicity 1. It is a well known fact (see [5, Theorem 1.3.1]) that $\theta_2 \geq 0$ and $\theta_n \leq -1$.

The following theorem [6, Theorem 9.1.3] will be of particular use in the rest of this chapter.

Theorem 4.25. *Let Γ be (n, k, λ, μ) strongly regular graph with adjacency matrix A. Let θ_2 and θ_n ($\theta_2 > \theta_n$) be the eigenvalues of A and let r, g be their respective multiplicities. Then*

(i) $\theta_2\theta_n = \mu - k$

(ii) $\theta_2 + \theta_n = \lambda - \mu$

(iii) $r, g = \frac{1}{2}\left(n - 1 \mp \dfrac{(\theta_2+\theta_n)(n-1)+2k}{\theta_2-\theta_n}\right)$

(iv) If θ_2 and θ_n are non-integral then $r = g$ and $(n, k, \lambda, \mu) = (4t+1,\ 2t,\ t-1,\ t)$ for some integer t.

Remark 4.26. *Since $\theta_2 \geq 0$ and $\theta_n \leq -1$ it follows that $\theta = 0$ if and only if $\mu = k$.*

Graphs satisfying the last condition from the previous theorem are called conference graphs. The Paley graphs belong to these class of graphs, but there are many further examples.

The following theorem [6, Theorem 3.5.2] gives us the well known Hoffman ratio bound for the independence number.

Theorem 4.27. *If* Γ *is a regular graph of non-zero degree* k*, then*

$$\alpha(\Gamma) \leq \frac{n}{1 + k/(-\theta_n)}. \tag{4.15}$$

If an independent set C *meets this bound, then every vertex not in* C *is adjacent to precisely* $-\theta_n$ *vertices of* C*.*

Note that since we proved that all components of $\Gamma - S$ are singletons, it is obvious that $\alpha(\Gamma) \geq \sigma(\Gamma - S) = |S| - 3 = \frac{n-3}{2}$.

Now we can prove the following lemma.

Lemma 4.28. *With reference to Notation 4.7 with additional assumption that* Γ *is primitive, we have* $n > 3\alpha(\Gamma)$*.*

Proof. By Lemma 4.23 we know that $\lambda \leq 1$. Depending on the value of λ we have the following two cases.
Case 1: $\lambda = 0$. The proof of this case is the same as the proof of [9, Lemma 14], so it will be omitted.
Case 2: $\lambda = 1$. The Hoffman-ratio bound states that $\alpha(\Gamma) \leq \dfrac{n}{1 + k/(-\theta_n)}$. Note that in this case the only conference graph is the Paley graph on 9 vertices, but in that case $k = 4$. Therefore, we may assume that all eigenvalues of Γ are integer. As Γ is primitive, case $\theta_2 = 0$ is not possible by Theorem 4.25 and Remark following it. So, we will consider cases $\theta_2 \geq 2$ and $\theta_2 = 1$.
If $\theta_2 \geq 2$ then, since $\theta_2(-\theta_n) = k - \mu < k$, we have $\frac{k}{-\theta_n} > 2$. Thus $\alpha(\Gamma) < \frac{n}{3}$.
If $\theta_2 = 1$ then $\theta_n = \mu - k$. We have $1 - \mu = \lambda - \mu = \theta_2 + \theta_n = 1 + \mu - k$, which implies $k = 2\mu$. By Lemma 4.23 we have that $\mu \in \{1, k-1\}$. In both cases we get $k = 2$. Contradiction. $\square$

Now we can finally prove that all primitive strongly regular graphs with $k \geq 6$ and an odd number of vertices are $2\frac{1}{2}$-extendable. The following theorem is the main result of this chapter.

Theorem 4.29. *Let* Γ *be a* (n, k, λ, μ) *primitive strongly regular graph with an odd number of vertices and* $k \geq 6$*. Then* Γ *is* $2\frac{1}{2}$*-extendable.*

Proof. Suppose to the contrary, that Γ is not $2\frac{1}{2}$-extendable. Then, by the previous results, we have

$$\frac{n-3}{2} \leq \alpha(\Gamma) < \frac{n}{3}. \tag{4.16}$$

This implies $n < 9$. Contradiction (with the fact that $|S| \geq 5$ and there are at least 4 singletons in $\Gamma - S$). □

In the following theorem we will summarize the results obtained in this chapter.

Theorem 4.30. *Let Γ be a (n, k, λ, μ) strongly regular graph. If Γ is a non-$2\frac{1}{2}$-extendable primitive graph, then it is isomorphic to the Paley graph on 9 vertices. If Γ is a non-$2\frac{1}{2}$-extendable connected imprimitive graph, then it is isomorphic to the complete multipartite graph $K_{3\times 3}$.*

5 Extendability of regular graphs with diameter 3

This chapter will be devoted to the extendability of matchings in regular graphs with diameter 3. We will consider graphs with an odd number of vertices and also graphs with an even number of vertices.
First we will prove some results which will hold for all regular graphs, and then we will observe graphs with an even number of vertices and graphs with an odd number of vertices separately.

5.1 Basic results

In this section we will give two results which will hold for both, graphs with an even number of vertices and graphs with an odd number of vertices. This results will be important for proving the results in the next sections.

Lemma 5.1. *Let Γ be a regular graph with valency k and diameter 3. Let $S \subset V(\Gamma)$ be such that $\Gamma - S$ is not connected and let C be a connected component of $\Gamma - S$. Then there is at most one component of $\Gamma - S$ containing a vertex with no neighbours in S.*

Proof. Suppose to the contrary that there are components C_1 and C_2 of $\Gamma - S$ containing vertices, say v_1 and v_2 (respectively), with no neighbours in S. Then the distance between vertices v_1 and v_2 is at least 4. Contradiction. □

The next Lemma is a generalization of Lemma 2.2 for graphs with diameter 3. The proof is the same as for graphs with diameter 2, so it will be omitted.

Lemma 5.2. *Let Γ be a regular graph with valency k and diameter 3. Let $S \subset V(\Gamma)$ be such that $\Gamma - S$ is not connected and let C be a component of $\Gamma - S$ such that each vertex in C has at least one neighbour in S. Then there are at least k edges between k and S.*

5.2 Graphs with an even number of vertices

This section will be devoted to the extendability of matchings in regular graphs with an even number of vertices and diameter 3. We will prove that all such graphs are 0-extendable. We will give a sufficient condition for 1-extendability of regular graphs with diameter 3. Also, we will give some examples of non-2-extendable regular and edge-regular graphs with an even number of vertices and diameter 3.

In the next theorem we will prove that all regular graphs with an even number of vertices and diameter 3 have a perfect matching.

Theorem 5.3. *Let Γ be a regular graph with an even number of vertices, valency k and diameter 3. Then Γ is 0-extendable.*

Proof. Suppose that Γ is not 0-extendable, that is, it does not contain a perfect matching. Then, by Tutte's result, there exists a set $S \subset V(\Gamma)$ such that

$$\sigma(\Gamma - S) > |S|. \tag{5.1}$$

Let $t = \sigma(\Gamma - S)$ and let us denote by f the number of edges between S and $\Gamma - S$. Obviously, $f \leq k|S|$. By Lemma 5.1 and Lemma 5.2 we have that $f > k(t-1)$. Therefore, $k(t-1) < k|S|$ which implies

$$t - 1 < |S|. \tag{5.2}$$

Since $|V(\Gamma)|$ is even, $|S|$ and t must have the same parity, so inequality (5.1) implies $|S| \leq t - 2$. This is in a contradiction with inequality (5.2). □

Proving that all regular graphs with an even number of vertices and diameter 3 are 1-extendable seems to be a difficult problem. In this master thesis we will give only one sufficient condition for 1-extendability of regular graphs with diameter 3. But before that let us fix some notation.

Notation 5.4. *Let Γ be a regular graph with valency k, an even number of vertices and diameter 3 which is not 1-extendable. Let $e = \{x, y\}$ be an edge of Γ such that $\Gamma' = \Gamma - \{x, y\}$ does not contain a perfect matching. By Tutte's result there exists $S' \subset V(\Gamma')$ such that $\sigma(\Gamma' - S') > |S'|$. Let $S = S' \cup \{x, y\}$. The connected components of $\Gamma - S$ are the same as connected components of $\Gamma' - S'$. In particular, $\sigma(\Gamma - S) = \sigma(\Gamma' - S')$.*

Now we can find the number of odd components in a regular graph with diameter 3 and even order.

Lemma 5.5. *With reference to Notation 5.4 we have that* $\sigma(\Gamma - S) = |S|$.

Proof. Let t denote the number of odd components of $\Gamma - S$. By Tutte's result, we have that $t > |S'| = |S| - 2$. Let $C_1, C_2, ..., C_t$ be odd components of $\Gamma - S$. As $|V(\Gamma)|$ is an even number, the cardinality of $S \cup C_1 \cup ... \cup C_t$ is even. Therefore, $|S|$ and $|C_1 \cup ... \cup C_t|$ have the same parity. Numbers $|C_1|$, $|C_2|$,...,$|C_t|$ are odd, so the parity of t is the same as the parity of $|C_1 \cup ... \cup C_t|$. Therefore, t and $|S|$ have the same parity.
Let us calculate the number of edges f between S and $\Gamma - S$. Since Lemma 5.1 and Lemma 5.2 hold (and Γ is connected), we have

$$f \geq k(t-1) + 1 > k(t-1). \tag{5.3}$$

On the other hand, since e is an edge in Γ, we have

$$f \leq k|S| - 2 < k|S|. \tag{5.4}$$

Combining these inequalities, we get

$$k(t-1) < f < k|S|, \tag{5.5}$$

which implies $t < |S| + 1$. This, together with $t > |S| - 2$ and the fact that $|S|$ and t are of the same parity, gives us $t = |S|$. □

The fact that diameter of Γ is equal to 3 gives a possibility that, with reference to Notation 5.4, there may exist a component of $\Gamma - S$ which contains a vertex with no neighbours in S. Now we can prove that if such a component does not exist, Γ is 1-extendable.

Theorem 5.6. *Let* Γ *be a regular graph with an even number of vertices and diameter 3. Let* $S \subset V(\Gamma)$ *be such that* $\Gamma - S$ *is not connected. Let each vertex in each odd component of* $\Gamma - S$ *has at least one neighbour in* S*. Then* Γ *is 1-extendable.*

Proof. Suppose that Γ is not 1-extendable. Then following Notation 5.4 and using Lemma 5.5 we have that $\sigma(\Gamma - S) = |S|$. Let f denote the number of edges between S and $\Gamma - S$. Since, by Lemma 5.2, each odd component of $\Gamma - S$ sends at least k edges to S, we have that $f \geq k|S|$. On the other hand, since e is an edge in S we have that $f \leq k|S| - 2$. Contradiction. □

Since we could not find any example of a non-1-extendable regular graph with an even number of vertices and diameter 3, we believe that all such graphs are 1-extendable. Neither, we were not able to prove similar result for the family of edge-regular graphs. Family of distance-regular graphs[1] was studied in [7]. It is proved that all distance-regular graphs with an even number of vertices and diameter greater than 2 are 1-extendable.

[1]this graphs are generalization of strongly regular graphs with diameter greater than 2.

5.2.1 Examples of non-2-extendable graphs

In this section we will give some examples of regular and edge-regular graphs with an even number of vertices and diameter 3 which are not 2-extendable.

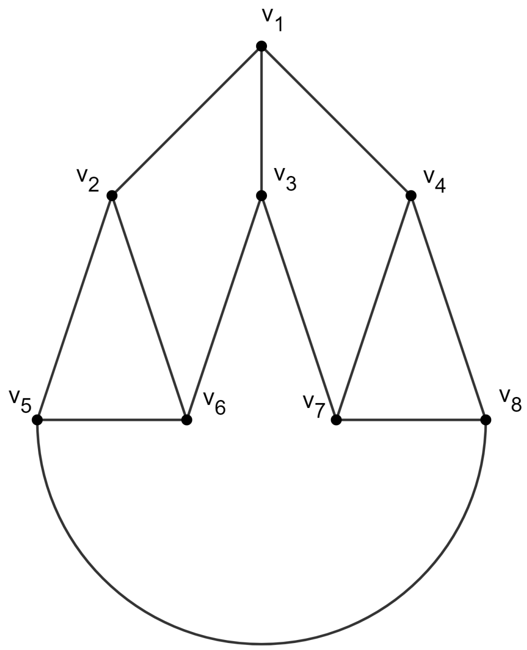

Figure 9: $(8, 3)$-regular graph

It is easy to see that a regular graph with diameter 3 from Figure 9 is not 2-extendable. Namely, if we take edges $\{v_5, v_6\}$ and $\{v_7, v_8\}$ it is obvious that exactly one of vertices v_2, v_3, v_4 can be covered by a matching. So, starting matching cannot be extended to a perfect matching.

Let us observe a graph from Figure 10. Obviously, this graph is regular graph with an even number of vertices and diameter 3. Let us take edges $\{v_5, v_6\}$ and $\{v_9, v_{10}\}$ for starting matching. In this case, at least one of vertices v_2 and v_4 will stay uncovered. Therefore, graph from Figure 10 is not 2-extendable.

Let us observe a graph from Figure 11. It is easy to see that this graph is edge-regular with diameter 3. Let us take edges $\{v_5, v_7\}$ and $\{v_6, v_8\}$ for starting matching. In this case, at least one of the vertices v_2, v_3 will stay uncovered. Therefore, graph from Figure 11 is not 2-extendable.

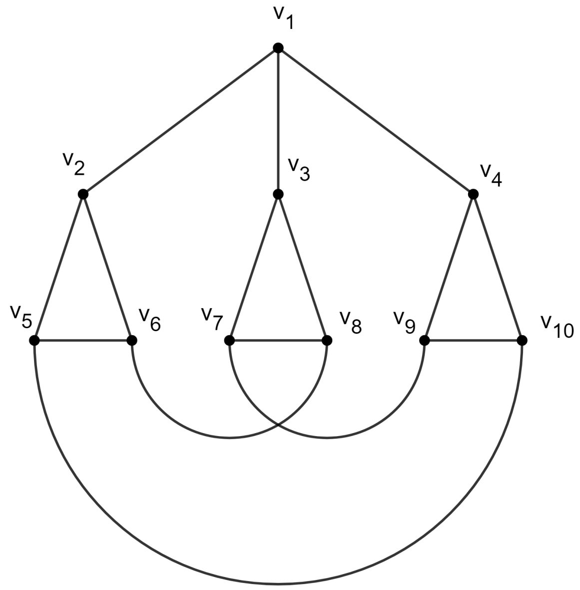

Figure 10: $(10, 3)$-regular graph

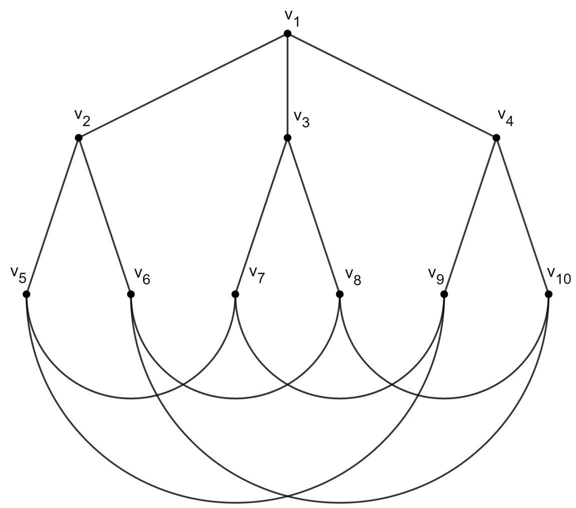

Figure 11: $(10, 3, 0)$-edge-regular graph

Graph from Figure 12 is a regular graph with an even number of vertices and diameter 3. Let us take edges $\{v_8, v_{10}\}$ and $\{v_6, v_{12}\}$ for starting matching. There are two possibilities to cover vertex v_9. WLOG let us cover vertex v_9 by an edge $\{v_7, v_9\}$. Then at least one of vertices v_2, v_3 will stay uncovered. Therefore, observed graph is not 2-extendable.

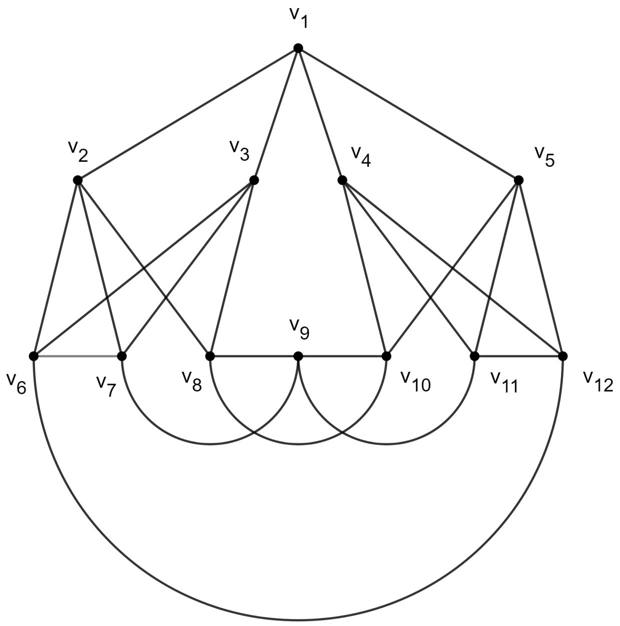

Figure 12: $(12, 3)$-regular graph

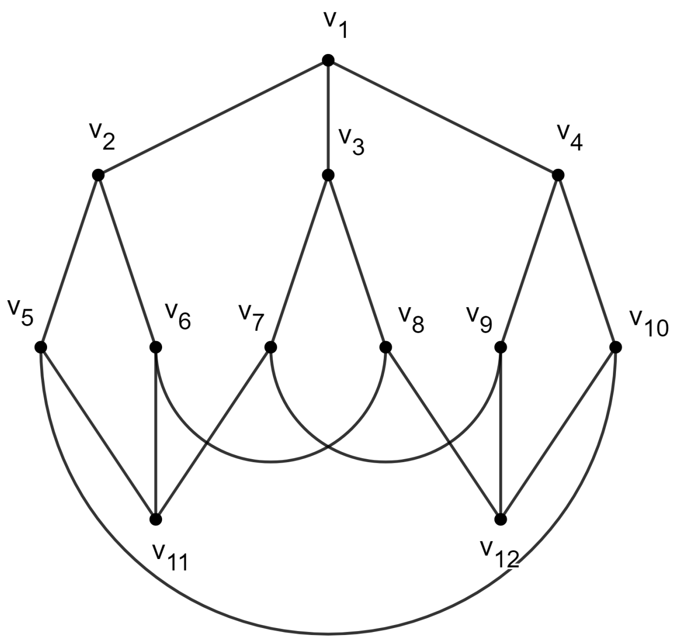

Figure 13: $(12, 3, 0)$-edge-regular graph

Let us observe a graph from Figure 13. It is easy to see that this graph is edge-regular with diameter 3. Let us take edges $\{v_6, v_8\}$ and $\{v_7, v_9\}$ for starting matching.

In this case, the only way to cover vertices v_3, v_{11} and v_{12} is to include edges $\{v_1, v_3\}$, $\{v_5, v_{11}\}$ and $\{v_{10}, v_{12}\}$ in the matching. But in this case vertices v_2 and v_4 will stay uncovered. So, observed graph is not 2-extendable.

As we have seen, there are some edge-regular graphs with an even number of vertices and diameter 3 which are not 2-extendable. Therefore, we can restrict our attention to the family of distance-regular graphs with diameter 3. But, the results for extendability of matchings in distance-regular graphs with an even number of vertices and diameter 3 are given in [10]. Therefore, we will continue studying extendability of graphs with an odd number of vertices.

5.3 Graphs with an odd number of vertices

This section will be devoted to the extendability of matchings in regular graphs with an odd number of vertices and diameter 3.
First we will prove that all regular graphs with an odd number of vertices and diameter 3 are $0\frac{1}{2}$-extendable and then we will give some examples of regular graphs with an odd number of vertices and diameter 3 which are not $1\frac{1}{2}$-extendable.

Theorem 5.7. *Let Γ be a regular graph with an odd number of vertices, valency k and diameter 3. Then Γ is $0\frac{1}{2}$-extendable.*

Proof. Suppose to the contrary that there exists a vertex x in $V(\Gamma)$ such that $\Gamma - x$ does not contain a perfect matching. Then by Tutte's result there exists a subset $S \subset V(\Gamma - x)$ such that $\sigma((\Gamma - x) - S) > |S|$. Let us denote $S' = S \cup \{x\}$. Let t denote the number of odd components of $\Gamma - S'$. Let f be the number of edges between S' and $\Gamma - S'$. Using Lemma 5.1 and Lemma 5.2 we will calculate the number of edges between S' and odd components. We get

$$k(t-1) < f \leq k|S'| = k(|S|+1), \tag{5.6}$$

which implies

$$t < |S'| + 1 = |S| + 2. \tag{5.7}$$

This, together with the fact that $t > |S|$, gives us

$$t = |S| + 1. \tag{5.8}$$

Let $C_1, C_2, ..., C_t$ be odd components of $\Gamma - S'$. Note that since $|V(\Gamma)|$ is odd, the cardinality of $S' \cup C_1 \cup C_2 \cup ... \cup C_t$ is odd. Therefore, $|S'|$ and $|C_1 \cup ... \cup C_t|$ have different parity. Since $|C_i|$ is odd for each $i \in \{1, 2, ..., t\}$, it follows that $|C_1 \cup C_2 \cup ... \cup C_t|$ and t have the same parity. Therefore, t and $|S'|$ have different parity. Hence $|S|$ and t have the same parity. Contradiction (with equality (5.8)). □

5.3.1 Examples of non-$1\frac{1}{2}$-extendable graphs

In this subsection we will give two examples of non-$1\frac{1}{2}$-extendable regular graphs with an odd number of vertices and diameter 3.

Let us observe a graph Γ from Figure 14. It is easy to see that this graph is regular with diameter 3 and an odd number of vertices. Let us observe $\Gamma - u_1$ and a matching containing an edge $\{u_6, u_7\}$. This matching cannot be extended to a perfect matching of $\Gamma - u_1$ since at least one of the vertices u_2, u_3 will stay uncovered.

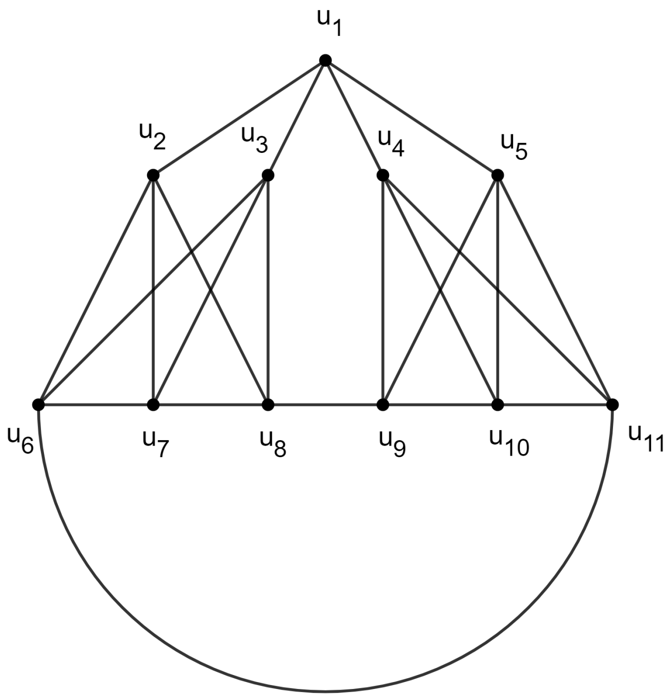

Figure 14: $(11, 4)$-regular graph

Let us observe a graph Γ from Figure 15. This graph is an edge-regular graph with an odd number of vertices and diameter 3. Let us observe $\Gamma - u_1$ and a matching containing an edge $\{u_8, u_9\}$. There are 2 possibilities to cover vertex u_2: to add an edge $\{u_2, u_6\}$ or an edge $\{u_2, u_7\}$. WLOG let us add an edge $\{u_2, u_6\}$ in the matching. Then the only way to cover vertex u_3 is to include an edge $\{u_3, u_7\}$ in the matching. Then we must include an edge $\{u_{10}, u_{11}\}$ (in order to cover vertex u_{11}). In this case vertex u_5 will stay uncovered. Therefore, observed graph is not $1\frac{1}{2}$-extendable.

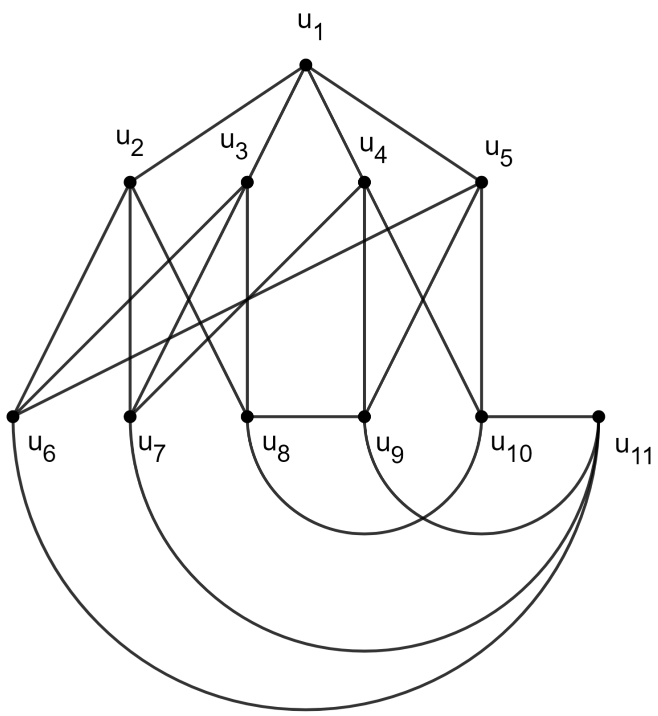

Figure 15: $(11, 4, 0)$-edge-regular graph

To conclude, in this chapter we studied matching extensions in regular graphs with diameter 3. For graphs with an even number of vertices we obtained a proof that all such graphs contain a perfect matching. Possible next step in further research in this area could be classifying all 1-extendable regular or edge-regular graphs with diameter 3.

For graphs with an odd number of vertices we proved that all such graphs are $0\frac{1}{2}$-extendable. We have also constructed an example of non-$1\frac{1}{2}$-extendable regular and an example of non-$1\frac{1}{2}$-extendable edge-regular graph. Possible next step in further work could be observing distance-regular graphs with diameter 3 and an odd number of vertices.

6 Conclusion

In this thesis, we were solving the problem of extendability of matchings in regular graphs with small diameter. Main focus was on the graphs with an odd number of vertices, while graphs with an even number of vertices were studied only in the last chapter. At the beginning of the thesis we gave basic definitions and a short overview of results from the literature. Also, we presented some examples and relationships between this problem in graphs with odd order and in graphs with even order.

Since extendability of matchings in graphs with diameter 1 is trivial, we started with observing $0\frac{1}{2}$-extendability of regular graphs with an odd number of vertices and diameter 2. We proved that all regular graphs with an odd number of vertices and diameter 2 are $0\frac{1}{2}$-extendable. Trying to answer the question are all such graphs $1\frac{1}{2}$-extendable, we constructed an infinite family of non-$1\frac{1}{2}$-extendable regular graphs with an odd number of vertices and diameter 2.
Later on, we restricted attention to the family of edge-regular graphs with an odd number of vertices and diameter 2. Main result of this part of the thesis was the proof that cycle on 5 vertices is the only non-$1\frac{1}{2}$-extendable edge-regular graph with an odd number of vertices and diameter 2.
After constructing several examples of non-$2\frac{1}{2}$-extendable edge-regular graphs, we tried to classify all such graphs, but that was a difficult problem. Therefore, we restricted attention to the family of strongly regular graphs. Using various results from the spectral graph theory, we were able to classify all non-$2\frac{1}{2}$-extendable connected strongly regular graphs with an odd number of vertices. Namely, we proved that all primitive strongly regular graphs except Paley graph on 9 vertices, are $2\frac{1}{2}$-extendable. Moreover, we proved that the complete multipartite graph $K_{3\times 3}$ is the only connected imprimitive strongly regular graph which is not $2\frac{1}{2}$-extendable.
Last part of this thesis was devoted to the matching extensions in regular graphs with diameter 3. In this part we studied also graphs with an even number of vertices. Using Tutte's result, we proved that all such graphs have a perfect matching. We gave a condition when all such graphs are 1-extendable. Also, we constructed several examples of non 2-extendable regular and edge-regular graphs with diameter 3 and even order.
For regular graphs with odd order and diameter 3 we proved that all such graphs have

a near-perfect matching, i.e. all of them are $0\frac{1}{2}$-extendable. We concluded this chapter with some examples of non-$1\frac{1}{2}$-extendable regular and edge-regular graphs with an odd number of vertices and diameter 3. Since the existence of some non-$1\frac{1}{2}$-extendable regular graphs with diameter 3 has been proved, one of the possible steps in further work could be classifying all such graphs or restricting attention to the family of distance-regular graphs.

In this master thesis we have successfully solved some of the open problems in the area of matching extensions. But there are many open problems in this area, giving the possibility of obtaining new research results, solving either partially or completely some of the open problems in future work.

7 Povzetek naloge v slovenskem jeziku

Teorija prirejanj že od nekdaj predstavlja eno izmed osnovnih področjih teorije grafov. V zadnjem obdobju pa se je znotraj teorije prirejan izjemno povečalo zanimanje za teorijo razširljivosti prirejanj.
Glavni predmet magistrskega dela je problem razširljivosti prirejanj v regularnih grafih z majhnim premerom.

Prirejanje v grafu je taka podmnožica njegovih povezav, da nobeni dve povezavi nimata skupnega krajišča. Prirejanje M je popolno, ce je vsako vozlissče grafa krajišče vsaj ene (in zato natanko ene) povezave iz M. Potreben in zadosten pogoj za obstoj popolnega prirejanja je že leta 1947 podal Tutte in je danes znan kot Tutte-ov izrek. Ta izrek pravi, da graf Γ vsebuje popolno prirejanje če in samo če je $\sigma(\Gamma - S) \leq |S|$ za vsako množico $S \subseteq V(\Gamma)$ (oznaka $\sigma(\Gamma - S)$ označuje število lihih povezanih komponent grafa $\Gamma - S$). Tutte-ov izrek je bil v tem magistrskem delu pogostokrat uporabljen.
Naj bo l nenegativno celo število. Za povezan graf Γ z množico vozlišč $V(\Gamma)$, kjer je $|V(\Gamma)| \geq 2l + 2$, pravimo, da je l-razširljiv, če vsebuje l-prirejanje, in če lahko vsako l-prirejanje v grafu Γ razširimo do popolnega prirejanja.

Leta 1980 je Plummer uvedel definicijo l-razširljivega grafa za grafe s sodim številom točk. Od takrat se je ta lastnost grafov veliko proučevala. V literaturi pa lahko zasledimo tudi kar nekaj del, ki se ukvarjajo s problemom l-razširljivosti znotraj posameznih grafovskih razredov. Na primer, v [9] in [10] je proučevan problem razširljivosti prirejanj v krepko regularnih in razdaljno regularnih grafih s sodim številom vozlišč. V [14] je podana klasifikacija povezavno-regularnih grafov s premerom 2, ki niso 2-razširljivi. Motivirani z zgoraj opisanimi rezultati za grafe s sodim številom vozlišč,, smo se odločili raziskat problem razširljivosti prirejanj v regularnih grafih z majhnim premerom in lihim številom vozlišč.
Definicijo $l\frac{1}{2}$-razširljivega grafa za grafe z lihim številom vozlišč je uvedel Yu (v [21]). Naj bo l nenegativno celo število. Povezan graf Γ z množico vozlišč $V(\Gamma)$, kjer je $|V(\Gamma)| \geq 2l+3$, je $l\frac{1}{2}$-razširljiv, če je za poljubno vozlišče $x \in V(\Gamma)$ graf $\Gamma - x$ l-razširljiv.

Na začetku magistrske naloge je predstavljeno nekaj osnovnih definicij in pojmov s področja teorije grafov. Problem razširljivosti prirejanj v regularnih grafih z majhnim premerom je formalno definiran in motivirano je raziskovanje problema v posebnih grafovskih razredih. Bistvo magistrskega dela se začne v drugem poglavju, kjer definiramo regularne grafe in dokažemo da so vsi regularni grafi s premerom 2 in lihim številom točk $0\frac{1}{2}$-razširljivi. Med pomembne rezultate tega poglavja sodi tudi konstrukcija neskončne družine regularnih grafov s premerom 2 in lihim številom vozlišč ki niso $1\frac{1}{2}$-razširljivi.
V tretjem poglavju so definirani povezavno-regularni grafi. Dokazan je izrek, ki pravi, da je cikel na 5 vozliščih edini povezavno-regularen graf s premerom 2 in lihim številom vozlišč ki ni $1\frac{1}{2}$-razširljiv.
V četrtem poglavju je pregled osnovnih rezultatov o povezavno-regularnih grafih, ki niso $2\frac{1}{2}$-razširljivi. Pokazano je, da obstajajo povezavno-regularni grafi s premerom 2 in lihim številom vozlišč ki niso $2\frac{1}{2}$-razširljivi. Kasneje, z uporabo različnih rezultatov spektralne teorije grafov, izpeljemo dokaz izreka, ki pravi, da so vsi krepko regularni grafi, $2\frac{1}{2}$-razširljivi (razen $K_{3\times 3}$ in Paley-ov graf na 9 vozliščih) .
Na koncu smo se ukvarjali z regularnimi grafi s premerom 3. Pokazali smo, da so vsi regularni grafi s premerom 3 in sodim številom vozlišč, 0-razširljivi. Pokazali smo tudi, da so vsi regularni grafi s premerom 3 in lihim številom vozlišč $0\frac{1}{2}$-razširljivi. Podani so tudi primeri regularnih grafov s premerom 3 ki niso $1\frac{1}{2}$-razširljivi.

V zaključku magistrske naloge so povzeti pridobljeni rezultati ter podane nekatere možnosti za nadaljnje raziskave.

8 Bibliography

[1] H. Alajbegović, A. Huskanović, Š. Miklavič, and P. Šparl, On the extendability of quasistrongly regular graphs with diameter 2. *Graphs Combin.* 34 (2018) 711–726.

[2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Second Edition, 1993.

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory.* Springer, 2010.

[4] A.E. Brouwer, *Parameters of Strongly Regular Graphs*, `http://www.win.tue.nl/ aeb/graphs/srg/srgtab.html`. (Viewed on: 06/02/2019.)

[5] A.E. Brouwer, A.M. Cohen, and A. Neiumaier, *Distance-Regular Graphs.* Springer, 1989.

[6] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs.* Springer, 2012.

[7] A.E. Brouwer and W.H. Haemers, Eigenvalues and perfect matchings. *Linear Algebra and its Applications* 395 (2005) 155–162.

[8] A.E. Brouwer and D.M. Mesner, The connectivity for strongly regular graph. *European J. Combinatorics* 6 (1985) 215–216.

[9] S.M. Cioabă and W. Li, The extendability of matchings in strongly regular graphs. *The Electronic Journal of Combinatorics* 21 (Issue 2, 2014) P2.34.

[10] S.M. Cioabă, J. Koolen, and W. Li, Max-cut and extendability of matchings in distance-regular graphs. *The Electronic Journal of Combinatorics* 62 (2017) 232–244.

[11] R. Diestel, *Graph Theory*, Springer-Verlag, Electronic Edition, 2000.

[12] C. Godsil and G. Royle, *Algebraic Graph Theory.* Springer, 2001.

[13] D.A. Holton and D. Lou, Matching Extensions of Strongly Regular Graphs. *Australasian Journal of Combinatorics* 6 (1992) 187–208.

[14] K. Kutnar, D. Marušič, Š. Miklavič, and P. Šparl, The classification of 2-extendable edge-regular graphs with diameter 2. *The Electronic Journal of Combinatorics* 26 (2019) P1.16.

[15] W. Li, *Algebraic methods in graph theory*, Ph.D. dissertation, University of Delaware, 2015.

[16] D. Lou and Q. Zhu, The 2-extendability of strongly regular graphs. *Discrete Math.* 148 (1996) 133–140.

[17] L. Lovász and M.D. Plummer, *Matching Theory*. North-Holland, 1986.

[18] Š. Miklavič and P. Šparl, On extendability of Deza graphs with diameter 2. *Discrete Math.* 338 (2015) 1416–1423.

[19] M.D. Plummer, On n-extendable graphs. *Discrete Math.* 31 (1980) 201–210.

[20] D.B. West, *Introduction to Graph Theory*, Pearson Education, Inc, Second Edition, 2001.

[21] Q. Yu, Characterizations of various matching extensions in graphs. *Australian Journal of Combinatorics* 7 (1993) 55–64.

[22] Q. Yu and G. Liu, *Graph Factors and Matching Extensions*. Springer, 2009.

[23] F. Zhang and H. Zhang, Construction for bicritical graphs and k-extendable bipartite graphs. *Discrete Math.* 306(13) (2006) 1415–1423.