UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN
INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA
(DOCTORAL THESIS)

STRUKTURNI REZULTATI O VOZLIŠČNO IN POVEZAVNO TRANZITIVNIH GRAFIH

(STRUCTURAL RESULTS ON VERTEX- AND EDGE-TRANSITIVE GRAPHS)

ALEJANDRA RAMOS RIVERA

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MENTOR: IZR. PROF. DR. PRIMOŽ ŠPARL

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Alejandra Ramos Rivera

Abstract

STRUCTURAL RESULTS ON VERTEX- AND EDGE-TRANSITIVE GRAPHS

The main theme of this PhD thesis are finite graphs admitting a considerable degree of symmetry. More precisely, we focus on graphs admitting a vertex- and edge-transitive group of automorphisms.

For a graph Γ and a subgroup $G \leq \mathrm{Aut}(\Gamma)$ the graph Γ is said to be G-vertex-transitive, G-edge-transitive and G-arc-transitive whenever the subgroup G acts transitively on the vertex set $V(\Gamma)$, the edge set $E(\Gamma)$ and the arc set $A(\Gamma)$ of Γ, respectively. We say that Γ is G-half-arc-transitive if it is G-vertex- and G-edge-transitive but not G-arc-transitive. In the case of $G = \mathrm{Aut}(\Gamma)$ we omit the prefix G and simply write vertex-transitive, edge-transitive, arc-transitive and half-arc-transitive.

Let Γ be a G-vertex- and G-edge-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$. Then two essentially different possibilities can occur:

(i) Γ is G-arc-transitive.

(ii) Γ is G-half-arc-transitive.

In the first main topic of this PhD thesis we focus on the situations from the above possibility (i). In particular, we apply the known results on arc-transitive graphs as a tool in the investigation of symmetries of certain maps.

Regarding possibility (ii), it is known that graphs admitting a half-arc-transitive group of automorphisms must have even valency. Since graphs of valency two are cycles, the smallest valency where the study of such graphs may not be trivial is four. In the second main topic of this PhD thesis we focus on tetravalent graphs admitting a half-arc-transitive action. In particular, we introduce a new parameter for such graphs, giving a better understanding of their structure. We study the properties of the graphs with respect to this parameter and use it to relate two important existing approaches for a systematic study of such graphs.

Finally, we focus on half-arc-transitive graphs with even valency greater than two. We generalize the well-known Bouwer graphs to obtain a much larger family of vertex- and edge-transitive graphs. We give a complete classification of its half-arc-transitive members. It turns out that this family contains almost all so-called tightly attached tetravalent half-arc-transitive graphs.

Math. Subj. Class (2010): 20B25, 05C25, 05C60, 51E30.

Key words: half-arc-transitive, Bouwer graphs, tightly attached, tetravalent, alternating cycle, alternating jump.

Izvleček

STRUKTURNI REZULTATI O VOZLIŠČNO IN POVEZAVNO TRANZITIVNIH GRAFIH

Glavna tema doktorske disertacije so končni grafi z visoko stopnjo simetrije. Natančneje, osredotočili se bomo na grafe, ki premorejo točkovno in povezavno tranzitivno grupo avtomorfizmov.

Za dano podgrupo Γ in podgrupa $G \leq \mathrm{Aut}(\Gamma)$ pravimo, da je graf Γ G-točkovno tranzitiven, G-povezavno tranzitiven, oziroma G-ločno tranzitiven, če podgrupa G deluje tranzitivno na množico točk $V(\Gamma)$, na množico povezav $E(\Gamma)$, oziroma na množico lokov $A(\Gamma)$ grafa Γ. Rečemo, da je graf Γ G-pol-ločno tranzitiven, če je G-točkovno in G-povezavno tranzitiven, ni pa G-ločno tranzitiven. V primeru, ko je $G = \mathrm{Aut}(\Gamma)$ opustimo predpono G in pišemo kar točkovno tranzitiven, povezavna tranzitivnost, ločno tranzitiven, oz. pol-ločno tranzitivnost.

Naj bo graf Γ G-točkovno in G-povezavno tranzitiven za neko podgrupo $G \leq \mathrm{Aut}(\Gamma)$. Potem drži ena od dveh različnih možnosti:

(i) graf Γ je G-ločno tranzitiven.

(ii) graf Γ je G-pol-ločno tranzitiven.

V prvem delu disertacije se osredotočimo na obravnavo možnosti (i). Že znane rezultate s področja ločno tranzitivnih grafov uporabimo kot orodje za proučevanje simetrij določenih zemljevidov.

Glede možnosti (ii) je znano, da morajo biti grafi, ki premorejo pol-ločno tranzitivno grupo avtomorfizmov, sode valence. Ker so grafi valence 2 cikli, je najmanjša valenca grafov, ki jih je smiselno proučevati, 4. V drugem delu disertacije se posvetimo tetravalentnim grafom, ki premorejo pol-ločno tranzitivno delovanje. Uvedemo nov parameter za takšne grafe, preko katerega lahko bolje razumemo njihovo strukturo. Proučujemo lastnosti grafov glede na novo uvedeni parameter in na ta način povežemo dva pomembna že obstoječa pristopa za sistematično obravnavo takšnih grafov.

Končno se posvetimo še pol-ločno tranzitivnim grafom sode valence večje od 2. Posplošimo dobro znano konstrukcijo Bouwerjevih grafov v precej večjo družino točkovno in povezavno tranzitivnih grafov. Podamo popolno klasifikacijo njenih pol-ločno tranzitivnih članov. Izkaže se, da ta družina vsebuje skoraj vse tako imenovane

tesno spete tetravalentne pol-ločno tranzitivne grafe.

Math. Subj. Class (2010): 20B25, 05C25, 05C60, 51E30.

Ključne besede: pol-ločna tranzitivnost, Bouwerjevi grafi, tesna spetost, tetravalentnost, alternirajoči cikel, alternirajoči skok.

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Chapter 1

Introduction

Graph Theory is one of the mathematical areas with a strong influence on the modern life functioning, having applications in computer science, biology, economic models, chemistry or social science as some examples. From the second half of the last century there has been an intensive development of graph theory with approaches from numerous areas of mathematics such as topology, algebra, combinatorics and geometry. One of the most useful tools to understand the structure of a graph has been the study of its symmetries.

It is the aim of this PhD thesis to study the structure of graphs admitting a considerable degree of symmetry. Specifically, we focus on graphs admitting a vertex- and edge-transitive group of automorphisms. This is a class of graphs that has been extensively studied for the past 70 years with hundreds of research papers being published (see for instance [21, 29, 38, 47] and the references cited therein).

We are interested in the study of the transitive action on the set of vertices and the set of edges of a graph not only for its entire automorphisms group, but also for the subgroups of its automorphisms group having this property. It is clear that such groups have a natural action on other sets related to the structure of the graph namely, on the sets of cycles or paths of certain length. However, the number of orbits of this action on these sets depends on the structure of the studied graph. Nevertheless, for the set of arcs of a graph (that is, ordered pairs of adjacent vertices) this is not the case. The number of orbits of this action on the arc set of the studied graph is always at most two. In other words, whenever a graph Γ is G-vertex- and G-edge-transitive for some $G \leq \mathrm{Aut}(\Gamma)$, then two essentially different possibilities can occur:

(i) Γ is G-arc-transitive.

(ii) Γ is G-half-arc-transitive.

For instance, it is known that for a cycle C_n, with $n \geq 3$, its automorphism group is of order $2n$ and is generated by a one-step "rotation" (of order n) and a "reflection" (of order 2). It follows that the graph C_n is $\mathrm{Aut}(C_n)$-arc-transitive. However, it is also $\langle\rho\rangle$-half-arc-transitive, where ρ is a one-step rotation of C_n. Therefore, even if the first of the two possibilities occurs, one can ask whether there is some other subgroup $H \leq \mathrm{Aut}(\Gamma)$ acting half-arc-transitively on Γ. Similarly, in the second of the two possibilities one can ask whether some other $H \leq \mathrm{Aut}(\Gamma)$ acts arc-transitively on Γ. It was the aim of our research to obtain structural results for graphs (and subgroups of their automorphism group) from each of these two possibilities and to address the above mentioned additional questions.

In the first main topic of this PhD thesis we focus on the situations from the above possibility (i). In particular, we are interested in the application of the known results on arc-transitive graphs as a tool in the investigation of symmetries of maps. In the late nineteen-nineties, Graver and Watkins [22] initiated the study of all edge-transitive maps and divided them into 14 different types or classes. Recently, Gareth Jones [26] revisited the study of such maps while working on the question of which groups can act as the automorphism group of an edge-transitive map and answering it for some specific classes of groups. This question naturally extends to the question of which graphs can be the underlying graph of an edge-transitive map. Moreover, in both cases, one can ask which are the classes of edge-transitive maps that can occur, how many maps of a given class are there and whether one can describe all of them. When studying edge-transitive maps it is natural to restrict to a subset of them: those whose automorphism group acts transitively on the set of arcs.

When studying arc-transitive maps one of the most useful of their properties is that their automorphism group has at most two orbits on the set of its flags (incident vertex-edge-face triples). Moreover, it is known that there are 5 classes of arc-transitive maps based on the local configuration of the flags and the orbits they belong to [23]. One of these classes consists of maps for which its automorphism group acts transitive on the set of its flags, reflexible maps, and the elements of the other four classes are the 2-orbit maps of classes known as 2, 2_0, 2_1 and $2_{\{0,1\}}$. The most studied arc-transitive maps so far are the reflexible ones and the maps contained the class 2. Furthermore, the maps in class 2_0 are closely related to the maps in class 2 via the Petrie dual operator (and their graphs and groups are the same). This leaves us with the maps of classes 2_1 and $2_{\{0,1\}}$, which are also related via the Petrie operator. In this thesis we thus restrict ourselves to one of this two classes and study maps of class $2_{\{0,1\}}$. It turns out that the smallest admissible valency for such maps is four. In Chapter 3 we investigate the connection of this class of arc-transitive maps to the structure of their underlying (arc-transitive) graph, with special emphasis on maps of valence four. Together with the results from [28], our analysis gives the main result of Chapter 3, namely a complete classification of arc-transitive maps whose underlying graphs are the well-known arc-transitive Rose window graphs.

In the second and third part of this doctoral thesis we focus on the situations from the above possibility (ii). That is, we study graphs admitting a half-arc-transitive group G of automorphism (G-HAT graphs for short).

The first result concerning graphs admitting a half-arc-transitive action was given by Tutte, who proved that the valency of such graphs must be even [55]. Since any 2-valent graph is a disjoint union of cycles (and possibly infinite paths), the smallest valency where the study of G-HAT graphs is not trivial is four. It is thus not surprising that the majority of papers on G-HAT graphs deal with the tetravalent ones. Despite the fact that numerous papers on the topic have been published in the last half a century the complete classification of tetravalent half-arc-transitive graphs appears to be a very difficult, if not impossible, problem. Nevertheless, this problem has been approached in many different ways and several important results have been obtained.

One of the most fruitful approaches to the study of structural properties of tetravalent graphs admitting a half-arc-transitive action was started 20 years ago by Marušič [32]. It is based on the investigation of certain cycles of such graphs called *alternating cycles* (or more generaly *G-alternating cycles* for a G-HAT graph). In a G-HAT graph Γ it is easy to see that all the G-alternating cycles have the same length and two non-disjoint G-alternating cycles always intersect in the same number of vertices. We call the half of the length of the G-alternating cycles the*G-radius* of Γ, $\mathrm{rad}_G(\Gamma)$, and the size of their nonempty intersection the *G-attachment number* of Γ, $\mathrm{att}_G(\Gamma)$. It is worth mentioning that tetravalent tightly G-attached graphs (that is, graphs such that $\mathrm{att}_G(\Gamma) = \mathrm{rad}_G(\Gamma)$) are already completely classified [32, 37, 53, 58]. The importance of these results stems from [37] where it was shown that each tetravalent G-HAT graph Γ is either tightly G-attached or arises as a certain cover from a loosely- or an antipodally-attached graph (that is, a graph with $\mathrm{att}_G(\Gamma) = 1$ or $\mathrm{att}_G(\Gamma) = 2$, respectively).

In [3], a new framework for a systematic study of tetravalent graphs admitting a half-arc-transitive group of automorphisms was proposed. It is based on the so-called normal quotients method, where smaller graphs with "the same" properties as the original graph are obtained by identifying the orbits of a non-transitive normal subgroup of the automorphism group of the studied graph (omitting parallel edges or loops). One thus aims to classify all "basic" graphs with respect to this quotienting procedure and then tries to determine how all larger graphs can be reconstructed from the minimal ones. Recently, some results regarding this approach have been obtained ([1, 2]).

In our research we consider tetravalent G-HAT graphs from both of the above points of view. Moreover, we improve some of the existing results and, in fact, bring together this two important approaches. To reach this objective, in Chapter 4 we introduce a new parameter for a tetravalent G-HAT graph Γ, *the alternating jump* of Γ with respect to the group G and give some of its basic properties. This parameter describes how two non-disjoint alternating cycles are attached to one another and gives a more detailed insight into the local structure of such graphs when compared to the one given by simply considering their radius and attachment number.

The alternating jump of Γ with respect to the group G turns out to be a very useful tool in the study of tetravalent G-HAT graphs. For instance, in all know examples of tetravalent HAT graphs (that is a G-HAT where G is its whole automorphism group) the attachment number $\mathrm{att}(\Gamma)$ divides the radius $\mathrm{rad}(\Gamma)$ of the graph. It is thus natural to ask whether $\mathrm{att}(\Gamma)$ divides $\mathrm{rad}(\Gamma)$ for all HAT graphs (see [49]). In [39, Theorem 1.2] an affirmative answer to this question ca be found for the antipodally-attached HAT graphs and, most recently, in [49, Theorem 2] for graphs with $\mathrm{rad}(\Gamma)$ an odd number. In order to approach this problem in general, we study the graph of G-alternating cycles of G-HAT graphs. In Chapter 4 we present several results on this graph, obtained by using the properties of the jump parameter, which enable us to make a considerable step towards the complete answer to the question mentioned above on whether for a tetravalent HAT graph $\mathrm{att}(\Gamma)$ divides rad(Γ) or not.

Finally, in the last part of this PhD thesis we focus on HAT graphs with valencies greater than four. In contrast to a large number of papers dealing with tetravalent HAT graphs, the papers investigating HAT graphs of all possible valencies are very rare (see for instance [5, 8] for some of the not so recent ones). This is most probably due to the fact that already with tetravalent HAT graphs there are many very difficult questions which at the moment we cannot answer. However, there has recently also been some progress on HAT graphs of all even valencies (see for instance [11, 25, 30]) where the generalizations of the concepts of alternating cycles, attachment number and radius from [32], proposed in [58], are investigated. The generalization of alternating cycles in such graphs are called alternets (which are not cycles). Similarly as for tetravalent HAT graphs, half of the size of an alternet its called the *radius* of the graph and two non-disjoint alternets always intersect in the same number of vertices, this number is called the *attachment number* of the graph.

In 1970, Bouwer [8] constructed an infinite family of vertex- and edge-transitive graphs that are now known as the Bouwer graphs $\mathcal{B}(k, m, n)$. The graph $\mathcal{B}(k, m, n)$ is a $2k$-valent graph of order mn^{k-1}. Bouwer showed that for any integer $k \geq 2$ the graph $\mathcal{B}(k, 6, 9)$ is HAT, thereby providing one example of a HAT graph for each even valence greater than 2. However, he did not consider the question of which of the other $\mathcal{B}(k, m, n)$ graphs are HAT nor whether for each k there exist infinitely many HAT graphs of valence $2k$. A recent complete classification of all HAT Bouwer graphs by Conder and Žitnik [11] provides answers to both of these questions. Nevertheless, it should be mentioned that the fact that for each $k \geq 2$ there exist infinitely many HAT graphs of valence $2k$, was already implicitly indicated by Alspach and Xu [5]. They classified all HAT graphs of order $3p$, where p is a prime. The infinitude of HAT graphs of valencies greater that 4 was later confirmed also by Li and Sim [31], who provided infinitely many HAT graphs (of various even valences) of prime power order.

It turns out that all of the Bouwer graphs, as well as the graphs from [5] and [31], are tightly attached (that is, the radius is equal to the attachment number). However, even by combining all the tetravalent members of these three families of graphs we get only a small part of the family of all tetravalent tightly attached HAT graphs (which have been completely classified by Marušič and Šparl [32, 53]). In Chapter 5 we generalize the family of Bouwer graphs to obtain a much larger family of vertex- and edge-transitive graphs of all possible even valences greater than 2. The generalization is very natural and contains almost all tetravalent tightly attached HAT graphs. We investigate the obtained family of generalized Bouwer graphs in great detail and give a complete classification of the HAT members and determine their automorphism groups.

The results presented in this PhD Thesis are contained in the following articles:

- A. Ramos-Rivera and P. Šparl. The classification of half-arc-transitive generalizations of Bouwer graphs. *European J. Combin.*, 64:88–112, 2017.

- I. Hubard, A. Ramos-Rivera and P. Šparl. Arc-transitive maps with underlying Rose Window graphs. Submitted.

- A. Ramos-Rivera and P. Šparl. New structural results on tetravalent half-arc-transitive graphs. *J. Comb. Theory, Ser. B*, 135:256-278, 2018.

Chapter 2

Definitions, notation and preliminary results

In this chapter we introduce the basic notation, concepts and terminology used in the thesis.

2.1 Groups

Let X be a nonempty set. We denote by $\mathrm{Sym}(X)$ the group of all permutations of X. In this thesis we let permutations act on the right, i. e., if τ and ρ are permutations of X, then by their product $\tau\rho$ we denote the permutation of X where we first apply τ and then ρ.

An *action* of a group G on the set X is a function $X \times G \to X$ which satisfies the following axioms:

- $x1_G = x$ for every $x \in X$, and
- $(xg)h = x(gh)$ for every $x \in X$ and for all $g, h \in G$,

where, for $x \in X$ and $g \in G$, the symbol xg denotes the image of $(x, g) \in X \times G$ in X under this function.

Let G be a group acting on a set X. For every $g \in G$ the mapping $\pi_g : X \to X$ defined by $x \mapsto xg$ is a permutation of X. The mapping $\varphi : g \mapsto \pi_g$ defines a homomorphism from G to $\mathrm{Sym}(X)$. This homomorphism is called *the permutation representation* of G induced by the action. The *kernel* $\mathrm{Ker}(\varphi) = \{g \in G \,|\, xg = x, \forall x \in X\}$ is called the kernel of the action, and if this is trivial then the action is said to be *faithful*.

Let $x \in X$. The set of all the elements of G mapping x to itself is denoted by G_x an is called the *stabilizer* of x in the group G. The set $xG = \{xg \mid g \in G\}$ is called the *G-orbit* of x. If $|G_x| = 1$ for all $x \in X$ we say that the action of G is *semiregular*. We shall say that the action of the group G is *transitive* if for each pair x, $y \in X$ there exists $g \in G$ such that $xg = y$. In other words, the action of G on X is transitive if $xG = X$ for any $x \in X$. If the action of G on X is both transitive and semiregular, then the action is called *regular* .

Let $\Delta \subseteq X$ and for each $g \in G$ define $\Delta g = \{xg \,|\, x \in \Delta\}$. Suppose that G is transitive on X. A nonempty subset Δ of X is called a *block* if for each $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \emptyset$. It follows from the definition that the whole set X and the singletons $\{x\}$, $x \in X$, are blocks. These are called *trivial blocks* and any other block is called *nontrivial.* We say that G is *primitive* if it has no nontrivial blocks; otherwise it is *imprimitive*. If Δ is a block for G, then the set $\beta = \{\Delta g \,|\, g \in G\}$ is a partition of the set X. This partition is called the *imprimitivity block system* of G induced by Δ.

Let G be a group acting on a set X and let Δ be a subset of X. Then the *pointwise stabilizer* of Δ in G is:

$$G_\Delta = \{g \in G \,|\, xg = x, \ ,\forall x \in \Delta\} = \bigcap_{x \in \Delta} G_x,$$

and the *setwise stabilizer* of Δ in G is:

$$G_{\{\Delta\}} = \{g \in G \,|\, \Delta g = \Delta\}.$$

The following theorem from [13, Theorem 1.6A] gives a natural source of imprimitivity block systems.

Theorem 2.1. *Let G be a group acting transitively on a set X, and let N be a nontransitive and nontrivial normal subgroup of G. Then the N-orbits form an imprimitivity block system for the action of G on X.*

We present the notation and definitions of special classes of groups and ring that will appear throughout the thesis.

- $\mathbb{Z}$, the aditive group of integers.
- $\mathbb{Z}_n$, the ring of integers modulo n, where n is an integer.
- $\mathbb{Z}_n^*$, the multiplicative group of units of the ring $\mathbb{Z}_n$.
- $\mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein 4-group
- D_n, the dihedral group of order $2n$.

For group theoretic concepts not defined here we refer to [17].

2.2 Graphs

Throughout this thesis all graph are assumed to be simple, finite, connected and undirected (but with an implicit orientation of the edges when appropriate). Let Γ be a graph and let $V(\Gamma)$ and $E(\Gamma)$ be the sets of its vertices and edges, respectively. Let $u, v \in V(\Gamma)$. If u and v are adjacent in Γ then we write $u \sim v$. The corresponding edge $\{u, v\}$ will usually be denoted by uv with the understanding that $uv = vu$. An *arc* of Γ is an ordered pair of vertices (u, v) such that $uv \in E(\Gamma)$ (each edge uv thus gives rise to two arcs (u, v) and (v, u)), and the set of all arc of Γ is denoted by $A(\Gamma)$.

In the case that we are working with a fixed orientation of the edges of Γ we indicate that the edge uv is oriented from u to v by $u \to v$ and say that u is the *tail* and v is the *head* of the (oriented) edge uv. We let $N(v)$ denote the set of all vertices adjacent to v, and the size of $N(v)$ is called the *valency* of the vertex v. If all the vertices of Γ have the same valency, we say that Γ is a *regular graph*. In this thesis we also use the term *tetravalent* for a regular graph of valency 4.

An *automorphism* γ of a graph Γ is a permutation of the set $V(\Gamma)$ such that $u \sim v$ if and only if $u\gamma \sim v\gamma$. The set of all the automorphism of a graph form a group and this group is denoted by $\mathrm{Aut}(\Gamma)$. For a subgroup $G \leq \mathrm{Aut}(\Gamma)$ the graph Γ is said to be *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if G acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, respectively. In particular, if G acts vertex- and edge-transitively but not arc-transitively, then Γ is said to be *G-half-arc-transitive*. In this case we also say that Γ admits a half-arc-transitive group of automorphisms. In the case of $G = \mathrm{Aut}(\Gamma)$, we omit the prefix G and simply write vertex-transitive, edge-transitive, arc-transitive and half-arc-transitive. As we pointed out in the introduction, if Γ is a G-vertex- and G-edge-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$, then two essentially different possibilities can occur:

(i) Γ is G-arc-transitive.

(ii) Γ is G-half-arc-transitive.

Let Γ_1 and Γ_2 be graphs. The *lexicographic product* of Γ_1 and Γ_2, denoted by $\Gamma_1[\Gamma_2]$ is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ in which two distinct vertices (u, v) and (u', v') are adjacent if and only if $u \sim u'$ in Γ_1, or $u = u'$ and $v \sim v'$ in Γ_2 (see Figure 2.1 for an example).

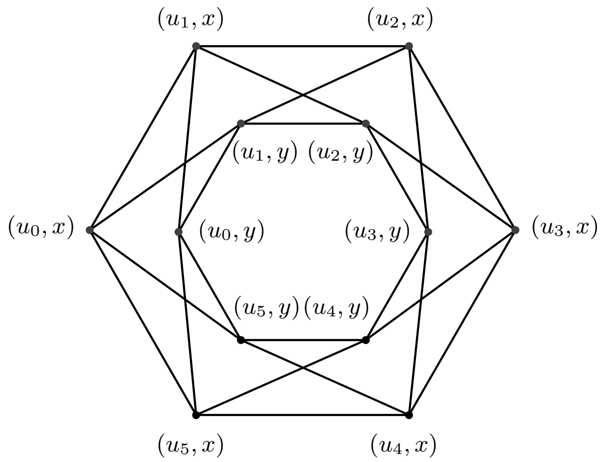


Figure 2.1: The lexicographic product of a cycle of length six, $(u_0, u_1, \ldots, u_5)$, and two disjoint vertices, x and y, $C_6[2K_1]$.

For the sake of completeness we also include the definition of a Cayley graph. Let G be a group and $S \subseteq G \setminus \{1_G\}$ such that $S = S^{-1}$. The *Cayley graph* $\mathrm{Cay}(G, S)$

is the graph whose vertex set is G and edge set is $\{\{x, sx\} \mid x \in G, s \in S\}$. By definition, $\mathrm{Cay}(G, S)$ has valency $|S|$ and it is connected if and only if $\langle S \rangle = G$, i. e., S generates G.

Chapter 3

Arc-transitive maps with underlying Rose window graphs

In this chapter we study arc-transitive graphs and use some known results about their properties to investigate symmetries of maps.

A *map* $\mathcal{M}$ is an embedding of a connected graph Γ on a compact surface S without boundary, in such a way that $S \setminus \Gamma$ is a disjoint union of simply connected regions. For example, the Platonic Solids can be regarded as maps on the sphere. The vertices and edges of the map are the same as those of its underlying graph, and the *faces* of the map are the simply connected regions obtained by removing the graph from the surface. An *automorphism* of a map is an automorphism of its underlying graph that also preserves its faces, and so the automorphism group of a map is a subgroup of the automorphism group of its underlying graph.

As mention in the introduction, in this thesis we focus on maps with the property that its automorphism group acts arc-transitively on its underlying graph. These maps are called *arc-transitive* maps. It is known that such maps can be divided into five classes, two of which have been extensively studied in the literature, namely the *reflexible* and *chiral* maps. Moreover, via a map operator, called *the Petrial dual*, the maps in one of the remaining these classes are put into a bijective correspondence with the chiral ones. We thus decide to investigate the two remaining classes that, until now, have not been studied a lot. It turns out that the smallest admissible valency of the underlying graph of such maps is 4. It thus seems natural to first study such maps. Since this two classes are also related via the Petrie dual operator, we focus in the study of one of them. One of the aims in this chapter is to present necessary and sufficient conditions for a tetravalent arc-transitive graph be the underlying graph of such maps.

Moreover, to continue our investigation of arc-transitive maps and apply the obtained results in the following sections, we describe the well-know family of tetravalent graphs called Rose Window graphs, whose arc-transitive members have already been classified into four subfamilies [27]. The reflexible and chiral maps with underlying Rose Window graphs were classified in [28]. The main goal of this chapter is to complete the classification of all arc-transitive maps with underlying Rose Window graphs.

3.1 Consistent cycles in arc-transitive graphs

When studying arc-transitive graphs investigation of certain cycles, called *consistent cycles*, may give an insight into the structure of the graph in question. The notion of consistent cycles in arc-transitive graphs was introduced by Conway in 1971 and has recently been studied in various families of graphs and other combinatorial structures (see for instance [6, 41, 42] and the references therein). In this section we give the definition of these cycles and use their properties for the study of G-arc-transitive graphs Γ where the group $G \leq \mathrm{Aut}(\Gamma)$ acts regularly on the set of arcs of Γ. The obtained results turn out to be a very useful tool for the study of a particular class of maps.

Let Γ be a graph admitting an arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$. A directed (but not rooted) cycle $\vec{C} = (v_0, v_1, \ldots, v_{r-1})$ of Γ is said to be *G-consistent* if there exists $g \in G$ mapping each v_i to v_{i+1} (where the indices are computed modulo r). For us a directed cycle is thus nothing but a connected subgraph of valence 2 together with one of its two possible orientations. In this case g is said to be a *shunt* of $\vec{C}$. Of course, the *inverse* $\vec{C}^{-1} = (v_0, v_{r-1}, v_{r-2}, \ldots, v_1)$ is G-consistent if and only if $\vec{C}$ is G-consistent. Thus an (undirected) cycle is said to be G-consistent if both of its two corresponding directed cycles are G-consistent.

Suppose $\vec{C}$ is a G-consistent directed cycle. It may happen that there is an automorphism in G, mapping $\vec{C}$ to $\vec{C}^{-1}$. In such a case we say that the underlying undirected cycle C of $\vec{C}$ is a *G-symmetric consistent*. Otherwise it is a *G-chiral consistent cycle*. It is well known and easy to see that G induces a natural action on the set of all G-consistent (directed) cycles. The above remarks thus imply that each G-orbit of G-symmetric consistent cycles corresponds to one G-orbit of G-consistent directed cycles, while each G-orbit of G-chiral consistent cycles corresponds to two such orbits. Moreover, the following has been proved in [42, Corollary 5.2].

Proposition 3.1. ([42]) *Let Γ be a graph of valency k admitting an arc-transitive group of automorphisms G and let s and c denote the numbers of G-orbits of G-symmetric and G-chiral consistent cycles, respectively. Then $s + 2c = k - 1$. In particular, if k is even then Γ contains at least one G-orbit of G-symmetric consistent cycles.*

The following example illustrates the above definitions and Proposition 3.1. See [42] for a more detailed explanation.

Example 3.2. Let Γ be the well-known Petersen graph with the labels of its vertices as in Figure 3.1. Then, following the notation from Proposition 3.1, $k = 3$. Let $G = \mathrm{Aut}(G)$. First, note that the permutation $\rho = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10) \in \mathrm{Aut}(\Gamma)$ and that it is a shunt, in particular, for the cycle $\vec{C}_1 = (1, 2, 3, 4, 5)$. Moreover, the involution $\tau = (2\ 5)(3\ 4)(7\ 10)(8\ 9) \in \mathrm{Aut}(\Gamma)$ and it maps $\vec{C}_1$ to $\vec{C}_1^{-1}$. Then $\vec{C}_1$ is an $\mathrm{Aut}(\Gamma)$-symmetric consistent cycle. It is not hard to see that the permutation $\gamma = (3\ 4\ 9\ 7\ 10\ 8)(2\ 5\ 6) \in \mathrm{Aut}(\Gamma)$, and so it is a shut for the cycle $\vec{C}_2 = (3, 4, 9, 7, 10, 8)$. Since $\vec{C}_2\tau = \vec{C}_2^{-1}$ and $\vec{C}_1$ and $\vec{C}_2$ have different lengths, it follows by Proposition 3.1 that $s = 2$ and $c = 0$.

As mentioned in the beginning of this chapter, we will study a certain family of arc-transitive maps. It turns out that the automorphism group of such maps is

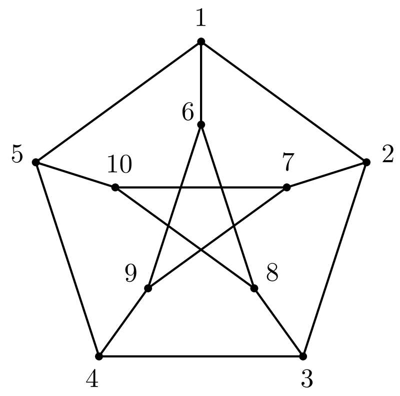


Figure 3.1: Petersen graph.

1-regular on its underlying graph, that is, it acts regularly on the set of its arcs. In the rest of this section we present some results that are useful for such situations. The following result was proved in [41].

Lemma 3.3. *Let Γ be a G-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $(u,v) \in A(\Gamma)$. For a G-orbit $\mathcal{A}$ of G-consistent directed cycles, let $\mathcal{B}_{\mathcal{A}}$ denote the set of all automorphisms $\gamma \in G$, such that $u\gamma = v$, and the orbit of u under γ is in $\mathcal{A}$. Let $G_{(u,v)}$ denote the G-stabilizer of the arc (u,v). Then the number of elements in $\mathcal{B}_{\mathcal{A}}$ is independent of $\mathcal{A}$, and is equal to the order of $G_{(u,v)}$.*

The next lemma is an immediate corollary of Lemma 3.3.

Lemma 3.4. *Let Γ be a graph admitting a 1-regular group of automorphisms G and let e be an edge of Γ. Then each G-orbit of G-consistent directed cycles of Γ contains exactly one G-consistent directed cycle containing e.*

In the case that the graph under consideration is tetravalent we can say more.

Lemma 3.5. *Let Γ be a tetravalent graph admitting a 1-regular group of automorphisms G. Then Γ has three G-orbits of G-consistent cycles, all of which are G-symmetric, if and only if the vertex stabilizers in G are isomorphic to the Klein 4-group.*

Proof. Let $v \in V(\Gamma)$ be a vertex of Γ. Since Γ is tetravalent and G is 1-regular the vertex stabilizer G_v is isomorphic either to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose first that $G_v = \langle \beta \rangle \cong \mathbb{Z}_4$, let u be a neighbor of v and let $u_i = u\beta^i$ for $i \in \mathbb{Z}_4$. Since G is 1-regular there exists an automorphism $\gamma \in G$ mapping the arc (u_0, v) to the arc (v, u_1). Then γ is a shunt of the directed G-consistent cycle $\vec{C} = (u_0, u_0\gamma, u_0\gamma^2, .., u_0\gamma^{r-1})$ containing (u_0, v, u_1), where r is the order of γ in G. If the underlying cycle C was G-symmetric, there would exist an automorphism $\delta \in G_v$ interchanging u_0 and u_1, which is impossible as $G_v = \langle \beta \rangle$. Thus C is G-chiral, and so Proposition 3.1 implies that Γ has one G-orbit of G-symmetric and one G-orbit of G-chiral consistent cycles.

Suppose now that $G_v \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and note that 1-regularity of G implies that the action of G_v on the four neighbors of v is transitive. Let $\vec{C} = (v_0, v_1, \ldots, v_{n-1})$ be a directed G-consistent cycle with $v = v_0$ and shunt γ. By assumption there exists $\beta \in G_v$ mapping v_1 to v_{n-1}. Since β is of order 2, it thus also maps v_{n-1} to v_1. Let $\vec{C}' = \vec{C}\beta$ and observe that both $\vec{C}^{-1}$ and $\vec{C}'$ contain the directed 2-path (v_1, v_0, v_{n-1}). Since G is 1-regular there exists a unique automorphism $\gamma \in G$ mapping the arc (v_1, v_0) to the arc (v_0, v_{n-1}), and so $\vec{C}'$ and $\vec{C}^{-1}$ have the same shunt in G (namely γ), implying that they coincide. Thus the underlying cycle of $\vec{C}$ is a G-symmetric consistent cycle, and so all G-consistent cycles of Γ are G-symmetric. $\square$

3.2 Maps

In this section we describe in detail the structure of a map and give the basic terminology and notation that we will be using through this chapter. For simplicity, we often refer to the vertices, edges and faces of a map as their 0-, 1-, and 2-faces, respectively.

Let $\mathcal{M}$ be a map. By selecting one point in the interior of each 1- and each 2-face of $\mathcal{M}$ we can identify an incident vertex-edge-face triple $\{v, e, f\}$ with the triangle(s) with vertices v and the chosen interior points of the edge e and the face f. By doing this everywhere on $\mathcal{M}$ we obtain a triangulation of the map, called the *barycentric subdivision* $\mathcal{BS}(\mathcal{M})$ of $\mathcal{M}$ (see Figure 3.2 for an example). If the triangles of $\mathcal{BS}(\mathcal{M})$, called *flags*, are then in one-to-one correspondence to the incident triples $\{v, e, f\}$, we say that the map $\mathcal{M}$ is *polytopal*. In such a case $\mathcal{M}$ can be regarded as an abstract polytope of rank 3 (in the sense of [40]). In this thesis we will only be dealing with polytopal maps. We therefore use the term *flag* both for the flags themselves and the corresponding incident triples $\{v, e, f\}$ of $\mathcal{M}$. We refer the reader to [20] for a detailed study of the polytopality of maps and their generalisations to higher dimensions as maniplexes.

It is convenient to colour the vertices of $\mathcal{BS}(\mathcal{M})$ with the colours 0, 1 and 2, whenever they represent a vertex, edge or face of $\mathcal{M}$, respectively.

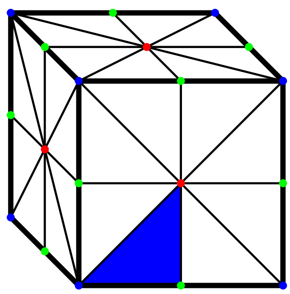


Figure 3.2: The barycentric subdivision of the cube.

For a given flag $\Phi \in \mathcal{BS}(\mathcal{M})$ corresponding to the incident triple $\{v, e, f\}$ we say that v is the *vertex*, e is the *edge* and f is the *face* of Φ, respectively, and that v, e

and f *belong* to Φ. Observe that given a flag $\Phi \in \mathcal{BS}(\mathcal{M})$, it shares its three sides with three other flags of $\mathcal{BS}(\mathcal{M})$. We shall denote then by Φ^0, Φ^1 and Φ^2, where Φ and Φ^i share the vertices of colours different from i (see Figure 3.3). The flags Φ and Φ^i are said to be *i-adjacent flags*.

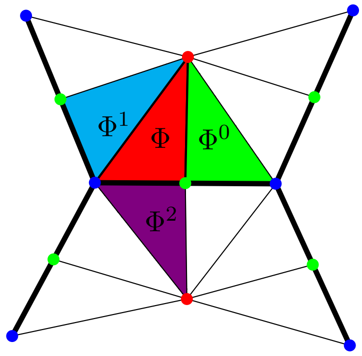


Figure 3.3: A *base* flag Φ of a map and its adjacent flags.

Given a sequence $i_0, i_1, \ldots, i_k$, with $i_j \in \{0,1,2\}$, we define inductively the flag $\Phi^{i_0,i_1,\ldots,i_k}$ as the i_k-adjacent flag to the flag $\Phi^{i_0,i_1,\ldots,i_{k-1}}$. Note that as each edge of $\mathcal{M}$ belongs to exactly four flags, we have that $\Phi^{0,2} = \Phi^{2,0}$ holds for every flag $\Phi \in \mathcal{F}(\mathcal{M})$, where $\mathcal{F}(\mathcal{M})$ denotes the set of all flags of $\mathcal{M}$. Note also that since the underlying graph Γ of $\mathcal{M}$ is connected, given any two flags $\Phi, \Psi \in \mathcal{F}(\mathcal{M})$ there exists a sequence $i_0, i_1, \ldots, i_k$, with $i_j \in \{0,1,2\}$ such that $\Psi = \Phi^{i_0,i_1,\ldots,i_k}$.

The group of all automorphisms of $\mathcal{M}$, denoted by $\mathrm{Aut}(\mathcal{M})$, has a natural action on the set of flags of $\mathcal{M}$. When studying this action the following straightforward observation can be very useful.

Lemma 3.6. *For each automorphism $\alpha \in \mathrm{Aut}(\mathcal{M})$, each flag $\Phi \in \mathcal{F}(\mathcal{M})$ and each $i \in \{0,1,2\}$ the i-adjacent flag of $\Phi\alpha$ is the α-image of the i-adjacent flag of Φ, that is $(\Phi\alpha)^i = \Phi^i\alpha$.*

Hence, the connectivity of $\mathcal{M}$ implies that the action of $\mathrm{Aut}(\mathcal{M})$ is free on $\mathcal{F}(\mathcal{M})$ (that is, no nonidentity automorphism of $\mathcal{M}$ fixes any flag). Therefore, the action of an automorphismof $\mathcal{M}$ is completely determined by its action on any given flag. Lemma 3.6 also implies the following.

Lemma 3.7. *Let $\mathcal{M}$ be a map and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be (possibly the same) orbits of the action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$. Suppose that $\Phi^i \in \mathcal{O}_2$ holds for some $\Phi \in \mathcal{O}_1$ and some $i \in \{0,1,2\}$. Then $\Psi^i \in \mathcal{O}_2$ for every $\Psi \in \mathcal{O}_1$.*

We shall say that an automorphism α of $\mathcal{M}$ is a *reflection* whenever there exists $\Phi \in \mathcal{F}(\mathcal{M})$ and $i \in \{0,1,2\}$ such that $\Phi\alpha = \Phi^i$. Note that by Lemma 3.6 each reflection is an involution. We say that α is a *one step rotation* at the face (resp. at the vertex) of Φ if $\Phi\alpha = \Phi^{0,1}$ or $\Phi\alpha = \Phi^{1,0}$ (resp. $\Phi\alpha = \Phi^{2,1}$ or $\Phi\alpha = \Phi^{1,2}$). Note that since the action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$ is free there can be at most one pair of mutually inverse one step rotations at any given vertex or face. The following observation is straightforward.

Lemma 3.8. *Let $\mathcal{M}$ be a map such that the underlying graph has no multiple edges or loops and has minimal degree at least* 3*. If for some face f of $\mathcal{M}$ there exists a one step rotation at f in* $\mathrm{Aut}(\mathcal{M})$ *then the traversal of f along its boundary visits each of its vertices and edges exactly once.*

Lemma 3.8 thus implies that maps with underlying simple graphs of minimal degree at least 3 that admit a one step rotation at every face are polytopal.

If the action of $\mathrm{Aut}(\mathcal{M})$ has k orbits on the flags of $\mathcal{M}$ we say that $\mathcal{M}$ is a *k-orbit map*. The 1-orbit maps are usually called *reflexible maps* in the literature. A map is reflexible if and only if given a *base flag* Φ there exist automorphisms α_i, $i \in \{0,1,2\}$, sending Φ to Φ^i. In such a case, the automorphism group of $\mathcal{M}$ is generated by α_0, α_1 and α_2.

The 2-orbit maps have been also studied, for example, in [16], [23] and [45]. There exist 7 classes of 2-orbit maps. Given $I \subsetneq \{0,1,2\}$ we say that a 2-orbit map is in *class* 2_I if for any given flag Φ we have that Φ^i is in the same $\mathrm{Aut}(\mathcal{M})$-orbit as Φ if and only if $i \in I$. By Lemma 3.7 and the fact that we are dealing with 2-orbit maps, this definition does not depend on the choice of the flag Φ, and thus the 7 classes are disjoint. We abbreviate $2_\emptyset$ by 2 and $2_{\{i\}}$ by 2_i for each of $i \in \{0,1,2\}$.

Maps in class 2 correspond to *chiral maps*, that is, maps that have two orbits on flags under its automorphism group in such a way that adjacent flags belong to different orbits. A map that is either reflexible or chiral is called a *rotary map*. Rotary maps are precisely the maps that admit one step rotations around each of its faces and each of its vertices.

A map is said to be *j-face transitive* if its automorphism group acts transitively on the j-faces and it is called *fully transitive* if it is j-face transitive for all $j \in \{0,1,2\}$. Rotary maps are examples of fully transitive maps. However, this is no longer true for all 2-orbit maps. In fact, a 2-orbit map is j-face transitive, for some $j \in \{0,1,2\}$, if and only if $I \neq \{0,1,2\} \setminus \{j\}$ (see [23, Theorem 5]). In particular this means that for each $j \in \{0,1,2\}$ there is exactly one class of 2-orbit maps that are not j-face transitive and hence there are just three classes of 2-orbit maps that are not fully transitive.

A *Petrie polygon* of a map $\mathcal{M}$ is a path P along the edges of the underlying graph Γ such that every two consecutive edges of P are consecutive edges on the same face of $\mathcal{M}$, but no three consecutive edges of P are consecutive edges on the same face of $\mathcal{M}$. For example, a Petrie polygon of a tetrahedron contains each of the four vertices of the tetrahedron exactly once. It is not difficult to see that every edge of $\mathcal{M}$ belongs to at most two Petrie polygons (see Figure 3.4). The map having Γ as the underlying graph and all the Petrie polygons of $\mathcal{M}$ as faces is called the *Petrial* or *Petrie dual* of $\mathcal{M}$, and shall be denoted by $\mathcal{M}^\pi$. It is well known (see, for example, [24]) that $(\mathcal{M}^\pi)^\pi \cong \mathcal{M}$ and that $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{M}^\pi)$. Hence, the Petrial of a k-orbit map is again a k-orbit map. Moreover, if $\mathcal{M}$ is a 2-orbit map in class 2_I, for some $I \subsetneq \{0,1,2\}$, then $\mathcal{M}^\pi$ is in class $2_{I'}$, where $I' = I \setminus \{0\}$, if $0 \in I$ and $2 \notin I$, $I' = I \cup \{0\}$ if $0, 2 \notin I$ and $I' = I$ if $2 \in I$ (see [24]).

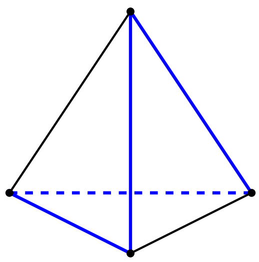


Figure 3.4: A Petrie polygon of a tetrahedron.

3.3 Arc-transitive maps

In [22] Graver and Watkins proposed the study of maps whose automorphism group acts transitively on the set of edges of the map. Rotary maps are examples of such maps, but not all edge-transitive maps are rotary. In fact, Graver and Watkins divided the study of edge-transitive maps into 14 different types or classes of such maps based on the local configuration of the flags and the orbits they belong to (see also [57]). Two of these 14 classes correspond to rotary maps. Even though most of the research on edge-transitive maps in the existing literature is focused on these two classes of rotary maps, some articles also investigated other classes (see for instance [44, 52]). In these articles questions like which edge-transitive maps of small genera exist, and which of the 14 types of edge-transitive maps can be realized by maps with an automorphism group abstractly isomorphic to a symmetric group, have been considered.

Recently, Jones (see [26]) extended the work of [52] while working on the question of which groups can act as the automorphism group of an edge-transitive map and answering it for some specific classes of groups. This question naturally extends to the question of which graphs can be the underlying graph of an edge-transitive map. When studying edge-transitive maps it is natural to restrict to a subset of them: the *arc-transitive maps*. Rotary maps are an example of such maps, but, again, there are others. For example, the cuboctahedron (or medial of the cube) seen as a map on the sphere is arc-transitive, but it is not a rotary map.

Let $\mathcal{M}$ be an arc-transitive map and let $\Phi = (v, e, f) \in \mathcal{F}(\mathcal{M})$. If u is the other vertex of the edge e, then there exists $\alpha \in \mathrm{Aut}(\mathcal{M})$, fixing the edge e while interchanging u and v. This means that α sends Φ to either Φ^0 or $\Phi^{0,2}$. Consequently, α sends Φ^2 to either $\Phi^{0,2}$ or Φ^0. Since e can be mapped by an automorphism of $\mathrm{Aut}(\mathcal{M})$ to any other edge of $\mathcal{M}$, the connectivity of $\mathcal{M}$ and Lemma 3.7 imply that there are at most two orbits of flags under the action of $\mathrm{Aut}(\mathcal{M})$.

Of course, if the action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$ has exactly one orbit, then $\mathcal{M}$ is reflexible. However, if there are two orbits, then the above remarks imply that $\mathcal{M}$ has to be in class 2_I for some $I \subset \{0, 1, 2\}$ with $2 \notin I$, that is, $\mathcal{M}$ belongs to one of the following four classes of maps: 2, 2_0, 2_1 or $2_{\{0,1\}}$.

As we mentioned before, the reflexible and chiral maps (rotary maps) have been extensively studied in the literature. Moreover, the maps in class 2_0 are strongly related to the chiral ones via the Petrie dual operator, since if $\mathcal{M}$ is a map in the class 2, then $\mathcal{M}^{\pi}$ is a map in the class 2_0 and vice versa (and their graphs and automorphisms groups are the same). This leaves us with the maps of classes 2_1 and $2_{\{0,1\}}$ which are also related via the Petrie operator. In the rest of this section we thus restrict ourselves to maps of class $2_{\{0,1\}}$.

Maps in class $2_{\{0,1\}}$ are *hereditary* in the sense that all the combinatorial symmetries of their faces can be extended to the entire map (see [43] for a study of hereditary polytopes). We remark that these maps are of type 2^* in the sense of [22].

If $\mathcal{M}$ is a map in class $2_{\{0,1\}}$, then for each flag Φ there exist (unique) automorphisms $\alpha_0(\Phi)$ and $\alpha_1(\Phi)$ sending Φ to the flags Φ^0 and Φ^1, respectively. By Lemma 3.6 the automorphism $\alpha = \alpha_1(\Phi^2)$ maps $\Phi\alpha = (\Phi^2\alpha)^2 = (\Phi^{2,1})^2 = \Phi^{2,1,2}$, and so there also exists an (unique) automorphism $\alpha_{212}(\Phi)$ of $\mathcal{M}$ sending a given flag Φ to $\Phi^{2,1,2}$. Whenever the flag Φ will be clear from the context we will write α_0, α_1 and α_{212} instead of $\alpha_0(\Phi)$, $\alpha_1(\Phi)$ and $\alpha_{212}(\Phi)$, respectively. In [23], it was shown that if $\mathcal{M}$ is a map in class $2_{\{0,1\}}$ and Φ is any flag of $\mathcal{M}$, then $\mathrm{Aut}(\mathcal{M}) = \langle \alpha_0(\Phi), \alpha_1(\Phi), \alpha_{212}(\Phi)\rangle$.

Recall that maps in class $2_{\{0,1\}}$ are not 2-face transitive. On the other hand, in view of the existence of automorphisms α_0 and α_1, all of the flags corresponding to a given face of such a map $\mathcal{M}$ are in the same $\mathrm{Aut}(\mathcal{M})$-orbit (and thus $\mathcal{M}$ is hereditary). Since there are only two orbits of flags this implies that there are two orbits of faces. In other words, there is no automorphism of $\mathcal{M}$ mapping a flag Φ to either Φ^2 or to $\Phi^{2,0}$. Since the action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$ is free, the group $\mathrm{Aut}(\mathcal{M})$ acts as a 1-regular group on the underlying graph of $\mathcal{M}$.

As we mention before, an example of a map of class $2_{\{0,1\}}$ is the cuboctahedron seen as a map on the sphere. In this case, it is clear that it is not face transitive since it has faces of two different lengths (see figure 3.5), in fact faces of the same are in the same $\mathrm{Aut}(\mathcal{M})$-orbit.

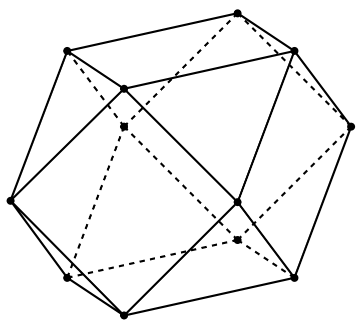


Figure 3.5: Cuboctahedron.

Let Φ be a flag of a map $\mathcal{M}$ of class $2_{\{0,1\}}$, v be the vertex belong to Φ and $\mathcal{O}_1$ and $\mathcal{O}_2$ be the two $\mathrm{Aut}(\mathcal{M})$-orbits of flags such that $\Phi \in \mathcal{O}_1$. Then the flag $\Phi^1 \in \mathcal{O}_1$ while $\Phi^{1,2}, \Phi^{1,2,1} \in \mathcal{O}_2$. If we continue taking the 1- and 2-adjacent flags we can see that faces of the two $\mathrm{Aut}(\mathcal{M})$-orbits alternated around the vertex v. It follows that the valency of the underlying graph of a map in class $2_{\{0,1\}}$ must be even (see figure 3.6). By the Petrie dual operator, this also holds for maps in the class 2_1.

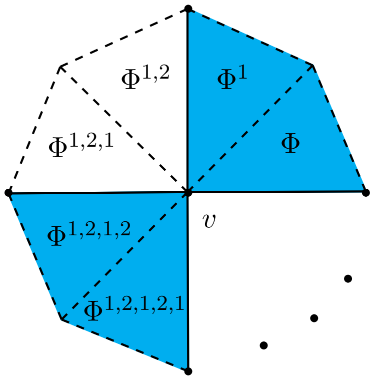


Figure 3.6: Local configuration around a vertex of a map of class $2_{\{0,1\}}$.

Let $\mathcal{M}$ be a map in class $2_{\{0,1\}}$, let n and m be the lengths of its faces from the two $\mathrm{Aut}(\mathcal{M})$-orbits and let $2q$ be the valency of each vertex. Then we say that the *type* of $\mathcal{M}$ is $\left\{ \begin{matrix} n \\ m \end{matrix}, 2q \right\}$ (the cuboctahedron is of type $\left\{ \begin{matrix} 3 \\ 4 \end{matrix}, 4 \right\}$). In this case the face stabilizers $\langle \alpha_0, \alpha_1 \rangle$ and $\langle \alpha_0, \alpha_{212} \rangle$ are isomorphic to D_n and D_m, respectively (assuming the base flag Φ is in a face of length n), while the vertex stabilizer $\langle \alpha_1, \alpha_{212} \rangle$ is isomorphic to D_q. Furthermore $\langle \alpha_1, \alpha_{212} \rangle$ acts transitively (in fact regularly) on the neighbours of the *base* vertex belonging to Φ. In Figure 3.7 the local configuration around the base flag Φ in a map of class $2_{\{0,1\}}$ is shown.

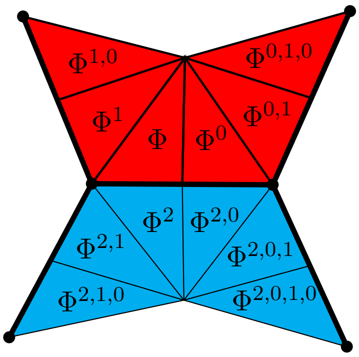


Figure 3.7: Local configuration in a map of class $2_{\{0,1\}}$.

In the rest of this section we give some results about maps in class $2_{\{0,1\}}$ related to their underlying graphs and their automorphisms group. Following [12], the next lemma is straightforward.

Lemma 3.9. *If $\mathcal{M}$ is a map such that there exists a flag Φ and automorphisms α_0, α_1 and α_{212} sending Φ to Φ^0, Φ^1 and $\Phi^{2,1,2}$, respectively, then $\mathcal{M}$ is either a 2-orbit map in class $2_{\{0,1\}}$ or it is a reflexible map.*

Let $\mathcal{M}$ be a map in class $2_{\{0,1\}}$ and let α_0, α_1 and α_{212} be the *distinguished generators* of $\mathrm{Aut}(\mathcal{M})$ with respect to some base flag Φ. Then the automorphism $\alpha_0\alpha_1$ acts as a 1-step rotation of the face belonging to Φ and $\alpha_0\alpha_{212}$ acts as a 1-step rotation of the face belonging to Φ^2. Moreover, if in addition we apply α_0 to any of these rotations we get a 1-step rotation in the "opposite direction" of the corresponding face.

We have therefore established the following lemma.

Lemma 3.10. *Let $\mathcal{M}$ be a map in class $2_{\{0,1\}}$ with the underlying graph Γ. Then the boundaries of faces of $\mathcal{M}$ are $\mathrm{Aut}(\mathcal{M})$-symmetric consistent cycles of Γ.*

Corollary 3.11. *Let $\mathcal{M}$ be a map in class $2_{\{0,1\}}$. Then no two faces of $\mathcal{M}$ share two consecutive edges.*

Proof. Since $\mathrm{Aut}(\mathcal{M})$ is 1-regular on the underlying graph, each face has an unique pair of mutually inverse one step rotations in $\mathrm{Aut}(\mathcal{M})$. Hence, if two faces share two common consecutive edges, the one step rotations for both faces coincide, implying that they share all its vertices and edges. But in this case the map is a reflexible map on the sphere with two faces, a contradiction. □

Note that the smallest admissible valency of the underlying graph of maps in class $2_{\{0,1\}}$ is 4. We finish this section with a result that is of great help when dealing with such maps.

Theorem 3.12. *Let Γ be a tetravalent graph admitting a 1-regular group of automorphisms G. Then:*

a) If Γ is the underlying graph of a map $\mathcal{M}$ in class $2_{\{0,1\}}$ with $\mathrm{Aut}(\mathcal{M}) = G$ then all orbits of G-consistent cycles of Γ are G-symmetric.

b) If $G = \mathrm{Aut}(\Gamma)$ and all orbits of G-consistent cycles of Γ are G-symmetric, then for any two orbits of G-consistent cycles of Γ there exists a map $\mathcal{M}$ in class $2_{\{0,1\}}$ with $\mathrm{Aut}(\mathcal{M}) = G$ and underlying graph Γ such that the boundaries of its faces are the members of these two orbits.

Proof. By Proposition 3.1 the graph Γ has three orbits of G-consistent directed cycles and at least one of the orbits of G-consistent cycles is G-symmetric.

Let us start by assuming that Γ is the underlying graph of a map $\mathcal{M}$ in class $2_{\{0,1\}}$ with $\mathrm{Aut}(\mathcal{M}) = G$. By Lemma 3.10 the faces of $\mathcal{M}$ are G-symmetric consistent cycles, and since G has two orbits on the set of faces of $\mathcal{M}$ this shows that Γ has at least two orbits of G-symmetric consistent cycles, implying that Γ in fact has three orbits of G-consistent cycles, all of which are G-symmetric.

For the second part of the theorem suppose $G = \mathrm{Aut}(\Gamma)$ and that all orbits of G-consistent cycles are G-symmetric. By Proposition 3.1 there are three of them; let $\mathcal{O}_1$ and $\mathcal{O}_2$ be any two. We now show that there is a map in class $2_{\{0,1\}}$ having the elements of $\mathcal{O}_1$ and $\mathcal{O}_2$ as faces. In fact, by Lemma 3.4, taking all the elements of both $\mathcal{O}_1$ and $\mathcal{O}_2$ as faces, we do get a map $\mathcal{M}$ with underlying graph Γ. We just need to show that this map $\mathcal{M}$ is in class $2_{\{0,1\}}$. Since we have chosen two complete orbits of G-consistent cycles as the faces of $\mathcal{M}$, every automorphism of Γ sends faces to faces, implying that it is an automorphism of $\mathcal{M}$, and so $\mathrm{Aut}(\Gamma) = G = \mathrm{Aut}(\mathcal{M})$. Moreover, since G is 1-regular, the map $\mathcal{M}$ is not reflexible. By Lemma 3.9 we thus only need to show that we can send a given flag Φ to the flags Φ^0, Φ^1 and $\Phi^{2,1,2}$.

Let $\Phi = (v, e, f_1)$ be a flag of $\mathcal{M}$ and let u be the other vertex of Γ, incident to e. Without loss of generality assume that f_1 belongs to $\mathcal{O}_1$. Denote by f_2 the unique (confront Lemma 3.4) face from $\mathcal{O}_2$ containing e. First, since f_1 is a G-symmetric consistent cycle there exists $\alpha_0 \in G$ fixing f_1 and e and interchanging u and v. We can therefore map Φ to its 0-adjacent flag Φ^0. Let $\beta, \beta' \in G$ be the shunts of f_1 and f_2, respectively, that send v to u. Then $\alpha_1 := \beta\alpha_0$ maps Φ to Φ^1. Finally, $\alpha_{212} := \beta'\alpha_0$ maps Φ to $\Phi^{2,1,2}$, proving that $\mathcal{M}$ is indeed in class $2_{\{0,1\}}$. □

Combining together Lemma 3.5 and Theorem 3.12 we have the following useful corollary for searching for possible examples of maps in class $2_{\{0,1\}}$.

Corollary 3.13. *Let Γ be a tetravalent graph with a 1-regular group of automorphisms. Then Γ is the underlying graph of a map of class $2_{\{0,1\}}$ if and only if the vertex stabilizers in $\mathrm{Aut}(\Gamma)$ are isomorphic to the Klein 4-group. Moreover, in this case there are three pairwise nonisomorphic such maps.*

3.4 Rose Window graphs

Another way to approach the investigation of maps in class $2_{\{0,1\}}$ is by asking the question of which graphs can be the underlying graphs of such maps. Of course, in this case we are looking for graphs having at least the necessary properties, that is being arc-transitive and having even valency. We then have to study their structure. In this section we describe a family of tetravalent graphs whose arc-transitive members have been classified.

In 2008 Wilson [59] introduced a family of tetravalent graphs now known as the Rose Window graphs. This class of graphs has been studied quite a lot and is now well understood (see for instance [15, 27, 28]). In [59] Wilson identified four specific subfamilies of Rose Window graphs (defined below) and proved that their members are arc-transitive. His conjecture that each edge-transitive Rose Window graph (which in the case of Rose Window graphs is equivalent to being arc-transitive) belongs to one of these four subfamilies was confirmed in 2010 by Kovács, Kutnar and Marušič [27].

Let $n \geq 3$ be an integer and let $1 \leq r \leq n-1$, with $r \neq n/2$, and $1 \leq a \leq n-1$ be integers. The *Rose Window graph* $R_n(a, r)$ is then the graph with vertex-set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ whose edge-set consists of four kinds of edges:

- the set of all *rim edges* $x_i x_{i+1}$, $i \in \mathbb{Z}_n$;
- the set of all *hub edges* $y_i y_{i+r}$, $i \in \mathbb{Z}_n$;
- the set of all *in-spokes* $x_i y_i$, $i \in \mathbb{Z}_n$;
- the set of all *out-spokes* $x_i y_{i-a}$, $i \in \mathbb{Z}_n$,

where all the indices are computed modulo n. It is clear that we can assume $a \leq n/2$ and $r < n/2$ (see Figure 3.8 for an example).

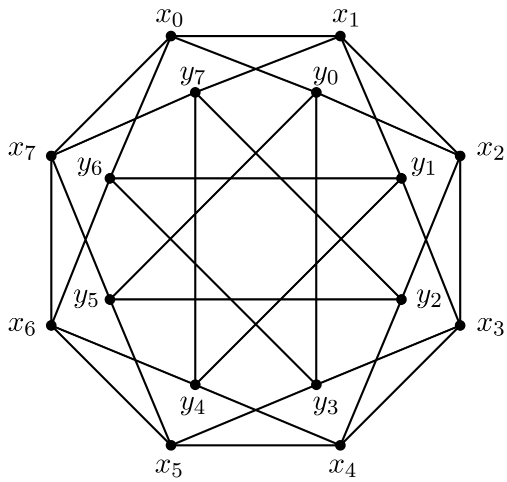


Figure 3.8: $R_8(2, 3)$.

Observe that the graph $R_n(a, r)$ admits the automorphisms

$$\rho = (x_0, x_1, \ldots, x_{n-1})(y_0, y_1, \ldots, y_{n-1}) \quad \text{and} \quad \mu, \tag{3.1}$$

where μ interchanges each x_i with x_{n-i} and each y_i with y_{n-i-a}. Note that the group $\langle \rho, \mu \rangle \cong D_n$ (the dihedral group of order $2n$) has two orbits on the vertex-set of $R_n(a, r)$.

We can now state the result [27, Corollary 1.3] giving a complete classification of arc-transitive Rose Window graphs.

Proposition 3.14. ([27]) *Let $n \geq 3$ be an integer and let $1 \leq r < n/2$ and $1 \leq a \leq n/2$ be integers. Then the Rose Window graph $R_n(a, r)$ is arc-transitive if and only if it belongs to one of the following four families:*

(i) $R_n(2, 1)$;

(ii) $R_{2m}(m - 2, m - 1)$;

(iii) $R_{2m}(2b, r)$, *where* $b^2 \equiv \pm 1 \pmod{m}$ *and either* $r = 1$ *or* $r = m - 1$, *in which case* m *must be even;*

(iv) $R_{12m}(3m + 2, 3m - 1)$ *or* $R_{12m}(3m - 2, 3m + 1)$.

We remark that in [27] and [28] the order of the families (iii) and (iv) (called (d) and (c) there) was reversed but we choose to stick with the order and names given in [59] where the families were first introduced.

3.5 The classification

As announced in the beginning of this chapter we now classify all maps of class $2_{\{0,1\}}$ whose underlying graph is a Rose Window graph. Since the chiral maps underlying Rose Window graphs have already been classified in [28] the remarks from Section 3.3 imply that this completes the classification of all arc-transitive maps corresponding to

Rose Window graphs. We analyze each of the four subfamilies from Proposition 3.14 in a separate subsection.

3.5.1 Family (i)

We start by considering the family of graphs $R_n(2,1)$, $n \geq 3$. The graph $R_n(2,1)$ is isomorphic to the lexicographic product $C_n[2K_1]$ of a cycle of length n with two independent vertices and is known also as the wreath graph. These graphs appear in various investigations of symmetries of graphs and have thus been studied in great detail before (see for instance [51] where certain generalizations, now know as the Praeger-Xu graphs, have been introduced and their automorphism groups determined).

Throughout this subsection let $\Gamma = R_n(2,1)$. For convenience we relabel the vertices of Γ in the following way. For each $i \in \mathbb{Z}_n$ we let $u_i = x_i$ and $v_i = y_{i-1}$. With this notation each pair of vertices u_i and v_i have the same neighborhood $\{u_{i\pm1}, v_{i\pm1}\}$ (see Figure 3.9 for an example). The permutations ρ and μ from (3.1) thus map in such a way that $u_i\rho = u_{i+1}$, $v_i\rho = v_{i+1}$, $u_i\mu = u_{-i}$ and $v_i\mu = v_{-i}$ for all $i \in \mathbb{Z}_n$. For each $i \in \mathbb{Z}_n$ let σ_i be the involution interchanging u_i and v_i and fixing all other vertices. Clearly $\sigma_i = \rho^{-i}\sigma_0\rho^i$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ hold for all $i,j \in \mathbb{Z}_n$. Moreover, it is well known that, unless $n = 4$ in which case $\Gamma \cong K_{4,4}$, the 2-element sets $\{u_i, v_i\}$ are blocks of imprimitivity for $\mathrm{Aut}(\Gamma) = \langle \rho, \mu, \sigma_0 \rangle$ which is thus of order $n2^{n+1}$ with $N = \langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle \lhd \mathrm{Aut}(\Gamma)$.

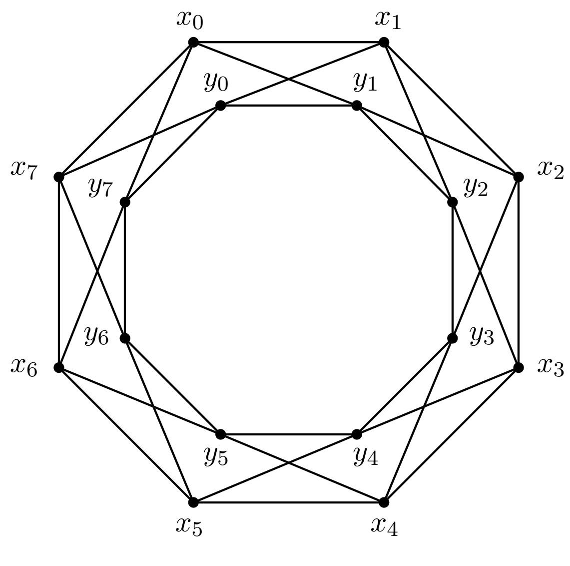


Figure 3.9: $R_8(2,1)$ with the relabeling of its vertices.

Since $R_4(2,1) \cong K_{4,4}$ is a bit special, we deal with it separately. Using a suitable computer package it is easy to see that there is exactly one map of class $2_{\{0,1\}}$ with $K_{4,4}$ as its underlying graph; the faces are of lengths 4 and 8. For the rest of this subsection we thus assume $n \neq 4$. Our approach in determining all maps of class $2_{\{0,1\}}$ on Γ is similar to the one taken in [28]. We first determine the structure of potential 1-regular subgroups of $\mathrm{Aut}(\Gamma)$ which could be the automorphism groups of such maps.

Suppose then that $\mathcal{M}$ is a class $2_{\{0,1\}}$ map with the underlying graph Γ and let $T = N \cap \mathrm{Aut}(\mathcal{M})$. Clearly, each $\sigma \in N$ can uniquely be expressed as $\sigma = \Pi_{j=0}^{n-1}\sigma_j^{i_j}$, where $i_j \in \{0,1\}$, and so we can denote each such σ with the corresponding n-tuple $(i_0, i_1, \ldots, i_{n-1})$.

Lemma 3.15. *We either have*

$$\begin{gathered} T = \{(0,0,\ldots,0),(0,1,1,0,1,1,\ldots,0,1,1),\\ (1,0,1,1,0,1,\ldots,1,0,1),(1,1,0,1,1,0,\ldots,1,1,0)\}, \end{gathered} \tag{3.2}$$

in which case $3 \mid n$, *or*

$$\begin{gathered} T = \{(0,0,\ldots,0),(0,1,0,1,\ldots,0,1),\\ (1,0,1,0,\ldots,1,0),(1,1,\ldots,1)\}, \end{gathered} \tag{3.3}$$

in which case $2 \mid n$. *In particular,* $\gcd(n,6) \neq 1$.

Proof. Since $\mathrm{Aut}(\mathcal{M})$ is 1-regular it easily follows that $T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $T = \{1, t_1, t_2, t_3\}$. The subgroup N is normal in $\mathrm{Aut}(\Gamma)$, implying that T is normal in $\mathrm{Aut}(\mathcal{M})$. Since $\mathrm{Aut}(\mathcal{M})$ is arc-transitive on Γ and the sets $\{u_i, v_i\}$ are blocks of imprimitivity for $\mathrm{Aut}(\mathcal{M})$, the group $\mathrm{Aut}(\mathcal{M})$ contains elements of the form $\rho\sigma$ and $\mu\sigma'$, where $\sigma, \sigma' \in N$. Since T is normal in $\mathrm{Aut}(\mathcal{M})$ and N is abelian we thus get $T = T^{\sigma\rho^{-1}} = T^{\rho^{-1}}$ and $T = T^{\sigma'\mu^{-1}} = T^{\mu^{-1}}$, implying that $T^\rho = T = T^\mu$.

Now, 1-regularity of $\mathrm{Aut}(\mathcal{M})$ implies that for any $1 \neq t = (i_0, i_1, \ldots, i_{n-1}) \in T$ we cannot have $i_j = i_{j+1} = 0$ for any $j \in \mathbb{Z}_n$. We can thus assume that $t_1 = (0, 1, i_2, i_3, \ldots, i_{n-3}, i_{n-2}, 1)$. Then $t_1^\rho = \rho^{-1}t_1\rho = (1, 0, 1, i_2, i_3, \ldots, i_{n-3}, i_{n-2}) \neq t_1$. We can assume $t_1^\rho = t_2$. Now, $t_2^\rho = (i_{n-2}, 1, 0, 1, i_2, i_3, \ldots, i_{n-3}) \neq t_2$, and so we either have $t_2^\rho = t_3$ (in which case $t_3^\rho = t_1$) or $t_2^\rho = t_1$ (in which case $t_3^\rho = t_3$). It is now clear that in the first case n is divisible by 3 and T is as in (3.2) and in the second case n is even and T is as in (3.3). □

We can now analyze the different possibilities for the faces of our map $\mathcal{M}$. By Proposition 3.1 the graph Γ has three orbits of $\mathrm{Aut}(\Gamma)$-consistent cycles. The representatives of the orbits are $(u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1})$, $(u_0, u_1, \ldots, u_{n-1})$ and (u_0, u_1, v_0, v_1), with shunts, $\rho\sigma_0$, ρ and $\sigma_1\mu\rho$, respectively. Thus, all the $\mathrm{Aut}(\Gamma)$-consistent cycles are clearly $\mathrm{Aut}(\Gamma)$-symmetric. By Lemma 3.10 the possible face lengths for $\mathcal{M}$ are 4, n and $2n$. Moreover, the following holds.

Lemma 3.16. *The map* $\mathcal{M}$ *has faces of two different lengths.*

Proof. Recall that $\mathcal{M}$, being a map of class $2_{\{0,1\}}$, admits a one-step rotation around each of its faces. Thus, by Lemma 3.8, each edge is on the boundary of two different faces of $\mathcal{M}$. Since $(u_i, u_{i+1}, v_i, v_{i+1})$ is clearly the only $\mathrm{Aut}(\Gamma)$-consistent 4-cycle containing any of the corresponding four edges (recall that $n \neq 4$), it is clear that each edge of Γ lies on at least one face of length greater than 4.

Suppose f is a face of $\mathcal{M}$ of length n. Since $|T| = 4$ and f clearly has exactly one vertex of each block of imprimitivity $B_i = \{u_i, v_i\}$ the set $\mathcal{O}_1 = fT = \{ft \mid t \in T\}$ is the orbit of f under the action of $\mathrm{Aut}(\mathcal{M})$. By way of contradiction suppose that the other $\mathrm{Aut}(\mathcal{M})$-orbit of faces of $\mathcal{M}$ also consists of faces of length n and let f' be any face from the second orbit $\mathcal{O}_2 = f'T$. Since f' corresponds to an $\mathrm{Aut}(\Gamma)$-consistent cycle it also contains exactly one vertex from each of the blocks B_i, and so there exists $\sigma \in N$ such that $f' = f\sigma$. But as N is abelian we get $\mathcal{O}_2 = f'T = f\sigma T = fT\sigma = \mathcal{O}_1\sigma$, and so σ interchanges the two orbits of faces of $\mathcal{M}$, implying that it is in fact an automorphism of $\mathcal{M}$. But then $\sigma \in T$, and so $\mathcal{O}_2 = \mathcal{O}_1\sigma = \mathcal{O}_1$, a contradiction.

A similar argument shows that $\mathcal{M}$ also cannot have all faces of length $2n$. $\square$

The analysis of possible maps $\mathcal{M}$ of class $2_{\{0,1\}}$ whose underlying graph is Γ is now straightforward. By Lemma 3.16 we either have an orbit of faces of length n (and either an orbit of faces of length 4 or faces of length $2n$) or one orbit of faces of length $2n$ and one orbit of faces of length 4. Moreover, Corollary 3.11 implies that the map is completely determined once we have chosen one orbit of faces of length n or $2n$ and decided on the length of the faces from the other orbit. Next, in view of the action of $\mathrm{Aut}(\Gamma)$ and the remarks from the paragraph preceding Lemma 3.16 we can assume that, in the case that $\mathcal{M}$ has faces of length n, one of them is $f = (u_0, u_1, \ldots, u_{n-1})$ while in the case it does not have faces of length n one of the faces of length $2n$ is $f = (u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1})$. The corresponding $\mathrm{Aut}(\mathcal{M})$ orbit of f is then completely determined by the action of T, which by Lemma 3.15 is also known (up to the two possibilities). Once the faces have been determined one only needs to check that we indeed have the required automorphisms of the map for Lemma 3.9 to apply. Note that, since the faces are of two different lengths, the obtained map cannot be reflexible, and is thus automatically of class $2_{\{0,1\}}$.

Theorem 3.17. *Let $\Gamma = R_n(2,1)$ be a Rose Window graph with $n \geq 3$. Then Γ is the underlying graph of a map $\mathcal{M}$ of class $2_{\{0,1\}}$ if and only if $\gcd(n,6) \neq 1$. Moreover, letting $n_0 \in \{0,2,3,4,6,8,9,10\}$ be the residue of n modulo 12 the following holds:*

(i) if $n = 4$, then Γ is the underlying graph of exactly one map of class $2_{\{0,1\}}$ with face lengths 4 and 8;

(ii) if $n_0 \in \{3,9\}$, then Γ is the underlying graph of a unique map in class $2_{\{0,1\}}$; the faces are of lengths 4 and n;

(iii) if $n_0 \in \{4,8\}$, then Γ is the underlying graph of two nonisomorphic maps of class $2_{\{0,1\}}$; one has faces of lengths 4 and n and the other has faces of lengths 4 and $2n$;

(iv) if $n_0 \in \{2,10\}$, then Γ is the underlying graph of three nonisomorphic maps of class $2_{\{0,1\}}$; one has faces of lengths 4 and n, one has faces of lengths 4 and $2n$, and one has faces of lengths n and $2n$;

(v) if $n_0 = 0$, then Γ is the underlying graph of three nonisomorphic maps of class $2_{\{0,1\}}$; two have faces of lengths 4 and n, and one has faces of lengths 4 and $2n$;

(vi) if $n_0 = 6$, then Γ is the underlying graph of four nonisomorphic maps of class $2_{\{0,1\}}$; two have faces of lengths 4 and n, one has faces of lengths 4 and $2n$, and one has faces of lengths n and $2n$.

Proof. The case $n = 4$ has been dealt with at the beginning of this section. For the rest of the proof we thus assume $n \neq 4$. By Lemma 3.15 at least one of 2 and 3 must divide n. We now consider the possibilities for the combinations of the lengths of faces of $\mathcal{M}$. We first analyze the possibility that $\mathcal{M}$ has faces of length n. Recall that we can assume that one of the n-faces is $f = (u_0, u_1, \ldots, u_{n-1})$. We now separate the argument for the two possibilities regarding the subgroup T.

Suppose first that T is as in (3.2) and recall that in this case 3 divides n. The four faces of length n are then (see Figure 3.10):

$$\begin{aligned} f &= (u_0, u_1, \ldots, u_{n-1}), \\ ft_1 &= (u_0, v_1, v_2, u_3, v_4, v_5, \ldots, u_{n-3}, v_{n-2}, v_{n-1}), \\ ft_2 &= (v_0, u_1, v_2, v_3, u_4, v_5, \ldots, v_{n-3}, u_{n-2}, v_{n-1}) \text{ and} \\ ft_3 &= (v_0, v_1, u_2, v_3, v_4, u_5, \ldots, v_{n-3}, v_{n-2}, u_{n-1}). \end{aligned}$$

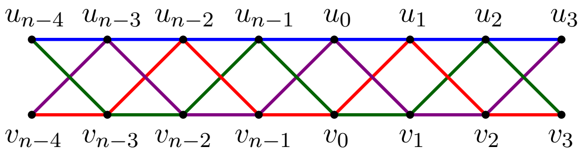


Figure 3.10: The n-faces of $\mathcal{M}$ in the case when T is as in (3.2).

Let f' be the face containing the edge u_0u_1, different from f. If it is not a 4-cycle then the fact that $3 \mid n$ and Corollary 3.11 imply that it is

$$(u_0, u_1, v_2, u_3, u_4, v_5, \ldots, u_{n-3}, u_{n-2}, v_{n-1}),$$

contradicting Lemma 3.16. Thus the non n-faces of $\mathcal{M}$ are the 4-cycles $(u_i, u_{i+1}, v_i, v_{i+1})$, $i \in \mathbb{Z}_n$. It remains to be shown that the resulting map $\mathcal{M}$ is indeed a map of class $2_{\{0,1\}}$. It is clear that $\rho, \mu \in \mathrm{Aut}(\mathcal{M})$. Let Φ be the flag corresponding to the vertex u_0, edge u_0u_1 and the 4-face (u_0, u_1, v_0, v_1). Then $\mu\rho = \alpha_0$, $t_1 = \alpha_1$ and $\mu = \alpha_{212}$, and so Lemma 3.9 implies that $\mathcal{M}$ is a map of class $2_{\{0,1\}}$.

Suppose now that T is as in (3.3) and recall that in this case n is even. The four faces of length n are then (see Figure 3.11):

$$\begin{aligned} f &= (u_0, u_1, \ldots, u_{n-1}), \\ ft_1 &= (u_0, v_1, u_2, v_3, \ldots, u_{n-2}, v_{n-1}), \\ ft_2 &= (v_0, u_1, v_2, u_3, \ldots, v_{n-2}, u_{n-1}) \text{ and} \\ ft_3 &= (v_0, v_1, \ldots, v_{n-1}). \end{aligned}$$

Let f' be the face containing the edge u_0u_1, different from f. If it is not a 4-face then its boundary contains the path $(u_0, u_1, v_2, v_3, u_4, u_5, \ldots, u_{n-3}, v_{n-2}, v_{n-1})$, and so Lemma 3.16 implies that $n \equiv 2 \pmod 4$. The two $2n$-faces are thus

$$\begin{aligned} f' &= (u_0, u_1, v_2, v_3, \ldots, u_{n-2}, u_{n-1}, v_0, v_1, \ldots, u_{n-4}, u_{n-3}, v_{n-2}, v_{n-1}) \text{ and} \\ f't_1 &= (u_0, v_1, v_2, u_3, u_4, \ldots, v_{n-1}, v_0, u_1, u_2, \ldots, v_{n-3}, v_{n-2}, u_{n-1}). \end{aligned}$$

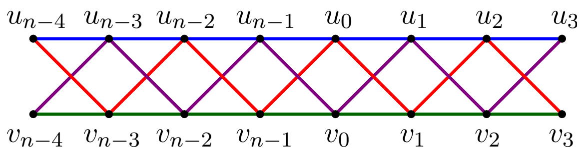


Figure 3.11: The faces of length n in the case that T is as in (3.3).

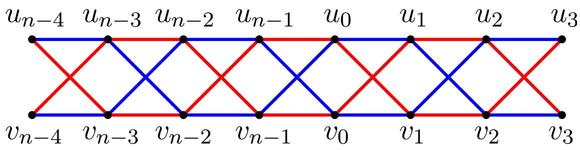


Figure 3.12: The $2n$-faces of the map that has $2n$-faces and n-faces.

Again (see Figure 3.12), $\rho, \mu \in \mathrm{Aut}(\mathcal{M})$. Letting Φ be the flag corresponding to the vertex u_0, edge u_0u_1 and the corresponding n-face, it is clear that $\mu\rho = \alpha_0$, $\mu = \alpha_1$ and $\mu t_1 = \alpha_{212}$, and so Lemma 3.9 implies that $\mathcal{M}$ is a map of class $2_{\{0,1\}}$. If however f' is a 4-face, then we get a map $\mathcal{M}$ with $\rho, \mu \in \mathrm{Aut}(\mathcal{M})$ and $\mu\rho = \alpha_0$, $t_1 = \alpha_1$ and $\mu = \alpha_{212}$, where Φ is the flag corresponding to the vertex u_0, edge u_0u_1 and the 4-face (u_0, u_1, v_0, v_1). Thus $\mathcal{M}$ is again a map of class $2_{\{0,1\}}$.

The case when $\mathcal{M}$ has faces of lengths 4 and $2n$ can be dealt with in a similar way. Letting $f = (u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1})$ be one of the faces of length $2n$ it is easy to see that Lemma 3.4 forces T to be as in (3.3), and so n is even. This time $\rho\sigma_0, \mu t_3\sigma_0 \in \mathrm{Aut}(\mathcal{M})$, and consequently $\mu t_3\sigma_0\rho\sigma_0 = \alpha_0$, $t_1 = \alpha_1$ and $\mu t_3\sigma_0 = \alpha_{212}$, where Φ is the flag corresponding to the vertex u_0, edge u_0u_1 and the 4-face (u_0, u_1, v_0, v_1). We therefore get a map of class $2_{\{0,1\}}$. Details are left to the reader.

To prove that all the obtained maps are pairwise nonisomorphic observe that this is clearly true if the two maps under consideration have faces of different lengths. As for the maps from items (v) and (vi) of the Theorem note that the maps with faces of lengths 4 and n corresponding to the case when T is as in (3.2) are such that a face of length n meets all other three faces of length n while in the case when T is as in (3.3) this does not hold, proving that the corresponding maps cannot be isomorphic. □

3.5.2 Family (ii)

We now consider the second family of arc-transitive Rose Window graphs, namely, the graphs of the form $R_{2n}(n+2, n+1)$. Using a suitable computer package one can verify that the graph $R_6(5,4)$ is the underlying graph of three maps of class $2_{\{0,1\}}$. Their face lengths are 3 and 4, 3 and 6 and 4 and 6, respectively. Similarly, the graph $R_8(6,5)$ is the underlying graph of exactly one map of class $2_{\{0,1\}}$. Its face lengths are 4 and 8. For the rest of this section we can thus assume $n \geq 5$.

The graph $\Gamma = R_{2n}(n+2, n+1)$ is isomorphic to the Praeger-Xu graph $C(2, n, 2)$

([51]; see also [59, Section 5]). For convenience we relabel the vertices of Γ by setting:

$$u_i = \begin{cases} x_i & ; \ 0 \le i \le n-2 \\ y_{n-2} & ; \ i = n-1 \end{cases}, \quad v_i = \begin{cases} y_{i-1} & ; \ 1 \le i \le n-2 \\ y_{2n-1} & ; \ i = 0 \\ x_{n-1} & ; \ i = n-1 \end{cases},$$

$$w_i = \begin{cases} y_{n+i-1} & ; \ 0 \le i \le n-2 \\ x_{2n-1} & ; \ i = n-1 \end{cases} \text{ and } \quad z_i = \begin{cases} x_{n+i} & ; \ 0 \le i \le n-2 \\ y_{2n-2} & ; \ i = n-1. \end{cases}$$

Note that now the indices for u_i, v_i, w_i and z_i can be taken in $\mathbb{Z}_n$, and we do so.

A presentation of Γ with respect to this relabeling, which we will be relying on in the reminder of this section, is given in Figure 3.13. The rim edges are colored red, the hub edges yellow, the in-spokes green and the out-spokes blue.

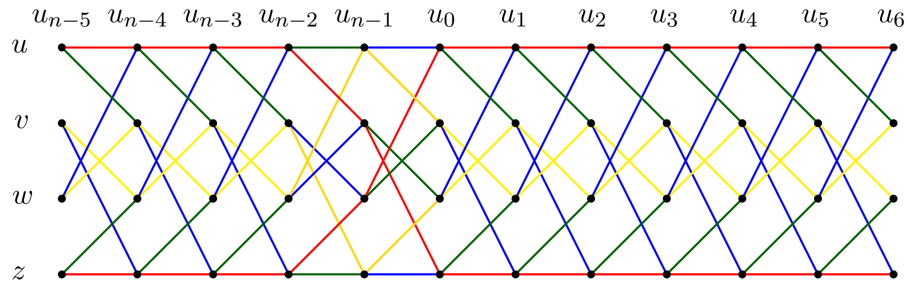


Figure 3.13: Relabeling of the vertices of the graph $R_{2n}(n+2, n+1)$.

Let us define $\sigma_i = (u_i, v_i)(w_i, z_i)(u_{i+1}, w_{i+1})(v_{i+1}, z_{i+1})$ for each $i \in \{0, 1, \ldots, n-1\}$, where the indices are taken in $\mathbb{Z}_n$. It is clear that the σ_i are automorphisms of Γ. (The σ_i correspond to the permutations ϵ_i of [28] and σ_i of [59]). Note that, for each i, $\sigma_i\sigma_{i+1} = \sigma_{i+1}\sigma_i$, and so $\sigma_i\sigma_j = \sigma_j\sigma_i$ holds for every $i, j \in \{0, 1, \ldots, n-1\}$. We denote the (elementary abelian) group generated by $\sigma_0, \ldots, \sigma_{n-1}$ by N.

Now, set $\alpha = \rho\sigma_{n-1}$ and observe that

$$\alpha = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})(w_0, w_1, \ldots, w_{n-1})(z_0, z_1, \ldots, z_{n-1}), \quad (3.4)$$

and thus, $\sigma_i = \sigma_0^{\alpha^i}$. This shows that the n-cycles $(u_0, u_1, \ldots, u_{n-1})$ and $(z_0, z_1, \ldots, z_{n-1})$ are $\mathrm{Aut}(\Gamma)$-consistent cycles of Γ. Next, let

$$\beta = \left(\prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (u_i, u_{n-i})(z_i, z_{n-i}) \right) (v_0, w_0) \left(\prod_{i=1}^{n-1} (v_i, w_{n-i}) \right). \quad (3.5)$$

We can think of β being the "twisted" reflection with respect to the "line through u_0 and z_0" in Figure 3.13, which interchanges the roles of the v_i and w_j vertices. It is easy to verify that $\eta = \sigma_1\sigma_2 \cdots \sigma_{n-2}\beta$, and so $\beta \in \mathrm{Aut}(\Gamma)$. Moreover, $\mathrm{Aut}(\Gamma) = \langle \rho, \mu, \sigma_0 \rangle = \langle \alpha, \beta, \sigma_0 \rangle$ (see also [59]).

The graph Γ has several blocks of imprimitivity for the action of its automorphism group. For instance, one can verify that the sets $B_i = \{u_i, v_i, w_i, z_i\}$ are blocks for each i (recall that $n > 4$). Consequently, the subsets $\{u_i, z_i\}$ and $\{v_i, w_i\}$ are also blocks for $\mathrm{Aut}(\Gamma)$ (since these are the only 2-subsets of vertices of B_i which do not lie on a common 4-cycle).

For the rest of this section we assume that Γ is the underlying graph of a map $\mathcal{M}$ of type $2_{\{0,1\}}$ and we let $T = N \cap \mathrm{Aut}(\mathcal{M})$. Since $\mathrm{Aut}(\mathcal{M})$ is 1-regular it follows that $|T| = 8$. Moreover, as in the case of Family (i) T is normal in $\mathrm{Aut}(\mathcal{M})$ and 1-regularity of $\mathrm{Aut}(\mathcal{M})$ implies that no nontrivial element of T fixes an arc of Γ. Observe that since each element of N fixes each B_i setwise any 2-element subset of B_i is a block of imprimitivity for the restriction of the action of N (and thus T) on B_i. This proves that any $t \in T$ fixing a vertex of B_i must fix B_i pointwise. Thus, an element $t \in T$ has one of the following actions on the block B_i:

$$1, \quad (u_i, v_i)(w_i, z_i), \quad (u_i, w_i)(v_i, z_i), \text{ or } \quad (u_i, z_i)(v_i, w_i). \tag{3.6}$$

For $t \in T$ we shall write $t = (j_0, j_1, \ldots, j_{n-1})$, where j_i is equal to 0, 1, 2 or 3, depending on whether the action of t on B_i is trivial, or is $(u_i, v_i)(w_i, z_i)$, $(u_i, w_i)(v_i, z_i)$ or $(u_i, z_i)(v_i, w_i)$, respectively. We can now determine all the possibilities for the subgroup T.

Lemma 3.18. *We either have*

$$\begin{gathered} T = \{(0,0,\ldots,0), (0,1,2,0,1,2,\ldots,0,1,2), \\ (2,0,1,2,0,1,\ldots,2,0,1), (1,2,0,1,2,0,\ldots,1,2,0), \\ (2,1,3,2,1,3,\ldots,2,1,3), (3,2,1,3,2,1,\ldots,3,2,1), \\ (1,3,2,1,3,2,\ldots,1,3,2), (3,3,\ldots,3)\}, \end{gathered} \tag{3.7}$$

in which case $3 \mid n$*, or*

$$\begin{gathered} T = \{(0,0,\ldots,0), (0,1,3,2,0,1,3,2,\ldots,0,1,3,2), \\ (2,0,1,3,2,0,1,3,\ldots,2,0,1,3), (3,2,0,1,3,2,0,1,\ldots,3,2,0,1), \\ (1,3,2,0,1,3,2,0,\ldots,1,3,2,0), (2,1,2,1,\ldots,2,1), \\ (1,2,1,2,\ldots,1,2), (3,3,\ldots,3)\}, \end{gathered} \tag{3.8}$$

in which case $4 \mid n$*. In particular,* $\gcd(n, 12) \neq 1$*.*

Proof. As in the proof of Lemma 3.15 we can show that the fact that T is a normal subgroup of $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{M})$ is 1-regular implies that $T^\alpha = T$ and $T^\beta = T$. That is, given $t \in T$, both t^α and t^β are in T. Note that if $t = (j_0, j_1, \ldots, j_{n-1})$, then $t^\alpha = (j_{n-1}, j_0, \ldots, j_{n-1})$, and $t^\beta = (k_0, k_1, k_2, \ldots, k_{n-1})$, where $k_i = j_{n-i}$ whenever $j_{n-i} \in \{0, 3\}$, $k_i = 2$ if $j_{n-i} = 1$ and $k_i = 1$ if $j_{n-i} = 2$.

Consider the block B_i and let $t \in T$. Observe that if $j_i = 0$, since the common neighbours of u_i and w_i are u_{i+1} and v_{i+1}, then $j_{i+1} \in \{0, 1\}$. Analogously if $j_i = 2$, then $j_{i+1} \in \{0, 1\}$, and if $j_i \in \{1, 3\}$, then $j_{i+1} \in \{2, 3\}$. We also note that 1-regularity of $\mathrm{Aut}(\mathcal{M})$ implies that the identity is the only element of T such that $j_i = j_{i+1} = 0$ holds for some i. In particular, for a non-identity $t \in T$ every 0 must be followed by a 1 and be preceded by a 2. Similarly, for any $t = (j_0, j_1, \ldots, j_{n-1})$ and $t' = (k_0, k_1, \ldots, k_{n-1})$ in T if for some i, $j_i = k_i$ and $j_{i+1} = k_{i+1}$, then $t = t'$.

Let $T = \{1, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$. Without loss of generality let

$$t_1 = (0, 1, i_2, i_3, \ldots, i_{n-2}, 2)$$

and $t_2 = t_1^\alpha$. Recall that $i_2 \in \{2,3\}$. We consider each of the two cases separately.

Case 1: $i_2 = 2$.
Then $t_1 = (0,1,2,i_3,\ldots,i_{n-2},2)$ and $t_1^\beta = (0,1,k_2,k_3,\ldots,k_{n-3},1,2)$, and so the above remarks imply that $t_1^\beta = t_1 = (0,1,2,i_3,\ldots,i_{n-3},1,2)$. Then $t_1^{\alpha^3} = (i_{n-3},1,2,0,1,2,i_3,\ldots,i_{n-4})$, and so $t_1^{\alpha^3} = t_1$, implying that n is divisible by 3 and $t_1 = (0,1,2,0,1,2,\ldots,0,1,2)$. The subgroup T is now completely determined:

$$\begin{array}{rcccl} & & t_1 & = & (0,1,2,0,1,2,\ldots,0,1,2),\\ t_1^\alpha & = & t_2 & = & (2,0,1,2,0,1,\ldots,2,0,1),\\ t_2^\alpha & = & t_3 & = & (1,2,0,1,2,0,\ldots,1,2,0),\\ t_1t_2 & = & t_4 & = & (2,1,3,2,1,3,\ldots,2,1,3),\\ t_4^\alpha & = & t_5 & = & (3,2,1,3,2,1,\ldots,3,2,1),\\ t_5^\alpha & = & t_6 & = & (1,3,2,1,3,2,\ldots,1,3,2),\\ t_3t_4 & = & t_7 & = & (3,3,3,3,3,3,\ldots,3,3,3). \end{array}$$

Case 2: $i_2 = 3$.
In this case we have $t_1 = (0,1,3,i_3,i_4,\ldots,i_{n-2},2)$. As before $t_1^\beta = t_1$, and so $t_1 = (0,1,3,i_3,\ldots,i_{n-3},3,2)$. Set $t_2 = t_1^\alpha = (2,0,1,3,i_3,\ldots,i_{n-3},3)$ and $t_3 = t_1^{\alpha^2} = (3,2,0,1,3,i_3,\ldots,i_{n-3})$. Then $t = t_1t_2t_3 = (1,3,2,j_3,j_4,\ldots,j_{n-1})$. Since $t_4 = t_1^{\alpha^3} = (i_{n-3},3,2,0,1,3,i_3,\ldots,i_{n-4})$ it follows that $i_{n-3} = 1$, and so $t_1^{\alpha^4} = t_1$, implying that n is divisible by 4 and $t_1 = (0,1,3,2,0,1,3,2,\ldots,0,1,3,2)$. The subgroup T is now completely determined:

$$\begin{array}{rcccl} & & t_1 & = & (0,1,3,2,0,1,3,2,\ldots,0,1,3,2),\\ t_1^\alpha & = & t_2 & = & (2,0,1,3,2,0,1,3,\ldots,2,0,1,3),\\ t_2^\alpha & = & t_3 & = & (3,2,0,1,3,2,0,1,\ldots,3,2,0,1),\\ t_3^\alpha & = & t_4 & = & (1,3,2,0,1,3,2,0,\ldots,1,3,2,0),\\ t_1t_2 & = & t_5 & = & (2,1,2,1,2,1,2,1,\ldots,2,1,2,1),\\ t_5^\alpha & = & t_6 & = & (1,2,1,2,1,2,1,2,\ldots,1,2,1,2),\\ t_1t_3 & = & t_7 & = & (3,3,3,3,3,3,3,3,\ldots,3,3,3,3). \end{array}$$

□

Recall that, by Lemma 3.10, the boundaries of faces of a map $\mathcal{M}$ of class $2_{\{0,1\}}$ are $\mathrm{Aut}(\mathcal{M})$-consistent cycles. By Proposition 3.1 the graph Γ has three orbits of $\mathrm{Aut}(\Gamma)$-consistent cycles. Since $\sigma_1\beta\alpha, \alpha, \rho \in \mathrm{Aut}(\Gamma)$ are shunts for the cycles

$$(u_0,u_1,w_0,v_1),\ (u_0,u_1,u_2,\ldots,u_{n-1}) \text{ and}$$

$$(u_0,u_1,\ldots,u_{n-3},u_{n-2},v_{n-1},z_0,z_1,\ldots,z_{n-3},z_{n-2},w_{n-1}),$$

respectively, these three cycles are representatives of the three orbits of $\mathrm{Aut}(\Gamma)$-consistent cycles. Therefore, the $\mathrm{Aut}(\Gamma)$-consistent cycles are of lengths 4, n and $2n$, implying that these are the only possible lengths of faces of $\mathcal{M}$. We remark that $n \neq 4$ implies that each edge of Γ is in exactly one 4-cycle. The proof of the following lemma is similar to that of Lemma 3.16 and is left to the reader (it again relies on the fact that any two $\mathrm{Aut}(\Gamma)$-consistent cycles of length n or $2n$ are permutable by an element of the subgroup N).

Lemma 3.19. *The map $\mathcal{M}$ has faces of two different lengths.*

We can now describe all maps $\mathcal{M}$ of type $2_{\{0,1\}}$ whose underlying graph is Γ.

Theorem 3.20. *Let $\Gamma = R_{2n}(n+2, n+1)$ be a Rose Window graph with $n \geq 3$. Then Γ is the underlying graph of a map $\mathcal{M}$ of class $2_{\{0,1\}}$ if and only if $\gcd(n, 12) > 2$. Moreover, letting $n_0 \in \{0, 3, 4, 6, 8, 9\}$ be the residue of n modulo 12, the following holds:*

(i) if $n = 4$, then Γ is the underlying graph of a unique map of class $2_{\{0,1\}}$ with face lengths 4 and 8;

(ii) if $n_0 \in \{3, 9\}$, then Γ is the underlying graph of three nonisomorphic maps of class $2_{\{0,1\}}$; one has faces of lengths 4 and n, one has faces of lengths 4 and $2n$, and one has faces of lengths n and $2n$;

(iii) if $n_0 \in \{4, 6, 8\}$, then Γ is the underlying graph of two nonisomorphic maps of class $2_{\{0,1\}}$; one has faces of lengths 4 and n, while the other has faces of lengths 4 and $2n$;

(iv) if $n_0 = 0$, then Γ is the underlying graph of four nonisomorphic maps of class $2_{\{0,1\}}$; two have faces of lengths 4 and n, and two has faces of lengths 4 and $2n$.

Proof. The cases $n = 3$ and $n = 4$ have been dealt with at the beginning of this section. For the rest of the proof we can thus assume $n > 4$. We distinguish the cases depending on whether $\mathcal{M}$ has n-faces or not.

Case 1: $\mathcal{M}$ has an orbit of n-faces.
Without loss of generality we can assume that one of the n-faces of $\mathcal{M}$ is $f = (u_0, u_1, \ldots, u_{n-1})$. As in the proof of Theorem 3.17, we deal with the two possibilities for T separately.

Subcase 1.1: T is as in (3.7), in which case 3 divides n.
By Lemma 3.18 the eight n-faces of $\mathcal{M}$ are as represented in Figure 3.14, that is:

$$\begin{array}{lllll}
f_0 & = & f & = & (u_0, u_1, \ldots, u_{n-1}), \\
f_1 & = & ft_1 & = & (u_0, v_1, w_2, u_3, v_4, w_5, \ldots, u_{n-3}, v_{n-2}, w_{n-1}), \\
f_2 & = & ft_2 & = & (w_0, u_1, v_2, w_3, u_4, v_5, \ldots, w_{n-3}, u_{n-2}, v_{n-1}), \\
f_3 & = & ft_3 & = & (v_0, w_1, u_2, v_3, w_4, u_5, \ldots, v_{n-3}, w_{n-2}, u_{n-1}), \\
f_4 & = & ft_4 & = & (w_0, v_1, z_2, w_3, v_4, z_5, \ldots, w_{n-3}, v_{n-2}, z_{n-1}), \\
f_5 & = & ft_5 & = & (z_0, w_1, v_2, z_3, w_4, v_5, \ldots, z_{n-3}, w_{n-2}, v_{n-1}), \\
f_6 & = & ft_6 & = & (v_0, z_1, w_2, v_3, z_4, w_5, \ldots, v_{n-3}, z_{n-2}, w_{n-1}), \\
f_7 & = & ft_7 & = & (z_0, z_1, \ldots, z_{n-3}, z_{n-1}).
\end{array} \tag{3.9}$$

The automorphism $\beta \in \mathrm{Aut}(\Gamma)$ from (3.5) preserves the set of n-cycles (3.9) since it fixes each of the faces f_0, f_1, f_5 and f_7, and interchanges f_2 with f_3 and f_4 with f_6. Since the other $\mathrm{Aut}(\mathcal{M})$-orbit of faces of $\mathcal{M}$ either consists of 4-faces or $2n$-faces and each edge of Γ lies on a unique 4-cycle (recall that $n > 4$), while Lemma 3.10 and Corollary 3.11 imply that the $2n$-faces are uniquely determined by the n-faces, the corresponding maps are completely determined (once we have decided for the

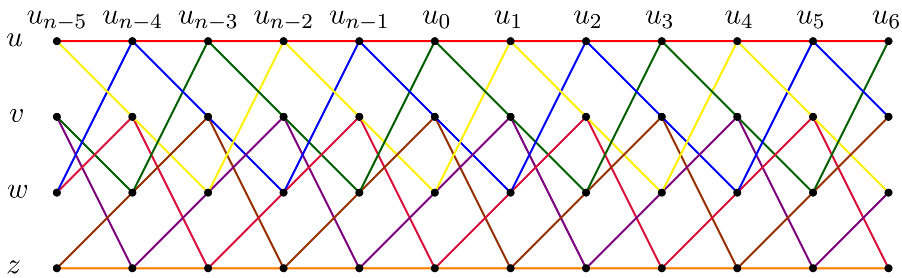


Figure 3.14: The n-faces of $\mathcal{M}$ in the case when T is as in (3.7).

length of the faces of the other $\mathrm{Aut}(\mathcal{M})$-orbit). Moreover, since the automorphism β preserves the set of the eight n-faces, it in fact follows that $\beta \in \mathrm{Aut}(\mathcal{M})$. We thus only need to check if the resulting maps are indeed of class $2_{\{0,1\}}$.

Suppose first that $\mathcal{M}$ has 4-faces (and the n-faces from (3.9)). We show that $\mathcal{M}$ is in class $2_{\{0,1\}}$ by exhibiting the automorphisms α_0, α_1 and α_{212}, with respect to some base flag, so that we can apply Lemma 3.9. Let Φ be the flag of $\mathcal{M}$ containing the vertex u_0, the edge u_0u_1 and the n-face f_0. Then $\Phi\beta\alpha = \Phi^0$, $\Phi\beta = \Phi^1$ and $\Phi t_1 = \Phi^{2,1,2}$, and so $\mathcal{M}$ is of class $2_{\{0,1\}}$ by Lemma 3.9.

Suppose now that $\mathcal{M}$ has $2n$-faces (and the n-faces from (3.9)). By the above remarks the $2n$-faces are completely determined. In fact, n has to be odd for this to be possible (that is $n \equiv 3 \pmod 6$) and in this case the four $2n$-faces are:

$$\begin{array}{l}
(u_0, u_1, v_2, z_3, z_4, w_5, \ldots, u_{n-3}, u_{n-2}, v_{n-1}, z_0, z_1, w_2, \ldots, u_{n-6}, u_{n-5}, v_{n-4}, z_{n-3}, z_{n-2}, w_{n-1}),\\
(u_0, v_1, z_2, z_3, w_4, u_5, \ldots, u_{n-3}, v_{n-2}, z_{n-1}, z_0, w_1, u_2, \ldots, u_{n-6}, v_{n-5}, z_{n-4}, z_{n-3}, w_{n-2}, u_{n-1}),\\
(v_0, z_1, z_2, w_3, u_4, u_5, \ldots, v_{n-3}, z_{n-2}, z_{n-1}, w_0, u_1, u_2, \ldots, v_{n-6}, z_{n-5}, z_{n-4}, w_{n-3}, u_{n-2}, u_{n-1}),\\
(v_0, w_1, v_2, w_3, \ldots, v_{n-1}, w_0, v_1, w_2, \ldots, v_{n-2}, w_{n-1}).
\end{array}$$

Again let Φ be the flag corresponding to the vertex u_0, edge u_0u_1 and the face f_0. As before we get $\Phi\beta\alpha = \Phi^0$, $\Phi\beta = \Phi^1$, while this time $\Phi t_1\beta = \Phi^{2,1,2}$, and so we can again apply Lemma 3.9 to show that $\mathcal{M}$ is of class $2_{\{0,1\}}$.

Subcase 1.2: T is as in (3.8), in which case 4 divides n.
By Lemma 3.18 the eight n-faces of $\mathcal{M}$ are as represented in Figure 3.15, that is:

$$\begin{array}{lllll}
f_0 & = & f & = & (u_0, u_1, \ldots, u_{n-1}),\\
f_1 & = & ft_1 & = & (u_0, v_1, z_2, w_3, u_4, v_5, z_6, w_7, \ldots, u_{n-4}, v_{n-3}, z_{n-2}, w_{n-1}),\\
f_2 & = & ft_2 & = & (w_0, u_1, v_2, z_3, w_4, u_5, v_6, z_7, \ldots, w_{n-4}, u_{n-3}, v_{n-2}, z_{n-1}),\\
f_3 & = & ft_3 & = & (z_0, w_1, u_2, v_3, z_4, w_5, u_6, v_7, \ldots, z_{n-4}, w_{n-3}, u_{n-2}, v_{n-1}),\\
f_4 & = & ft_4 & = & (v_0, z_1, w_2, u_3, v_4, z_5, w_6, u_7, \ldots, v_{n-4}, z_{n-3}, w_{n-2}, u_{n-1}),\\
f_5 & = & ft_5 & = & (w_0, v_1, w_2, v_3, \ldots, w_{n-2}, v_{n-1}),\\
f_6 & = & ft_6 & = & (v_0, w_1, v_2, w_3, \ldots, v_{n-2}, w_{n-1}),\\
f_7 & = & ft_7 & = & (z_0, z_1, \ldots, z_{n-1}).
\end{array} \tag{3.10}$$

Again, β from (3.5) preserves the set of eight n-cycles (3.10), implying that $\beta \in \mathrm{Aut}(\mathcal{M})$. We show that the other faces of $\mathcal{M}$ must be of length 4. Namely, if this was not the case then, by Corollary 3.11, the other face containing the edge

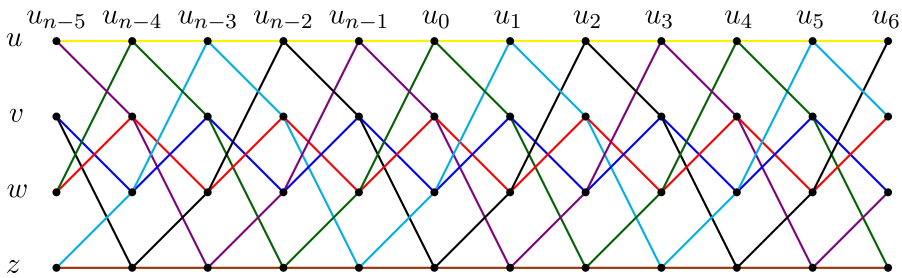


Figure 3.15: The n-faces of $\mathcal{M}$ in the case when T is as in (3.8).

u_0u_1 would have to be

$$(u_0, u_1, v_2, w_3, u_4, u_5, v_6, w_7, \ldots, u_{n-4}, u_{n-3}, v_{n-2}, w_{n-1}),$$

which is of length n, contradicting Lemma 3.19. Thus $\mathcal{M}$ contains 4-faces (and the n-faces from (3.10)). Let Φ be the flag of $\mathcal{M}$ containing the vertex u_0, the edge u_0u_1 and the n-face f_0, and observe that $\Phi\beta = \Phi^1$, $\Phi\beta\alpha = \Phi^0$ and $\Phi t_1 = \Phi^{2,1,2}$. Thus Lemma 3.9 implies that $\mathcal{M}$ is in class $2_{\{0,1\}}$.

Case 2: $\mathcal{M}$ has faces of lengths 4 and $2n$.
Recall that any automorphism of Γ preserves the set of 4-cycles of Γ, and so an automorphism of Γ is an automorphism of $\mathcal{M}$ if and only if it preserves the set of $2n$-faces. Without loss of generality we can assume that

$$f = (u_0, u_1, \ldots, u_{n-2}, v_{n-1}, z_0, z_1, \ldots, z_{n-2}, w_{n-1})$$

is one of the faces of $\mathcal{M}$. Note that $f\mu = f$ and $f\rho = f$ (in fact, ρ is a shunt for f). Since μ and ρ both normalize the subgroup T from Lemma 3.18, it follows that $\mu, \rho \in \mathrm{Aut}(\mathcal{M})$. We again distinguish the two possibilities for the subgroup T.

Subcase 2.1: T is as in (3.7), in which case 3 divides n.
The $2n$-faces are then (see Figure 3.16):

$$\begin{aligned}
f &= (u_0, u_1, u_2, \ldots, u_{n-2}, v_{n-1}, z_0, z_1, \ldots, z_{n-2}, w_{n-1}),\\
ft_1 &= (u_0, v_1, w_2, u_3, v_4, w_5, \ldots, v_{n-2}, z_{n-1}, z_0, w_1, v_2, z_3, \ldots, w_{n-2}, u_{n-1}),\\
ft_2 &= (w_0, u_1, v_2, w_3, u_4, v_5, \ldots u_{n-2}, u_{n-1}, v_0, z_1, w_2, v_3, z_4 \ldots, z_{n-2}, z_{n-1}),\\
ft_3 &= (v_0, w_1, u_2, v_3, w_4, u_5, \ldots, w_{n-2}, v_{n-1}, w_0, v_1, z_2, w_3, v_4, \ldots, v_{n-2}, w_{n-1}).
\end{aligned}$$

Let Φ be the flag of $\mathcal{M}$ containing the vertex u_0, the edge u_0w_{n-1} and the $2n$-face f. Then $\Phi\rho\mu = \Phi^0$, $\Phi\mu = \Phi^1$ and $\Phi t_1 = \Phi^{2,1,2}$, and so Lemma 3.9 implies that $\mathcal{M}$ is of class $2_{\{0,1\}}$.

Subcase 2.2: T is as in (3.8), in which case 4 divides n.
The $2n$-faces are then (see Figure 3.17):

$$\begin{aligned}
f &= (u_0, u_1, u_2, \ldots, u_{n-2}, v_{n-1}, z_0, z_1, \ldots, z_{n-2}, w_{n-1}),\\
ft_1 &= (u_0, v_1, z_2, w_3, u_4, v_5, \ldots, z_{n-2}, z_{n-1}, z_0, w_1, u_2, v_3, \ldots, u_{n-2}, u_{n-1}),\\
ft_2 &= (w_0, u_1, v_2, z_3, w_4, u_5, \ldots v_{n-2}, w_{n-1}, v_0, z_1, w_2, u_3, v_4 \ldots, w_{n-2}, v_{n-1}),\\
ft_3 &= (v_0, w_1, v_2, w_3, v_4, w_5, \ldots, v_{n-2}, z_{n-1}, w_0, v_1, w_2, v_3, w_4, \ldots, w_{n-2}, u_{n-1}).
\end{aligned}$$

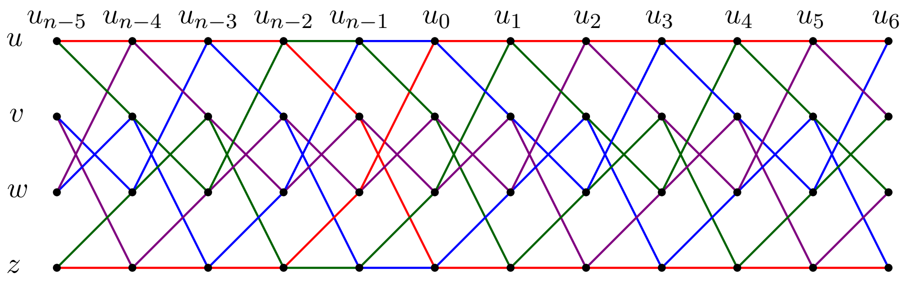


Figure 3.16: The $2n$-faces of $\mathcal{M}$ in the case when T is as in (3.7).

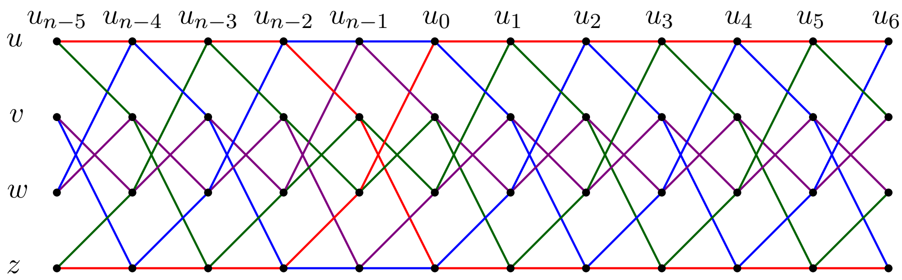


Figure 3.17: The $2n$-faces of $\mathcal{M}$ in the case when T is as in (3.8).

Letting Φ be as above we again find that $\Phi\rho\mu = \Phi^0$, $\Phi\mu = \Phi^1$ and $\Phi t_1 = \Phi^{2,1,2}$ (note however, that the t_1 now differs from the one in the previous paragraph). Thus Lemma 3.9 implies that $\mathcal{M}$ is a map of class $2_{\{0,1\}}$.

The proof that all of the obtained maps are pairwise nonisomorphic is similar to the one in the proof of Theorem 3.17. $\square$

3.5.3 Family (iii)

Combining together the results of [28] and [59] with Corollary 3.13 the classification of maps of class $2_{\{0,1\}}$ whose underlying graphs belong to family (iii) from Proposition 3.14 is straightforward.

Theorem 3.21. *Let $\Gamma = R_{2m}(2b, r)$, where $b^2 \equiv \pm 1 \pmod m$ and either $r = 1$, or $r = m-1$ with m even, be such that Γ does not belong to any of the families (i) and (ii) from Proposition 3.14. Then Γ is the underlying graph of a map of class $2_{\{0,1\}}$ if and only if $b^2 \equiv 1 \pmod m$ in which case there are exactly three pairwise nonisomorphic such maps.*

Proof. By [28, Proposition 3.8] the automorphism group of Γ is 1-regular and is generated by ρ, μ and σ, where ρ and μ are as in (3.1) and σ is as in [59, page 16]. It was pointed out in [59] that in the case of $b^2 \equiv -1 \pmod m$ the vertex stabilizers are cyclic (and thus isomorphic to $\mathbb{Z}_4$) while in the case of $b^2 \equiv 1 \pmod m$ they are isomorphic to the Klein 4-group. We can thus apply Corollary 3.13. $\square$

3.5.4 Family (iv)

The fourth family of edge-transitive Rose Window graphs consists of the graphs $R_{12m}(3m+2, 3m-1)$ and $R_{12m}(3m-2, 3m+1)$, where $m \geq 1$. As in [59], using the fact that $R_n(a, r) \cong R_n(-a, r) \cong R_n(a, -r)$, we can denote these graphs as $R_{12m}(3d+2, 9d+1)$, where $d = m$ or $d = -m$ (modulo $12m$). In [59] the following automorphism σ of $\Gamma = R_{12m}(3d+2, 9d+1)$ has been identified (recall that $a = 3d+2$):

$$x_i^\sigma = \begin{cases} x_i & ; \ i \equiv 0 \pmod 3 \\ y_{i-1} & ; \ i \equiv 1 \pmod 3 \\ y_{i+1-a} & ; \ i \equiv 2 \pmod 3 \end{cases} \quad \text{and } y_i^\sigma = \begin{cases} x_{i+1} & ; \ i \equiv 0 \pmod 3 \\ x_{i-1+a} & ; \ i \equiv 1 \pmod 3 \\ y_{i+6d} & ; \ i \equiv 2 \pmod 3. \end{cases}$$

Moreover, it was shown that whenever $m \equiv 2 \pmod 4$, setting $b = d+1$, an additional automorphism τ of Γ exists:

$$x_i^\tau = \begin{cases} x_{bi} & ; \ i \equiv 0 \pmod 3 \\ y_{bi-b} & ; \ i \equiv 1 \pmod 3 \\ x_{bi+b-1} & ; \ i \equiv 2 \pmod 3 \end{cases} \quad \text{and } y_i^\tau = \begin{cases} x_{bi+1} & ; \ i \equiv 0 \pmod 3 \\ y_{4+bi-4b} & ; \ i \equiv 1 \pmod 3 \\ y_{bi+b-1} & ; \ i \equiv 2 \pmod 3. \end{cases}$$

Observe that $a = 3b-1$, $r = 4-3b$ and $3b^2 \equiv 3 \pmod{12m}$, and so $a \equiv 2 \pmod 3$ and $r \equiv 1 \pmod 3$. It was shown in [28] that $\mathrm{Aut}(\Gamma) = \langle \rho, \mu, \sigma, \tau \rangle$, whenever $m \equiv 2 \pmod 4$, and $\mathrm{Aut}(\Gamma) = \langle \rho, \mu, \sigma \rangle$ otherwise, where ρ and μ are as in (3.1). This enables us to classify the maps of class $2_{\{0,1\}}$ with underlying graphs from family (iv).

Theorem 3.22. *Let $\Gamma = R_{12m}(3d+2, 9d+1)$ be a Rose Window graph, where $d = m$ or $d = 11m$. Then*

(i) if $m \not\equiv 2 \pmod 4$, Γ is the underlying graph of exactly three nonisomorphic maps of class $2_{\{0,1\}}$,

(ii) if $m \equiv 2 \pmod 4$, Γ is the underlying graph of exactly two nonisomorphic maps of class $2_{\{0,1\}}$.

Proof. We first deal with the case when $m \not\equiv 2 \pmod 4$. By [28, Proposition 3.5] the automorphism group $\mathrm{Aut}(\Gamma)$ is 1-regular in this case and is isomorphic to $\langle \rho, \mu, \sigma \rangle$. Note that $\mathrm{Aut}(\Gamma)_{x_0} = \langle \sigma, \mu \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and so the vertex stabilizers in $\mathrm{Aut}(\Gamma)$ are isomorphic to the Klein 4-group. We can thus apply Corollary 3.13 to prove that Γ is the underlying graph of three pairwise nonisomorphic maps of class $2_{\{0,1\}}$.

For the rest of the proof we can thus assume that $m \equiv 2 \pmod 4$. Recall that in this case $\mathrm{Aut}(\Gamma) = \langle \rho, \mu, \sigma, \tau \rangle$ is arc-transitive with vertex-stabilizers of order 8. In particular, $\mathrm{Aut}(\Gamma)_{x_0} = \langle \mu, \sigma, \tau \rangle \cong D_4$, the dihedral group of order 8. It is easy to see that σ commutes with both μ and τ, while $\tau\mu\tau = \mu\sigma$ and $\tau\rho\tau = \rho\sigma$.

Suppose $\mathcal{M}$ is a map in class $2_{\{0,1\}}$ with underlying graph Γ and recall that the automorphism group $\mathrm{Aut}(\mathcal{M})$ is then 1-regular on Γ. By Theorem 3.12 the boundaries of the faces of $\mathcal{M}$ are $\mathrm{Aut}(\mathcal{M})$-symmetric consistent cycles, and so Lemma 3.5 implies that the vertex stabilizers in $\mathrm{Aut}(\mathcal{M})$ are isomorphic to the Klein 4-group (which of course must be transitive on the neighbourhood of the fixed vertex). It is easy to see that the only subgroup of $\mathrm{Aut}(\Gamma)_{x_0} = \langle \mu, \sigma, \tau \rangle$, transitive on the set of four neighbours of x_0 and isomorphic to the Klein 4-group is $\langle \mu, \sigma \rangle$. Thus, $\mathrm{Aut}(\mathcal{M})$ is a transitive index 2 subgroup of $\mathrm{Aut}(\Gamma)$ containing the subgroup $\langle \mu, \sigma \rangle$.

We claim that $H_1 = \langle \mu, \sigma, \rho \rangle$ and $H_2 = \langle \mu, \sigma, \tau\rho \rangle$ are the only two such subgroups. Since $\mathrm{Aut}(\mathcal{M})$ must be vertex-transitive, it has to contain an element $\gamma \in \mathrm{Aut}(\Gamma)$ mapping x_0 to x_1. But $\mathrm{Aut}(\Gamma)$ is vertex-transitive with $\mathrm{Aut}(\Gamma)_{x_0} = \langle \mu, \sigma, \tau \rangle$, and so the fact that τ normalizes $\langle \mu, \sigma \rangle$ implies that $\gamma \in \langle \mu, \sigma \rangle \rho \cup \langle \mu, \sigma \rangle \tau\rho$. It follows that $\mathrm{Aut}(\mathcal{M})$ could only be one of H_1 and H_2. We next prove that H_1 and H_2 are indeed of index 2 in $\mathrm{Aut}(\Gamma)$ (since they contain $\langle \mu, \sigma \rangle$ and an element mapping x_0 to x_1, they are both arc-transitive on Γ). Note that the fact that each H_i acts arc-transitively implies that H_i is of index 2 in $\mathrm{Aut}(\Gamma)$ or $H_i \cong \mathrm{Aut}(\Gamma)$. It thus suffices to find an element of $\mathrm{Aut}(\Gamma)$ which is not contained in H_i.

We first deal with H_1. To this end we identify four cycles of Γ, each of length $12m$. Observe that $a + 1 = 3d + 3$, implying that $\gcd(12m, a + 1) = 3$ (recall that $d = m$ or $d = -m$ and m is even). Therefore, the cycle

$$C_1 = (x_0, x_1, y_1, x_{a+1}, x_{a+2}, y_{a+2}, x_{2(a+1)}, \ldots, y_{-a})$$

is indeed of length $12m$. Observe that C_1 is an H_1-consistent cycle with a shunt $\sigma\rho$. Let $C_2 = C_1\rho$ and $C_3 = C_1\rho^2$ be the images of C_1 under ρ and ρ^2, respectively, and note that C_2 and C_3 are thus also H_1-consistent cycles of Γ. Finally, observe that $\gcd(12m, r) = 1$ and let $C_4 = (y_0, y_r, y_{2r}, \ldots, y_{-r})$. Of course, C_4 also is an H_1-consistent cycle with a shunt ρ^r. It is easy to see that each of the generators ρ, σ and μ of H_1 preserves the set $\{C_1, C_2, C_3, C_4\}$ setwise (ρ fixes C_4 and permutes C_1, C_2 and C_3, σ interchanges C_1 with C_3 and C_2 with C_4, while μ interchanges C_1 with C_3 and fixes both C_2 and C_4). Since τ clearly does not map C_1 to any of the cycles C_i, this implies that $\tau \notin H_1$, and so H_1 is indeed of index 2 in $\mathrm{Aut}(\Gamma)$.

The proof that H_2 is also of index 2 in $\mathrm{Aut}(\Gamma)$ is similar. First we observe that $\gcd(12m, 2 + 2r + a) = \gcd(12m, 9d + 6) = 6$, and so the cycle

$$C_1 = (x_0, x_1, x_2, y_2, y_{2+r}, y_{2+2r}, x_{2+2r+a}, x_{3+2r+a}, x_{4+2r+a}, \ldots, y_{-a})$$

is of length $12m$ and is an H_2-consistent cycle with a shunt $\sigma\tau\rho$. Setting $C_2 = C_1\mu$, $C_3 = C_1\tau\rho$ and $C_4 = C_1\tau\rho\sigma$ we get three other H_2-consistent cycles. We can then verify that each of the generators $\tau\rho$, σ and μ of H_2 preserve the set $\{C_1, C_2, C_3, C_4\}$ setwise Since ρ clearly does not map C_1 to any of the cycles C_i, this implies that $\rho \notin H_2$, and so H_2 is indeed of index 2 in $\mathrm{Aut}(\Gamma)$.

We now classify the maps $\mathcal{M}$ of class $2_{\{0,1\}}$ with underlying graph Γ and automorphism group isomorphic to H_1 or H_2 separately. We first deal with the maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = H_1$. By Lemma 3.10 the boundaries of the faces of $\mathcal{M}$ are H_1-symmetric consistent cycles. Since H_1 is 1-regular, there are exactly three orbits, say $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$, of H_1-consistent directed cycles of Γ and by Lemma 3.4 there is exactly one H_1-consistent directed cycle containing the arc (x_{-1}, x_0) from each of these three orbits. Let $\vec{C}_1$, $\vec{C}_2$ and $\vec{C}_3$ be the corresponding H_1-consistent directed cycles with $\vec{C}_i \in \mathcal{O}_i$. With no loss of generality we can assume $\vec{C}_1$ contains the 2-arc (x_{-1}, x_0, x_1), $\vec{C}_2$ contains the 2-arc (x_{-1}, x_0, y_0), and $\vec{C}_3$ contains the 2-arc (x_{-1}, x_0, y_{9d-2}). Observe that ρ maps the arc (x_{-1}, x_0) to the arc (x_0, x_1), and so the fact that H_1 is 1-regular implies that ρ is a shunt of $\vec{C}_1$ in H_1. Similarly, $\rho\sigma$ is a shunt of $\vec{C}_2$ in H_1 and $\rho\sigma\mu$ is a shunt of $\vec{C}_3$ in H_1. Note that by Lemma 3.4 any pair of orbits $\mathcal{O}_i$, $i \in \{1,2,3\}$, determines a map $\mathcal{M}$ with underlying graph Γ, such that $H_1 \leq \mathrm{Aut}(\mathcal{M})$. We thus only have to check which pairs of orbits are such that $\mathrm{Aut}(\mathcal{M}) = H_1$ and not $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(\Gamma)$ (which occurs if and only if $\tau \in \mathrm{Aut}(\mathcal{M})$). Since $\tau\rho\tau = \rho\sigma$ and $\tau\rho\mu\sigma\tau = \rho\mu\sigma$ it follows that τ interchanges the H_1-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ and fixes the orbit $\mathcal{O}_3$ (in fact, it fixes the cycle $\vec{C}_3$). Thus $\tau \in \mathrm{Aut}(\mathcal{M})$ if and only if the boundaries of faces of $\mathcal{M}$ are all the members of the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. Consequently, $\mathcal{M}$ is of class $2_{\{0,1\}}$ if and only if the boundaries of its faces are all the members of $\mathcal{O}_3$ and one of $\mathcal{O}_1, \mathcal{O}_2$. Since τ fixes $\mathcal{O}_3$ and interchanges $\mathcal{O}_1$ and $\mathcal{O}_2$, this proves that H_1 gives rise to exactly one map of class $2_{\{0,1\}}$, up to isomorphism.

In a similar way one can prove that there is exactly one map of class $2_{\{0,1\}}$ with underlying graph Γ and automorphism group H_2.

It is clear that the two maps of class $2_{\{0,1\}}$, corresponding to H_1 and H_2, respectively, are not isomorphic since a corresponding isomorphism would have to be an automorphism of Γ, and so Γ is the underlying graph of exactly two maps of class $2_{\{0,1\}}$. $\square$

Chapter 4

Tetravalent G-half-arc-transitive graphs

In the last two chapters graphs admitting a half-arc-transitive group of automorphisms, that is a subgroup of their automorphism group acting transitively on its vertices and its edges but not on its arcs, are investigated. In this chapter we focus on the tetravalent ones.

The first result concerning graphs admitting a half-arc-transitive action was given by Tutte [55], who proved that the valency of such graphs must be even. Since any connected 2-valent graph is a cycle, the smallest interesting valency for the study of graphs admitting a half-arc-transitive group of automorphisms is four. It is thus not surprising that the majority of papers on such graphs deal with the tetravalent ones. Despite the fact that numerous papers on the topic have been published in the last half a century the complete classification of tetravalent half-arc-transitive graphs appears to be a very difficult, if not impossible, problem. As a result various restricted subproblems have been considered and different general approaches to the study of tetravalent graphs admitting a half-arc-transitive group of automorphisms have been proposed. For instance, the graphs of orders of specific types such as p^3, p^4, p^5, pq, $3p$, $4p$, $2pq$, etc., where p and q are prime numbers, have been classified, some even for all valencies (see for instance [5, 9, 14, 18, 19, 30, 56, 60]). The vertex-stabilizers in tetravalent graphs admitting a half-arc-transitive group of automorphisms and the connection of such graphs to maps have also been studied (see for instance [10, 33, 34, 35, 46]). Recently, Potočnik, Spiga and Verret constructed a census [48] of all tetravalent graphs admitting a half-arc-transitive group action up to order 1000. Since we will be referring to some of the graphs from the census we mention that they have names of the form HAT$[n, i]$ or GHAT$[n, i]$, where n is the order of the graph and the prefix G indicates that the full automorphism group of the graph acts arc-transitively.

In this chapter we review two frameworks for a systematic study of all tetravalent graphs admitting a half-arc-transitive group of automorphisms. In the first of these frameworks certain cycles, together with a given orientation of the edges of such graphs are investigated, and in the second one smaller graphs with "the same" properties as the original graph are obtained by identifying the orbits of a non-transitive normal subgroup of the automorphism group of the studied graph. One of the aims of this chapter is to show that, in fact, these two approaches are strongly related. To this purpose, we first define a new parameter for such graphs, which we call the jump parameter. This parameter gives a further insight into the structure of the studied graphs. We study some of the properties of the jump parameter. The obtained results enable us to establish the mentioned link between the two frameworks, and let us improve some of the existing results for tetravalent graphs admitting a half-arc-transitive group of automorphisms.

4.1 Alternating cycles

In this section we present the framework for the study of structural properties of tetravalent graphs admitting a half-arc-transitive action started 20 years ago by Marušič [32]. As mentioned in the introduction, this approach is one of the most general and fruitful for the study of such graphs. It is based on the investigation of certain cycles called *alternating cycles*. This concept was later generalized by Wilson in [58] to graphs admitting a half-arc-transitive group of automorphisms of larger valences (see also [25]). Since we also deal with such graphs later in this thesis. we decide to follow [58] to introduce the main concepts of this framework, and only then focus on the tetravalent ones.

Let Γ be a G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$. It is easy to see that the action of G on $A(\Gamma)$ has two paired orbits. Let $\mathcal{O}_G$ be one of them. Then for each edge of Γ the orbit $\mathcal{O}_G$ contains exactly one of the two arcs corresponding to this edge, and so $\mathcal{O}_G$ gives rise to an orientation of the edges of Γ, preserved by the action of G. We say that the orientation $\mathcal{O}_G$ is *G-induced* and denote the corresponding oriented graph by $\overrightarrow{\Gamma}_G$. We indicate that the edge uv is oriented from u to v by $u \to v$ and say that u is the *tail* and v is the *head* of the (oriented) edge uv (and of the arc (u, v)). We say that two edges uv and $u'v'$ of Γ are *related* if the corresponding oriented edges in $\overrightarrow{\Gamma}_G$ share a common head or a common tail. Thus, if $u \to v$ and $u' \to v'$ in $\overrightarrow{\Gamma}_G$ then uv and $u'v'$ are related if and only if $u = u'$ or $v = v'$ (but are not related if $u' = v$ or $v' = u$). The transitive hull of this relation, called the *reachability relation* on Γ, is of course an equivalence relation on $E(\Gamma)$. We call the subgraphs of Γ (as well as of $\overrightarrow{\Gamma}_G$), corresponding to the equivalence classes of this relation, the *G-alternets* of Γ (and of $\overrightarrow{\Gamma}_G$). For an G-alternet A its *head-set* consists of all the heads of the edges from A while its *tail-set* consists of all the tails of the edges from A. The size of (any) head-set is called the *G-radius* of Γ and is denoted by $\mathrm{rad}_G(\Gamma)$. It turns out that all G-alternets of Γ have the same number of vertices. The non empty intersections of the head-set of a G-alternet with the tail-set of a G-alternet (possible the same) are called the *G-attachment sets* of Γ, and the size of any G-attachment set is called the *G-attachment number* of Γ and is denoted by $\mathrm{att}_G(\Gamma)$. If Γ has at least two G-alternets and if for two G-alternets, sharing a common vertex, the head-set of one coincides with the tail-set of the other, we say that Γ is *tightly G-attached*.

Following the notation of Section 3.5.1, let $\Gamma = R_8(2, 1)$. It is easy to see that the group $G = \langle \rho, \sigma_0 \rangle$ is a half-arc-transitive group of automorphisms of Γ (where ρ is the on step rotation mapping each x_i to x_{i+1} and y_i to y_{i+1}, and $\sigma_0 = (x_0\ y_0)$). Fixing one of the two G-induced orientation we obtain that $\mathrm{rad}_G(\Gamma) = 2$, $\mathrm{att}_G(\Gamma) = 2$ and Γ is tightly G-attached (see Figure 4.1).

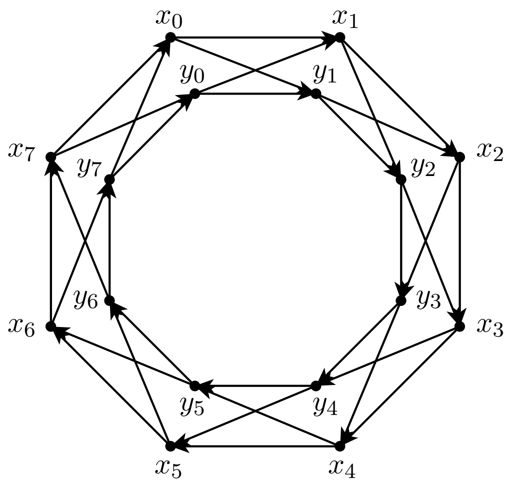


Figure 4.1: One of the two oriented graph of $R_8(2,1)$ corresponding to the action of $G = \langle \rho, \sigma_0 \rangle$.

As in the example above, in the case of Γ being a tetravalent graph the G-alternets turn out to be cycles and we call them *G-alternating cycles* (as were defined in [32]). Note that in a G-alternating cycle each pair of its consecutive edges have opposite orientations in $\mathcal{O}_G$, that is, traversing the cycle we alternate between traveling with and against the orientation of the edges from $\mathcal{O}_G$. In this case, the G-radius is half of the length of any G-alternating cycle.

Let us point out a useful fact that we use recurrently in this chapter. Since the orientation of the edges of Γ is given by the action of G, the elements of G permute the G-alternating cycles, and so G induces a natural action on the set of all G-alternating cycles as well as on the set of the G-attachment sets.

A half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$ of a tetravalent graph Γ may have large vertex-stabilizers (see for instance [33, 35]). However, by the following result of [37] which we will be using in this chapter, this cannot occur if the G-attachment number $\mathrm{att}_G(\Gamma)$ is at least 3.

Proposition 4.1. *[37, Lemma 3.5.] Let Γ be a tetravalent graph admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$. If $\mathrm{att}_G(\Gamma) \geq 3$ then $G_v \cong \mathbb{Z}_2$ for all $v \in V(\Gamma)$.*

Before the last comment of this section we give the definition of a well-known family of graphs which we may refer to in the rest of this chapter.

Let n be an integer and let $S \subset \mathbb{Z}_n$ be an inverse closed subset of the additive group $\mathbb{Z}_n$ of residue classes modulo n, where $0 \notin S$. The *circulant graph* $\mathrm{Circ}_n(S)$ is the graph with vertex set $\mathbb{Z}_n$ in which two vertices $i, j \in \mathbb{Z}_n$ are adjacent if and only if $j - i \in S$. In other words it is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$.

It was proved in [32] that if Γ is a tetravalent graph admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$, then $\mathrm{att}_G(\Gamma) \leq \mathrm{rad}_G(\Gamma)$ or $\mathrm{att}_G(G) = 2\mathrm{rad}_G(\Gamma)$ must hold. Graphs for which the latter equality holds were characterised explicitly also in [32] and turn out to be some particular arc-transitive circulants. More precisely, $\Gamma \cong \mathrm{Circ}_{2r}(\{\pm 1, \pm s\})$, where $r = \mathrm{rad}_G(\Gamma)$ is an integer and $s \in \mathbb{Z}_{2r}^* \setminus \{\pm 1\}$ such that $s^2 - 1 \equiv 0$ modulo $2r$, Γ has exactly two G-alternating cycles and each of it contains every vertex of Γ.

As an example let $\Gamma = \mathrm{Circ}_8(\pm 1, \pm 3)$. Let $\rho = (0\ 2\ 4\ 6)(1\ 3\ 5\ 7)$, $\tau = (1\ 7)(2\ 6)(3\ 5)$, $\gamma = (0\ 5\ 4\ 1)(2\ 3\ 6\ 7)$ and $G = \langle \rho, \tau, \sigma \rangle$. It is easy to see that Γ is G-half-arc-transitive, but it is arc-transitive. Fixing one of the two G-induced orientations of Γ one can observe that in fact Γ has exactly two G-alternating cycles and that $\mathrm{rad}_G(\Gamma) = 4$ and $\mathrm{att}_G(\Gamma) = 8$ (see Figure 4.2).

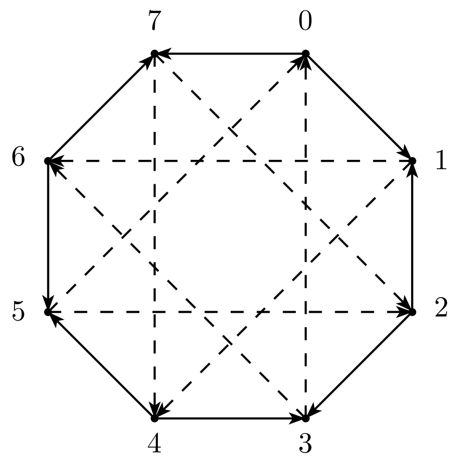


Figure 4.2: One of the two corresponding oriented graph of $\mathrm{Circ}_8(\pm 1, \pm 3)$ by G.

4.2 The Quotient and Normal Quotient graphs

We now focus on two specific kinds of quotients of tetravalent graphs admitting a half-arc-transitive group of automorphisms. First we define the quotient graph $\Gamma_{\mathcal{B}}$ from [37] (the graph $\Gamma_{\mathcal{B}}$ was denoted by Γ_{Σ} in [37]).

Construction 4.2 ([37])**.** Let Γ be a tetravalent graph admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$ and $\ell = 2r/a$. Define

$$s = \begin{cases} \ell/2 & : \quad \text{if } \ell \text{ is even;} \\ \ell & : \quad \text{if } \ell \text{ is odd.} \end{cases}$$

Let $C = (u_0, u_1, \ldots, u_{2r-1})$ be a G-alternating cycle. For each $u_i \in V(C)$ set $B_s(C; u_i) = \{u_{i+2sj} \colon 0 \leq j < a\}$ (where the indices are computed modulo $2r$). Let $u = u_i \in V(C)$ for some $i \in \{0, 1, \ldots, 2r-1\}$ and note that if ℓ is even the set $B_s(C; u)$ is precisely the G-attachment set containing u, while if ℓ is odd the G-attachment set containing u is $B_s(C; u) \cup B_s(C, u_{i+\ell})$. Moreover, if ℓ is odd and u is the tail of the two arcs of C incident to it, then all of the vertices in $B_s(C;u)$ also have this property, while all of the vertices in $B_s(C;u_{i+\ell})$ are heads of the two arcs of C incident to them. The set $B_s(C;u)$ thus forms a block of imprimitivity for G, and so $\mathcal{B} = \{B_s(C;u)\gamma\colon \gamma \in G\}$ is an imprimitivity block system for G (note that it does not depend on the choice of the G-alternating cycle C nor the vertex u of C). The graph $\Gamma_\mathcal{B}$ is then defined as the quotient graph of Γ with respect to $\mathcal{B}$, whose vertex set coincides with $\mathcal{B}$ with two different blocks B and B' from $\mathcal{B}$ being adjacent whenever there is a pair of adjacent vertices $u \in B$ and $v \in B'$.

The study of alternating cycles and their intersections has been the topic of several papers with most of the attention given to the *loosely*, *antipodally* and *tightly G-attached* graphs in which $\mathrm{att}_G(\Gamma)$ attains one of the extremal values 1, 2 or $\mathrm{rad}_G(\Gamma)$, respectively (see for instance [32, 39, 49]). The importance of these three special cases is based on the results of [37], where it was proved that each tetravalent G-half-arc-transitive graph Γ, where $G \leq \mathrm{Aut}(\Gamma)$, is either tightly G-attached or admits an imprimitivity block system for G such that the corresponding quotient graph $\Gamma_\mathcal{B}$ is loosely or antipodally attached. The tightly G-attached graphs have been classified ([32, 37, 53, 58]), while in the remaining cases there is still a lot of work to be done. We would like to point out however, that in [37] it was not known whether the mentioned imprimitivity block system for G, giving rise to $\Gamma_\mathcal{B}$, can be obtained as the set of orbits of a normal subgroup of G or not.

This question is of great importance. Namely, in [3] a new framework for a systematic study of tetravalent graphs admitting a half-arc-transitive group of automorphisms was proposed. It is based on the well-known method of taking normal quotients, which has already led to important results in the study of graphs possessing a considerable degree of symmetry. For instance, its use in the context of s-arc-transitive graphs was initiated by Praeger in 1993 [50]. The main idea in our setting is that whenever a half-arc-transitive subgroup $G \leq \mathrm{Aut}(\Gamma)$ for a tetravalent graph Γ has a normal subgroup N with at least three orbits, the quotient graph with respect to the orbits of N is again a tetravalent graph (provided that it is not a cycle with 'doubled edges') admitting a half-arc-transitive group of automorphisms (a quotient group of G). One thus aims to classify all 'basic' examples (not admitting such normal subgroups) and to understand how the remaining graphs can be reconstructed from the basic ones (see [3] for details). Recently, some results of this kind have been obtained in [1, 2].

4.3 The alternating jump parameter

In this section we introduce a new parameter for tetravalent graphs admitting a half-arc-transitive action called the *alternating jump* and give some of its basic properties. This parameter describes how two non-disjoint alternating cycles are attached to one another and gives a more detailed insight into the structure of such graphs when compared to the one given by simply considering their radius and attachment number. The obtained results enable us to link the above mentioned approaches from [32, 37] and [3], in particular to prove that the imprimitivity block system giving rise to the above mentioned quotient graph $\Gamma_\mathcal{B}$ from [37] is in fact obtained by orbits of a normal cyclic subgroup (see Theorem 4.16), which thus links the quotients of [37] to those of [3].

Before defining the alternating jump parameter, we first describe two families of tetravalent graphs which contain all the tetravalent tightly attached half-arc-transitive graphs. We will be referring to them quite often in this section, and as well in the rest of this chapter. As explained before such graphs are of great importance in the study of tetravalent graphs admitting a half-arc-transitive group of automorphism and have been classified in [32, 37, 53, 58]. The definition is given in two parts depending on the parity of the radius of the graphs, where we follow [53].

Construction 4.3 ([53])**.** For each $m \geq 3$, $r \geq 3$ odd, $q \in \mathbb{Z}_r^*$, where $q^m = \pm 1$, let $\mathcal{X}_o(m, r; q)$ be the graph with vertex set $V = \{u_i^j : i \in \mathbb{Z}_m, j \in \mathbb{Z}_r\}$ and edges defined by the following adjacencies:

$$u_i^j \sim \; u_{i+1}^{j \pm q^i}; \quad i \in \mathbb{Z}_m, j \in \mathbb{Z}_r.$$

Construction 4.4 ([53])**.** Let $m \geq 4$ and $r \geq 4$ be even, and for each $q \in \mathbb{Z}_r^*$, $t \in \mathbb{Z}_n$ satisfying

$$q^m = 1,\ t(q-1) = 0 \text{ and } 1 + q + \cdots + q^{m-1} + 2t = 0,$$

let $\mathcal{X}_e(m, r; q, t)$ be the graph with vertex set $V = \{u_i^j : i \in \mathbb{Z}_m, j \in \mathbb{Z}_r\}$ and edges defined by the following adjacencies:

$$u_i^j \sim \begin{cases} u_{i+1}^j, u_{i+1}^{j+q^i}; & i \in \mathbb{Z}_m \backslash \{m-1\}, j \in \mathbb{Z}_r \\ u_0^{j+t}, u_0^{j+q^{m-1}+t}; & i = m-1, j \in \mathbb{Z}_r. \end{cases}$$

For instance, it turns out that the smallest half-arc-transitive graph, the well-known Doyle-Holt graph, then Γ is tightly attached with radius 9 (see Figure 4.3) and is isomorphic to $\mathcal{X}_o(3, 9; 2)$ (see [4]).

In order to start the description of the mentioned alternating jump parameter we first make a careful study of the structure of the attachment sets in a tetravalent graph admitting a half-arc-transitive group of automorphisms.

Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$. Fix one of the two paired orientations of the edges of Γ, induced by the action of G, and let $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$. For a vertex $v \in V(\Gamma)$ let $C = (u_0, u_1, \ldots, u_{2r-1})$ and $C' = (v_0, v_1, \ldots, v_{2r-1})$ be the two G-alternating cycles containing v, where $u_0 = v_0 = v$ and v is the tail of the two arcs of C, incident to it. By [32, Lemma 2.6], the G-attachment set $V(C) \cap V(C')$ containing v, which we abbreviate by $C \cap C'$ throughout this chapter, is

$$C \cap C' = \{u_{i\ell} : 0 \leq i < a\} = \{v_{i\ell} : 0 \leq i < a\}, \tag{4.1}$$

where $\ell = 2r/a$. Define

$$q_t(v) = \min\{q : v_{q\ell} \in \{u_\ell, u_{-\ell}\}\} \text{ and } q_h(v) = \min\{q : u_{q\ell} \in \{v_\ell, v_{-\ell}\}\},$$

where in the case of $a = 1$ this is understood as $q_t(v) = q_h(v) = 0$. Observe that, by definition, $q_t(v), q_h(v) \leq a/2$. Moreover, since G acts vertex- and edge-transitively

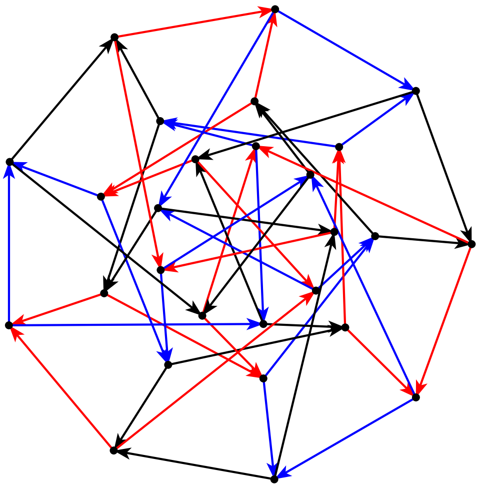


Figure 4.3: The alternating cycles in one of the two corresponding oriented graph of the Doyle-Holt graph.

on Γ, the parameters $q_t(v)$ and $q_h(v)$ do not depend on the choice of the vertex v. Note also that taking the other of the two G-induced orientations of the edges of Γ reverses the roles of $q_t(v)$ and $q_h(v)$. For each tetravalent G-half-arc-transitive graph Γ we can thus define $Q_G(\Gamma) = \{q_t, q_h\}$, where $q_t = q_t(v)$ and $q_h = q_h(v)$ for some $v \in V(\Gamma)$ with respect to one of the two G-induced orientations of the edges of Γ. For instance, in the case of loosely G-attached graphs, that is when $\mathrm{att}_G(\Gamma) = 1$, we have $Q_G(\Gamma) = \{0\}$, and in the case of antipodally G-attached graphs, that is when $\mathrm{att}_G(\Gamma) = 2$, we have $Q_G(\Gamma) = \{1\}$.

The following results show that the parameters q_t and q_h give us a lot of information about how two G-alternating cycles of Γ with a non empty intersection are attached to one another and that the two parameters are nicely related.

Lemma 4.5. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$ and $\ell = 2r/a$. Fix one of the two G-induced orientations of the edges of Γ and let $v \in V(\Gamma)$. Let $C = (u_0, u_1, \ldots, u_{2r-1})$ and $C' = (v_0, v_1, \ldots, v_{2r-1})$ be the two G-alternating cycles containing v, where $u_0 = v_0 = v$, v is the tail of the two arcs of C incident to it, and $u_\ell = v_{q_t\ell}$. Then $u_{i\ell} = v_{iq_t\ell}$ holds for each $0 \leq i < a$. Similarly, depending on whether $v_\ell = u_{q_h\ell}$ or $v_\ell = u_{-q_h\ell}$ holds, we have that $v_{i\ell} = u_{iq_h\ell}$ holds for each $0 \leq i < a$ or $v_{i\ell} = u_{-iq_h\ell}$ holds for each $0 \leq i < a$.*

Proof. Observe that if $a \leq 2$, there is nothing to prove. We can thus assume that $a \geq 3$, and so Proposition 4.1 applies. Let γ be the unique nontrivial element of G_{u_ℓ} and observe that, since G is edge-transitive on Γ, it does not fix any of the neighbors of u_ℓ. Then γ fixes both C and C' setwise, and so the restriction of its action to C (respectively C') is the reflection with respect to u_ℓ (respectively $v_{q_t\ell}$). Thus $u_{2\ell} = u_0\gamma = v_0\gamma = v_{2q_t\ell}$. We can now continue inductively to see that $u_{i\ell} = v_{iq_t\ell}$ holds for each $0 \leq i < a$. The second part of the lemma can be proved analogously. $\square$

Lemma 4.6. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$, let $a = \mathrm{att}_G(\Gamma)$ and let $Q_G(\Gamma) = \{q_t, q_h\}$. Then* $\gcd(a, q_t) = \gcd(a, q_h) = 1$ *and* $q_t q_h \equiv \pm 1 \pmod{a}$.

Proof. We can again assume that $a \geq 3$. Adopt the notation of Lemma 4.5. By Lemma 4.5 we have that $u_{i\ell} = v_{iq_t\ell}$ holds for each $0 \leq i < a$. Thus (7.1) implies that $\{v_{iq_t\ell} : 0 \leq i < a\} = \{v_{i\ell} : 0 \leq i < a\}$, and so $\gcd(a, q_t) = 1$. A similar argument shows that $\gcd(a, q_h) = 1$ as well. To see that $q_t q_h \equiv \pm 1 \pmod{a}$ observe that Lemma 4.5 implies that $u_{q_h\ell} = v_{q_h q_t \ell}$ holds (recall that $q_h < a$). By definition of q_h we thus get $v_{\pm\ell} = v_{q_h q_t \ell}$, that is $q_h q_t \ell \equiv \pm \ell \pmod{2r}$. Since $2r = a\ell$, we obtain $q_h q_t \equiv \pm 1 \pmod{a}$. □

We remark that both $q_h q_t \equiv 1 \pmod{a}$ and $q_h q_t \equiv -1 \pmod{a}$ can occur. For instance one can verify that in the Doyle-Holt graph, $\mathcal{X}_o(3, 9; 2)$, we have $a = 9$ and $Q = \{2, 4\}$, and so $q_t q_h \equiv -1 \pmod{a}$, while for both of the graphs $\mathcal{X}_e(4, 20; 3, 0)$ and $\mathcal{X}_e(4, 20; 3, 10)$ we have $a = 20$ and $Q = \{3, 7\}$, and so $q_h q_t \equiv 1 \pmod{a}$. Other examples of both possibilities ($q_t q_h \equiv 1 \pmod{a}$ or $q_t q_h \equiv -1 \pmod{a}$), including non-tightly attached ones, can be found by going through the census [48].

Observe that Lemma 4.6 implies that in fact $q_h, q_t < a/2$ holds unless $a = 2$ in which case of course $q_h = q_t = 1 = a/2$. Moreover, Lemma 4.6 implies that q_t is uniquely expressible as the smaller of the elements $\pm q_h^{-1}$ in $\mathbb{Z}_a$ and that, conversely, q_h is the smaller of the elements $\pm q_t^{-1}$ in $\mathbb{Z}_a$. We can thus define the parameter $\mathrm{jmp}_G(\Gamma) = \min(Q_G(\Gamma))$ of a tetravalent G-half-arc-transitive graph Γ where $G \leq \mathrm{Aut}(\Gamma)$. We call $\mathrm{jmp}_G(\Gamma)$ the *G-alternating jump* of Γ. In the case that $G = \mathrm{Aut}(\Gamma)$ we abbreviate $\mathrm{jmp}_{\mathrm{Aut}(\Gamma)}(\Gamma)$ to $\mathrm{jmp}(\Gamma)$ and speak of the *alternating jump* of Γ. Since $Q_G(\Gamma)$ does not depend of the choice of the G-induced orientation of Γ, this also holds for the G-alternating jump of Γ.

Note that Lemma 4.6 implies that the circulants $\mathrm{Circ}_a(\{\pm 1, \pm q_t\})$ and $\mathrm{Circ}_a(\{\pm 1, \pm q_h\})$ are isomorphic, so we say that $\mathrm{Circ}_a(\{\pm 1, \pm \mathrm{jmp}_G(\Gamma)\})$ is the *associated circulant* of the pair (Γ, G) (see Figure 4.4 for an example), where in the case of $a = 1$ we disregard the loop and consider the associated circulant simply as a one-vertex graph. In the case of $a \leq 2$ the circulant is somewhat degenerate but in general it is tetravalent if and only if $\mathrm{jmp}_G(\Gamma) \neq 1$. This brings us to the following natural question.

Question 4.7. Let $a \geq 3$ be an integer and $1 \leq q < a/2$ be coprime to a. Does there exist a tetravalent graph Γ admitting a half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$ such that the associated circulant of the pair (Γ, G) is isomorphic to $\mathrm{Circ}_a(\{\pm 1, \pm q\})$?

We remark that the associated circulant of a pair (Γ, G), where Γ is a tetravalent G-half-arc-transitive graph may or may not be arc-transitive. For instance, for the half-arc-transitive graph $\mathcal{X}_o(3, 13; 3)$ the attachment number and the alternating jump are 13 and 3, respectively (see Proposition 4.10), and one can easily check that the circulant $\mathrm{Circ}_{13}(\{\pm 1, \pm 3\})$ is not arc-transitive. On the other hand, one can verify that the half-arc-transitive graph HAT[500, 6] from the census [48] has attachment number 5 and the alternating jump parameter 2, and so the associated circulant $\mathrm{Circ}_5(\{\pm 1, \pm 2\})$, which is the complete graph on 5 vertices, is arc-transitive. Nevertheless, we think it is worth studying whether the fact that the associated circulant of a pair (Γ, G) is or is not arc-transitive has any implications on the graph Γ. Note

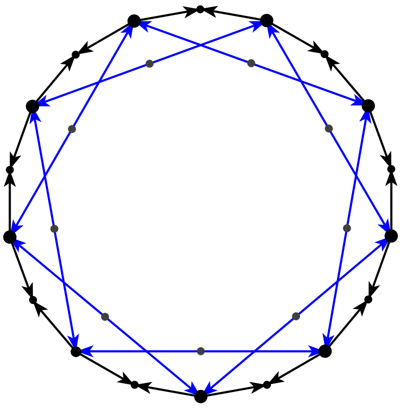


Figure 4.4: Two G-alternating cycles of the Doyle-Holt graph. It's clear that its associated circulant is isomorphic to the graph $\mathrm{Circ}_9(\{\pm 1, \pm 2\})$.

that in the case of $\mathrm{att}_G(\Gamma) = 2\mathrm{rad}_G(\Gamma)$, the associated circulant is in fact isomorphic to Γ.

It is well known that a tetravalent half-arc-transitive graph of given order is not uniquely determined by its radius and attachment number. For instance, [32, Theorem 3.4] implies that the graphs $\mathcal{X}_o(6, 13; 2)$ and $\mathcal{X}_o(6, 13; 3)$ are both half-arc-transitive with radius 13 and attachment number 13. It is not difficult to prove that they are not isomorphic. One of the ways to see this is by inspecting their alternating jump parameter (denoted by q in the remainder of this paragraph). Namely, it can directly be verified (but see also Proposition 4.10) that the graph $\mathcal{X}_o(6, 13; 2)$ has $q = 2$, while the graph $\mathcal{X}_o(6, 13; 3)$ has $q = 3$. This shows that the parameter q does give a further refinement in the classification of all tetravalent half-arc-transitive graphs. Unfortunately, even the triple (r, a, q) does not uniquely determine a tetravalent half-arc-transitive graph of a given order. For instance, [53, Theorem 1.3, Proposition 9.1] imply that the graphs $\mathcal{X}_e(4, 20; 3, 0)$ and $\mathcal{X}_e(4, 20; 3, 10)$ are both half-arc-transitive with radius 20 and attachment number 20, but are not isomorphic. However, both also have $q = 3$, which thus shows that nonisomorphic tetravalent half-arc-transitive graphs with the same order, radius, attachment number and alternating jump parameter exist. Nevertheless, we show in the remainder of this chapter that the alternating jump parameter does give a very useful insight into the structure of tetravalent graphs admitting a half-arc-transitive group of automorphisms.

4.4 The alternating jump of tightly attached graphs

The majority of the examples from the the previous section are all tightly attached half-arc-transitive graphs. The fact that their alternating jump parameter is actually one of their defining parameters is not a coincidence. In this section we show that the alternating jump parameter of tetravalent graphs admitting a half-arc-transitive group action relative to which they are tightly attached is determined by their defining parameters from the classifications given in [32, 53]. Depending on whether the corresponding radius is odd or even, respectively, the classification of such graphs is given in the following two propositions, which can be extracted from [32, Propositions 3.2, 3.3] and [53, Theorem 1.2], respectively (see also [37, Theorem 4.5] and [58, Theorem 8.1]).

Proposition 4.8. *[32] A tetravalent graph Γ admits a half-arc-transitive group of automorphisms relative to which it is tightly attached of odd radius r if and only if $\Gamma \cong \mathcal{X}_o(m,r;q)$ for some integer $m \geq 3$ and some $q \in \mathbb{Z}_r^*$ with $q^m \equiv \pm 1 \pmod{r}$.*

Proposition 4.9. *[53] A tetravalent graph Γ admits a half-arc-transitive group of automorphisms relative to which it is tightly attached of even radius r if and only if either $r = 2$ and Γ is isomorphic to a lexicographic product of a cycle with $2K_1$, or $r \geq 4$ and $\Gamma \cong \mathcal{X}_e(m,r;q,t)$, where $m \geq 4$ is even and $q \in \mathbb{Z}_r^*$, $t \in \mathbb{Z}_r$ are such that $q^m = 1$, $t(q-1) = 0$ and $1 + q + \cdots + q^{m-1} + 2t = 0$.*

Before we determine the alternating jump parameter of the graphs Γ from the above two propositions we fix some notation. We first point out that the action of the corresponding half-arc-transitive group G of automorphisms, both in the graphs $\mathcal{X}_o(m,r;q)$ and the graphs $\mathcal{X}_e(m,r;q,t)$, is such that the corresponding G-attachment sets are $\Gamma_i = \{u_i^j \in V(\Gamma) \colon j \in \mathbb{Z}_r\}$, where $i \in \mathbb{Z}_m$ (see [32, 53] for details). In particular, in one of the two G-induced orientations of the edges of Γ all edges are oriented from Γ_i to Γ_{i+1} for all $i \in \mathbb{Z}_m$. In the proof of the following result we always choose this orientation. The notation $\min_r\{\pm q, \pm q^{-1}\}$ in the statement of the following proposition stands for the minimal integer from the set $\{0, 1, \ldots, r-1\}$, which is congruent to one of $q, -q, q^{-1}, -q^{-1}$ modulo r, where q^{-1} is the inverse of q modulo r.

Proposition 4.10. *Let Γ be a tetravalent graph admitting a half-arc-transitive group G of automorphisms relative to which it is tightly attached with radius $r \geq 3$. Then $\mathrm{jmp}_G(\Gamma) = \min_r\{\pm q, \pm q^{-1}\}$, where $\Gamma \cong \mathcal{X}_o(m,r;q)$ or $\Gamma \cong \mathcal{X}_e(m,r;q,t)$, depending on whether r is odd or even, respectively.*

Proof. Recall that, since the G-attachment sets are blocks of imprimitivity for the action of G on Γ, the radius r divides $|V(\Gamma)|$. Let m be such that $|V(\Gamma)| = mr$ and note that, by [32, Proposition 2.4], we have $m \geq 3$. We separate the proof depending on the parity of r.

Suppose first that r is odd. In this case Proposition 4.8 implies that $\Gamma \cong \mathcal{X}_o(m,r;q)$ for some $q \in \mathbb{Z}_r^*$ such that $q^m = \pm 1$. Choose the orientation of Γ described in the paragraph preceding Proposition 4.10 and let $v = u_1^0$. In order to determine $\mathrm{jmp}_G(\Gamma)$ we find the values $q_t(v)$ and $q_h(v)$ (recall that the parameters q_t and q_h do not depend on the choice of the vertex v). By definition of the graph $\mathcal{X}_o(m,r;q)$ the two G-alternating cycles containing v are

$$C = (u_1^0, u_2^q, u_1^{2q}, u_2^{3q}, \ldots, u_1^{r-2q}, u_2^{r-q}) \text{ and } C' = (u_1^0, u_0^1, u_1^2, u_0^3, \ldots, u_1^{r-2}, u_0^{r-1}).$$

Since r is odd and q is coprime to r, the corresponding G-attachment set $C \cap C'$ is then

$$C \cap C' = \{u_1^{2jq} \colon j \in \mathbb{Z}_r\} \cap \{u_1^{2j} \colon j \in \mathbb{Z}_r\} = \{u_1^j \colon j \in \mathbb{Z}_r\}.$$

It follows that $q_t(v) = \min\{s\colon u_1^{2s} \in \{u_1^{2q}, u_1^{-2q}\}\}$, which implies $q_t(v) = \min\{q, -q\}$. Therefore, Lemma 4.6 implies $q_h(v) \in \{q^{-1}, -q^{-1}\}$. By definition of $q_h(v)$ it follows that $q_h(v) = \min\{q^{-1}, -q^{-1}\}$, and so $\mathrm{jmp}_G(\Gamma) = \min_r\{\pm q, \pm q^{-1}\}$, as claimed.

Suppose now that r is even. Since $r \geq 3$ Proposition 4.9 implies that $\Gamma \cong \mathcal{X}_e(m, r; q, t)$ for some $q \in \mathbb{Z}_r^*$ and $t \in \mathbb{Z}_r$ such that $q^m = 1$, $t(q-1) = 0$ and $1 + q + \cdots + q^{m-1} + 2t = 0$. Then, choosing the orientation of Γ described in the paragraph preceding Proposition 4.10, the two G-alternating cycles containing the vertex $v = u_1^0$ are

$$C = (u_1^0, u_2^q, u_1^q, u_2^{2q}, u_1^{2q} \ldots, u_1^{r-q}, u_2^0) \text{ and } C' = (u_1^0, u_0^0, u_1^1, u_0^1, \ldots, u_1^{r-1}, u_0^{r-1}).$$

A similar argument as in the case of r being odd proves that $\mathrm{jmp}_G(\Gamma) = \min_r\{\pm q, \pm q^{-1}\}$. □

By [32, Proposition 4.1] for any $m \geq 3$, any odd $r \geq 3$ and any $q \in \mathbb{Z}_r$ such that $q^m = \pm 1$ the graph $\mathcal{X}_o(m, r; q)$ is isomorphic to each of the graphs $\mathcal{X}_o(m, r; -q)$, $\mathcal{X}_o(m, r; q^{-1})$ and $\mathcal{X}_o(m, r; -q^{-1})$. Similarly, [53, Proposition 3.9] shows that for any pair of even integers $m, r \geq 4$ and any $q, t \in \mathbb{Z}_r$ with $q^m = 1$, $t(q-1) = 0$ and $1 + q + \cdots + q^{m-1} + 2t = 0$ the graph $\mathcal{X}_e(m, r; q, t)$ is isomorphic to each of the graphs $\mathcal{X}_e(m, r; q^{-1}, t)$, $\mathcal{X}_e(m, r; -q, t + q + q^3 + \cdots + q^{m-1})$ and $\mathcal{X}_e(m, r; -q^{-1}, t + q + q^3 + \cdots + q^{m-1})$. We thus have the following corollary.

Corollary 4.11. *Let Γ be a tetravalent graph admitting a half-arc-transitive group G of automorphisms relative to which it is tightly attached with radius $r \geq 3$. Let m be such an integer that the order of Γ is mr and let $q = \mathrm{jmp}_G(\Gamma)$. Then, one of (i) and (ii) below holds, depending on whether r is odd or even, respectively:*

(i) $\Gamma \cong \mathcal{X}_o(m, r; q)$.

(ii) $\Gamma \cong \mathcal{X}_e(m, r; q, t)$, *where $t \in \mathbb{Z}_r$ is one of the two solutions of the equation* $1 + q + \cdots + q^{m-1} + 2t = 0$.

The above corollary shows that, at least for tetravalent graphs admitting a half-arc-transitive group of automorphisms relative to which they are tightly attached, the order, together with the corresponding radius and alternating jump parameter (almost) completely determine the graph. More precisely, they determine it completely in the case of odd radius while in the case of even radius at most two such graphs can exist.

We want to point out however, that a given tetravalent graph Γ may admit more than one half-arc-transitive group of automorphisms (even such that it is tightly attached with respect to more than one of them). Moreover, the corresponding radius (and/or alternating jump parameter) does not need to be the same for all such subgroups of $\mathrm{Aut}(\Gamma)$. For instance in [37, Example 2.1] (see also [36]) it was shown that the well-known lexicographic products $C_{2r}[2K_1]$, where $r \geq 3$ (graphs $R_{2r}(2, 1)$ from section 3.5.1), admit two different half-arc-transitive groups of automorphisms relative to which they are tightly attached. For one of them the corresponding radius is 2 while for the other one it is r. There are other such examples that can be found in [37] but one can also find others by going through the census from [48]. For instance, the graph GHAT[294; 1] from the census admits three different half-arc-transitive groups of automorphisms. It is loosely attached with radii 3 and 7 for two of them and is antipodally attached with radius 3 for the third one. Of course, if we restrict to half-arc-transitive graphs (where the full automorphism group is half-arc-transitive), these phenomena cannot occur.

4.5 The graph of alternating cycles and the quotient graph $\mathbf{\Gamma_{\mathcal{B}}}$

There is another possibility for the study of tetravalent graphs Γ admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$, also based on the G-alternating cycles. Instead of considering the quotient graph $\Gamma_{\mathcal{B}}$ one can study the graph $\mathrm{Alt}_G(\Gamma)$ of G-alternating cycles introduced in [49]. Employing the properties of the alternating jump parameter we determine the kernel of the natural action of G on $\mathrm{Alt}_G(\Gamma)$ (see Theorem 4.13) and show that this kernel coincides with the kernel of the natural action of G on $\Gamma_{\mathcal{B}}$ as well as to the kernel of its action on the set of all G-attachment sets of Γ (see Theorem 4.15). The group G induces a natural vertex- and edge-transitive action on both $\Gamma_{\mathcal{B}}$ and $\mathrm{Alt}_G(\Gamma)$. We consider the question of when the graphs $\Gamma_{\mathcal{B}}$ and $\mathrm{Alt}_G(\Gamma)$ are half-arc-transitive and when they are arc-transitive, and we indicate some connections between these two graphs.

We first recall the definition of the graph of alternating cycles $\mathrm{Alt}_G(\Gamma)$ from [49]. Let Γ be a tetravalent graph admitting a half-arc-transitive group of automorphisms G. The vertex set of $\mathrm{Alt}_G(\Gamma)$ is then the set of all G-alternating cycles of Γ with two such cycles being adjacent whenever they have a non-empty intersection. Of course, G has a natural action on the graph $\mathrm{Alt}_G(\Gamma)$. In fact, it was proved in [49, Proposition 4] that the induced action is vertex- and edge-transitive and is half-arc-transitive if and only if $\mathrm{att}_G(\Gamma)$ divides $\mathrm{rad}_G(\Gamma)$. As we mentioned before, one of the aims of this section is to determine the kernel $K_G(\mathrm{Alt}_G(\Gamma))$ of this action. Observe that $K_G(\mathrm{Alt}_G(\Gamma))$ is the normal subgroup of G consisting of all elements fixing each G-alternating cycle (throughout the rest of this chapter we say that an automorphism of Γ *fixes* a given cycle if it maps this cycle as a subgraph to itself). The following lemma gives a sufficient condition for an element of G to be contained in $K_G(\mathrm{Alt}_G(\Gamma))$.

Lemma 4.12. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ such that $3 \leq a < r$, where $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. Suppose $\gamma \in G$ fixes some G-alternating cycle C, as well as all the G-attachment sets containing vertices of C. Then $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$. Moreover, there exists an integer k, $0 \leq k \leq a/2$, such that the action of γ on C is a $k\ell$-step rotation, where $\ell = 2r/a$.*

Proof. Denote $C = (u_0, u_1, \ldots, u_{2r-1})$, fix one of the two G-induced orientations of the edges of Γ and let $Q_G(\Gamma) = \{q_t, q_h\}$. Since γ fixes C, as well as all the G-attachment sets containing vertices of C, there exists a unique $0 \leq k < a$ such that $u_0\gamma = u_{k\ell}$, where $\ell = 2r/a$ (recall that the G-attachment sets are of the form given in (7.1)). In fact, by exchanging the roles of u_j and u_{-j} for each j if necessary, we can assume $k \leq a/2$. By [32, Proposition 2.4] the G-alternating cycles of Γ are all induced. Since u_1 is a vertex of the induced cycle C and it is adjacent to u_0, the image $u_1\gamma$ is one of the vertices $u_{k\ell+1}$ and $u_{k\ell-1}$. Moreover, since $a < r$, we have

$\ell \geq 3$, and so the G-attachment set containing u_1, namely $\{u_{1+i\ell}\colon 0 \leq i < a\}$, does not contain $u_{k\ell-1}$. It follows that $u_1\gamma = u_{k\ell+1}$, and so the restriction of the action of γ to C is the $k\ell$-step rotation such that $u_j\gamma = u_{j+k\ell}$ for all $0 \leq j \leq 2r-1$.

Let $C' = (v_0, v_1, \ldots, v_{2r-1})$ be one of the G-alternating cycles having a non-empty intersection with C. Without loss of generality assume $v_0 = u_0$ and $u_\ell = v_{q\ell}$, where q is one of q_t and q_h, depending on whether u_0 is the tail of the two arcs of C incident to it or not, respectively. By Lemma 4.5 we have $u_{i\ell} = v_{iq\ell}$ for all $0 \leq i < a$, and so $v_0\gamma = u_0\gamma = u_{k\ell} = v_{kq\ell}$. Recall that $v_\ell = u_{q^{-1}\ell}$, where q^{-1} is the inverse of q modulo a. Thus

$$v_\ell\gamma = u_{q^{-1}\ell}\gamma = u_{q^{-1}\ell+k\ell} = u_{(q^{-1}+k)\ell} = v_{(q^{-1}+k)q\ell} = v_{\ell+kq\ell}.$$

Since $\ell < r$ and γ fixes C' (since it fixes C and the attachment set $C \cap C'$), it is now clear that the restriction of the action of γ to C' is the $kq\ell$-step rotation such that $v_j\gamma = v_{j+kq\ell}$ for all $0 \leq j \leq 2r-1$. It follows that γ fixes all of the G-attachment sets containing vertices of C', and so the assumptions of the lemma hold for C' and γ as well. By connectedness γ fixes each G-alternating cycle of Γ, and so $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$. $\square$

We can now (almost) completely determine the kernel $K_G(\mathrm{Alt}_G(\Gamma))$ for a tetravalent graph Γ admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$. We use D_r to denote the dihedral group of order $2r$.

Theorem 4.13. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. Let $K = K_G(\mathrm{Alt}_G(\Gamma))$ be the kernel of the action of G on the graph $\mathrm{Alt}_G(\Gamma)$ of G-alternating cycles of Γ. Then one of the following holds:*

- *(i) $a = 2r$, in which case $\Gamma \cong \mathrm{Circ}_{2r}(\{\pm 1, \pm q\})$ where $q^2 \equiv \pm 1 (\mathrm{mod}\, 2r)$, and $K \cong D_r$;*
- *(ii) $a = r = 2$, in which case Γ is the lexicographic product $C_n[2K_1]$ for some integer n, and K is isomorphic to a subgroup of the elementary abelian group $\mathbb{Z}_2^n$;*
- *(iii) $a = r > 2$ and $K \cong D_a$;*
- *(iv) $a < r$ with $a \mid r$ and $K \cong \mathbb{Z}_a$, unless possibly K is trivial with $a = 2$;*
- *(v) $a < r$ with $a \nmid r$ and $K \cong \mathbb{Z}_{a/2}$.*

Moreover, if $a = 2$ and K is trivial, then $\tilde{G} = G \times \langle \tau \rangle$ acts half-arc-transitively on Γ with $K_{\tilde{G}}(\mathrm{Alt}_{\tilde{G}}(\Gamma)) \cong \mathbb{Z}_2$, where τ is the automorphism of Γ interchanging each vertex with its antipodal counterpart on both of the G-alternating cycles containing it.

Proof. By [32, Proposition 2.4] the graph $\mathrm{Alt}_G(\Gamma)$ has at least three vertices unless $a = 2r$, in which case $\Gamma \cong \mathrm{Circ}_{2r}(\{\pm 1, \pm q\})$ for some q with $q^2 \equiv \pm 1 (\mathrm{mod}\, 2r)$, there are exactly two G-alternating cycles (each containing all of the vertices of Γ) and $K = D_r$ is the dihedral group of order $2r$ (while G itself is an extension of K by $\mathbb{Z}_2$). Moreover, the case $a = r$, that is when Γ is a tightly G-attached graph, is settled by

[32, Proposition 3.1] since in this case the kernel K clearly coincides with the kernel of the action of G on the set of all G-attachment sets. For the rest of the proof we can thus assume that Γ has at least three G-alternating cycles and that $1 \leq a < r$. Observe that in this case the action of K on Γ is semiregular since $a < r$ implies that no two neighbors of a vertex of Γ can belong to the same pair of G-alternating cycles.

Next, observe that if $a = 1$, that is if Γ is loosely G-attached, the kernel K is trivial. We can thus further assume that $a \geq 2$. For the rest of the proof we adopt the notation from Lemma 4.5. We set $\ell = 2r/a$, fix one of the two G-induced orientations of the edges of Γ, set $Q_G(\Gamma) = \{q_t, q_h\}$ and we let $v \in V(\Gamma)$. Furthermore, we let $C = (u_0, u_1, \ldots, u_{2r-1})$ and $C' = (v_0, v_1, \ldots, v_{2r-1})$ be the two G-alternating cycles containing $v = u_0 = v_0$ where v is the tail of the two arcs of C incident to it and $u_\ell = v_{q_t \ell}$.

Suppose first that $a = 2$. If r is odd (in which case $a \nmid r$), then u_0 is the tail, while u_r is the head of the two arcs of C incident to u_0 and u_r, respectively. It follows that no element of K can map u_0 to u_r, and so K is trivial in this case. If however r is even, then the fact that the intersection of any two adjacent G-alternating cycles is a pair of antipodal vertices on both of them implies that the automorphism τ, mapping each vertex to its antipodal counterpart on both G-alternating cycles containing it (see [49, Proposition 7]), is the only possible nontrivial element of K. Depending on whether τ is or is not contained in G, the kernel K is either the cyclic group of order 2 or is trivial, respectively. Moreover, if $\tau \notin G$ then the fact that τ centralizes G implies that $\tilde{G} = \langle G, \tau \rangle = G \times \langle \tau \rangle$. Since r is even, it is clear that $\tilde{G}$ acts half-arc-transitively on Γ (giving rise to the same alternating cycles as G) and that $\langle \tau \rangle = K_{\tilde{G}}(\mathrm{Alt}_{\tilde{G}}(\Gamma))$.

We are left with the case $3 \leq a < r$. Recall that the vertex u_ℓ is the tail of the two arcs of C incident to it if and only if ℓ is even, which occurs if and only if a divides r. Since $a \geq 3$, Proposition 4.1 implies that G acts regularly on the edge set of Γ. If ℓ is even let $\rho \in G$ be the unique element mapping the arc (u_0, u_1) to the arc $(u_\ell, u_{\ell+1})$ and if ℓ is odd let $\rho \in G$ be the unique element mapping the arc (u_0, u_1) to $(u_{2\ell}, u_{2\ell+1})$. It is clear that the restriction of the action of ρ to C is an ℓ-step or 2ℓ-step rotation, depending on whether a divides r or not, respectively. In both cases ρ fixes C as well as all of the ℓ attachment sets containing the vertices of C. Thus ρ satisfies the conditions of Lemma 4.12, and so $\rho \in K$. Since $C \cap C' = \{u_{i\ell} \colon 0 \leq i < a\}$ and the action of K on Γ is semiregular, it follows that $K = \langle \rho \rangle$, which completes the proof. □

We point out that for $a = 2$ and r even both possibilities from item (iv) of the above theorem can indeed occur. Going through the census of all tetravalent graphs of order up to 1000, admitting a half-arc-transitive group of automorphisms [48], one can verify that in most cases where $a = 2$ and r is even the kernel $K_G(\mathrm{Alt}_G(\Gamma))$ is nontrivial (and is thus of order 2). However, there are examples where it is in fact trivial. The smallest such example in the above mentioned census is the graph $\Gamma = \mathrm{GHAT}[162, 1]$ of order 162 whose automorphism group $\mathrm{Aut}(\Gamma)$ is of order 1296. The group $\mathrm{Aut}(\Gamma)$ acts arc-transitively on Γ but it has precisely two (normal) subgroups G_1 and G_2 of orders 324 and 648, respectively, acting half-arc-transitively on Γ, for both of which the corresponding radius is 6 and the attachment number is 2. However, for the group G_1 the corresponding kernel $K_G(\mathrm{Alt}_G(\Gamma))$ is trivial, while for the group G_2, it is of order 2. Of course, since G_1 and G_2 are the only half-arc-transitive subgroups of $\mathrm{Aut}(\Gamma)$, Theorem 4.13 implies that $G_2 = G_1 \times \langle\tau\rangle$. Nevertheless, the following problem naturally arises.

Problem 4.14. Classify all tetravalent graphs Γ, admitting a half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$ with $\mathrm{att}_G(\Gamma) = 2$ and $\mathrm{rad}_G(\Gamma)$ even, such that $K_G(\mathrm{Alt}_G(\Gamma))$ is trivial. In particular, does a half-arc-transitive graph Γ admitting such a half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$ exist?

Before stating and proving the next theorem in which we compare the kernel of the action of G on $\mathrm{Alt}_G(\Gamma)$ and on $\Gamma_{\mathcal{B}}$ we fix the following notation. Let $\mathcal{A}$ be the set of all G-attachment sets of Γ (note that $\mathcal{A} = \mathcal{B}$ if and only if a divides r). Then G has a natural action on the graph $\Gamma_{\mathcal{B}}$ as well as on the set $\mathcal{A}$. Let $K_G(\Gamma_{\mathcal{B}})$ and $K_G(\mathcal{A})$ be the kernels of these actions. The following result shows that these two kernels are closely related to the kernel $K_G(\mathrm{Alt}_G(\Gamma))$.

Theorem 4.15. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. Let $\Gamma_{\mathcal{B}}$ be as in Construction 4.2 and let $\mathcal{A}$ be the partition of $V(\Gamma)$ described in the preceding paragraph. If $a < 2r$, then*

$$K_G(\Gamma_{\mathcal{B}}) = K_G(\mathrm{Alt}_G(\Gamma)) = K_G(\mathcal{A}).$$

Proof. Let $\alpha \in K_G(\mathrm{Alt}_G(\Gamma))$. Then α fixes each G-alternating cycle, and so it also fixes all the G-attachment sets since they are the intersections of G-alternating cycles. Thus $K_G(\mathrm{Alt}_G(\Gamma)) \subseteq K_G(\mathcal{A})$.

Suppose now there exists $\gamma \in K_G(\mathcal{A}) \backslash K_G(\mathrm{Alt}_G(\Gamma))$. Let $\ell = 2r/a$, let $C = (u_0, u_1, \ldots, u_{2r-1})$ be a G-alternating cycle such that $C\gamma \neq C$ and let $C' = (v_0, v_1, \ldots, v_{2r-1})$ be the other G-alternating cycle containing the vertex u_0, where $u_0 = v_0$. Since $\gamma \in K_G(\mathcal{A})$, it fixes the G-attachment set $C \cap C'$ setwise, and so it interchanges the cycles C and C'. Note that $a < 2r$ implies that the vertex u_1 is not contained in C' but in some G-alternating cycle C'' with $C'' \neq C$ and $C'' \neq C'$. But since γ also fixes setwise the G-attachment set $C \cap C''$ we get $C \cap C'' = C\gamma \cap C''\gamma = C' \cap C''\gamma$, which is impossible. Therefore, $K_G(\mathrm{Alt}_G(\Gamma)) = K_G(\mathcal{A})$, as claimed.

To complete the proof we need to verify that $K_G(\Gamma_{\mathcal{B}}) = K_G(\mathcal{A})$. Of course, if a divides r then ℓ is even in which case $\mathcal{B}$ coincides with $\mathcal{A}$, and so there is nothing to prove. We can thus assume that $a \nmid r$. In this case each G-attachment set is the disjoint union of two elements of $\mathcal{B}$, and so it is clear that $K_G(\Gamma_{\mathcal{B}}) \leq K_G(\mathcal{A})$. By the first part of the proof it thus suffices to prove that $K_G(\mathrm{Alt}_G(\Gamma)) \leq K_G(\Gamma_{\mathcal{B}})$. Let $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$ and let $C = (u_0, u_1, \ldots, u_{2r-1})$ be a G-alternating cycle. Let $u = u_i$ for some $i \in \{0, 1, \ldots, 2r-1\}$ and let C' be the other G-alternating cycle containing u. Since in this case ℓ is odd, $C \cap C' = B_s(C; u_i) \cup B_s(C, u_{i+\ell})$. Since γ fixes both C and C', each $u_{i+j\ell}$ is mapped by γ to a vertex $u_{i+j'\ell}$ such that j' has the same parity as j (otherwise γ interchanges C and C'). Therefore γ fixes each of $B_s(C; u_i)$ and $B_s(C, u_{i+\ell})$ setwise, and so it fixes all the elements of $\mathcal{B}$ setwise. It thus follows that $K_G(\mathrm{Alt}_G(\Gamma)) \leq K_G(\Gamma_{\mathcal{B}})$, which completes the proof. $\square$

As was pointed out in the Introduction, the quotient graph $\Gamma_{\mathcal{B}}$ from Construction 4.2 is of great importance in the study of tetravalent graphs admitting a half-arc-transitive group of automorphisms. Namely, by [37, Theorem 3.6] a tetravalent graph Γ admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$ with $\mathrm{att}_G(\Gamma) \neq 2\mathrm{rad}_G(\Gamma)$ is either tightly G-attached or the quotient graph $\Gamma_{\mathcal{B}}$ is a tetravalent graph admitting a half-arc-transitive action of a quotient group of G, relative to which it is loosely or antipodally attached. Of course, this quotient group is precisely the group $G/K_G(\Gamma_{\mathcal{B}})$. Combining together Theorems 4.13 and 4.15 we can make an important improvement of [37, Theorem 3.6]. Namely, we now know that the imprimitivity block system $\mathcal{B}$ actually coincides with the set of orbits of the cyclic normal subgroup $K_G(\Gamma_{\mathcal{B}})$ of G (recall that in the case that $\mathrm{rad}_G(\Gamma)$ is even, $\mathrm{att}_G(\Gamma) = 2$ and $K_G(\mathrm{Alt}_G(\Gamma))$ is trivial, we first need to enlarge the group G to $G \times \langle\tau\rangle$). We thus obtain the following result.

Theorem 4.16. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ such that $a \neq 2r$, where $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. In the case that r is even, $a = 2$ and $K_G(\Gamma_{\mathcal{B}})$ is trivial replace G by $\tilde{G}$ from Theorem 4.13. Let $\mathcal{B}$ and $\Gamma_{\mathcal{B}}$ be as in Construction 4.2. Then one the following holds:*

(i) *$a = r$, that is Γ is tightly G-attached;*

(ii) *$a < r$ and the partition $\mathcal{B}$ coincides with the orbits of the cyclic normal subgroup $K_G(\Gamma_{\mathcal{B}})$ which is of order a or $a/2$, depending on whether a divides r or not, respectively. Moreover, letting $\bar{G} = G/K_G(\Gamma_{\mathcal{B}})$, the quotient graph $\Gamma_{\mathcal{B}}$ is a tetravalent $\bar{G}$-half-arc-transitive graph which is loosely $\bar{G}$-attached or antipodally $\bar{G}$-attached, depending on whether a divides r or not, respectively.*

Theorem 4.16 provides the link between the frameworks for a systematic study of tetravalent graphs admitting a half-arc-transitive group of automorphism, started in [3, 37]. Namely, for a tetravalent graph Γ admitting a half-arc-transitive group of automorphism G such that Γ does not have one of the three special attachment numbers (1,2 or $\mathrm{rad}_G(\Gamma)$), Theorem 4.16 ensures the existence of a non-trivial cyclic normal subgroup of G such that the quotient graph with respect to its orbits is again a tetravalent graph admitting a half-arc-transitive group of automorphism with attachment number 1 or 2. Therefore, in order to describe or classify all the basic graphs in the sense of [3] the tetravalent graphs admitting a half-arc-transitive group of automorphisms relative to which they are loosely or antipodally attached will have to be thoroughly investigated.

In the remainder of this section we give two additional results on the graphs $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$, corresponding to a tetravalent G-half-arc-transitive graph Γ where $G \leq \mathrm{Aut}(\Gamma)$. Moreover, by making a careful analysis of the graphs from the census [48] using a suitable package such as Magma [7] several interesting questions about these two graphs arise which provide a possible direction for future research. We formulate some of these questions and give a few examples of graphs with interesting properties.

Let Γ be a tetravalent G-half-arc-transitive graph where $G \leq \mathrm{Aut}(\Gamma)$ and let $a = \mathrm{att}_G(\Gamma)$ and $r = \mathrm{rad}_G(\Gamma)$. As already pointed out the group G induces a natural action on both $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$. The quotient group $G/K_G(\mathrm{Alt}_G(\Gamma))$ acts vertex- and edge-transitively on $\mathrm{Alt}_G(\Gamma)$ and this action is arc-transitive if and only if a does not divide r and is half-arc-transitive otherwise (see [49, Proposition 4]). Of course, even if a does divide r the graph $\mathrm{Alt}_G(\Gamma)$ may be arc-transitive. A similar situation holds for $\Gamma_{\mathcal{B}}$, but in this case $G/K_G(\Gamma_{\mathcal{B}})$ always acts half-arc-transitively. As we now prove, a natural way for $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ to be arc-transitive is that there is an automorphism of Γ, interchanging the two G-induced orientations of the edges of Γ.

Proposition 4.17. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$. If Γ is tightly G-attached or $\mathrm{Aut}(\Gamma)$ contains an element interchanging the two G-orbits on $A(\Gamma)$, then $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ are arc-transitive graphs.*

Proof. Note that if Γ is tightly G-attached, then $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$, are both isomorphic to the cycle of length $|V(\Gamma)|/\mathrm{rad}_G(\Gamma)$. For the rest of the proof we can thus assume that Γ is not tightly G-attached.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the two G-orbits on $A(\Gamma)$. Suppose that $\mathrm{Aut}(\Gamma)$ contains an element α interchanging $\mathcal{O}_1$ and $\mathcal{O}_2$ and let $(u,v) \in \mathcal{O}_1$. Since $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ admit a vertex- and edege-transitive action, it suffices to find an element of their automorphism groups which interchanges two adjacent vertices. Moreover, in terms of vertices and edges the G-alternating cycles with respect to $\mathcal{O}_1$ coincide with the G-alternating cycles with respect to $\mathcal{O}_2$, and so α induces an automorphism of both $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$. Let $C_1 = (u_0, u_1, \dots, u_{2r-1})$ and $C_2 = (v_0, v_1, \dots, v_{2r-1})$ be the two G-alternating cycles such that $u_0 = v_0 = u$ and $u_1 = v$. Note that there exists $\beta \in G$ such that $(u_0, u_1)\alpha\beta = (u_0, v_1)$. Then $\alpha\beta$ maps C_1 to C_2, and since it fixes u_0, it maps C_2 to C_1, proving that $\mathrm{Alt}_G(\Gamma)$ is arc-transitive. Similarly, there exists $\gamma \in G$ such that $(u_0, u_1)\alpha\gamma = (u_1, u_0)$. Then $\alpha\gamma$ clearly interchanges the adjacent vertices of $\Gamma_{\mathcal{B}}$ containing the vertices u_0 and u_1, respectively. Therefore the graph $\Gamma_{\mathcal{B}}$ is also arc-transitive. $\square$

Let us mention, that the situation from Proposition 4.17 seems to be quite common. Namely, by going through the census of all arc-transitive tetravalent graphs Γ admitting a half-arc-transitive group G of automorphisms up to order 1000, one finds that most of them admit automorphisms interchanging the two G-orbits on $A(\Gamma)$. Nevertheless, there are examples where this does not hold. For instance, the arc-transitive graph $\Gamma = \mathrm{GHAT}[21, 1]$ is tightly attached with respect to a suitable half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$, but there is no element of $\mathrm{Aut}(\Gamma)$ interchanging the corresponding orbits on $A(\Gamma)$. The same occurs for the graph $\mathrm{GHAT}[252, 14]$, which turns out to be the smallest example with this property that is not tightly attached (it is loosely attached).

We remark that each of the graphs $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ can be arc-transitive even if no automorphism of Γ interchanges the two G-orbits on $A(\Gamma)$. This turns out to be the case also for the above mentioned graphs $\mathrm{GHAT}[21, 1]$ and $\mathrm{GHAT}[252, 14]$. In fact, by going through the census from [48] one finds that for all arc-transitive tetravalent graphs admitting a half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$ up to order 1000, the graphs $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ are both arc-transitive. This suggests the following natural question.

Question 4.18. Is it true that if Γ is an arc-transitive tetravalent graph admitting a half-arc-transitive group $G \leq \mathrm{Aut}(\Gamma)$, the graphs $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ are both arc-transitive?

It is also interesting to investigate what can be said about arc-transitivity or half-arc-transitivity of the graphs $\mathrm{Alt}_G(\Gamma)$ and $\Gamma_{\mathcal{B}}$ in the case that Γ is half-arc-transitive. Going through the census of all tetravalent half-arc-transitive graphs of order up to 1000 one finds that there are examples Γ for which $\mathrm{Alt}(\Gamma)$ and/or $\Gamma_{\mathcal{B}}$ are half-arc-transitive. For instance, for $\Gamma = \mathrm{HAT}[500;6]$ (which has $\mathrm{rad}(\Gamma) = 25$ and $\mathrm{att}(\Gamma) = 5$) the graph $\Gamma_{\mathcal{B}}$ is half-arc-transitive while $\mathrm{Alt}(\Gamma)$ is arc-transitive. For $\Gamma = \mathrm{HAT}[600;4]$ (which has $\mathrm{rad}(\Gamma) = 6$ and $\mathrm{att}(\Gamma) = 2$) the graphs $\Gamma_{\mathcal{B}}$ and $\mathrm{Alt}(\Gamma)$ are both half-arc-transitive. Finally, for $\Gamma = \mathrm{HAT}[84;1]$ (which has $\mathrm{rad}(\Gamma) = 14$ and $\mathrm{att}(\Gamma) = 7$) both $\Gamma_{\mathcal{B}}$ and $\mathrm{Alt}(\Gamma)$ are arc-transitive. The census contains no example Γ for which $\mathrm{Alt}(\Gamma)$ would be half-arc-transitive but the quotient graph $\Gamma_{\mathcal{B}}$ would be arc-transitive. We thus pose the following natural question.

Question 4.19. Does there exist a tetravalent half-arc-transitive graph Γ such that $\mathrm{Alt}(\Gamma)$ is half-arc-transitive but $\Gamma_{\mathcal{B}}$ is arc-transitive?

The following result, which shows that the graph of alternating cycles is independent of taking quotients with respect to $\mathcal{B}$, can be of help when dealing with the above two questions.

Proposition 4.20. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ such that $\mathrm{att}_G(\Gamma) < \mathrm{rad}_G(\Gamma)$. Then $\mathrm{Alt}_G(\Gamma) \cong \mathrm{Alt}_{\overline{G}}(\Gamma_{\mathcal{B}})$, where $\overline{G} = G/K_G(\Gamma_{\mathcal{B}})$.*

Proof. Note that if Γ is loosely G-attached then $\Gamma \cong \Gamma_{\mathcal{B}}$, so there is nothing to prove. We can thus assume that $a = \mathrm{att}_G(\Gamma) > 1$. We now construct an isomorphism $\psi\colon \mathrm{Alt}_G(\Gamma) \to \mathrm{Alt}_{\overline{G}}(\Gamma_{\mathcal{B}})$. For each $v \in V(\Gamma)$ define B_v to be the element of $\mathcal{B}$ containing v. Let $C = (u_0, u_1, \ldots, u_{2r-1}) \in V(\mathrm{Alt}_G(\Gamma))$, where $r = \mathrm{rad}_G(\Gamma)$, be a G-alternating cycle of Γ and let $\ell = 2r/a$. Suppose first that a divides r. Then for each $0 \leq i \leq \ell - 1$ we have $B_{u_i} = \{u_{i+j\ell} \mid 0 \leq j \leq a-1\}$. In this case $(B_{u_0}, B_{u_1}, B_{u_2}, \ldots, B_{u_{\ell-1}})$ is a $\overline{G}$-alternating cycle of $\Gamma_{\mathcal{B}}$, which we set as the ψ-image of C. Suppose now that a does not divide r and recall that in this case a is even. Then for each $0 \leq i \leq 2\ell - 1$ we have $B_{u_i} = \{u_{i+2j\ell} \mid 0 \leq j \leq a/2 - 1\}$. In this case $(B_{u_0}, B_{u_1}, B_{u_2}, \ldots, B_{u_{2\ell-1}})$ is a $\overline{G}$-alternating cycle of $\Gamma_{\mathcal{B}}$, which we set as the ψ-image of C. It is easy to see that in both cases the mapping ψ is a bijection. Since adjacency in the graph of alternating cycles is given by non-empty intersection, ψ is clearly an isomorphism. $\square$

Note that Proposition 4.20 implies that in the case that the answer to Question 4.18 is in the affirmative, the answer to Question 4.19 is negative. Namely, suppose that the answer to Question 4.18 is in the affirmative. If Γ is half-arc-transitive but $\Gamma_{\mathcal{B}}$ is arc-transitive, then either Γ is tightly attached (in which case $\mathrm{Alt}(\Gamma)$ is a cycle and is thus arc-transitive), or Proposition 4.20 implies that $\mathrm{Alt}(\Gamma) \cong \mathrm{Alt}_{\overline{G}}(\Gamma_{\mathcal{B}})$, which, by assumption is arc-transitive.

4.6 The graphs with $\mathrm{att}_G(\Gamma) \nmid \mathrm{rad}_G(\Gamma)$

Let Γ be a tetravalent graph admitting a half-arc-transitive group G of automorphisms and let $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. It is well know that a divides $2r$. However, it may happen that a does not divide r (see for instance [49], where the examples with $r = 3$ and $a = 2$ were characterized). Nevertheless, for all of the known examples where a does not divide r, the graph Γ is in fact arc-transitive. This has led to the question, posed as Question 1 in [49], of whether in each tetravalent half-arc-transitive graph the attachment number divides the radius. In this section we use the alternating jump parameter and some of the results from the previous sections to address this question.

We first prove that in the case that a does not divide r, the set $Q_G(\Gamma)$ in fact consists of a single number, that is, the parameters q_t and q_h from Section 4.3 coincide. Recall that the number ℓ was defined as $\ell = 2r/a$, and so a does not divide r if and only if ℓ is odd.

Lemma 4.21. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$ and $q = \mathrm{jmp}_G(\Gamma)$. If a does not divide r and $a \neq |V(\Gamma)|$ then $q^2 \equiv \pm 1 \pmod a$, and so $Q_G(\Gamma) = \{q\}$.*

Proof. Fix one of the two G-induced orientations of the edges of Γ and let $v \in V(\Gamma)$. Let $q_t = q_t(v)$ and $q_h = q_h(v)$, so that $Q_G(\Gamma) = \{q_t, q_h\}$ (see Section 4.3). Let $C = (u_0, u_1, \ldots, u_{2r-1})$ and $C' = (v_0, v_1, \ldots, v_{2r-1})$ be the two G-alternating cycles containing v, where $u_0 = v_0 = v$, v is the tail of the two arcs of C, incident to it, and $u_\ell = v_{q_t\ell}$, where $\ell = 2r/a$. Observe that, since $a \nmid r$ but $a \mid 2r$, the number ℓ is odd. It follows that $u_{i\ell}$ is the tail of the two arcs of C, incident to it, if and only if i is even. In particular, u_0 is the tail and u_ℓ is the head of the two arcs of C, incident to u_0 and u_ℓ, respectively. By Lemma 4.5 we have that $u_{2\ell} = v_{2q_t\ell}$, and so $q_h(u_\ell) = q_t$ (recall that $u_\ell = v_{q_t\ell}$ and $q_t \leq a/2$). Thus $q_h = q_t = q$, and so Lemma 4.6 implies $q^2 \equiv \pm 1 \pmod a$, as claimed. □

The above lemma has the following useful corollary, describing the action of certain elements of the kernel $K_G(\mathrm{Alt}_G(\Gamma))$.

Corollary 4.22. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ such that $3 \leq a < r$, where $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. Suppose a does not divide r, let $q = \mathrm{jmp}_G(\Gamma)$ and let $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$ be a nontrivial element of the kernel of the action of G on the graph $\mathrm{Alt}_G(\Gamma)$. Then there exists an integer k, $0 < k \leq a/2$, such that for any two non-disjoint G-alternating cycles of Γ the restriction of the action of γ to them is a $k\ell$-step rotation on one of them and a $qk\ell$-step rotation on the other, where $\ell = 2r/a$.*

Proof. Let C be a G-alternating cycle of Γ. Note that the assumption $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$ implies that γ fixes C, as well as all the G-attachment sets containing the vertices of C. The existence of a suitable $0 < k \leq a/2$ such that γ acts as $k\ell$-step rotation on C is thus ensured by Lemma 4.12. Moreover, the proof of that lemma also shows that for each of the G-alternating cycles C', having a non-empty intersection with C, the restriction of the action of γ to C' is a $qk\ell$-step rotation. We can now repeat the same argument for each C'. Since γ acts as a $qk\ell$-step rotation on C', it acts as a $k\ell$-step rotation on each of the G-alternating cycles, having a non-empty intersection with C' (recall that $q^2 \equiv \pm 1 \pmod{a}$ holds by Lemma 4.21). By connectedness it now follows that for each pair of non-disjoint G-alternating cycles of Γ the restriction of the action of γ on them is a $k\ell$-step rotation on one of them and a $qk\ell$-step rotation on the other one. □

We next show that in the case when a does not divide r, the graph $\mathrm{Alt}_G(\Gamma)$ is necessarily bipartite, unless possibly if $q = 1$ or $q = a/2 - 1$.

Lemma 4.23. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ such that $a \neq |V(\Gamma)|$ and that a does not divide r, where $r = \mathrm{rad}_G(\Gamma)$ and $a = \mathrm{att}_G(\Gamma)$. If $\mathrm{jmp}_G(\Gamma) \notin \{1, a/2 - 1\}$, then the graph $\mathrm{Alt}_G(\Gamma)$ is bipartite.*

Proof. Denote $q = \mathrm{jmp}_G(\Gamma)$ and recall that Lemma 4.21 implies that $Q_G(\Gamma) = \{q\}$. Suppose $q \notin \{1, a/2 - 1\}$. Then Lemma 4.6 implies that $a \geq 3$. Let $C = (u_0, u_1, \ldots, u_{2r-1})$ be a G-alternating cycle of Γ and let $\gamma \in G$ be the unique (see Proposition 4.1) element mapping u_0 to $u_{2\ell}$ and u_1 to $u_{2\ell+1}$, where $\ell = 2r/a$ (observe that such a γ does indeed exist since 2ℓ is even). Since $a \geq 3$ we have $2\ell < 2r$, and so γ is a nontrivial automorphism. It is clear that γ is a 2ℓ-step rotation of C, and so it fixes C as well as all the G-attachment sets containing the vertices of C. By Lemma 4.12 we have $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$, and so Corollary 4.22 implies that for any pair of non-disjoint G-alternating cycles of Γ the restriction of the action of γ to one of them is a 2ℓ-step rotation while the restriction of its action to the other is a $2q\ell$-step rotation. It now only remains to see that a 2ℓ-step rotation of a cycle of length $2r$ is not a $2q\ell$-step rotation of this cycle (in any of the two possible directions). If this was true, then one of $2\ell \equiv 2q\ell \pmod{2r}$ and $2\ell \equiv -2q\ell \pmod{2r}$ would have to hold, implying that one of $2\ell(q-1)$ and $2\ell(q+1)$ is divisible by $2r = a\ell$. However, since by assumption $1 < q < a/2 - 1$ holds, none of these is possible. This shows that $\mathrm{Alt}_G(\Gamma)$ is bipartite with the two bipartition sets coinciding with the set of all G-alternating cycles of Γ on which γ acts as a 2ℓ-step rotation or $2q\ell$-step rotation, respectively. □

We remark that tetravalent graphs Γ, admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$ such that $\mathrm{att}_G(\Gamma)$ does not divide $\mathrm{rad}_G(\Gamma)$ and $\mathrm{Alt}_G(\Gamma)$ is not bipartite do exist. For instance, the results of [49] show that any (non-bipartite) 2-arc-transitive cubic graph is the graph $\mathrm{Alt}_G(\Gamma)$ for some tetravalent graph Γ admitting a half-arc-transitive group of automorphisms G for which $\mathrm{att}_G(\Gamma) = 2$ and $\mathrm{rad}_G(\Gamma) = 3$, thus providing infinitely many examples Γ with the above mentioned situation. Of course, since $\mathrm{att}_G(\Gamma) = 2$ we have $\mathrm{jmp}_G(\Gamma) = 1$ for all of these cases. We know of no example however, where $\mathrm{jmp}_G(\Gamma) = \mathrm{att}_G(\Gamma)/2 - 1 > 1$ holds, which thus leads us to the following question.

Question 4.24. Does there exist a tetravalent graph Γ, admitting a half-arc-transitive group of automorphisms $G \leq \mathrm{Aut}(\Gamma)$, such that $a \neq |V(\Gamma)|$, $a \nmid r$ and $q = a/2 - 1 > 1$, where $a = \mathrm{att}_G(\Gamma)$, $r = \mathrm{rad}_G(\Gamma)$ and $q = \mathrm{jmp}_G(\Gamma)$, but the graph $\mathrm{Alt}_G(\Gamma)$ is not bipartite?

We now give a result which is a considerable improvement of the results of [49] towards the answer to Question 1 of [49]. Moreover, if one is able to show that the answer to Question 4.24 is negative, the next theorem will in fact almost completely solve Question 1 from [49], since the only remaining case will be the one when $\mathrm{att}_G(\Gamma) = 4$.

Theorem 4.25. *Let Γ be a tetravalent G-half-arc-transitive graph for some $G \leq \mathrm{Aut}(\Gamma)$ and let $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$ and $q = \mathrm{jmp}_G(\Gamma)$. Suppose a does not divide r and $4 < a < r$. If $q = 1$ or the graph $\mathrm{Alt}_G(\Gamma)$ is bipartite, then there exists an automorphism ρ of Γ, fixing all of the G-alternating cycles of Γ and acting as a $2r/a$-step rotation on at least one of them. Consequently, the graph Γ is arc-transitive. In particular, if $4 < a < r$, a does not divide r and $q \neq a/2 - 1$, then Γ is arc-transitive.*

Proof. Fix one of the two G-induced orientations of the edges of Γ. Since a does not divide r, the number $\ell = 2r/a$ is odd and Lemma 4.21 implies that $Q_G(\Gamma) = \{q\}$. Moreover, for each G-alternating cycle $C = (u_0, u_1, \ldots, u_{2r-1})$, where u_0 is the tail of the two arcs of C, incident with it, the vertices of the form $u_{2i\ell}$ are the tails of the two arcs of C, incident with them, while the vertices of the form $u_{(2i+1)\ell}$ are the heads of the two arcs of C, incident with them. Now, choose a G-alternating cycle C and, as in the proof of Lemma 4.23, let $\gamma \in K_G(\mathrm{Alt}_G(\Gamma))$ be such that its restriction to C is a 2ℓ-step rotation. By Corollary 4.22 the restriction of the action of γ to any two non-disjoint G-alternating cycles is a 2ℓ-step rotation on one of them and a $2q\ell$-step rotation on the other. In what follows we show that we can construct an automorphism ρ of Γ such that $\rho^2 = \gamma$, having all the required properties.

To this end we first give two different labels to each vertex of Γ. Before doing this we number the G-alternating cycles by denoting them with $C_1, C_2, \ldots, C_s$ and we choose a certain subset I of the index set $S = \{1, 2, \ldots, s\}$ in the following way. If $q = 1$ then set $I = S$. Suppose now that $q > 1$. By assumption $\mathrm{Alt}_G(\Gamma)$ is bipartite in this case, and so the argument from the previous paragraph implies that the restriction of the action of γ to the G-alternating cycles from one of the two sets of bipartition of $\mathrm{Alt}_G(\Gamma)$ is a 2ℓ-step rotation while its restriction to the G-alternating cycles from the other set of bipartition is a $2q\ell$-step rotation. If $q \neq a/2 - 1$, then a 2ℓ-step rotation is different from a $2q\ell$-step rotation (in any of the two possible directions). In this case let I be the set of all $i \in S$ for which the restriction of the action of γ on C_i is a 2ℓ-step rotation. If however $q = a/2 - 1$ then simply choose one of the two sets of bipartition of $\mathrm{Alt}_G(\Gamma)$ and let I be the set of the indices of the G-alternating cycles C_i belonging to it. Observe that, in any case, for each $i \in I$ the restriction of the action of γ to C_i is a 2ℓ-step rotation while for each $i \in S \setminus I$ the restriction of γ to C_i is a $2q\ell$-step rotation.

We now label the vertices of each C_i by v^i_j, $j \in \mathbb{Z}_{2r}$, in such a way that v^i_j and v^i_{j+1} are consecutive vertices of C_i for each j and that $v^i_j\gamma = v^i_{j+2\ell}$ for each $j \in \mathbb{Z}_{2r}$ whenever $i \in I$, while $v^i_j\gamma = v^i_{j+2q\ell}$ for each $j \in \mathbb{Z}_{2r}$ whenever $i \in S \setminus I$. Observe that, since q is coprime to a, each of $4\ell \equiv 0 \pmod{2r}$ and $4q\ell \equiv 0 \pmod{2r}$ contradicts $a > 4$, and so the above described labeling is unique up to cyclic rotations. Note however, that in this way each vertex received two different labels, one for each G-alternating cycle it belongs to.

We are now ready to define the mapping ρ, satisfying all of the properties from the statement of the theorem. We set

$$v_j^i\rho = \begin{cases} v_{j+\ell}^i & ; \quad i \in I \\ v_{j+q\ell}^i & ; \quad i \in S \setminus I. \end{cases}$$

To prove that ρ is a well-defined mapping let v be a vertex of Γ. Without loss of generality assume it belongs to the G-alternating cycles C_1 and C_2. We thus have $v = v_{j_1}^1 = v_{j_2}^2$ for some $j_1, j_2 \in \mathbb{Z}_{2r}$. We can further assume that we have $v_j^1\gamma = v_{j+2\ell}^1$ and $v_j^2\gamma = v_{j+2q\ell}^2$ for each $j \in \mathbb{Z}_{2r}$ (note that if $q = 1$ then $2q\ell = 2\ell$). Now, since C_1 and C_2 are non-disjoint and $Q_G(\Gamma) = \{q\}$, it follows that $v_{j_1+\ell}^1 \in \{v_{j_2+q\ell}^2, v_{j_2-q\ell}^2\}$. If $v_{j_1+\ell}^1 = v_{j_2-q\ell}^2$, then Lemma 4.5 implies that $v_{j_1+2\ell}^1 = v_{j_2-2q\ell}^2$. However, as $v_{j_1+2\ell}^1 = v_{j_1}^1\gamma = v\gamma = v_{j_2}^2\gamma = v_{j_2+2q\ell}^2$, this yields $v_{j_2-2q\ell}^2 = v_{j_2+2q\ell}^2$. But then $4q\ell \equiv 0 \pmod{a\ell}$, contradicting $a > 4$ (recall that q is coprime to a). It thus follows that $v_{j_1+\ell}^1 = v_{j_2+q\ell}^2$, and so ρ is well defined, as claimed. That ρ is indeed an automorphism of Γ is now clear from the definition. Moreover, it preserves each G-alternating cycle but does not respect the G-induced orientation of the edges of Γ. It follows that Γ is arc-transitive. The last part of the theorem is now an immediate consequence of Lemma 4.23. □

Let us wrap up the chapter with the following remark. If the remaining cases, not covered by Theorem 4.25, can be taken care of to give an affirmative answer to Question 1 of [49], Theorem 4.16 will have further implications for a systematic study of all tetravalent half-arc-transitive graphs. Namely, since the tightly attached graphs have already been classified, it will remain to classify the tetravalent graphs admitting a loosely attached half-arc-transitive action and to determine how to construct all other tetravalent half-arc-transitive graphs as cyclic covers of them. The difficult problem of classifying all tetravalent graphs admitting a loosely attached half-arc-transitive action, which was proposed already by Wilson [58], is thus one of the central problems to be considered in future investigations on tetravalent half-arc-transitive graphs.

Chapter 5

The classification of half-arc-transitive generalizations of Bouwer graphs

In this chapter we focus on half-arc-transitive graphs of all possible valences. As mentioned before, half-arc-transitive graphs must have even valency, and the smallest possible valency for a half-arc-transitive graph is four.

In 1970 I. Z. Bouwer [8] constructed the following infinite family of graphs and proved that they are vertex- and edge-transitive.

Construction 5.1. Let $m \geq 3$, $n \geq 2$, $k \geq 2$ be integers such that $2^m \equiv 1$ modulo n. The Bouwer graph $\mathcal{B}(k,m,n)$ is defined to have vertex set

$$V(\mathcal{B}(k,m,n)) = \{(a;\boldsymbol{b}) \mid a \in \mathbb{Z}_m, \boldsymbol{b} \in \mathbb{Z}_n^{k-1}\}$$

with adjacency defined by the following rule:

$$(a;\boldsymbol{b}) \sim \begin{cases} (a+1;\boldsymbol{b}) & ; \quad 0 \leq a < m-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (a+1;\boldsymbol{b}+2^a\boldsymbol{e_i}) & ; \quad 0 \leq a < m-1,\ 1 \leq i \leq k-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1}, \end{cases}$$

where $\boldsymbol{e_i} \in \mathbb{Z}_n^{k-1}$ is the i-th standard vector and $\lambda \boldsymbol{v}$ denotes the usual scalar multiplication in the $\mathbb{Z}_n$-module $\mathbb{Z}_n^{k-1}$.

Note that the graphs $\mathcal{B}(k,m,n)$ (originally denoted by $X(N,m,n)$) have even valency $2k$. In [8] Bouwer also proved that the graph $\mathcal{B}(k,6,9)$ is not arc-transitive, providing an example of a half-arc-transitive graph for each even valency greater than 2.

We recall some of the concepts from the previous chapter that we shall use. Let Γ be a half-arc-transitive graph (with any even valency) and fix one of the two $\mathrm{Aut}(\Gamma)$-induced orientations. Then the alternets of Γ are the subgraphs of Γ (as well as of $\overrightarrow{\Gamma}$) corresponding to the equivalence classes of the reachability relation (the transitive hull of the relation where two edges uv and $u'v'$ of Γ are related if the corresponding oriented edges in $\overrightarrow{\Gamma}$ share a common head or a common tail). We point out that in general the alternets are not necessarily cycles as when the studied graph is tetravalent. Recall that the radius of Γ is denoted by $\mathrm{rad}(\Gamma)$ (and is the size of (any) head-set) and that if Γ has at least two alternets and if for any two alternets with a non-empty intersection the head-set of one coincides with the tail-set of the other, then Γ is tightly attached.

Of course half-arc-transitive graphs can be very far from being tightly attached. In fact, even half-arc-transitive graphs for which the intersection of two non-disjoint alternets contains only one vertex exist (see for instance [48] or [38] where an infinite family of such graphs has been constructed). Nevertheless, it seems that the tightly attached half-arc-transitive graphs have the least complicated structure, and so it is reasonable to first try to understand these graphs. As we pointed out in the previous chapter, for the tetravalent half-arc-transitive graphs only the tightly attached ones have been completely classified so far (see [32, 53]).

It is the aim of this chapter to construct an infinite family of tightly attached half-arc-transitive graphs of all even valences by generalizing the Bouwer graphs and moreover, prove that this family of graphs contains almost all tetravalent tightly attached half-arc-transitive graphs. Throughout of this chapter we write HAT for short instead of "half-arc-transitive".

5.1 The generalized Bouwer graphs

In this section we introduce the family of graphs that will play a central role in the rest of this chapter and prove some facts about their symmetry. Throughout the chapter we will constantly be dealing with elements of the ring $\mathbb{Z}_n$. All the calculations with such elements will thus be made modulo n.

Construction 5.2. Let $m \geq 3$, $n \geq 2$, $k \geq 2$ be integers and let $r \in \mathbb{Z}_n^*$, $t \in \mathbb{Z}_n$ be such that

$$r^m = 1,\ tr = t \text{ and } 1 + r + \cdots + r^{m-1} + kt = 0. \tag{5.1}$$

The generalized Bouwer graph $\mathcal{GB}(m, n, k; r, t)$ is defined to have vertex set

$$V(\mathcal{GB}(m, n, k; r, t)) = \{(a; \boldsymbol{b}) \mid a \in \mathbb{Z}_m, \boldsymbol{b} \in \mathbb{Z}_n^{k-1}\}$$

with adjacency defined by the following rule:

$$(a;\boldsymbol{b}) \sim \begin{cases} (a+1;\boldsymbol{b}) & ;\ 0 \leq a < m-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ = (a+1;\boldsymbol{b} + r^a \boldsymbol{e_i}) & ;\ 0 \leq a < m-1,\ 1 \leq i \leq k-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (0;\boldsymbol{b} + t\boldsymbol{1}) & ;\ a = m-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (0;\boldsymbol{b} + r^{m-1}\boldsymbol{e_i} + t\boldsymbol{1}) & ;\ a = m-1,\ 1 \leq i \leq k-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1}. \end{cases}$$

As in Construction 5.1 $\boldsymbol{e_i} \in \mathbb{Z}_n^{k-1}$ is the i-th standard vector, $\lambda \boldsymbol{v}$ denotes the usual scalar multiplication in the $\mathbb{Z}_n$-module $\mathbb{Z}_n^{k-1}$ and $\boldsymbol{1} = \boldsymbol{e_1} + \boldsymbol{e_2} + \cdots + \boldsymbol{e_{k-1}}$ is the all-ones vector.

Observe that the graph $\mathcal{GB}(m, n, k; r, t)$ is regular of valence $2k$. This family of graphs is a generalization of the Bouwer graphs $\mathcal{B}(k, m, n)$ [8] which coincide with the graphs $\mathcal{GB}(m, n, k; 2, 0)$ where $2^m = 1$ in $\mathbb{Z}_n$ (observe that the condition $1 + 2 + 2^2 + \cdots + 2^{m-1} = 0$ is implied by $2^m = 1$). Moreover, if $r = 2$, the condition

$t(r-1)=0$ is in fact $t=0$, and so the Bouwer graphs are precisely the generalized Bouwer graphs $\mathcal{GB}(m,n,k;r,t)$ with $r=2$. We would like to point out that the classification of HAT Bouwer graphs was given by Conder and Žitnik in [11] by the following thoerem.

Theorem 5.3. *[11] Let $m \geq 3$, $n \geq 3$ and $k \geq 2$ be integers such that $2^m = 1$ holds in $\mathbb{Z}_n$. Then the Bouwer graph $\mathcal{B}(k,m,n)$ is half-arc-transitive unless $n=3$ or one of the following possibilities occurs:*

- *$(k,n)=(2,5)$,*
- *$(k,m,n) \in \{(2,3,7),(2,6,7),(2,6,21)\}$.*

Remark 5.4. Note that $2^2=1$ holds in $\mathbb{Z}_n$ when $n=3$ while in all the remaining arc-transitive cases $k=2$ has to hold, and so the corresponding graphs are tetravalent. Since in all these cases n is odd ($n \in \{5,7,21\}$), one can thus also compare this result with [32, Theorem 3.4]. Therefore, the only arc-transitive Bouwer graphs of valence greater than 4 are the ones where the condition $r^2 \neq 1$ fails. As we prove in Theorem 5.30 this is also the only possibility that a generalized Bouwer graph of valence at least 6 is arc-transitive.

We now introduce some terminology and notation regarding the $\mathcal{GB}(m,n,k;r,t)$ graphs that will be used throughout the rest of the chapter. Let $\Gamma = \mathcal{GB}(m,n,k;r,t)$. For each $e \in E(\Gamma)$ we define its *label* $\mathcal{L}(e) \in \{0,1,\ldots,k-1\}$ in the following way. For the edges $e=(a;\boldsymbol{b})(a+1;\boldsymbol{b})$, where $0 \leq a \leq m-2$, and $e=(m-1;\boldsymbol{b})(0;\boldsymbol{b}+t\mathbf{1})$ we set $\mathcal{L}(e)=0$, while for any $1 \leq i \leq k-1$ we set the label of the edges $e=(a;\boldsymbol{b})(a+1;\boldsymbol{b}+r^a\boldsymbol{e_i})$, where $0 \leq a \leq m-2$, and of $e=(m-1;\boldsymbol{b})(0;\boldsymbol{b}+r^{m-1}\boldsymbol{e_i}+t\mathbf{1})$ to $\mathcal{L}(e)=i$. In this way each edge is given one of the labels from 0 to $k-1$. For each $0 \leq i \leq k-1$ we let $E_i = \{e \in E(\Gamma) \mid \mathcal{L}(e)=i\}$ and we denote $\mathcal{E} = \{E_i \mid i \in \{0,1,\ldots,k-1\}\}$. Clearly, $\mathcal{E}$ is a partition of $E(\Gamma)$ into k subsets of size mn^{k-1}. For each $a \in \mathbb{Z}_m$ set $\Gamma_a = \{(a;\boldsymbol{b}) \in V(\Gamma) \mid \boldsymbol{b} \in \mathbb{Z}_n^{k-1}\}$ and denote by $[\Gamma_a,\Gamma_{a+1}]$ the subgraph of Γ induced on the union $\Gamma_a \cup \Gamma_{a+1}$.

We investigate give a complete classification of the HAT members of the family of the generalized Bouwer graphs. The following is our main result

Theorem 5.5. *Let $m \geq 3$, $n \geq 2$, $k \geq 2$ be integers and let $r \in \mathbb{Z}_n^*$, $t \in \mathbb{Z}_n$ be such that $r^m=1$, $tr=t$ and $1+r+\cdots+r^{m-1}+kt=0$. Then the graph $\Gamma = \mathcal{GB}(m,n,k;r,t)$ is a Cayley graph of a metabelian group admitting a half-arc-transitive subgroup of automorphisms with respect to which it is tightly attached. Moreover, Γ is half-arc-transitive unless*

- *$r^2=1$ or*
- *$k=2$ and one of the following possibilities occurs:*
    - *$r^2=-1$.*
    - *$(m,n;r,t) \in \{(3,7;2,0),(3,7;4,0)\}$.*
    - *$(m,n)=(6,7n_0)$ for some $n_0 \geq 1$ with $7 \nmid n_0$, and there exists a unique solution $r' \in \{r,-r,r^{-1},-r^{-1}\}$ of the equation $2-r'-r'^2=0$ with $r' \equiv 5 \pmod 7$ and $2+r'+t'=0$, where $t'=t$ in the case that $r' \in \{r,r^{-1}\}$ and $t'=t+r+r^3+r^5$ in the case that $r' \in \{-r,-r^{-1}\}$.*

5.1.1 Vertex and edge-transitivity

We now prove that the graphs $\mathcal{GB}(m,n,k;r,t)$ are vertex- and edge-transitive. For the rest of this subsection let $\Gamma = \mathcal{GB}(m,n,k;r,t)$, where $m \geq 3$, $n \geq 2$, $k \geq 2$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ are as in (5.1). We first define certain permutations of $V(\Gamma)$ which ensure vertex-transitivity of Γ. In fact, as we show in Lemma 5.6 they guarantee that Γ is in fact a Cayley graph of a metabelian group.

Let $\sigma\colon V(\Gamma) \to V(\Gamma)$ be the permutation defined by the rule:

$$(a;\boldsymbol{b})\sigma = \begin{cases} (a+1; r\boldsymbol{b}) & ; \quad 0 \leq a \leq m-2, \ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (0; r\boldsymbol{b} + t\boldsymbol{1}) & ; \quad a = m-1, \ \boldsymbol{b} \in \mathbb{Z}_n^{k-1}. \end{cases} \tag{5.2}$$

Moreover, for each $1 \leq i \leq k-1$ let $\rho_i\colon V(\Gamma) \to V(\Gamma)$ be the permutation defined by the rule:

$$(a;\boldsymbol{b})\rho_i = (a; \boldsymbol{b} + \boldsymbol{e_i}). \tag{5.3}$$

Lemma 5.6. *The permutations σ and ρ_i, $1 \leq i \leq k-1$, from (5.2) and (5.3) are automorphisms of $\Gamma = \mathcal{GB}(m,n,k;r,t)$. Moreover, the subgroup $R = \langle \rho_1, \rho_2, \ldots, \rho_{k-1}\rangle \cong \mathbb{Z}_n^{k-1}$ of $\mathrm{Aut}(\Gamma)$ acts transitively on each of the sets Γ_a and the subgroup $T = \langle \sigma, R\rangle$ acts regularly on $V(\Gamma)$. In addition, $\sigma^{-1}\rho_i\sigma = \rho_i^r$ holds for each i, and so Γ is a Cayley graph of the metabelian group T. Moreover, each of the sets E_i is T-invariant, that is, each of the automorphisms from T preserves the label of each of the edges of Γ.*

Proof. That each of the permutations ρ_i is an automorphism of Γ is clear from the definition. It is also clear that R is isomorphic to the direct product of the subgroups generated by the ρ_i. Moreover, R clearly acts transitively on each of the sets Γ_a and preserves the label of each of the edges of Γ.

To prove that σ also preserves adjacency of Γ observe first that σ clearly maps the edges of label $0 \leq i \leq k-1$ from $[\Gamma_a, \Gamma_{a+1}]$, where $0 \leq a \leq m-3$, to the edges of label i from $[\Gamma_{a+1}, \Gamma_{a+2}]$. Taking into account that $tr = t$ and $r^m = 1$ it is easy to verify that σ also maps the edges from $[\Gamma_{m-2}, \Gamma_{m-1}]$ and $[\Gamma_{m-1}, \Gamma_0]$ to edges of Γ and preserves their labels. We leave the details to the reader.

Observe finally that σ cyclically permutes the sets Γ_a, that is $\Gamma_a\sigma = \Gamma_{a+1}$ for all $a \in \mathbb{Z}_m$. It thus follows that T is vertex-transitive and that each label class E_i is T-invariant. That $\sigma^{-1}\rho_i\sigma = \rho_i^r$ holds for all $1 \leq i \leq k-1$ can be verified directly from (5.2) and (5.3). Moreover, $\sigma^m = (\rho_1\rho_2\cdots\rho_{k-1})^t$ holds, and so $T/R \cong \mathbb{Z}_m$ and the group T is a regular metabelian group. □

We note that since $\sigma^m = (\rho_1\rho_2\cdots\rho_{k-1})^t$, the order of σ is $m\frac{n}{\gcd(n,t)}$. Moreover, the orbits of R coincide with the sets Γ_a and the quotient multi graph Γ/R of Γ with respect to the orbits of R is the m-cycle with k edges between every pair of adjacent vertices.

To prove that Γ is also edge-transitive we introduce two additional permutations of $V(\Gamma)$. Let $\theta\colon V(\Gamma) \to V(\Gamma)$ be the permutation defined by the rule:

$$(a;\boldsymbol{b})\theta = (a; (b_2, b_3, \ldots, b_{k-1}, b_1)) \ ; \ a \in \mathbb{Z}_m, \ \boldsymbol{b} \in \mathbb{Z}_n^{k-1}, \tag{5.4}$$

that is, θ cyclically permutes the coordinates of the vector $\boldsymbol{b}$. Moreover, let $\tau\colon V(\Gamma) \to V(\Gamma)$ be the permutation of $V(\Gamma)$ defined by the rule:

$$(a;\boldsymbol{b})\tau = (a;(1+r+\ldots+r^{a-1}-S(\boldsymbol{b}),b_2,\ldots,b_{k-1}))\ ;\ a\in\mathbb{Z}_m,\ \boldsymbol{b}\in\mathbb{Z}_n^{k-1}, \quad (5.5)$$

where $S(\boldsymbol{b}) = b_1 + b_2 + \cdots + b_{k-1}$ is the sum of the components of $\boldsymbol{b}$ and it is understood that $(0;\boldsymbol{b})\tau = (0;(-S(\boldsymbol{b}),b_2,\ldots,b_{k-1}))$.

Lemma 5.7. *The permutations θ and τ from (5.4) and (5.5) are automorphisms of $\Gamma = \mathcal{GB}(m,n,k;r,t)$ and the group $H = \langle\theta,\tau\rangle$ acts as the full symmetric group S_k on $\mathcal{E}$. Furthermore, the group $G = \langle T, H\rangle$, where T is as in Lemma 5.6, acts half-arc-transitively on Γ and the partitions $\mathcal{E}$ of $E(\Gamma)$ and $\{\Gamma_0,\Gamma_1,\ldots,\Gamma_{m-1}\}$ of $V(\Gamma)$ are both G-invariant. Moreover, Γ is tightly attached with respect to the action of G.*

Proof. We first prove that $\tau\in\mathrm{Aut}(\Gamma)$. It is clear that τ is a bijection, but the proof that it preserves adjacencies requires some work. We take some $e\in[\Gamma_a,\Gamma_{a+1}]$ and prove that $e\tau\in E(\Gamma)$ where we separate the argument depending on whether $a\notin\{0,m-1\}$, $a=0$ or $a=m-1$, and depending on whether $\mathcal{L}(e)$ is 0, 1 or at least 2. Let $e=uv$, where $u=(a;\boldsymbol{b})$ and $v\in\Gamma_{a+1}$. Write $b_1' = 1+r+\ldots+r^{a-1}-S(\boldsymbol{b})$ and $\boldsymbol{b'} = (b_1',b_2,\ldots,b_{k-1})$. We point out the arguments in the cases when $a\neq 0$ and leave the case $a=0$ to the reader.

Suppose first that $1\le a\le m-2$. If $\mathcal{L}(e)=0$, that is $v=(a+1;\boldsymbol{b})$, then

$$e\tau = (a;\boldsymbol{b'})(a+1;(b_1'+r^a,b_2,\ldots,b_{k-1}))\in E(\Gamma) \text{ and } \mathcal{L}(e\tau)=1.$$

If $\mathcal{L}(e)=1$, that is $v=(a+1;\boldsymbol{b}+r^a\boldsymbol{e_1})$, then $r^a - S(\boldsymbol{b}+r^a\boldsymbol{e_1}) = -S(\boldsymbol{b})$, and so we get

$$e\tau = (a;\boldsymbol{b'})(a+1;(b_1',b_2,\ldots,b_{k-1}))\in E(\Gamma) \text{ and } \mathcal{L}(e\tau)=0.$$

Finally, if $\mathcal{L}(e)\ge 2$, that is $v=(a+1;\boldsymbol{b}+r^a\boldsymbol{e_i})$, where $i\ge 2$, then

$$e\tau = (a;\boldsymbol{b'})(a+1;(b_1',b_2\ldots,b_{i-1},b_i+r^a,b_{i+1},\ldots,b_{k-1}))\in E(\Gamma) \text{ and } \mathcal{L}(e\tau)=i=\mathcal{L}(e).$$

Suppose now that $a=m-1$. Note that in this case $b_1' = 1+r+\cdots+r^{m-2}-S(\boldsymbol{b})$, and so (5.1) implies $b_1' = -S(\boldsymbol{b})-r^{m-1}-kt$. Now, if $\mathcal{L}(e)=0$, that is $v=(0;\boldsymbol{b}+t\boldsymbol{1})$, then

$$e\tau = (m-1;\boldsymbol{b'})(0;(-S(\boldsymbol{b})-(k-1)t,b_2+t,\ldots,b_{k-1}+t))\in E(\Gamma) \text{ and } \mathcal{L}(e\tau)=1.$$

If $\mathcal{L}(e)=1$, that is $v=(0;\boldsymbol{b}+r^{m-1}\boldsymbol{e_1}+t\boldsymbol{1})$, then

$$e\tau = (m-1;\boldsymbol{b'})(0;(-S(\boldsymbol{b})-r^{m-1}-(k-1)t,b_2+t,\ldots,b_{k-1}+t))\in E(\Gamma) \text{ and } \mathcal{L}(e\tau)=0.$$

Finally, if $\mathcal{L}(e)\ge 2$, that is $v=(0;\boldsymbol{b}+r^{m-1}\boldsymbol{e_i}+t\boldsymbol{1})$, where $i\ge 2$, then

$$e\tau = (m-1;\boldsymbol{b'})(0;(-S(\boldsymbol{b})-r^{m-1}-(k-1)t,b_2+t,\ldots,b_{i-1}+t,b_i+r^{m-1}+t,b_{i+1}+t,\ldots,b_{k-1}+t))\in E(\Gamma)$$

and $\mathcal{L}(e\tau)=i=\mathcal{L}(e)$.

In a similar fashion one can see that also in the case $a=0$ the automorphism τ interchanges the labels 0 and 1 of the edges and leaves all other labels fixed. Thus $\tau\in\mathrm{Aut}(\Gamma)$ with $E_0\tau=E_1$, $E_1\tau=E_0$ and $E_i\tau=E_i$ for all $i\ge 2$.

Since θ cyclically permutes the coordinates of the vector $\boldsymbol{b}$ it is clear that $\theta \in \mathrm{Aut}(\Gamma)$. It is also clear that $E_0\theta = E_0$, $E_{k-1}\theta = E_1$ and $E_i\theta = E_{i+1}$ for all $1 \leq i \leq k-1$. Since T clearly acts transitively (in fact regularly) on each of the sets E_i this proves that the group $G = \langle T, H\rangle$ acts edge-transitively on Γ and that the partitions $\mathcal{E}$ and $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1}\}$ are G-invariant. Moreover, the action of G on Γ is clearly half-arc-transitive (with either all of the edges oriented from Γ_a to Γ_{a+1} for all a or vice versa). Since the actions of τ and θ on $\mathcal{E}$ are $(E_0\, E_1)$ and $(E_1\, E_2 \ldots E_{k-1})$, respectively, it follows that G acts as the full symmetric group S_k on $\mathcal{E}$, as claimed. Since the head-sets (and the tail-sets) clearly coincide with the sets Γ_i, the graph Γ is tightly attached with respect to the action of G. □

5.1.2 Arc-transitivity and isomorphisms

In the previous subsection we proved that all $\mathcal{GB}(m,n,k;r,t)$ graphs admit a HAT subgroup of automorphisms. To obtain the complete classification of HAT $\mathcal{GB}(m,n,k;r,t)$ graphs we thus have to determine which of them are indeed HAT graphs and which are arc-transitive. We first present a rather obvious sufficient condition for $\mathcal{GB}(m,n,k;r,t)$ to be arc-transitive. We will prove later that, unless $k = 2$, the condition is in fact also necessary (see Theorem 5.30).

Proposition 5.8. *Let $m \geq 3$, $n \geq 2$, $k \geq 2$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). If $r^2 = 1$, then the graph $\mathcal{GB}(m,n,k;r,t)$ is arc-transitive.*

Proof. Denote $\Gamma = \mathcal{GB}(m,n,k;r,t)$ and suppose $r^2 = 1$. Since $r^m = 1$ this forces m to be even unless $r = 1$. In any case $r^{m-i} = r^i$ holds for all $i \in \mathbb{Z}_m$. By Lemma 5.7 we know that the group G acts half-arc-transitively on Γ, and so it suffices to find an automorphism of Γ interchanging a pair of adjacent vertices. To this end let φ be the permutation of $V(\Gamma)$ defined by the rule:

$$(a;\boldsymbol{b})\varphi = \begin{cases} (m-a; -r\boldsymbol{b} - t\mathbf{1}) & ;\quad a \neq 0 \\ (0; -r\boldsymbol{b}) & ;\quad a = 0. \end{cases}$$

To prove that φ preserves adjacency let $e = uv \in E(\Gamma)$ where $u = (a;\boldsymbol{b})$ and $v \in \Gamma_{a+1}$. Several different cases need to be considered, depending on whether $a \in \{0, m-1\}$ or not and whether $\mathcal{L}(e) = 0$ or not. In fact, as $rt = t$ and $r^2 = 1$, it is straightforward to verify that φ is an involution, and so it suffices to only consider the possibilities when $0 \leq a \leq m/2$. We present the argument for the cases when $1 \leq a \leq m/2$ and leave the case $a = 0$ to the reader.
Suppose first that $\mathcal{L}(e) = 0$, that is $v = (a+1;\boldsymbol{b})$. Then

$$e\varphi = (m-a; -r\boldsymbol{b} - t\mathbf{1})(m-a-1; -r\boldsymbol{b} - t\mathbf{1}) \in E(\Gamma).$$

Suppose now that $\mathcal{L}(e) \geq 1$, that is $v = (a+1; \boldsymbol{b} + r^a\boldsymbol{e_i})$ for some $1 \leq i \leq k-1$. Then

$$e\varphi = (m-a; -r\boldsymbol{b} - t\mathbf{1})(m-a-1; -r\boldsymbol{b} - r^{a+1}\boldsymbol{e_i} - t\mathbf{1}) \in E(\Gamma)$$

since $r^{m-a-1} = r^{a+1}$.
Then $\varphi \in \mathrm{Aut}(\Gamma)$. However, since the automorphism $\varphi\sigma$, where σ is as in (5.2), interchanges the pair of adjacent vertices $(0;\mathbf{0})$ and $(1;\mathbf{0})$, the graph Γ is arc-transitive, as claimed. □

Remark 5.9. In order to ensure $r^2 \neq 1$ also $2(r-1) \neq 0$ must hold. Namely, if $2(r-1) = 0$ then either $r = 1$ or n is even and $r - 1 = n/2$. But in this case r is odd (since $r \in \mathbb{Z}_n^*$), and so $r - 1$ is even, implying that $(r-1)^2 = 0$. Consequently $r^2 = 1$.

In the remainder of the chapter we undertake a thorough analysis of the $\mathcal{GB}(m, n, k; r, t)$ graphs. When doing so it will be beneficial to take into account some isomorphisms between these graphs which we now record.

Proposition 5.10. *Let $m \geq 3$, $n \geq 2$, $k \geq 2$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). Then the graphs $\mathcal{GB}(m, n, k; r, t)$ and $\mathcal{GB}(m, n, k; r^{-1}, t)$ are isomorphic.*

Proof. Denote $\Gamma = \mathcal{GB}(m, n, k; r, t)$ and $\Gamma' = \mathcal{GB}(m, n, k; r^{-1}, t)$ and let $\Psi \colon V(\Gamma) \to V(\Gamma')$ be the mapping defined by the rule:

$$(a; \boldsymbol{b})\Psi = \begin{cases} (m - a; -r\boldsymbol{b} - t\mathbf{1}) & ; \quad a \neq 0 \\ (0; -r\boldsymbol{b}) & ; \quad a = 0. \end{cases}$$

It is clear that Ψ is a bijection, and so we only need to prove that it preserves the adjacencies. To this end let $e = uv$ where $u = (a; \boldsymbol{b})$ and $v \in \Gamma_{a+1}$. As in the proof of Proposition 5.8 we distinguish cases depending on whether $a \in \{0, m-1\}$ or not and whether $\mathcal{L}(e) = 0$ or not. This time we deal with the cases when $\mathcal{L}(e) \geq 1$ and leave the case $\mathcal{L}(e) = 0$ to the reader. Denote $\mathcal{L}(e) = i$.
Suppose first that $a = 0$, that is $v = (1; \boldsymbol{b} + \boldsymbol{e_i})$. Then

$$e\Psi = (0; -r\boldsymbol{b})(m-1; -r\boldsymbol{b} - r\boldsymbol{e_i} - t\mathbf{1}) \in E(\Gamma')$$

since $(r^{-1})^{m-1} = r$.
Suppose now that $1 \leq a \leq m-2$, that is $v = (a+1; \boldsymbol{b} + r^a \boldsymbol{e_i})$. Then

$$e\Psi = (m-a; -r\boldsymbol{b} - t\mathbf{1})(m-a-1; -r\boldsymbol{b} - r^{a+1}\boldsymbol{e_i} - t\mathbf{1}) \in E(\Gamma')$$

since $(r^{-1})^{m-a-1} = r^{a+1}$.
Finally, suppose $a = m-1$, that is $v = (0; \boldsymbol{b} + r^{m-1}\boldsymbol{e_i} + t\mathbf{1})$. Then, since $r^m = 1$ and $tr = t$,

$$e\Psi = (1; -r\boldsymbol{b} - t\mathbf{1})(0; -r\boldsymbol{b} - \boldsymbol{e_i} - t\mathbf{1}) \in E(\Gamma').$$

Therefore Ψ is an isomorphism, and so $\mathcal{GB}(m, n, k; r, t) \cong \mathcal{GB}(m, n, k; r^{-1}, t)$. $\square$

Remark 5.11. Note that Lemma 5.7 implies that the graph $\mathcal{GB}(m, n, k; r, t)$, where $m \geq 3$, $n \geq 2$, $k \geq 2$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ are as in (5.1), is either HAT or arc-transitive. Moreover, if it is HAT, it is necessarily tightly attached. We can thus use the classification of tetravalent tightly attached HAT graphs [32, 53] to determine which of the $\mathcal{GB}(m, n, 2; r, t)$ graphs are HAT (see the proof of Theorem 7 at the end of Section 5.4). Taking into account also Theorem 5.3 and Proposition 5.10 we will thus assume $k \geq 3$ and $r, r^{-1} \neq 2$ in the remainder of this chapter, unless otherwise specified. Observe that this implies $n \geq 7$.

5.2 Cycles and 2-paths

In this section we introduce the terminology regarding the 2-paths of $\mathcal{GB}(m,n,k;r,t)$ which plays a crucial role in the proof of Theorem 5.30 that almost all such graphs are HAT. These notions were first introduced in [32] and were later successfully used in certain classifications of tetravalent HAT graphs (see [53, 54]). In our setting the idea of the proof of half-arc-transitivity of the $\mathcal{GB}(m,n,k;r,t)$ graphs (when $k \geq 3$) is based on the fact that these graphs contain the so-called generic 6-cycles (defined later in this section) and then a careful analysis of all possible 6-cycles in the graph, together with their interplay with the 2-paths of the graph, reveals that such graphs cannot be arc-transitive (unless of course $r^2 = 1$).

We begin with a simple observation concerning cycles in $\Gamma = \mathcal{GB}(m,n,k;r,t)$. The existence of a particular cycle in Γ of course imposes certain conditions on the parameters m, n, k, r and t. In fact, recall that each edge, say $e = (a;\boldsymbol{b})(a+1;\boldsymbol{b'})$ has a unique label $\mathcal{L}(e)$ which, if $\mathcal{L}(e) \neq 0$, was defined in such a way that $\boldsymbol{b'} = \boldsymbol{b} + r^a \boldsymbol{e}_{\mathcal{L}(e)}$ (if $a \neq m-1$) or $\boldsymbol{b'} = \boldsymbol{b} + r^{m-1}\boldsymbol{e}_{\mathcal{L}(e)} + t\mathbf{1}$ (if $a = m-1$). Thus, by traversing an edge e the vector $\boldsymbol{b}$ either does not change (if $\mathcal{L}(e) = 0$) or changes only in the coordinate corresponding to the nonzero label of e (except when we traverse the edges from $[\Gamma_{m-1}, \Gamma_0]$ in which case in addition either all of the coordinates increase or they all decrease by t). Therefore, the existence of a particular cycle in Γ in fact imposes $k-1$ conditions on m, n, k, r and t (one for each of the $k-1$ coordinates), each of the form

$$\delta_{i,0} + \delta_{i,1}r + \cdots + \delta_{i,m-1}r^{m-1} + \delta t = 0. \tag{5.6}$$

Here, $\delta_{i,j}$ is the number of i-labeled edges (of the cycle) from $[\Gamma_j, \Gamma_{j+1}]$, that we traverse in the direction from Γ_j to Γ_{j+1}, minus the number of i-labeled edges from $[\Gamma_j, \Gamma_{j+1}]$, that we traverse in the direction from Γ_{j+1} to Γ_j. Similarly, δ is the number of all edges (of the cycle) from $[\Gamma_{m-1}, \Gamma_0]$, that we traverse in the direction from Γ_{m-1} to Γ_0, minus the number of all the edges from $[\Gamma_{m-1}, \Gamma_0]$, that we traverse in the direction from Γ_0 to Γ_{m-1}. Note that the coefficient δ does not depend on the label i. An immediate consequence of this fact is the following useful observation.

Lemma 5.12. *Let $m \geq 3$, $n \geq 2$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). If $r \neq 1$ then no cycle of $\mathcal{GB}(m,n,k;r,t)$ can contain exactly one edge of some label $0 \leq i \leq k-1$.*

Proof. Denote $\Gamma = \mathcal{GB}(m,n,k;r,t)$. Let C be a cycle of Γ and e be an edge of C such that $\mathcal{L}(e) = i$ for some $0 \leq i \leq k-1$. By way of contradiction suppose that e is the only edge of C with label i. By Lemma 5.7 we can assume that $e = (0;\mathbf{0})(1;\boldsymbol{e_1})$ and C contains no other edges of label 1. The corresponding condition (5.6) for label 1 is thus $1 + \delta t = 0$. As $t(r-1) = 0$, multiplication by $r-1$ yields $r-1 = 0$, contradicting the assumption $r \neq 1$. □

Following [32] we now introduce some terminology concerning 2-paths and cycles of the graphs $\Gamma = \mathcal{GB}(m,n,k;r,t)$. Let $P = (u,v,w)$ be a 2-path of Γ. If the endvertices of P belong to the same set Γ_a we call it an *anchor*. More precisely, if for some a we have that $u, w \in \Gamma_a$ and $v \in \Gamma_{a+1}$, then P is a *positive anchor*, and it is a *negative anchor* otherwise. If P in a non-anchor, then it is a *glide* if both of its edges have the same label and is a *zig-zag* otherwise (see Figure 5.1).

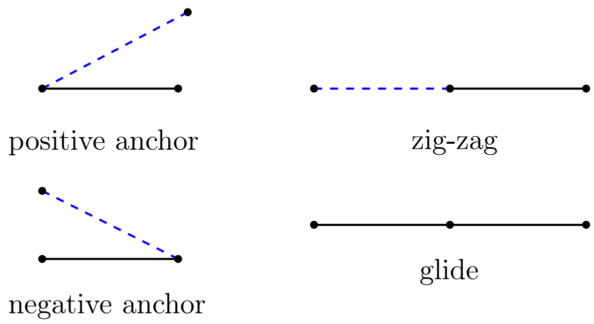


Figure 5.1: The two types of anchors, a glide and a zig-zag.

Let C be a cycle of Γ of length d. To each vertex v of C we assign one of the symbols a or n, depending on whether the corresponding 2-path in C with v as its internal vertex is an anchor or not, respectively. (Even though we are using n to represent a non-anchor and at the same time as one of the parameters of the graphs $\mathcal{GB}(m, n, k; r, t)$ this should cause no confusion.) In such a way the cycle C is assigned a sequence of length d with elements from the set $\{a, n\}$. We let the equivalence class of all sequences obtained from the above sequence by a reflection or a cyclic rotation be the *trace* of C. We say that the cycle C is *coiled* if none of its 2-paths is an anchor (and so its trace is n^d), and is *non-coiled* otherwise.

We now show that the graphs $\mathcal{GB}(m, n, k; r, t)$ contain certain 6-cycles which de not depend on the values of the parameters m, n, k, r and t. Let $m \geq 3$, $n \geq 2$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) and let $\Gamma = \mathcal{GB}(m, n, k; r, t)$. Let

$$C = ((0; \mathbf{0}), (1; \mathbf{0}), (0; -\boldsymbol{e_1}), (1; \boldsymbol{e_2} - \boldsymbol{e_1}), (0; \boldsymbol{e_2} - \boldsymbol{e_1}), (1; \boldsymbol{e_2}), (0; \mathbf{0}))\,.$$

Clearly, C consists of six pairwise distinct vertices, and so it is a 6-cycle of Γ. Moreover, the edges of C have three different labels and each antipodal pair of its edges has the same label. Observe that we needed $k \geq 3$ to be able to get the cycle C. We call the elements of the G-orbit of C, where G is as in Lemma 5.7, the *generic* 6-cycles of Γ. Note that the edge set of each generic 6-cycle is contained in some $[\Gamma_a, \Gamma_{a+1}]$, its edges have three different labels and each antipodal pair of its edges has the same label. The nature of the action of G implies the following straightforward lemma.

Lemma 5.13. *Let $m \geq 3$, $n \geq 2$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). For each vertex v of the graph $\mathcal{GB}(m, n, k; r, t)$ and each $0 \leq i_1 < i_2 < i_3 \leq k-1$ there exist exactly six generic 6-cycles containing v and having edges of labels i_1, i_2, i_3. Moreover, letting $0 \leq a \leq m-1$ be such that $v \in \Gamma_a$, three of these generic 6-cycles are contained in $[\Gamma_a, \Gamma_{a+1}]$ and the remaining three are contained in $[\Gamma_{a-1}, \Gamma_a]$.*

Remark 5.14. Observe that on any cycle of $\mathcal{GB}(m, n, k; r, t)$ the positive and negative anchors alternate. Therefore the number of anchors of a cycle is always even with half of them being positive and the other half negative.

We now show that if $r^2 \neq 1$ the generic 6-cycles are almost always the girth cycles of Γ.

Lemma 5.15. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). If $r^2 \neq 1$, then the shortest non-coiled cycle of $\Gamma = \mathcal{GB}(m,n,k;r,t)$ is of length at least 5. Moreover, the girth of Γ is 6, unless possibly if $m \leq 5$ and Γ contains coiled m-cycles or $m = 3$ and Γ contains 5-cycles of trace n^3a^2.*

Proof. Let C be a non-coiled cycle of Γ of smallest possible length d. Since C is non-coiled the Remark 5.14 implies that it contains at least two anchors, and so $d \geq 4$. If $d = 4$, then C is of trace a^4 or $anan$ (note that the trace a^2n^2 is not possible). By Lemma 5.12 the edges of C have exactly two labels. In view of the action of the group G trace a^4 would imply $n = 2$, while trace $anan$ would imply $1 \pm r = 0$, contradicting $r^2 \neq 1$. Therefore $d > 4$. Moreover, if $d = 5$ then clearly $m = 3$ must hold and C is of trace n^3a^2. Since Γ contains generic 6-cycles, this completes the proof. □

5.3 The 6-cycles

In this section we undertake a complete analysis of the possible "types" of 6-cycles of the $\mathcal{GB}(m,n,k;r,t)$ graphs. Throughout this section we let $m \geq 3$, $n \geq 7$, $k \geq 3$, $t \in \mathbb{Z}_n$ and $r \in \mathbb{Z}_n^*$ be as in (5.1) and we let $\Gamma = \mathcal{GB}(m,n,k;r,t)$. In accordance with Remark 5.11 we also assume $r, r^{-1} \neq 2$ and $r^2 \neq 1$.

We first determine all possible traces that 6-cycles of Γ can have. We then analyze 6-cycles of each of the possible traces to determine the conditions on the parameters m, n, k, r and t under which they exist.

Lemma 5.16. *The only possible traces of 6-cycles of Γ are*

- a^6,
- an^2an^2,
- a^3nan,
- a^2n^4,
- n^6.

Moreover, the traces a^2n^4 and n^6 are only possible if $m = 4$ and $m \in \{3, 6\}$, respectively.

Proof. Since the number of anchors in a cycle of Γ is even, there are only 8 possible traces of 6-cycles. Moreover, since $m \geq 3$, the traces a^4n^2, a^2na^2n and an^3an are clearly not possible. The last claim of the lemma is now also clear. □

We call the 6-cycles of traces an^2an^2, a^3nan and a^2n^4 the 6-cycles of types 1, 2 and 3, respectively. In Subsection 5.3.4 we in fact show that under our assumptions ($n \geq 7$ and $r^2 \neq 1$) the 6-cycles of type 3 do not exist.

We now analyze each of the 5 possible traces in separate subsections. We first deal with the noncoiled 6-cycles (the first four traces). For the next four subsections let

$$C = (x_0, x_1, x_2, x_3, x_4, x_5) \tag{5.7}$$

be a noncoiled 6-cycle of Γ. Since the negative and positive anchors alternate on cycles of Γ the cycle C contains at least one negative anchor. By Lemma 5.7 we can thus assume that (x_5, x_0, x_1) is a negative anchor and in particular that $x_0 = (0; \mathbf{0})$, $x_1 = (1; \boldsymbol{e_1})$ and $x_5 = (1; \mathbf{0})$, so that $\mathcal{L}(x_0x_1) = 1$ and $\mathcal{L}(x_0x_5) = 0$. Observe that, by Lemma 5.12, the 6-cycle C contains edges of at most three different labels, and so (since it is not coiled) it either contains edges of two or of three different labels.

5.3.1 The 6-cycles with trace a^6: generic

Assume that C has trace a^6. Since (x_1, x_0, x_5) is a negative anchor, it follows that x_2, $x_4 \in \Gamma_0$ and $x_3 \in \Gamma_1$. Moreover, since all of the 2-paths of C are anchors, no two consecutive edges of C have the same label. Since $n \neq 3$ the cycle C cannot have edges of just two labels, and so it has edges of exactly three labels, two of each label. Without loss of generality we can assume that the third label is 2. Then $\mathcal{L}(x_4x_5) = 2$, since otherwise the condition (5.6) for label 1 would yield $n = 2$. Similarly $\mathcal{L}(x_2x_3) \notin \{1, 2\}$, and so $\mathcal{L}(x_2x_3) = 0$. Therefore $\mathcal{L}(x_1x_2) = 2$, $\mathcal{L}(x_3x_4) = 1$, and so C is a generic 6-cycle. We therefore have the following result.

Lemma 5.17. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). Then the only 6-cycles of $\mathcal{GB}(m, n, k; r, t)$ of trace a^6 are the generic ones.*

5.3.2 The 6-cycles with trace an^2an^2: type 1

Since the trace of C is an^2an^2, we have that $x_2, x_4 \in \Gamma_2$ and $x_3 \in \Gamma_3$ (see Figure 5.2 and note that in the case of $m = 3$ this means $x_3 \in \Gamma_0$).

Suppose first that $\mathcal{L}(x_1x_2) = \mathcal{L}(x_4x_5)$. Observe that, when traversing C in the direction from x_0 to x_1, the traversal of x_1x_2 adds r while the traversal of x_4x_5 subtracts r in the same coordinate (since the edges have the same label). These two edges thus have no influence on the corresponding conditions (5.6) for such 6-cycles (for any of the labels). But then precisely one of the edges x_2x_3 and x_3x_4 has label 0 and the other has label 1. Since $r^2 \neq 1$ the only possibility is that $\mathcal{L}(x_2x_3) = \mathcal{L}(x_0x_1) = 1$ and $\mathcal{L}(x_3x_4) = \mathcal{L}(x_0x_5) = 0$. We can thus only choose the label of x_1x_2 (and x_4x_5) which can be arbitrary (also 0 and 1). In any case the condition under which such 6-cycles exist is $r^2 = -1$. In the case that $\mathcal{L}(x_1x_2) \in \{0, 1\}$ we say that C is of type 1.1 and we say that it is of type 1.2 otherwise (see Figure 5.2).

Suppose now that $\mathcal{L}(x_1x_2) \neq \mathcal{L}(x_4x_5)$. Observe that $r^2 \neq 1$ implies $r \pm 1 \neq 0$ and $r \pm r^2 \neq 0$, and so there are more than two edges of C with label $\mathcal{L}(x_1x_2)$ and more than two edges of C with label $\mathcal{L}(x_4x_5)$. Thus the edges of C have just two labels (namely 0 and 1), and so we have precisely four possibilities depending on which of the edges x_1x_2 and x_4x_5 is of label 0 and which of the edges x_2x_3 and x_3x_4 is of label 0. The four possibilities with the corresponding conditions are:

- $\mathcal{L}(x_1x_2) = \mathcal{L}(x_2x_3) = 0$ which occurs if and only if $1 - r - r^2 = 0$.
- $\mathcal{L}(x_1x_2) = \mathcal{L}(x_3x_4) = 0$ which occurs if and only if $1 - r + r^2 = 0$.
- $\mathcal{L}(x_2x_3) = \mathcal{L}(x_4x_5) = 0$ which occurs if and only if $1 + r - r^2 = 0$.
- $\mathcal{L}(x_3x_4) = \mathcal{L}(x_4x_5) = 0$ which occurs if and only if $1 + r + r^2 = 0$.

We call the 6-cycles of type 1 corresponding to the above four conditions 6-cycles of types 1.3, 1.4, 1.5 and 1.6, respectively (see also Figure 5.2). In Figure 5.2 and throughout the rest of the chapter the edges with the same color (or type of the lines - solid, doted or dashed) represent edges of the same label.

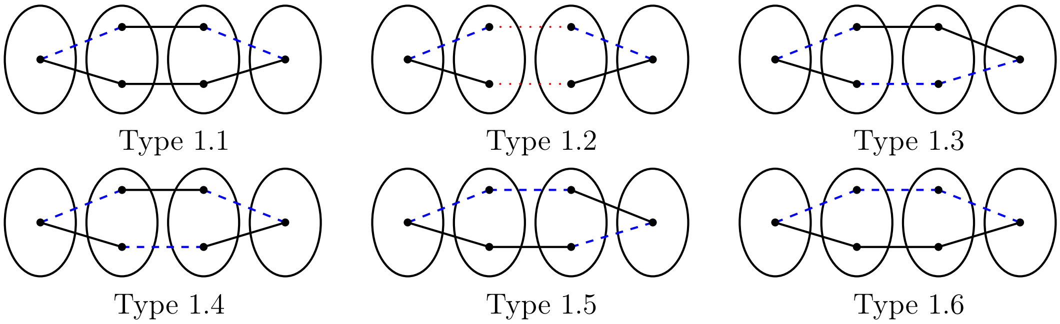


Figure 5.2: The 6-cycles of type 1.

As mentioned above the vertex x_3 might in general be equal to x_0 if $m = 3$, in which case C would not be a 6-cycle. However, it is easy to see that since $r^2 \neq 1$ this cannot happen, unless possibly if $1 + r + r^2 + t = 0$. Namely, since $r^3 = 1$, the condition $r^2 = -1$ would for instance imply $r = -1$, while $1 - r + r^2 + t = 0$ would imply $2(r - 1) = 0$ (multiply by $r^2 - 1$ and recall that $tr = t$). By the Remark 5.9 this is impossible. The remaining two possibilities are left to the reader. Note also that if $m = 3$ and $1 + r + r^2 + t = 0$, then Lemma 5.7 implies that $t = 0$ must also hold. This proves the following.

Lemma 5.18. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$. Then the graph $\mathcal{GB}(m, n, k; r, t)$ contains 6-cycles of type 1 if and only if at least one of the following conditions holds:*

- $1 + r^2 = 0$, *in which case the 6-cycles of types 1.1 and 1.2 exist.*
- $1 - r - r^2 = 0$, *in which case the 6-cycles of type 1.3 exist.*
- $1 - r + r^2 = 0$, *in which case the 6-cycles of type 1.4 exist.*
- $1 + r - r^2 = 0$, *in which case the 6-cycles of type 1.5 exist.*
- $1 + r + r^2 = 0$ *and if* $m = 3$ *then* $t \neq 0$, *in which case the 6-cycles of type 1.6 exist.*

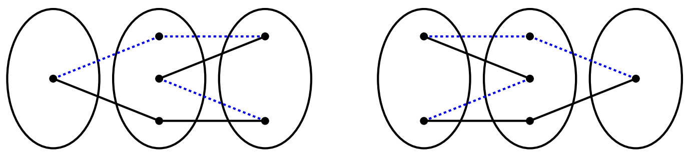
5.3.3 The 6-cycles with trace a^3nan: type 2

Note that in this case we have two options depending on whether the anchor surrounded by the two non-anchors is negative or positive. In the first case (we will say that such 6-cycles are of type 2.1) we can assume that the middle vertex of this anchor is x_0, and so $x_3 \in \Gamma_1$ and $x_2, x_4 \in \Gamma_2$. In the second case (we will say that such 6-cycles are of type 2.2) we can assume that the middle vertex of this positive anchor is x_2, and so $x_4 \in \Gamma_0$, $x_3 \in \Gamma_1$ and $x_2 \in \Gamma_2$ (see Figure 5.3).

We first prove that C contains only edges of the labels 0 and 1. By Lemma 5.12 the only other possibility is that C has edges of three different labels, two of each label. Suppose this is possible and take one of the two edges of the anchor, surrounded by two non-anchors, with nonzero label. Since there is only one more edge of C with this label, the corresponding condition (5.6) for this label is $1 \pm r = 0$ or $\pm 1 + r = 0$, all of which contradict $r^2 \neq 1$. Thus C only has edges of labels 0 and 1, and so the labels 0 and 1 alternate on the four consecutive edges of C corresponding to the part giving rise to a^3 in its trace.

Suppose first that C is of type 2.1. There are just two possibilities for C, depending on $\mathcal{L}(x_1x_2)$. If $\mathcal{L}(x_1x_2) = 0$, then $x_2 = (2; \boldsymbol{e_1})$, $x_3 = (1; (1-r)\boldsymbol{e_1})$ and $x_4 = (2; (1-r)\boldsymbol{e_1})$, and so $\mathcal{L}(x_4x_5) = 1$ implies that $1 - 2r = 0$. However, in this case $r^{-1} = 2$, which was assumed not to hold. Therefore $\mathcal{L}(x_1x_2) = 1$, and so $x_2 = (2; (1+r)\boldsymbol{e_1})$, $x_3 = (1; (1+r)\boldsymbol{e_1})$ and $x_4 = (2; (1+2r)\boldsymbol{e_1})$. Since $\mathcal{L}(x_4x_5) = 0$ this implies $1 + 2r = 0$.

Suppose next that C is of type 2.2. Then $x_4 = (0; -\boldsymbol{e_1})$ and $x_5 = (1; -\boldsymbol{e_1})$. Again there are just two possibilities for C, depending on $\mathcal{L}(x_1x_2)$. If $\mathcal{L}(x_1x_2) = 0$, then $x_2 = (2; \boldsymbol{e_1})$, and so $\mathcal{L}(x_2x_3) = 1$ implies that $1 - r = -1$, that is $r = 2$, which was assumed not to hold. Thus $\mathcal{L}(x_1x_2) = 1$, and so $x_2 = (2; (1+r)\boldsymbol{e_1})$. Then $\mathcal{L}(x_2x_3) = 0$ implies that $1 + r = -1$, and so $r = -2$ (see Figure 5.3).



Type 2.1 Type 2.2

Figure 5.3: The 6-cycles of type 2.

We finally observe that we cannot have 6-cycles of types 2.1 as well as 6-cycles of type 2.2. Namely, if this was the case then both $2r = -1$ and $r = -2$ hold, and so $n = 3$, a contradiction. This proves the following.

Lemma 5.19. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. Then $\mathcal{GB}(m,n,k;r,t)$ contains 6-cycles of type 2 if and only if one of the following conditions holds:*

- $1 + 2r = 0$, *in which case the 6-cycles of type 2.1 exist.*
- $2 + r = 0$, *in which case the 6-cycles of type 2.2 exist.*

Moreover, at most one of the two conditions can hold.

Remark 5.20. Note that if Γ contains 6-cycles of type 2.1, that is if $1 + 2r = 0$ holds, then $r^{-1} + 2 = 0$ holds. Therefore the graph $\mathcal{GB}(m,n,k;r^{-1},t)$, which is isomorphic to Γ by Proposition 5.10, contains 6-cycles of type 2.2. For the rest of the chapter we thus make the following agreement. If Γ contains 6-cycles of type 2, then we assume $r = -2$. Note that multiplying any of the conditions for 6-cycles of types 1.1–1.6 by r^{-2} we get one of the conditions for 6-cycles of types 1.1–1.6 (but perhaps for a different type) for the graph $\mathcal{GB}(m,n,k;r^{-1},t)$. Thus Γ contains 6-cycles of type 1 if and only if $\mathcal{GB}(m,n,k;r^{-1},t)$ does.

We now prove that, except in one very specific case, no two different conditions for the existence of 6-cycles of types 1.1–2.2 can hold simultaneously (note that the conditions for the 6-cycles of types 1.1 and 1.2 are the same).

Lemma 5.21. *Let $m \geq 3$, $k \geq 3$, $n \geq 7$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. Then at most one of the conditions for the existence of 6-cycles of types 1.1–2.2 of $\Gamma = \mathcal{GB}(m,n,k;r,t)$ can hold unless $\Gamma \cong \mathcal{GB}(6m_0,7,k;5,0)$ where $m = 6m_0$, in which case the only 6-cycles of types 1.1–2.2 that exist in Γ are the ones of types 1.4 and 2.2.*

Proof. Suppose first that Γ contains 6-cycles of type 2. According to the Remark 5.20 we can thus assume $r = -2$. It is now easy to check that each of the conditions for 6-cycles of types 1.1–1.6 contradicts $n \geq 7$, except for $1 - r + r^2 = 0$, in which case $n = 7$, $r = 5$ and $t = 0$ (since $t(r-1) = 0$). Since $r^m = 1$ also has to hold, $m = 6m_0$ for some $m_0 \geq 1$, and so $\Gamma = \mathcal{GB}(6m_0,7,k;5,0)$.

Suppose now that Γ contains no 6-cycles of type 2. To complete the proof we only have to verify that no two different conditions for 6-cycles of types 1.1–1.6 can hold simultaneously. It is clear that $1 + r^2 = 0$ prevents any of the conditions for 6-cycles of types 1.3–1.6 to also hold (since otherwise we either get $r = 2$, $r = 0$ or $r = -2$, none of which is possible). On the other hand, any two of the conditions for 6-cycles of types 1.3–1.6 imply $2r^a = 0$ for some $a \in \{0,1,2\}$, which again contradicts $n \geq 7$. □

5.3.4 The 6-cycles with trace a^2n^4: type 3

Note that, by Lemma 5.16, $m = 4$ holds in this case. By Lemma 5.7 we can assume that $x_2 \in \Gamma_2$, $x_3 \in \Gamma_3$ and $x_4 \in \Gamma_0$ (see Figure 5.4).

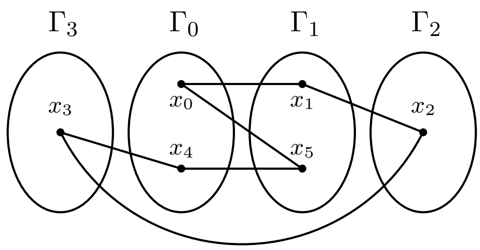


Figure 5.4: The 6-cycles of type 3.

We first show that all edges of C are of labels 0 and 1. Suppose to the contrary that C contains edges of three labels, two of each one (confront Lemma 5.12). Observe that the condition $1 + r + t = 0$ would contradict $r^2 \neq 1$ (multiply by $r - 1$), and so none of the edges x_1x_2 and x_3x_4 can have the same label as x_0x_1 or x_4x_5. It follows that $\mathcal{L}(x_4x_5) = 1$ (since otherwise $\mathcal{L}(x_1x_2) = \mathcal{L}(x_3x_4) = 0$, contradicting the fact that we only have two edges of label 0), and so $2 + t = 0$ holds. But this is also not possible, since otherwise multiplication by $r - 1$ yields $2(r-1) = 0$.

Therefore C contains only edges of labels 0 and 1, and so the condition (5.6) for label 2 (recall that $k \geq 3$) gives $t = 0$. Note that this implies $\mathcal{L}(x_4x_5) = 1$. Let now $\eta \in H$, where H is as in Lemma 5.7, be the automorphism interchanging the labels 0 and 2 and let $C' = C\eta$. Then C' is a 6-cycle of type 3 and it cannot have just two edges of label 2 since otherwise the corresponding condition (5.6) for label 2 would be $-1 + r^a = 0$ for some $a \in \{1, 2, 3\}$, contradicting $r^2 \neq 1$. This finally proves that C' contains three edges of label 1 and three of label 2 (otherwise the condition (5.6) for label 1 is $2 + t = 0$, which we already know is not possible). We have three possibilities depending on which of the edges x_1x_2, x_2x_3 and x_3x_4 is the remaining edge of label 1. It is easy to see that each one of them contradicts $n \geq 7$. For instance, if $\mathcal{L}(x_3x_4) = 1$, then the conditions (5.6) for labels 1 and 2 are

$$2 + r^3 = 0 \quad \text{and} \quad -1 + r + r^2 = 0,$$

respectively. Thus $2r = -1$ (recall that $r^4 = 1$), and so $0 = 2(-1+r+r^2) = -2-1-r$, that is $r = -3$. But then $n = 5 < 7$, a contradiction. We leave the other two possibilities to the reader. This proves the following lemma.

Lemma 5.22. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$. Then $\mathcal{GB}(m, n, k; r, t)$ contains no 6-cycle of type 3.*

5.3.5 The 6-cycles with trace n^6: coiled

In this subsection let $C = (x_0, x_1, x_2, x_3, x_4, x_5)$ be a coiled 6-cycle of Γ. By Lemma 5.16 either $m = 3$ or $m = 6$ must hold. We investigate each of the possibilities separately.

We first deal with the possibility $m = 3$. By Lemma 5.12 the 6-cycle C contains edges of at most three different labels. If C contains just two edges of some particular nonzero label, then the corresponding condition (5.6) is either of the form $2r^a + 2t = 0$ or $r^a + r^{a'} + 2t = 0$ for some $0 \leq a < a' \leq 2$. Multiplying by an appropriate power of r if necessary we thus get $2 + 2t = 0$ or $1 + r + 2t = 0$. But then multiplication by $r - 1$ yields $2(r - 1) = 0$ or $r^2 - 1 = 0$, both of which contradict $r^2 \neq 1$.

This proves (using τ if necessary) that either all of the edges of C have the same label or C has edges of two labels, three of each one. As in Subsection 5.3.4 (since $k \geq 3$) this implies $2t = 0$. Now, if C has edges of just one label, then (using τ if necessary) we get the condition $2 + 2r + 2r^2 = 0$. However, for C to actually be a 6-cycle $1 + r + r^2 + t \neq 0$ must hold and (again using τ if necessary) $t \neq 0$. It follows that n is even and $t = n/2 = 1 + r + r^2 + t$, implying that $1 + r + r^2 = 0$. But since n is even, $r \in \mathbb{Z}_n^*$ must be odd, and so $1 + r + r^2$ is odd, a contradiction. Thus there are no coiled 6-cycles with edges of just one label.

The 6-cycle C therefore contains edges of two labels, three of each one. If all of the edges of a given label are consecutive, we again get the condition $1 + r + r^2 = 0$ (recall that $2t = 0$). But as before, for C to indeed be a 6-cycle, $1 + r + r^2 + t \neq 0$ must hold, and so again $t = n/2$ with n even, leading to a contradiction. It is easy to see that the only other possibility is that the labels of edges on C alternate. Namely, if two consecutive edges on C of one label were "surrounded" by edges of the other label, we would get (multiplying by r if necessary) that both $2+r = 0$ and $1+2r = 0$

must hold, which contradicts $n \geq 7$. The only coiled 6-cycles that can exist in case of $m = 3$ are thus the ones on which two labels alternate and they in fact do exist if and only if $2t = 0$ and $1 + r + r^2 = 0$. The reader will check that in this case x_3 cannot be equal to x_0 so that we indeed get 6-cycles. We say that the corresponding 6-cycles are of type $c.1$ (see Figure 5.5).

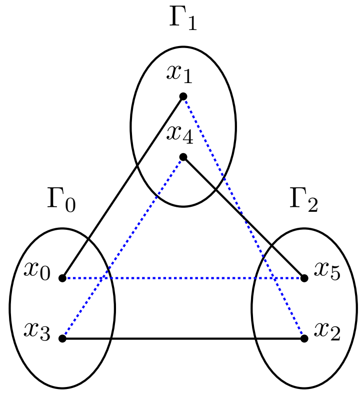


Figure 5.5: The coiled 6-cycles when $m = 3$.

We now consider the possibility $m = 6$. Note that $|V(C) \cap \Gamma_a| = 1$ for all $0 \leq a \leq 5$. In fact, we can assume that $x_a \in \Gamma_a$ for all $0 \leq a \leq 5$. As before, if C does not contain edges of all the possible k labels, then $t = 0$ must hold.

We first give a necessary and sufficient condition that C contains precisely two edges of a given nonzero label. Suppose that for some $i \in \{0, 1, \ldots, k-1\}$ the 6-cycle C contains exactly two edges with label i. If they are not antipodal on C, then they are either consecutive or at distance 2 on C. In the first case the corresponding condition (5.6) is $1 + r + t = 0$, and so multiplication by $r - 1$ yields $r^2 = 1$, a contradiction. In the second case we get $1 + r^2 + t = 0$, and so multiplication by $r^2 - 1$ yields $r^4 = 1$, a contradiction (since $r^6 = 1$). Therefore, if C contains exactly two edges of some (nonzero) label, they are antipodal on C.

Recall that, by Lemma 5.12, C contains edges of at most three different labels. If it contains edges of three labels, then by the above remarks each pair of edges with the same label is antipodal on C, and so we get the condition $1 + r^3 + t = 0$. We say that the corresponding 6-cycles are of type c.2.

If all of the edges of C are of the same label then $t = 0$ and $1+r+r^2+r^3+r^4+r^5 = 0$. Note that, since $1 + r + r^2 + r^3 + r^4 + r^5 + kt = 0$, the latter condition is implied by $t = 0$. We say that the corresponding 6-cycles are of type c.3.

Suppose finally that C contains edges of exactly two labels (recall that this implies $t = 0$). By Lemma 5.7 we can assume the two labels are 1 and 2. If for one of them there are just two edges of this label on C then they are antipodal and we get the condition $1 + r^3 = 0$. We say that the corresponding 6-cycles are of type c.4. We are left with the possibility that C contains three edges of label 1 and three edges of label 2. If the edges of label 1 (and thus also of label 2) are consecutive, we get the condition $1 + r + r^2 = 0$ (we say that the corresponding 6-cycles are of type c.5). If the two labels alternate on C then we get the condition $1 + r^2 + r^4 = 0$ (we say that the corresponding 6-cycles are of type c.6). The only other possibility would imply that both $1+r+r^3=0$ (for one label) and $1+r+r^4=0$ (for the other label) must hold. But then $r=1$, a contradiction. This finally proves the following lemma (see also Figure 5.6).

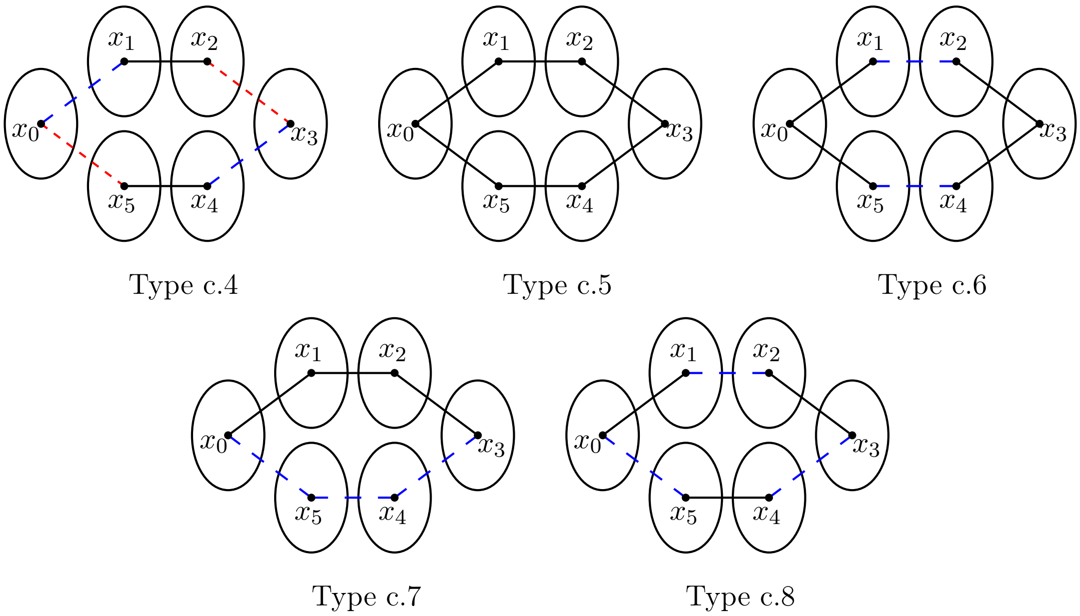


Figure 5.6: The coiled 6-cycles when $m=6$.

Lemma 5.23. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) where $r^2 \neq 1$. Then the graph $\mathcal{GB}(m,n,k;r,t)$ contains coiled 6-cycles if and only if at least one of the following conditions holds:*

- *$m=3$, $2t=0$ and $1+r+r^2=0$, in which case coiled 6-cycles of type c.1 exist.*
- *$m=6$ and $1+r^3+t=0$, in which case coiled 6-cycles of type c.2 exist.*
- *$m=6$ and $t=0$, in which case coiled 6-cycles of type c.3 exist.*
- *$m=6$, $t=0$ and $1+r^3=0$, in which case coiled 6-cycles of type c.4 exist.*
- *$m=6$, $t=0$ and $1+r+r^2=0$, in which case coiled 6-cycles of type c.5 exist.*
- *$m=6$, $t=0$ and $1+r^2+r^4=0$, in which case coiled 6-cycles of type c.6 exist.*

We end this section with the result in which we state the number of 6-cycles of a given type that each (negative) anchor, glide or zig-zag lies on (if such 6-cycles exist). From the above investigation of all the possible types of 6-cycles the proof is straightforward. We illustrate the argument for the 6-cycles of type 1.1 and leave the other types to the reader. By Lemma 5.18 the 6-cycles of type 1.1 exist if and only if $1+r^2=0$. Moreover, from Figure 5.2 we can see that the edges of such a 6-cycle are of two different labels. It is thus clear that a given negative anchor (by which we have already chosen the two labels) is contained on two 6-cycles of type 1.1 (we only need to choose the label of the two edges of the 6-cycle in question, not belonging to an anchor). Similarly, a given zig-zag (again, the labels have been chosen by it) is contained on two 6-cycles of type 1.1 (we need to choose which of the two edges of the zig-zag belongs to an anchor of the 6-cycle in question). Finally, a glide is contained in $2(k-1)$ different 6-cycles of type 1.1, since we need to choose one of the $k-1$ other labels, as well as which of the two edges of the glide belongs to an anchor of the 6-cycle in question.

Proposition 5.24. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. Then the only* 6*-cycles that the graph $\mathcal{GB}(m,n,k;r,t)$ can possibly contain are the generic ones and the ones of types* 1.1–1.6, 2.1–2.2 *and* c.1–c.6. *Moreover the corresponding conditions under which such* 6*-cycles exist, together with the number of such* 6*-cycles containing any given (negative) anchor, glide or zig-zag, are as given in Table 5.1.*

trace	type	condition	anchor	glide	zig-zag
a^6	generic	none	$k-2$	0	0
an^2an^2	1.1	$1+r^2=0$	2	$2(k-1)$	2
	1.2	$1+r^2=0$	$k-2$	0	$2(k-2)$
	1.3	$1-r-r^2=0$	1	$k-1$	1
	1.4	$1-r+r^2=0$	1	0	2
	1.5	$1+r-r^2=0$	1	$k-1$	1
	1.6	$1+r+r^2=0$, if $m=3$ then $t\neq 0$	1	$2(k-1)$	0
a^3nan	2.1	$1+2r=0$	2	$k-1$	0
	2.2	$r=-2$	2	$k-1$	0
n^6	c.1	$m=3$, $2t=0$, $1+r+r^2=0$	0	0	1
	c.2	$m=6$, $1+r^3+t=0$	0	0	$k-2$
	c.3	$m=6$, $t=0$	0	1	0
	c.4	$m=6$, $t=0$, $1+r^3=0$	0	$k-1$	2
	c.5	$m=6$, $t=0$, $1+r+r^2=0$	0	$2(k-1)$	1
	c.6	$m=6$, $t=0$, $1+r^2+r^4=0$	0	0	1

Table 5.1: Counting the number of 6-cycles containing a specific anchor, glide or zig-zag.

5.4 Half-arc-transitivity of $\mathcal{GB}(m,n,k;r,t)$

In this section we finally prove Theorem 7. Throughout this section we let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. We denote $\Gamma = \mathcal{GB}(m,n,k;r,t)$ and let $G = \langle T, H \rangle$ be as in Lemma 5.7.

The proof of half-arc-transitivity of Γ is divided into three cases where we first deal with the case that Γ contains no coiled 6-cycles and then separately analyze the possibilities $m=3$ and $m=6$. But first we need the following two results which prove that Γ could only be arc-transitive if the sets Γ_a, $a \in \mathbb{Z}_m$, were not blocks of imprimitivity for the whole automorphism group $\mathrm{Aut}(\Gamma)$ in which case the anchors and zig-zags would be in the same $\mathrm{Aut}(\Gamma)$-orbit.

Proposition 5.25. *Let $\Gamma = \mathcal{GB}(m, n, k; r, t)$ where $m \geq 3$, $k \geq 3$, $n \geq 7$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ are as in (5.1) with $r^2 \neq 1$. Suppose that the sets Γ_a, $a \in \mathbb{Z}_m$, are blocks of imprimitivity for* $\mathrm{Aut}(\Gamma)$. *Then Γ is half-arc-transitive.*

Proof. Suppose to the contrary that Γ is arc-transitive. Let $v = (0; \mathbf{0})$ and $\psi \in \mathrm{Aut}(\Gamma)_v$ be such that $(1; \mathbf{0})\psi = (m-1; -t\mathbf{1})$. Since the sets Γ_a are blocks for $\mathrm{Aut}(\Gamma)$, $\Gamma_0\psi = \Gamma_0$ and $\Gamma_1\psi = \Gamma_{m-1}$. Moreover, Lemma 5.17 implies that ψ maps generic 6-cycles to generic 6-cycles. Recall that, by Lemma 5.7, the subgroup $H \leq \mathrm{Aut}(\Gamma)_v$ acts as the full symmetric group S_k on $\mathcal{E}$, and so we can assume that for each $0 \leq i \leq k-1$ the automorphism ψ maps the unique edge of label i in $[\Gamma_0, \Gamma_1]$, incident to v, to the unique edge of label i in $[\Gamma_{m-1}, \Gamma_0]$, incident to v.

For each pair of vertices $w, z \in \Gamma_0 \cup \Gamma_1$ let $d_{0,1}(w, z)$ be the length of a shortest wz-path in the (connected) subgraph $[\Gamma_0, \Gamma_1]$ (of course $d_{0,1}(w, w) = 0$). Further, for each $uw = e \in [\Gamma_0, \Gamma_1]$, where $u \in \Gamma_0$ and $w \in \Gamma_1$, let $d_v(e) = max\{d_{0,1}(v, u), d_{0,1}(v, w)\}$. Note that $d_{0,1}(v, u)$ is even and $d_{0,1}(v, w)$ is odd. We claim that ψ preserves the labels of all the edges $e = uw$ in $[\Gamma_0, \Gamma_1]$. We prove this by induction on $d_v(e)$.

First, if $d_v(e) = 1$, then $u = v$, and so ψ preserves the label of e by hypothesis. If $d_v(e) = 2$, then $d_{0,1}(v, w) = 1$ and $d_{0,1}(v, u) = 2$. Let $z \in N(v) \cap \Gamma_1$ be the unique vertex with $\mathcal{L}(vz) = \mathcal{L}(uw)$. Note that $z \neq w$ and that u is the only neighbor $y \neq v$ of w in Γ_0 such that the path (z, v, w, y) is not contained on a generic 6-cycle of Γ. Since $\mathcal{L}(vw) = \mathcal{L}(v(w\psi))$ and $\mathcal{L}(vz) = \mathcal{L}(v(z\psi))$, it follows that $u\psi$ is the only neighbor $y' \neq v$ of $w\psi$ in Γ_0 such that the path $(z\psi, v, w\psi, y')$ is not contained on a generic 6-cycle of Γ. It follows that $\mathcal{L}((u\psi)(w\psi)) = \mathcal{L}(v(z\psi)) = \mathcal{L}(uw)$, and so $\mathcal{L}(e\psi) = \mathcal{L}(e)$. Suppose now that for some $\ell \geq 2$ the automorphism ψ preserves the labels of all edges $e' \in [\Gamma_0, \Gamma_1]$ with $d_v(e') \leq \ell$ and let $e \in [\Gamma_0, \Gamma_1]$ be such that $d_v(e) = \ell + 1$. Let $P = (v, v_1, v_2, \ldots, v_\ell, v_{\ell+1})$ be a $vv_{\ell+1}$-path in $[\Gamma_0, \Gamma_1]$ of length $\ell + 1$ such that $e = v_\ell v_{\ell+1}$. Note that for each $e' \in [\Gamma_0, \Gamma_1]$, incident to $v_{\ell-1}$, we have $d_v(e') \leq \ell$, and so $\mathcal{L}(e'\psi) = \mathcal{L}(e')$ holds. Since $d_{v_{\ell-1}}(e) = 2$, we can apply a similar argument as in the case $d_v(e) = 2$ to prove that ψ preserves the label of e. Therefore, by induction, ψ preserves the labels of all the edges in $[\Gamma_0, \Gamma_1]$, as claimed. Observe that this completely determines the action of ψ on $[\Gamma_0, \Gamma_1]$. Namely, since this subgraph is connected, we just take an arbitrary path in $[\Gamma_0, \Gamma_1]$ from $(0; \mathbf{0})$ to the vertex, whose image we want to determine, and then follow the corresponding path in $[\Gamma_{m-1}, \Gamma_0]$ from $(0; \mathbf{0})$ with the same sequence of labels. This way we find that $(1; \mathbf{0})\psi = (m-1; -t\mathbf{1})$ and $(1; r\boldsymbol{e_1})\psi = (m-1; -r \cdot r^{m-1}\boldsymbol{e_1} - t\mathbf{1}) = (m-1; -\boldsymbol{e_1} - t\mathbf{1})$. Since $(2; r\boldsymbol{e_1})$ is a common neighbor of $(1; \mathbf{0})$ and $(1; r\boldsymbol{e_1})$, it thus follows that $(2; r\boldsymbol{e_1})\psi \in \Gamma_{m-2}$ is a common neighbor of $(m-1; -t\mathbf{1})$ and $(m-1; -\boldsymbol{e_1} - t\mathbf{1})$. Since the vectors $-t\mathbf{1}$ and $-\boldsymbol{e_1} - t\mathbf{1}$ differ only in the first component, it is clear that either $-r^{m-2} = -1$ or $r^{m-2} = -1$ has to hold. Since the former case contradicts $r^2 \neq 1$, we must have that $r^2 = -1$. In particular, $\mathcal{L}((m-1; -t\mathbf{1})((2; r\boldsymbol{e_1})\psi)) = 0$. However, we can now repeat the same argument for the vertices $(1; \mathbf{0})$ and $(1; r\boldsymbol{e_2})$ and their neighbor $(2; r\boldsymbol{e_2})$ to get that also $\mathcal{L}((m-1; -t\mathbf{1})((2; r\boldsymbol{e_2})\psi)) = 0$ has to hold, which is of course impossible since ψ is a bijection. $\square$

Lemma 5.26. *Let $m \geq 3$, $k \geq 3$, $n \geq 7$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$. If the graph $\Gamma = \mathcal{GB}(m,n,k;r,t)$ is arc-transitive, then for each negative anchor (u,v,w) there exists $\alpha \in \mathrm{Aut}(\Gamma)$ fixing u and v and mapping this negative anchor to a zig-zag such that $\mathcal{L}(vw) = \mathcal{L}(v(w\alpha))$. Similarly, for each positive anchor (u,v,w) there exists $\alpha' \in \mathrm{Aut}(\Gamma)$ fixing u and v and mapping this positive anchor to a zig-zag such that $\mathcal{L}(vw) = \mathcal{L}(v(w\alpha'))$.*

Proof. First note that, by Lemma 5.7, it suffices to prove the result for the anchors (u,v,w) with $v = (0;\mathbf{0})$. Now, since $r^2 \neq 1$, Proposition 5.25 implies that at least one of the sets Γ_a, $a \in \mathbb{Z}_m$, is not a block of imprimitivity for $\mathrm{Aut}(\Gamma)$, and so Lemma 5.6 implies that none of them is. There thus exists some $\beta \in \mathrm{Aut}(\Gamma)$ such that $\Gamma_1\beta \neq \Gamma_1$ and $\Gamma_1\beta \cap \Gamma_1 \neq \emptyset$.

Choose an arbitrary negative anchor (u,v,w) with $v = (0;\mathbf{0})$. Since the subgraph $[\Gamma_0,\Gamma_1]$ is connected there exist $x,y \in \Gamma_1$ such that $x\beta \in \Gamma_1$, $y\beta \notin \Gamma_1$ and (x,z,y) is a negative anchor for some $z \in \Gamma_0$. Then either $z\beta \in \Gamma_0$ or $z\beta \in \Gamma_2$. In the later case let $\beta' \in G$ be such that $z\beta\beta' = z$ and note that then $y\beta\beta' \in \Gamma_1$ and $x\beta\beta' \in \Gamma_{m-1}$. Letting $\beta'' \in G$ be an automorphism fixing z and interchanging x and y we can thus replace β by $\beta''\beta\beta'$ to obtain an automorphism mapping x to Γ_1, z to Γ_0 and y to Γ_{m-1}. Lemma 5.7 thus implies that we can find some $\tilde{\beta} \in \mathrm{Aut}(\Gamma)$ fixing the vertices u and v and mapping w to Γ_{m-1}.

We claim that we can in fact assume $(u,v,w\tilde{\beta})$ is a zig-zag. Namely, if it is a glide then take any $w' \in (N(v) \cap \Gamma_1)\backslash\{u,w\}$ (note that since $k \geq 3$ the vertex w' does exist) and observe that $w'\tilde{\beta} \in \Gamma_{m-1} \cup \Gamma_1$. Now, if $w'\tilde{\beta} \in \Gamma_{m-1}$, then $\tilde{\beta}$ maps the negative anchor (u,v,w') to the zig-zag $(u,v,w'\tilde{\beta})$ while if $w'\tilde{\beta} \in \Gamma_1$ the negative anchor (w',v,w) is mapped by $\tilde{\beta}$ to the zig-zag $(w'\tilde{\beta},v,w\tilde{\beta})$. Applying additional automorphisms from G if necessary we can thus indeed find $\alpha \in \mathrm{Aut}(\Gamma)$, mapping the negative anchor (u,v,w) to a zig-zag with u and v being fixed (see Figure 5.7). Moreover, by Lemma 5.7 we can even assume that $\mathcal{L}(vw) = \mathcal{L}(v(w\alpha))$, as claimed.

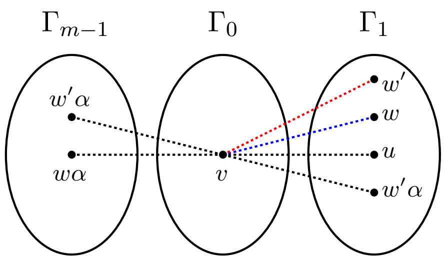


Figure 5.7: A negative anchor can be mapped to an appropriate zig-zag.

An analogous argument shows that the same can be done for any positive anchor. □

We are now ready to prove that the graph Γ is HAT. As mentioned at the beginning of this section we separate the proof in three parts. In all three cases the proof is by contradiction and relies heavily on the information from Table 5.1. We first deal with the case when no coiled 6-cycle exists.

Lemma 5.27. *Let $m \geq 3$, $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. If $\Gamma = \mathcal{GB}(m, n, k; r, t)$ contains no coiled 6-cycles then it is half-arc-transitive. In particular, if $m \notin \{3, 6\}$, then Γ is half-arc-transitive.*

Proof. Suppose to the contrary that Γ is arc-transitive. By Lemma 5.26 there exists $\alpha \in \mathrm{Aut}(\Gamma)$ fixing $v = (0; \mathbf{0})$ and $x_1 = (1; \boldsymbol{e}_1)$, while mapping the negative anchor $P = (x_0, v, x_1)$, where $x_0 = (1; \mathbf{0})$, to the zig-zag $(x_0\alpha, v, x_1)$ with $\mathcal{L}((x_0\alpha)v) = 0$.

Since, by assumption, there are no coiled 6-cycles in Γ, Proposition 5.24 implies that the only non-generic 6-cycles that can possibly exist in Γ are the ones of types 1.1–2.2. Since the zig-zag $P\alpha$ is contained in as many 6-cycles as the negative anchor P, it follows that Γ contains at least one non-generic 6-cycle. Moreover, by Lemma 5.21 precisely one of the conditions for 6-cycles of types 1.1–2.2 holds. Namely, otherwise 6-cycles of types 1.4 and 2.2 exist, and so Proposition 5.24 implies that each zig-zag is contained on two 6-cycles (of type 1.4) while each negative anchor is contained on $k-2+1+2 \geq 4$ different 6-cycles, a contradiction. This further implies that none of the two conditions for 6-cycles of type 2 can hold, since none of the corresponding 6-cycles contains zig-zags. Therefore, precisely one of the conditions for 6-cycles of type 1.1–1.6 holds.

Suppose first that $1 + r^2 = 0$ holds. It follows that each negative anchor, as well as each zig-zag, is contained on precisely $2k - 2$ different 6-cycles. Observe that x_0 has exactly $2k - 2$ different neighbors $w \neq v$ such that the 3-path (x_1, v, x_0, w) is contained in at least one 6-cycle of Γ ($k-2$ in Γ_0 and k in Γ_2). It follows that $x_0\alpha$ also has exactly $2k - 2$ different neighbors $w' \neq v$ such that the 3-path $(x_1, v, x_0\alpha, w')$ is contained in at least one 6-cycle of Γ. However, it is easy to check that $x_0\alpha$ has exactly k such neighbors (one in Γ_{m-2} and $k - 1$ in Γ_0), and so $2k - 2 = k$, contradicting $k \geq 3$ (see Figure 5.8).

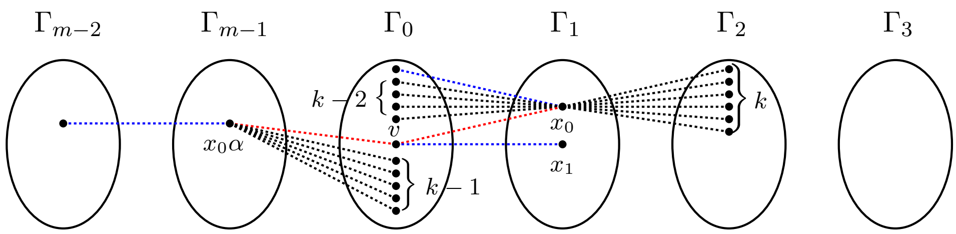


Figure 5.8: The neighbors of x_0 and $x_0\alpha$ giving rise to suitable 3-paths.

We are left with the possibility that $r^2 \neq -1$ and precisely one of the conditions for 6-cycles of types 1.3–1.6 holds. Therefore, the number of 6-cycles containing P is $k - 1 \geq 2$, and so the condition for 6-cycles of type 1.4 must hold. It follows that $k = 3$ and Γ has no 6-cycles containing glides. Let $C_1 = (x_0, v, x_1, x_2, x_3, x_4)$ and $C_2 = (x_0, v, x_1, x_5, x_6, x_7)$ be the two 6-cycles containing P, where C_1 is the generic one (see Figure 5.9). Since $P\alpha$ is a zig-zag, the 6-cycles $C_1\alpha$ and $C_2\alpha$ are both of type 1.4. Therefore the pair $\{x_4, x_7\}$ is mapped to the pair $\{x', x''\}$ of the unique two neighbors of $x_0\alpha$ such that $\mathcal{L}(x'(x_0\alpha)) = \mathcal{L}(x''(x_0\alpha)) = 1$. But then the zig-zag (x_4, x_0, x_7) is mapped to the glide $(x', x_0\alpha, x'')$, a contradiction since glides are in a different $\mathrm{Aut}(\Gamma)$-orbit than zig-zags. $\square$

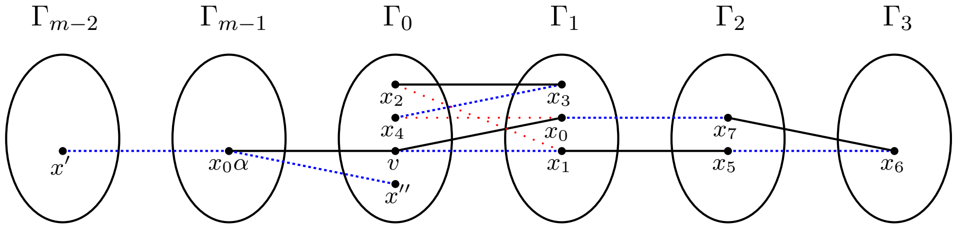


Figure 5.9: The situation in the case that the 6-cycles of type 1.4 exist.

We next deal with the case $m = 3$.

Lemma 5.28. *Let $\Gamma = \mathcal{GB}(3,n,k;r,t)$ where $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ are as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. Then Γ is half-arc-transitive.*

Proof. In view of Lemma 5.27 we can assume that Γ contains coiled 6-cycles. By Lemma 5.23 both $2t = 0$ and $1 + r + r^2 = 0$ hold. Consequently, the proof of Lemma 5.21 implies that the only 6-cycles of types 1.1–2.2 that Γ can possibly contain are the ones of type 1.6 (which occurs if and only if $t \neq 0$).

Now, if $t \neq 0$, then Proposition 5.24 implies that each negative anchor of Γ lies on $k-1 \geq 2$ different 6-cycles, while each zig-zag lies on just one (coiled) 6-cycle, and so Lemma 5.26 implies that Γ is HAT. We are thus left with the possibility that $t = 0$, in which case Γ only has generic 6-cycles and those of type c.1. By Proposition 5.24 each negative anchor lies on $k-2$ (generic) 6-cycles, each zig-zag lies on a single (coiled) 6-cycle, and no 6-cycle of Γ contains glides. In view of Lemma 5.26 we can thus assume $k = 3$. Note also that each automorphism of Γ maps glides to glides.

By way of contradiction suppose Γ is arc-transitive. By Lemma 5.26 there exists some $\alpha \in \mathrm{Aut}(\Gamma)$, fixing $v = (0;\mathbf{0})$ and $x_1 = (1;\boldsymbol{e_1})$ while mapping the negative anchor $P = (x_1, v, x_0)$, where $x_0 = (1;\mathbf{0})$, to the zig-zag $(x_0\alpha, v, x_1)$ with $\mathcal{L}((x_0\alpha)v) = 0$. Let $C_1 = (x_0, v, x_1, x_2, x_3, x_4)$ be the generic 6-cycle containing P and let $C_2 = (y_0, v, y_1, y_2, y_3, y_4)$ be the coiled 6-cycle containing $P\alpha$, where $y_i = x_i\alpha$ for all $0 \leq i \leq 4$ (note that $y_1 = x_1$). Since (y_0, v, x_0) is a glide, the above remarks imply that $y_0\alpha = x_0$, and so α interchanges C_1 and C_2, and thus also x_i and y_i for each $0 \leq i \leq 4$.

Let (v, z_1, z_2, v) be the 3-cycle of label 2 such that $z_1 \in \Gamma_1$ and $z_2 \in \Gamma_2$ (see Figure 5.10). Of course, α either fixes both z_1 and z_2, or interchanges them. With no loss of generality we can assume it interchanges them (otherwise replace α by $(\alpha\beta)^2$, where $\beta \in H$ from Lemma 5.7 interchanges the labels 0 and 2). The negative anchor $Q = (z_1, v, x_0)$ is then mapped to the positive anchor (z_2, v, y_0), and so the generic 6-cycle containing Q is mapped to the generic 6-cycle containing $Q\alpha$. In particular, the vertex $u \in \Gamma_0 \cap N(x_0)$ with $\mathcal{L}(x_0u) = 1$ is mapped to the unique vertex in $\Gamma_0 \cap N(y_0)$ such that $\mathcal{L}(y_0(u\alpha)) = 1$. But then the positive anchor (u, x_0, x_4) is mapped to the glide $(u\alpha, y_0, y_4)$ (with label 1), a contradiction.

□

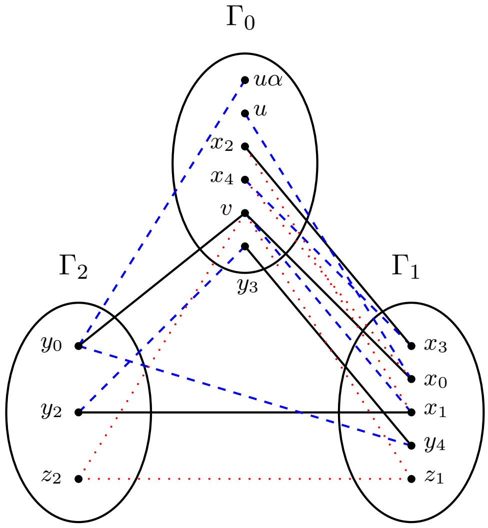


Figure 5.10: The situation in the case that the 6-cycles of type $c.1$ exist.

We finally deal with the most difficult case, that is when $m = 6$. The proof is rather long and technical since there are several possibilities that need to be analyzed.

Lemma 5.29. *Let $\Gamma = \mathcal{GB}(6, n, k; r, t)$ where $n \geq 7$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ are as in (5.1) with $r^2 \neq 1$ and $r, r^{-1} \neq 2$. Then Γ is half-arc-transitive.*

Proof. In view of Lemma 5.27 we can assume that Γ contains coiled 6-cycles. By way of contradiction suppose Γ is arc-transitive. By Lemma 5.26 there exists some $\alpha \in \mathrm{Aut}(\Gamma)$, fixing $v = (0; \mathbf{0})$ and $x_1 = (1; \boldsymbol{e_1})$ and mapping the negative anchor $P = (x_1, v, x_0)$, where $x_0 = (1; \mathbf{0})$, to the zig-zag $(x_0\alpha, v, x_1)$, where $\mathcal{L}((x_0\alpha)v) = 0$.

We first claim that the only 6-cycles of type 1 that Γ can possibly contain are the ones of type 1.6. The condition $r^6 = 1$ and $n \geq 7$ clearly prevents $1 + r^2 = 0$ to hold. Similarly, if $1 - r - r^2 = 0$, then $r^4 = 2 - 3r$, and so $1 = r^6 = 5 - 8r$, implying that $8r = 4$. Then $8 - 8r = 8r^2 = 4r$, and so $12r = 8$. It is now easy to see that either $n = 8$ or $n = 4$, both of which contradict $r^2 \neq 1$. A similar argument shows that the condition $1 + r - r^2 = 0$ is also not possible. Suppose finally that $1 - r + r^2 = 0$ holds. Multiplication by $1 + r$ yields $1 + r^3 = 0$ while multiplication by t yields $tr^2 = 0$, and so $t = 0$. Since $r^3 = -1$ we have that $1 + r^2 + r^4 = 1 - r + r^2 = 0$, and so coiled 6-cycles of types c.2, c.3, c.4 and c.6 all exist. However, then Proposition 5.24 implies that a zig-zag lies on at least $k + 3$ different 6-cycles while a negative anchor lies on at most $k + 1$ different 6-cycles of Γ, contradicting the assumption that α maps the negative anchor P to a zig-zag. This proves our claim that the only possible 6-cycles of type 1 are the ones of type 1.6. Note that in the case that these 6-cycles do exist $r^6 = 1$ implies $1 - r^3 = 0$.

By Lemma 5.21, Proposition 5.24 and the above claim it is clear that the negative anchor P lies in $k - 2$, $k - 1$ or k different 6-cycles of Γ. Let us denote this number by ℓ. We analyze each of the three possibilities for ℓ separately. Before doing so we fix some notation. Since $k \geq 3$ there exists $x_2 = (1; \boldsymbol{e_2})$. For each $0 \leq i \leq 2$ let

$y_i \in \Gamma_5 \cap N(v)$ be the unique vertex such that $\mathcal{L}(vy_i) = i$. Thus, for each $0 \le i \le 2$ the 2-path (y_i, v, x_i) is a glide with both edges of label i. Recall also that $x_0\alpha = y_0$.

Case 1: $\ell = k$.
By Proposition 5.24 and the Remark 5.20 we can assume that $r = -2$ and that none of the conditions for 6-cycles of types 1.1–1.6 holds. Since the zig-zag $P\alpha$ is contained on k different 6-cycles and the condition for 6-cycles of type c.5 implies the one for 6-cycles of type 1.6, coiled 6-cycles of types c.2 and c.4 must exist. But then $t = 0$ and $1 + r^3 = 0$, and so $r = -2$ implies $n = 7$. However, in this case the 6-cycles of type 1.4 also exist, a contradiction.

Case 2: $\ell = k - 1$.
By Proposition 5.24 and the above claim 6-cycles of type 1.6 exist, that is $1 + r + r^2 = 0$, and consequently also $r^3 = 1$. Now, $1 + r^3 = 2 \neq 0$ and moreover $1 + r^3 + t = 2 + t \neq 0$, since otherwise multiplication by $r - 1$ yields $2(r - 1) = 0$. Thus the only types of coiled 6-cycles that can exist are c.3, c.5 and c.6. Since $P\alpha$ lies on $k - 1$ different 6-cycles, coiled 6-cycles of types c.5 and c.6 exist, and so also those of type c.3 do. Thus $k = 3$ and $t = 0$.

Observe that the number of 6-cycles containing a glide is 9, implying that α maps glides to glides. However, since (y_0, v, x_0) is a glide, this implies $y_0\alpha = x_0$, and so the 6-cycle C of type c.5, containing the zig-zag (y_0, v, x_1), is mapped to the (unique) generic 6-cycle containing the anchor P. But then the four glides of C are mapped to anchors, a contradiction.

Case 3: $\ell = k - 2$.
By Proposition 5.24 we only have generic and coiled 6-cycles in this case. Moreover, since each zig-zag has to lie in exactly $k-2$ (coiled) 6-cycles we either have that only coiled 6-cycles of type c.2 exist (note that the 6-cycles of type c.3 cannot exist in this case since otherwise also those of type c.4 would) or only coiled 6-cycles of types c.3 and c.6 exist (note that the condition for type c.4 implies the one for type c.2 while the condition for type c.5 implies the one for type 1.6) in which case $k = 3$ clearly has to hold. In fact, $k = 3$ must hold also in the case that only coiled 6-cycles of type c.2 exist, since otherwise one of the conditions (5.6) for a given coiled 6-cycle is $t = 0$. We analyze both possibilities.

Subcase 3.1: The only coiled 6-cycles are the ones of type c.2.
In this situation each anchor and each zig-zag lie on a unique 6-cycle while no 6-cycle contains glides. It follows that $\mathrm{Aut}(\Gamma)$ has two orbits on the set of all 2-paths of Γ, one consisting of all glides and the other consisting of all anchors and all zig-zags.

Since glides are mapped to glides, $y_0\alpha = x_0$ and α fixes both x_1 and y_1. Then either α fixes both x_2 and y_2 or interchanges them. With no loss of generality we can assume that the later holds since otherwise we can replace α with $(\alpha\beta)^2$, where $\beta \in G$ is the automorphism fixing v, preserving label 1 and interchanging the labels 0 and 2 (recall that G is as in Lemma 5.7). Now, let $u, w \in \Gamma_0$ be the neighbors of x_1 and y_1, respectively, such that $\mathcal{L}(ux_1) = \mathcal{L}(wy_1) = 2$, let $u_1 \in \Gamma_5$ be the neighbor of u such that $\mathcal{L}(uu_1) = 1$ and let $w_1 \in \Gamma_1$ be the neighbor of w with $\mathcal{L}(ww_1) = 1$. Observe that $u_1 = (5; (1 - r^5)\boldsymbol{e_1} - \boldsymbol{e_2} - t\boldsymbol{1})$ and $w_1 = (1; (1 - r^5)\boldsymbol{e_1} + r^5\boldsymbol{e_2})$, and so $z = (0; (1 - r^5)\boldsymbol{e_1} + (-1 + r^5)\boldsymbol{e_2})$ is a common neighbor of u_1 and w_1 with $\mathcal{L}(u_1z) =$

$\mathcal{L}(zw_1) = 2$ (see Figure 5.11). Therefore, $(y_1, v, x_1, u, u_1, z, w_1, w)$ is an 8-cycle of Γ. Since (x_0, v, x_1, u) is contained in a generic 6-cycle and $x_0\alpha = y_0$, it is clear that $u\alpha \in \Gamma_2$ with $\mathcal{L}(x_1(u\alpha)) = 2$. Similarly $w\alpha \in \Gamma_4$ with $\mathcal{L}(y_1(w\alpha)) = 2$. Moreover, the unique neighbor $z' \in \Gamma_0$ of x_1 with $\mathcal{L}(z'x_1) = 0$ is interchanged with the unique neighbor of x_1 in Γ_2 with $\mathcal{L}(x_1(z'\alpha)) = 0$, and so the 3-path $(u_1, u, x_1, z'\alpha)$, which lies on a 6-cycle of type c.2, is mapped to a 3-path on a 6-cycle of type c.2, implying that $u_1\alpha \in \Gamma_3$. An analogous argument shows that $w_1\alpha \in \Gamma_3$. However, the glide (u_1, z, w_1) would then have to be mapped to a 2-path with both endvertices in Γ_3, which clearly cannot be a glide, a contradiction.

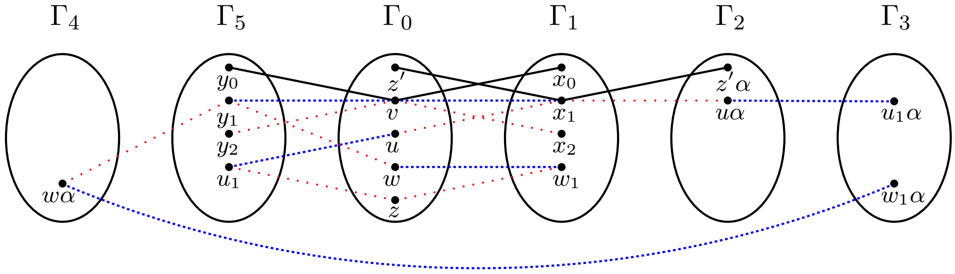


Figure 5.11: The situation in Subcase 3.1.

Subcase 3.2: The coiled 6-cycles of Γ are of types c.3 and c.6.
Recall that in this situation the number of 6-cycles containing a glide, a zig-zag and an anchor is one. Then we have two possibilities. Either glides and anchors are in the same $\mathrm{Aut}(\Gamma)$-orbit or not.

Suppose first they are not. Then glides are mapped to glides, and so we can again assume that α interchanges x_0 with y_0 and x_2 with y_2. Letting C be the unique (generic) 6-cycle containing P, it is clear that $C\alpha$ is of type c.6, and so the labels of its edges alternate between 0 and 1. In particular, letting $u \in \Gamma_0$ be as above we have that $u\alpha \in \Gamma_2$ with $\mathcal{L}(x_1(u\alpha)) = 0$. Let $z_1, z_2 \in \Gamma_0 \cap N(x_0)$ be such that $\mathcal{L}(z_i x_0) = i$ (see Figure 5.12). Then the 3-path (x_2, v, x_0, z_1) is contained on a generic 6-cycle, and so since (x_2, v, x_0) is mapped to a positive anchor, $z_1\alpha \in \Gamma_0 \cap N(y_0)$ with $\mathcal{L}(y_0(z_1\alpha)) = 1$. Likewise, the 3-path (x_1, v, x_0, z_2) is contained on a generic 6-cycle, and so $z_2\alpha \in \Gamma_4$ with $\mathcal{L}(y_0(z_2\alpha)) = 1$. But then the anchor (z_1, x_0, z_2) is mapped to a glide, a contradiction.

Suppose now that $\mathrm{Aut}(\Gamma)$ has just one orbit on the set of all 2-paths of Γ. Then there exists some $\beta \in \mathrm{Aut}(\Gamma)$ fixing both v and x_1 while mapping x_0 to y_1. Before continuing let us denote by $C = (v_0, v_1, v_2, v_3, v_4, v_5)$, where $v_0 = v$, $v_1 = x_1$ and $v_5 = x_0$, the generic 6-cycle containing the 2-path (x_0, v, x_1). Since β fixes v_0 and v_1 and maps v_5 to y_1 it is clear that $C\beta$ is of type c.3 with all of its edges being of label 1. Note that this implies $v_i\beta \in \Gamma_i$ for all $0 \leq i \leq 5$. For each $0 \leq i \leq 5$ we also let $u_i \in \Gamma_0 \cup \Gamma_1$ be the unique neighbor of v_i, different from v_{i-1} and v_{i+1}, where indices are computed modulo 6 (see Figure 5.13). We distinguish two possibilities depending on whether $x_2\beta \in \Gamma_1$ or not.

First assume that $x_2\beta \in \{x_0, x_2\}$. We can then assume that $x_2\beta = x_2$ (otherwise replace β by $\beta\gamma$ where $\gamma \in G$ fixes v and interchanges the labels 0 and 2). This

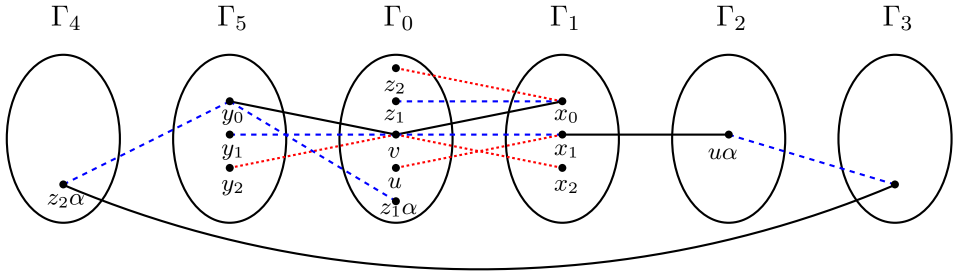


Figure 5.12: The situation in Subcase 3.2 with glides not being in the same Aut(Γ)-orbit as anchors.

completely determines the action of β on the vertices u_i, $0 \leq i \leq 5$. We already know that $u_0\beta = u_0$ (since $u_0 = x_2$). Next, since (u_0, v_0, v_1, u_1) is on a generic 6-cycle and β fixes u_0, v_0 and v_1, it also fixes u_1. Then, since (u_1, v_1, v_2, u_2) is on a generic 6-cycle and $(u_1, v_1, v_2)\beta = (u_1, v_1, v_2\beta)$ is a zig-zag with $\mathcal{L}(v_1(v_2\beta)) = 1$, it follows that $u_2\beta \in \Gamma_3$ with $\mathcal{L}((v_2\beta)(u_2\beta)) = 0$. In a similar way we then find that $u_3\beta \in \Gamma_2$ with $\mathcal{L}((v_3\beta)(u_3\beta)) = 2$, then $u_4\beta \in \Gamma_5$ with $\mathcal{L}((v_4\beta)(u_4\beta)) = 2$, and finally that $u_5\beta \in \Gamma_4$ with $\mathcal{L}((v_5\beta)(u_5\beta)) = 0$. However, this implies that the 3-path (u_0, v_0, v_5, u_5), which is on a generic 6-cycle, is mapped to the 3-path $(u_0, v_0, v_5\beta, u_5\beta)$ which is not contained on any 6-cycle (since it has edges of three different labels), a contradiction.

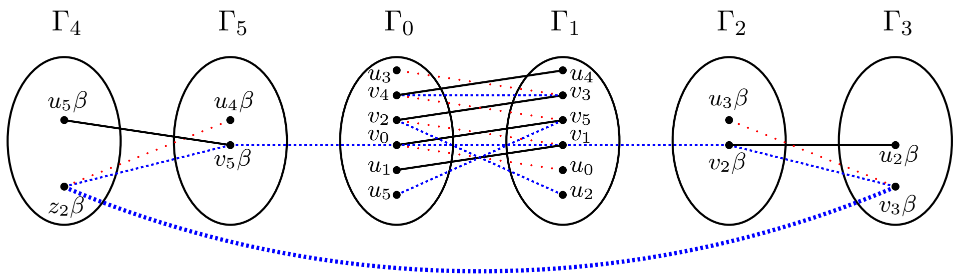


Figure 5.13: The situation in Subcase 3.2 with all 2-paths in the same Aut(Γ)-orbit.

We are left with the possibility that $x_2\beta \in \{y_0, y_2\}$. Similarly as before we can assume that $x_2\beta = y_2$. An analogous argument as above then shows that $u_1\beta \in \Gamma_2$ with $\mathcal{L}(v_1(u_1\beta)) = 2$, $u_2\beta \in \Gamma_1$ with $\mathcal{L}((v_2\beta)(u_2\beta)) = 0$, $u_3\beta \in \Gamma_4$ with $\mathcal{L}((v_3\beta)(u_3\beta)) = 0$, $u_4\beta \in \Gamma_3$ with $\mathcal{L}((v_4\beta)(u_4\beta)) = 2$, and finally that $u_5\beta \in \Gamma_0$ with $\mathcal{L}((v_5\beta)(u_5\beta)) = 2$. However, the 3-path (u_0, v_0, v_5, u_5), which is on a generic 6-cycle, is then mapped to the 3-path $(u_0\beta, v_0, v_5\beta, u_5\beta)$, which is not on any 6-cycle (since it is in $[\Gamma_5, \Gamma_0]$ but contains two edges of label 2). $\square$

Combining together Theorem 5.3, Propositions 5.8 and 5.10, and Lemmas 5.27, 5.28 and 5.29 we have the classification of all HAT generalized Bouwer graphs of valence at least 6.

Theorem 5.30. *Let $m \geq 3$, $n \geq 2$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1). Then the graph $\mathcal{GB}(m,n,k;r,t)$ is half-arc-transitive if and only if $r^2 \neq 1$ and is arc-transitive otherwise.*

Taking into account the results of [32, 53] we can now also prove Theorem 7.

Proof (of Theorem 7). That $\Gamma = \mathcal{GB}(m,n,k;r,t)$ is a Cayley graph of a metabelian group has been established in Lemma 5.7. Moreover, it admits a half-arc-transitive group with respect to which it is tightly attached. It thus remains to determine whether Γ is HAT or not. The case $k \geq 3$ has been covered by Theorem 5.30, so that we can assume $k = 2$ for the rest of the proof. Moreover, in view of Lemma 5.7 we only need to determine whether Γ is arc-transitive or not.

Now, if n is even, then $r \in \mathbb{Z}_n^*$ implies that r is odd, and so $1+r+\cdots+r^{m-1}+2t = 0$ forces m to be even. In this case [53, Theorem 1.3] is directly applicable. We thus only need to consider the case when n is odd. We show that in this case $\Gamma \cong \mathcal{X}_o(m,n;r)$ (from construction 4.3). Take the mapping $\Theta\colon V(\Gamma) \to V(\mathcal{X}_o(m,n;r))$ defined by the rule (note that since $k = 2$ the vertices of Γ are of the form $(a;j)$, where $a \in \mathbb{Z}_m$, $j \in \mathbb{Z}_n$)

$$(a;j)\Theta = v_a^{2j-1-r-\cdots-r^{a-1}},$$

where it is understood that $(0;j)\Theta = v_0^{2j}$. Since n is odd, $2 \in \mathbb{Z}_n^*$, and so Θ is clearly a bijection. It is easy to verify that Θ preserves adjacency on each $[\Gamma_a, \Gamma_{a+1}]$, where $0 \leq a \leq m-2$. Let us check that it also preserves adjacency on $[\Gamma_{m-1}, \Gamma_0]$. To this end let $j \in \mathbb{Z}_n$ be arbitrary and consider the two neighbors $w_0 = (0; j+t)$ and $w_1 = (0; j + r^{m-1} + t)$ of $w = (m-1; j)$. Since $1 + r + \cdots + r^{m-1} + 2t = 0$, it follows that $w\Theta = v_{m-1}^{2j+r^{m-1}+2t}$. Then both $w_0\Theta = v_0^{2j+2t}$ and $w_1\Theta = v_0^{2j+2r^{m-1}+2t}$ are neighbors of $w\Theta$ in $\mathcal{X}_o(m,n;r)$, and so Θ is an isomorphism of graphs. We can now apply [32, Theorem 3.4].

It is clear that we only need to make sure that item (iii) from [32, Theorem 3.4] corresponds to the third item of the $k = 2$ part in Theorem 7. To this end assume $m = 6$ and $n = 7n_0$, where n_0 is odd with $7 \nmid n_0$, and that $r' \in \{r, -r, r^{-1}, -r^{-1}\}$ is such that $2 - r' - r'^2 = 0$ and $r' \equiv 5 \pmod 7$. It is easy to see that then $1 + r' + \cdots + r'^5 = 9r' - 3$ holds. Moreover, since n is odd it follows that $2 + r' + t' = 0$ if and only if $2(2 + r' + t') = 0$. From the definition of t' it is clear that $2t' = -(1 + r' + \cdots + r'^5)$ (in the case that $r' \in \{-r, -r^{-1}\}$ we have $2t' = 2t + 2r + 2r^3 + 2r^5 = -1 + r - r^2 + r^3 - r^4 + r^5$), and so $2(2 + r' + t') = 4 + 2r' + 3 - 9r' = 7(1 - r')$. It is thus clear that $7(r' - 1) = 0$ (as required by [32, Theorem 3.4]) if and only if $2 + r' + t' = 0$, which completes the proof. □

It follows that the family of $\mathcal{GB}(m,n,k;r,t)$ graphs contains all tetravalent tightly attached HAT graphs, except for the non-Cayley ones. In fact, the following holds.

Proposition 5.31. *Let Γ be a connected tetravalent tightly attached half-arc-transitive graph. Then Γ is isomorphic to some $\mathcal{GB}(m,n,2;r,t)$ if and only if it is a Cayley graph which occurs if and only if it is not isomorphic to some $\mathcal{X}_o(m,n;r)$ with m even, n odd and $r \in \mathbb{Z}_n^*$ such that $r^m = -1$ but $r^2 \neq -1$.*

Proof. Suppose first that Γ is of even radius. By [53] it is isomorphic to some $\mathcal{GB}(m,n,2;r,t)$ with m and n both even, and so Theorem 7 guarantees that Γ is a Cayley graph (this also follows from [53, Theorem 2.1], which ensures that the subgroup $\langle\rho,\sigma\rangle$ is regular).

Suppose now that Γ is of odd radius. By [32] it is isomorphic to some $\Gamma = \mathcal{X}_o(m,n;r)$ where n is odd and $r \in \mathbb{Z}_n$ satisfies $r^m = \pm 1$. Let now ρ, σ and τ be as in [32], that is $v_a^j\rho = v_a^{j+1}$, $v_a^j\sigma = v_{a+1}^{rj}$ and $v_a^j\tau = v_a^{-j}$ for all $a \in \mathbb{Z}_m$, $j \in \mathbb{Z}_n$. The subgroup $\langle\rho,\sigma\rangle$ is clearly transitive and moreover, $\sigma^m = 1$ if and only if $r^m = 1$ while $\sigma^m = \tau$ whenever $r^m = -1$. Thus, if $r^m = 1$, the graph Γ is a Cayley graph of the regular group $\langle\rho,\sigma\rangle$. Moreover, since $\mathcal{X}_o(m,n;r) \cong \mathcal{X}_o(m,n;-r)$ (see [32, Proposition 4.1]), the graph Γ is a Cayley graph also in the case that $r^m = -1$ with m odd.

We claim that if $r^m = -1$ and m is even, the graph Γ is not a Cayley graph. To see this note first that it was shown in [32] that $\mathrm{Aut}(\Gamma) = \langle\rho,\sigma,\tau\rangle$. Thus Γ is a Cayley graph if and only if we can find an index 2 transitive subgroup of $\mathrm{Aut}(\Gamma)$ (which is then regular). Since σ and $\tau\sigma$ are the only two automorphisms of Γ mapping v_0^0 to v_1^0, one of them must be included in such a subgroup. But $\sigma^m = \tau$ fixes a vertex, and so σ is excluded. However, as σ and τ commute and m is even, $(\tau\sigma)^m = \sigma^m$, and so $\mathrm{Aut}(\Gamma)$ contains no regular subgroup. By Theorem 7 it follows that in this case Γ is not isomorphic to a $\mathcal{GB}(m,n,2;r,t)$ graph.

To complete the proof we now only have to verify that if $r^m = 1$ the graph $\mathcal{X}_o(m,n;r)$ is isomorphic to $\mathcal{GB}(m,n,2;r,t)$ for some t. Recall that n is odd and set $t = \frac{n-1}{2}(1 + r + \cdots + r^{m-1})$. Clearly $1 + r + \cdots + r^{m-1} + 2t = 0$ and since $r^m = 1$, we also have $t(r-1) = 0$. We can thus construct the graph $\mathcal{GB}(m,n,2;r,t)$, which, by the proof of Theorem 7, is isomorphic to $\mathcal{X}_o(m,n;r)$ and the proof is complete. □

5.5 The automorphism group and isomorphisms

To conclude this chapter we determine the automorphism group of the HAT $\mathcal{GB}(m,n,k;r,t)$ graphs and all possible isomorphisms among them. Since these questions have been answered for the tetravalent graphs in [32, 53], we only deal with the case $k \geq 3$ here. Throughout this section let $\Gamma = \mathcal{GB}(m,n,k;r,t)$ be HAT and let $G \leq \mathrm{Aut}(\Gamma)$ be as in Lemma 5.7. First we need a lemma.

Lemma 5.32. *Let $m \geq 3$, $n \geq 5$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$. Then the only automorphism of the graph $\mathcal{GB}(m,n,k;r,t)$ fixing the vertex $(0;\mathbf{0})$ and each of its neighbors in Γ_1 is the identity.*

Proof. Let $\Gamma = \mathcal{GB}(m,n,k;r,t)$ and let $\alpha \in \mathrm{Aut}(\Gamma)$ be such that it fixes the vertex $(0;\mathbf{0})$ and all of its neighbors in Γ_1. By Theorem 7 the graph Γ is HAT and by Lemma 5.7 its alternets coincide with the induced subgraphs $[\Gamma_a,\Gamma_{a+1}]$, $a \in \mathbb{Z}_m$. The sets Γ_a are then of course blocks of imprimitivity for $\mathrm{Aut}(\Gamma)$. A similar argument as in the proof of Proposition 5.25 shows that α preserves the labels of the edges in $[\Gamma_0,\Gamma_1]$, and so it fixes each vertex of Γ_0, as well as of Γ_1.

Observe that for each $a \in \mathbb{Z}_m$ no two vertices of Γ_a have more than one common neighbor in Γ_{a-1} (otherwise $n = 2$ would have to hold), and so each vertex of Γ_a is uniquely determined by its neighbors in Γ_{a-1}. Thus, α also fixes each vertex of Γ_2. Inductively we find that α is the identity. □

Since the group G acts half-arc-transitively on Γ with the vertex-stabilizer acting as the full symmetric group S_k on its outneighbors, this yields the following result.

Theorem 5.33. *Let $m \geq 3$, $n \geq 5$, $k \geq 3$, $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ be as in (5.1) with $r^2 \neq 1$. Then $\mathrm{Aut}(\mathcal{GB}(m,n,k;r,t)) = G$, where G is as in Lemma 5.7. In particular, $|\mathrm{Aut}(\mathcal{GB}(m,n,k;r,t))| = mn^{k-1}k!$.*

We now show that the only pairs of isomorphic HAT $\mathcal{GB}(m,n,k;r,t)$ graphs, where $k \geq 3$, are the ones given in Proposition 5.10.

Proposition 5.34. *Let $\Gamma = \mathcal{GB}(m,n,k;r,t)$ and $\Gamma' = \mathcal{GB}(m',n',k';r',t')$ with $n,n' \geq 5$, $k,k' \geq 3$, $r \in \mathbb{Z}_n^*$, $r' \in \mathbb{Z}_{n'}^*$, $t \in \mathbb{Z}_n$ and $t' \in \mathbb{Z}_{n'}$ be as in (5.1). Suppose $r^2 \neq 1$ and $r'^2 \neq 1$. Then $\Gamma \cong \Gamma'$ if and only if $(m,n,k,t) = (m',n',k',t')$ and either $r = r'$ or $r = r'^{-1}$.*

Proof. In view of Proposition 5.10 we only need to show that if $\Gamma \cong \Gamma'$ then the above condition holds. Suppose then that Γ and Γ' are isomorphic and let $\Gamma_a = \{(a;\boldsymbol{b}) \in V(\Gamma)\}$, where $0 \leq a \leq m-1$, let $\Gamma'_a = \{(a;\boldsymbol{b})' \in V(\Gamma')\}$, where $0 \leq a \leq m'-1$, and let $\alpha\colon V(\Gamma) \to V(\Gamma')$ be an isomorphism with $(0;\mathbf{0})\alpha = (0;\mathbf{0})'$. Since isomorphic graphs have the same valence and order, it follows that $k = k'$ and $mn^{k-1} = m'n'^{k'-1}$. Next, observe that, by Theorem 7, both Γ and Γ' are HAT graphs with the alternets of Γ and Γ' being the induced subgraphs $[\Gamma_a, \Gamma_{a+1}]$ and $[\Gamma'_a, \Gamma'_{a+1}]$, respectively. Therefore, $m = m'$ (the number of alternets) and consequently also $n = n'$.

Recall that the sets Γ_a and Γ'_a are blocks of imprimitivity for the groups $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma')$, respectively. As in the proof of Proposition 5.25 we can thus assume that α preserves the labels of all the edges in $[\Gamma_0, \Gamma_1]$ in the sense that for each $e \in [\Gamma_0, \Gamma_1]$ the label of $e\alpha$ in Γ' equals that of e in Γ, that is $\mathcal{L}(e\alpha) = \mathcal{L}(e)$. Since $(0;\mathbf{0})\alpha = (0;\mathbf{0})'$ we have $\Gamma_0\alpha = \Gamma'_0$, and so either $\Gamma_1\alpha = \Gamma'_1$ or $\Gamma_1\alpha = \Gamma'_{m-1}$. As in the proof of Proposition 5.25 we find that the action of α on $[\Gamma_0, \Gamma_1]$ is completely determined. In fact, we claim it is completely determined on the whole graph Γ.

Suppose first that $\Gamma_1\alpha = \Gamma'_1$ (and so $\Gamma_a\alpha = \Gamma'_a$ for all $a \in \mathbb{Z}_m$) and let $v = (2;\boldsymbol{b}) \in \Gamma_2$ be arbitrary. Of course, $(a;\boldsymbol{b})\alpha = (a;\boldsymbol{b})'$ holds for all $(a;\boldsymbol{b}) \in [\Gamma_0, \Gamma_1]$. Then, since $(1;\boldsymbol{b})$, $(1;\boldsymbol{b} - r\boldsymbol{e_1})$ and $(1;\boldsymbol{b} - r\boldsymbol{e_2})$ are all neighbors of v, the vertex $v\alpha \in \Gamma'_2$ is a common neighbor of the vertices $(1;\boldsymbol{b})'$, $(1;\boldsymbol{b} - r\boldsymbol{e_1})'$ and $(1;\boldsymbol{b} - r\boldsymbol{e_2})'$. Since $\boldsymbol{b}$ differs from each of $\boldsymbol{b} - r\boldsymbol{e_1}$ and $\boldsymbol{b} - r\boldsymbol{e_2}$ in exactly one component (to each in a different one), it is clear that $\mathcal{L}((1;\boldsymbol{b})'(v\alpha)) = 0$, and so $v\alpha = (2;\boldsymbol{b})'$. Note that this also forces $r' = r$. We can now proceed inductively to prove that in fact $(a;\boldsymbol{b})\alpha = (a;\boldsymbol{b})'$ holds for all the vertices of Γ, and so $t' = t$ also has to hold.

Suppose now that $\Gamma_1\alpha = \Gamma'_{m-1}$ (and so $\Gamma_a\alpha = \Gamma'_{m-a}$ for all $a \in \mathbb{Z}_m$). Similarly as above we first find that $(0;\boldsymbol{b})\alpha = (0;-r'^{m-1}\boldsymbol{b})'$ and $(1;\boldsymbol{b})\alpha = (m-1;-r'^{m-1}\boldsymbol{b} - t'\mathbf{1})'$ holds for all the vertices of Γ_0 and Γ_1, respectively. As above we then prove that for each $(2;\boldsymbol{b}) \in \Gamma_2$ we get $(2;\boldsymbol{b})\alpha = (m-2;-r'^{m-1}\boldsymbol{b} - t'\mathbf{1})'$. It follows that $r' = r^{-1}$. We can then again proceed inductively to verify that for each vertex $(a;\boldsymbol{b})$ of Γ with $a \geq 1$ we get $(a;\boldsymbol{b})\alpha = (m-a;-r\boldsymbol{b} - t'\mathbf{1})'$. The neighbor $(m-1;-t\mathbf{1})$ of $(0;\mathbf{0})$ is then mapped to $(1;rt\mathbf{1} - t'\mathbf{1}) = (1;(t-t')\mathbf{1})$, and so $t = t'$ has to hold. □

Using Theorem 7 and Proposition 5.34 it is now easy to compile the list of all HAT $\mathcal{GB}(m,n,k;r,t)$ graphs (up to isomorphism) up to some reasonable order. We collect the number of such graphs (column $\#\mathcal{GB}$) for all even valences between 6 and 16 up to order a million in Table 5.2 (of course, since $3 \cdot 5^8 > 10^6$, there are no such HAT $\mathcal{GB}(m,n,k;r,t)$ graphs with $k \geq 9$). To indicate that the family of generalized Bouwer graphs is indeed much larger than the family of Bouwer graphs we also give the number of HAT Bouwer graphs (column $\#\mathcal{B}$) up to this order. For each valence we also indicate the order of the smallest HAT Bouwer graph (column min $\mathcal{B}$) and the order of the smallest HAT $\mathcal{GB}(m,n,k;r,t)$ graph that is not a Bouwer graph (column min $\mathcal{GB} \setminus \mathcal{B}$).

valence	$\#\mathcal{GB}$	$\#\mathcal{B}$	min $\mathcal{B}$	min $\mathcal{GB} \setminus \mathcal{B}$
6	119 347	23 541	100	294
8	7 499	3 458	500	2 058
10	813	576	2 500	14 406
12	119	101	12 500	100 842
14	19	18	62 500	705 894
16	3	3	312 500	4 941 258
total	127 800	27 697	-	-

Table 5.2: The number of HAT $\mathcal{GB}(m,n,k;r,t)$ and $\mathcal{B}(k;m,n)$ graphs and smallest examples.

Chapter 6

Conclusions

We conclude this PhD Thesis with a few observations and give some suggestions for future research. We also pose some open problems that were not mentioned in the previous chapters.

The tools used in this thesis extend from group theory, algebraic graph theory and to purely combinatorial techniques. The implementation of computer algebraic tools, such as Magma, were used for the analysis of particular cases and testing results.

Arc-transitive maps with underlying Rose window graphs. In Chapter 3 we studied the structure of tetravalent arc-transitive graphs and used their properties to investigate symmetries of arc-transitive maps in class $2_{\{0,1\}}$. The results of this chapter complete the classification of all arc-transitive maps with underlying Rose Window graphs.

The results for the structure of the underlying graphs of maps in class $2_{\{0,1\}}$ can be used on graphs having the required properties to obtain a bigger family of examples of these maps and get a better understanding of their properties. For instance, if Γ is a tetravalent G-half-arc-transitive graph for $G \leq \mathrm{Aut}(\Gamma)$, such that Γ is arc-transitive with $[\mathrm{Aut}(\Gamma), G] = 2$ and $\mathrm{att}_G(\Gamma) > 2$, then $\mathrm{Aut}(\Gamma)$ acts 1-regularly on Γ. Now, observe that the vertex stabilizers of the action of $\mathrm{Aut}(\Gamma)$ are of order four. In the case of the vertex stabilizers being isomorphic to the Klein 4-group, by Corollary 3.13 we obtain that Γ is an underlying graph of three pairwise nonisomorphic maps in the class $2_{\{0,1\}}$ and also, by the Petrie operator, three pairwise nonisomorphic maps in class 2_1. By going through the census [48] one can find various examples of that kind of graphs, with the smallest one being the graph $\mathrm{GHAT}[15, 1]$.

Tetravalent graphs admitting a half-arc-transitive group of automorphisms. The results of Chapter 4 represent a significant contribution to a topic of research that has been active in mathematical community in the last decades. Moreover, they establish a link between two important frameworks for a systematic study of all tetravalent graphs admitting a half-arc-transitive group of automorphisms. In addition, a considerable step towards the complete answer to the question of whether the attachment number necessarily divides the radius in tetravalent half-arc-transitive graphs is made. However, the following cases remain open for this problem.

Problem 6.1. Let Γ be a tetravalent half-arc-transitive graph such that $\mathrm{att}(\Gamma)$ is an even integer greater than two and $\mathrm{jmp}(\Gamma) = \mathrm{att}(\Gamma)/2 - 1$. Prove that $\mathrm{att}(\Gamma) \mid \mathrm{rad}(\Gamma)$ holds.

A number of questions are posed throughout Chapter 4, but there are several other curiosities one can observe by studying the census of all tetravalent half-arc-transitive graphs up to order 1000 from [48]. We mention just two of them. For all graphs Γ from the census such that $\mathrm{Alt}(\Gamma)$ is also half-arc-transitive the graph Γ is either loosely attached or $\mathrm{jmp}(\Gamma) = 1$. Similarly, whenever $\Gamma_{\mathcal{B}}$ is half-arc-transitive, $|Q(\Gamma)| = 1$ holds.

We thus pose the following two questions.

Question 6.2. Does there exist a tetravalent half-arc-transitive graph Γ such that $\mathrm{Alt}(\Gamma)$ is half-arc-transitive but $\mathrm{att}(\Gamma), \mathrm{jmp}(\Gamma) > 1$?

Question 6.3. Does there exist a tetravalent half-arc-transitive graph Γ such that $\Gamma_{\mathcal{B}}$ is half-arc-transitive and $|Q(\Gamma)| \neq 1$?

Generalized Bouwer graphs. In Chapter 5 we worked with half-arc-transitive graphs of valencies greater than four, which have not been extensively studied in the literature. We constructed an infinite family of half-arc-transitive graphs, containing almost all tightly attached tetravalent half-arc-transitive graphs.

As we pointed out all of the half-arc-transitive generalized Bouwer graphs are tightly attached but since also the graphs from [5], which are clearly not isomorphic to any generalized Bouwer graphs (since they are of order $3p$, where p is a prime), are tightly attached, the class of all tightly attached half-arc-transitive graphs is even larger. We thus propose the following problem.

Problem 6.4. Classify all tightly attached half-arc-transitive graphs.

By Theorem 5.33 a vertex stabilizer of the whole automorphism group of a half-arc-transitive $\mathcal{GB}(m, n, k; r, t)$ graph acts as the full symmetric group on the outneighbors of the vertex. It can be seen that this does not hold for the graphs from [5] or [31]. It is thus natural to pose the following problem.

Problem 6.5. Determine whether there exists a tightly attached half-arc-transitive graph Γ of valence at least 6 such that a vertex stabilizer in $\mathrm{Aut}(\Gamma)$ acts as the full symmetric group on the outneighbors of the vertex but Γ is not a generalized Bouwer graph. Can a Cayley graph Γ with this property also be found? If so, classify all such graphs.

The reason why we are also asking for Cayley graphs with the above property is that we think the answer to the general question is in the affirmative. Namely, already with tetravalent examples we saw that the $\mathcal{X}_o(m, n; r)$ graphs with $r^m = -1$ and m even were not generalized Bouwer graphs. However, they were also not Cayley.

Finally, most of the half-arc-transitive graphs of valences at least 6 that we know of are either tightly attached or have only one alternet. It is thus natural to consider the following problem.

Problem 6.6. For each valence $2k \geq 6$, each integer $r \geq 3$ and each divisor a of r construct a half-arc-transitive graph of valence $2k$, radius r and attachment number a, or prove that for some triples (k, r, a) such graphs do not exist.

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Chapter 7

Povzetek v slovenskem jeziku

7.1 Uvod

Glavna tema doktorske disertacije so grafi z visoko stopnjo simetrije. Osredotočili smo se namreč na grafe, katerih grupe avtomorfizmov so vozliščno in povezavno tranzitivne. Ta razred grafov so zadnjih 70 let obširno preučevali. Objavljenih je bilo na stotine raziskovalnih člankov (glej na primer članke [21, 29, 38, 47] in reference, ki jih le-ti navajajo).

Za dani graf Γ z $V(\Gamma)$, $E(\Gamma)$ in $A(\Gamma)$ po vrsti označimo množico vozlišč, množico povezav in množico lokov (tj. množico vseh urejenih parov sosednjih vozlišč). Polno grupo avtomorfizmov grafa Γ (tj. grupo vseh tistih permutacij množice vozlišč, ki ohranjajo sosednost med vozlišči) bomo označevali z $\mathrm{Aut}(\Gamma)$. Pripomnimo še, da so v tej disertaciji vsi grafi končni, enostavni in neusmerjeni (tam kjer je to smiselno, pa imajo implicitno podano orientacijo na povezavah). Naj bo Γ graf in naj bosta $u, v \in V(\Gamma)$. Če sta u in v sosednji vozlišči grafa Γ, pišemo $u \sim v$. Pripadajočo povezavo $\{u, v\}$ bomo običajno označili z uv, pri čemer je samoumevno, da je $uv = vu$.

Pravimo, da je graf Γ *G-vozliščno tranzitiven, G-povezavno tranzitiven* oziroma *G-ločno tranzitiven*, če podgrupa $G \leq \mathrm{Aut}(\Gamma)$ deluje tranzitivno na $V(\Gamma)$, $E(\Gamma)$ oziroma $A(\Gamma)$. Nadalje je Γ *G-pol-ločno tranzitiven* (v nadaljevanju bomo uporabljali okrajšavo G-PLT) če je G-vozliščno in G-povezavno tranzitiven, toda ni G-ločno tranzitiven. V primeru, ko bo $G = \mathrm{Aut}(\Gamma)$, bomo predpono G opustili in uporabljali izraze vozliščno tranzitiven, povezavno tranzitiven, ločno tranzitiven in pol-ločno tranzitiven.

Naj bo Γ G-vozliščno in G-povezavno tranzitiven graf za neko podgrupo $G \leq \mathrm{Aut}(\Gamma)$. V tem primeru lahko nastopita dve bistveno različni možnosti:

(i) Γ je G-ločno tranzitiven;

(ii) Γ je G-pol-ločno tranzitiven.

Seveda si lahko tudi pri prvi izmed obeh možnosti zastavimo vprašanje, če obstaja kakšna podgrupa $H \leq \mathrm{Aut}(\Gamma)$, ki na grafu Γ deluje pol-ločno tranzitivno. Podobno si lahko pri drugi možnosti zastavimo vprašanje, če katera druga podgrupa $H \leq \mathrm{Aut}(\Gamma)$ deluje ločno tranzitivno na grafu Γ. V doktorski disertaciji proučujemo strukturne lastnosti grafov (in podgrup njihovih grup avtomorfizmov) v obeh opisanih primerih ter ob tem obravnavamo tudi zgoraj omenjeni vprašanjij.

Pri prvem osrednjem delu doktorske disertacije se osredotočimo na vprašanja, ki izhajajo iz zgornje možnosti (i). Bolj konkretno, zanima nas, kako lahko lastnosti ločno tranzitivnih grafov uporabimo kot orodje pri raziskovanju simetrijskih lastnosti zemljevidov. *Zemljevid* $\mathcal{M}$ je takšna vložitev povezanega grafa Γ v kompaktno ploskev S brez roba, da je $S \setminus \Gamma$ disjunktna unija enostavno povezanih območij. Platonska telesa, na primer, lahko obravnavamo kot zemljevide na sferi. *Vozlišča* in *povezave* zemljevida so kar vozlišča in povezave pripadajočega temeljnega grafa, *lica* zemljevida pa so enostavno povezana območja, ki jih dobimo z odstranitvijo grafa s ploskve. *Avtomorfizmi* zemljevida $\mathcal{M}$ so tisti avtomorfizmi vloženega grafa Γ, ki ohranjajo lica zemljevida $\mathcal{M}$. Pravimo, da je zemljevid *povezavno tranzitiven* oziroma *ločno tranzitiven*, če njegova grupa avtomorfizmov deluje povezavno tranzitivno oziroma ločno tranzitivno na njegovem temeljnem grafu. Konec devetdesetih let dvajsetega stoletja sta Graver in Watkins [22] začela s sistematičnim proučevanjem vseh povezavno tranzitivnih zemljevidov. Nedavno je nekatere nove smernice pri proučevanju takšnih zemljevidov postavil Gareth Jones [26], ko je predlagal klasifikacijo teh zemljevidov bodisi glede na njihove grupe avtomorfizmov ali pa glede na njihove grafe. Naravni korak v smeri klasifikacije povezavno tranzitivnih zemljevidov je seveda študij ločno tranzitivnih primerkov.

Naj bo dan zemljevid $\mathcal{M}$. Njegova grupa avtomorfizmov tedaj naravno deluje na množici njegovih praporov, ki jo označimo s $\mathcal{F}(\mathcal{M})$. *Prapor* je trojica, ki jo tvorijo vozlišče, povezava in lice, ki so med seboj incidenčni. Ni težko dokazati, da ima grupa avtomorfizmov ločno tranzitivnega zemljevida $\mathcal{M}$ največ dve orbiti na množici $\mathcal{F}(\mathcal{M})$. Naj bo $\mathcal{M}$ ločno tranzitiven zemljevid. V tem primeru je $\mathcal{M}$ *regularen zemljevid*, če ima $\mathrm{Aut}(\mathcal{M})$ eno samo orbito na $\mathcal{F}(\mathcal{M})$; sicer je $\mathcal{M}$ *zemljevid z dvema orbitama*. Znano je, da je mogoče ločno tranzitivne zemljevide razdeliti v 5 razredov glede na lokalno konfiguracijo praporov in orbit, ki jim ti prapori pripadajo [23]. Eden izmed razredov vsebuje vse regularne zemljevide, torej so v ostalih štirih razredih zemljevidi z dvema orbitama. Izmed vseh ločno tranzitivnih zemljevidov so bili do sedaj največje pozornosti deležni regularni zemljevidi ter tako imenovani *kiralni zemljevidi*, ki so vsebovani v enem izmed štirih razredov zemljevidov z dvema orbitama. Po drugi strani pa zemljevidi v razredu $2_{\{0,1\}}$ (zemljevid, pri katerem je vsak prapor v isti $\mathrm{Aut}(\mathcal{M})$-orbiti kot njegov 0- in 1-sosedni prapor, njegov 2-sosedni prapor pa je v drugi $\mathrm{Aut}(\mathcal{M})$-orbiti) zaenkrat v obstoječi literaturi še niso bili obširno obravnavani. Izkaže se, da je najmanjša možna stopnja takšnih zemljevidov štiri. V doktorski disertaciji raziskujemo povezavo med tem razredom ločno tranzitivnih zemljevidov in strukturo njihovih temeljnih grafov. Večji poudarek je na zemljevidih stopnje štiri. Skupaj z rezultati iz članka [28] smo uspeli narediti popolno klasifikacijo ločno tranzitivnih zemljevidov, katerih temeljni grafi so ločno tranzitivni rozetni grafi (opisani so v razdelku 7.2).

V drugem in tretjem delu doktorske disertacije se osredotočimo na vprašanja, ki izhajajo iz zgornje možnosti (ii). Proučevali smo torej grafe, ki dopuščajo polločno tranzitivno grupo avtomorfizmov G (z okrajšavo G-PLT). Pri vsakem G-PLT grafu Γ delovanje podgrupe G porodi dve orientaciji (ki sta nasprotni ena drugi) na povezavah. Izberimo si eno od teh orientacij in označimo pripadajoči usmerjen graf z $\overrightarrow{\Gamma}_G$. Oznaka $u \to v$ pomeni, da je povezava uv usmerjena od vozlišča u proti v. Vozlišče u bomo v tem primeru imenovali *rep* (usmerjene) povezave uv, vozlišče v pa *glava* povezave uv. Pri študiju takšnih grafov je v pomoč vpogled v njihove strukturne lastnosti, ki jih porodi delovanje podgrupe G. V članku [32] so bili vpeljani alternirajoči cikli, spojne množice, polmer in spojno število PLT grafov stopnje 4. V nadaljevanju tega odstavka se bomo zgledovali po članku [?], kjer so bili ti koncepti posplošeni na G-PLT grafe večjih stopenj (glej tudi [25]). Pravimo, da sta povezavi uv in $u'v'$ grafa Γ *združeni*, če imata pripadajoči usmerjeni povezavi v $\overrightarrow{\Gamma}_G$ skupno glavo ali skupen rep. Naj v digrafu $\overrightarrow{\Gamma}_G$ velja $u \to v$ in $u' \to v'$. Potem sta uv in $u'v'$ združeni natanko tedaj, ko je $u = u'$ ali $v = v'$ (nista pa združeni, če je $u' = v$ ali $v' = u$). Tranzitivna ogrinjača te relacije, ki ji bomo rekli *relacija dosegljivosti* na Γ, je seveda ekvivalenčna relacija na $E(\Gamma)$. Podgrafi grafa Γ (kot tudi digrafa $\overrightarrow{\Gamma}_G$), ki ustrezajo ekvivalenčnim razredom relacije dosegljivosti, se imenujejo *G-alterneti* grafa Γ (oziroma digrafa $\overrightarrow{\Gamma}_G$). *Množica glav* alterneta A sestoji iz glav vseh povezav v A, *množica repov* pa iz repov vseh povezav v A. Velikost (katerekoli) množice glav se imenuje *G-polmer* grafa Γ, označevali pa ga bomo z $\mathrm{rad}_G(\Gamma)$. Naj ima graf Γ vsaj dva G-alterneta. Če za dva G-alterneta s skupnim vozliščem velja, da je množica glav pri enem enaka množici repov pri drugem, pravimo, da je graf Γ *tesno G-spet*.

Prve rezultate o grafih, ki dopuščajo pol-ločno tranzitivna delovanja, je objavil Tutte. Dokazal je, da mora biti stopnja takšnih grafov sodo število. Ker je graf stopnje 2 pravzaprav disjunktna unija ciklov, postane obravnava G-PLT grafov netrivialna šele, ko imamo opravka z grafi stopnje vsaj štiri. Zato ni presenetljivo, da večina člankov o G-PLT grafih obravnava prav grafe stopnje 4. Drugi del doktorske disertacije smo posvetili takšnim grafom. V preteklosti so se raziskovalci študija teh grafov lotili na mnogo različnih načinov in prišli do nekaj pomembnih rezultatov. Eno od najbolj prodornih študij je začel Marušič v članku [32]. Lotil se je študija strukture alternetov štirivalentnih G-pol-ločno tranzitivnih grafov. V tem primeru (ko je graf Γ stopnje 4) se izkaže, da so G-alterneti pravzaprav cikli, ki jim pravimo *G-alternirajoči cikli* grafa Γ. Podobno kot velja v splošnem za G-alternete, je enostavno videti, da se dva G-alternirajoča cikla grafa Γ, ki imata skupen presek, vedno sekata v istem številu vozlišč. Temu številu pravimo *G-spojno število* grafa Γ in ga označimo z $\mathrm{att}_G(\Gamma)$. Velja omeniti, da so štirivalentni tesno G-speti grafi že popolnoma klasificirani [32, 37, 53, ?]. Pomembnost omenjenih rezultatov izhaja iz članka [37], kjer je bilo pokazano, da je vsak štirivalenten G-PLT graf bodisi testno G-spet ali pa nastopi kot krovni graf nad nekim šibko spetim ali nekim antipodno-spetim grafom (tj. nad grafom, ki ima spojno število 1 ali 2).

V članku [3] je predlagan nov okvir, ki odpira možnost klasifikacije grafov stopnje 4, ki dopuščajo pol-ločno tranzitivno delovanje. Temelji na tako imenovani metodi normalnih kvocientov, kjer v preučevanem grafu identificiramo orbite netranzitivne podgrupe edinke grupe avtomorfizmov (pri tem izpustimo morebitne vzporedne povezave in zanke) in tako dobimo manjše grafe, ki imajo "iste" lastnosti kot prvotni graf. Poanta je v tem, da bi najprej klasificirali vse "minimalne" grafe, ki jih lahko dobimo pri opisanem kvocientnem postopku, nato pa bi poskušali ugotoviti, kako lahko večje grafe rekonstruiramo iz minimalnih. Nedavno so bili objavljeni nekateri rezultati v smeri predlaganega pristopa ([1], [2]). Pri naši raziskavi preučujemo štirivalentne G-PLT grafe iz obeh prej omenjenih vidikov. Še več, izboljšamo nekatere obstoječe rezultate in v bistvu združimo ta dva pomembna pristopa. Da bi dosegli zastavljeni cilj, vpeljemo nov parameter štirivalentnega G-PLT grafa Γ in sicer *skok* grafa Γ glede na grupo G (definicija je podana v razdelku 7.2). Ta parameter omogoča boljše razumevanje strukture preučevanih grafov. Natančneje, skok grafa Γ nam da več informacij o tem, kako se prepletata dva G-alternirajoča cikla z nepraznim presekom. Izkaže se, da je skok grafa Γ glede na grupo G zelo uporabno orodje pri študiju štirivalentnih G-PLT grafov.

Še en možen pristop k študiju strukture štirivalentnega G-PLT grafa Γ je konstrukcija njegovega *grafa G-alternirajočih ciklov*, ki ga označimo z $\mathrm{Alt}_G(\Gamma)$. To je graf, čigar vozlišča so vsi G-alternirajoči cikli grafa Γ. Dve vozlišči sta sosednji, če in samo če imata pripadajoča cikla neprazen presek. Znanih je tudi nekaj rezultatov o povezavi med G-spojnim številom in G-polmerom štirivalentnega G-PLT grafa. Na primer, v članku [32] je dokazano, da $\mathrm{att}_G(\Gamma)$ deli $2\mathrm{rad}_G(\Gamma)$. Poleg tega pri vseh znanih primerih štirivalentnih PLT grafov velja, da spojno število deli tudi polmer grafa. Samo po sebi se torej zastavlja vprašanje, če $\mathrm{att}(\Gamma)$ deli $\mathrm{rad}(\Gamma)$ pri vseh PLT grafih (glej [49]). V [39, Theorem 1.2] je bilo potrjeno, da to velja za PLT grafe Γ, ki imajo $\mathrm{att}(\Gamma) = 2$. Nedavno je bilo to dejstvo potrjeno še za grafe Γ, kjer je $\mathrm{rad}(\Gamma)$ liho število [49, Theorem 2]. V doktorski disertaciji smo dokazali več rezultatov o grafu $\mathrm{Alt}_G(\Gamma)$, ki so nam omogočili velik korak proti popolnemu odgovoru na zgoraj zastavljeno vprašanje, če v PLT grafih $\mathrm{att}(\Gamma)$ res deli $\mathrm{rad}(\Gamma)$.

V zadnjem delu doktorske disertacije smo se osredotočili na PLT grafe, ki imajo stopnjo večjo od štiri. V nasprotju z velikim številom člankov, ki se ukvarjajo s štirivalentnimi PLT grafi, je bilo doslej o raziskavah PLT grafov večjih valenc obljavjenih bistveno manj prispevkov (glej na primer [5, 8]). To je najverjetneje posledica dejstva, da je že pri štirivalentnih PLT grafih še vedno odprtih mnogo težkih vprašanj, na katera še ne znamo odgovoriti. Kljub temu je bilo v zadnjem času opaziti nekaj napredka pri PLT grafih višjih stopenj (glej na primer [11, 25, **?**]).

Leta 1970 je Bouwer [8] konstruiral neskončno družino vozliščno in povezavno tranzitivnih grafov, ki so danes znani kot Bouwerjevi grafi $\mathcal{B}(k, m, n)$. Graf $\mathcal{B}(k, m, n)$ je stopnje $2k$ in reda mn^{k-1}. Bouwer je pokazal, da je za vsak $k \geq 2$ graf $\mathcal{B}(k, 6, 9)$ pol-ločno tranzitiven, in tako zagotovil obstoj po enega primera PLT grafa za vsako sodo valenco večjo od 2. Ni pa se lotil vprašanja, kateri od preostalih $\mathcal{B}(k, m, n)$ grafov so PLT, niti se ni vprašal, če za vsak k obstaja neskončno mnogo PLT grafov valence $2k$. Nedavno sta Conder in Žitnik [11] izdelala popolno klasifikacijo vseh PLT Bouwerjevih grafov in s tem odgovorila na obe vprašanji. Omeniti je treba, da sta obstoj neskončno mnogo PLT grafov stopenj $2k$ za $k > 2$ implicitno nakazala že Alspach in Xu [5]. Klasificirala sta vse PLT grafe reda $3p$, kjer je p praštevilo. Tudi Li in Sim [31] sta našla neskončno mnogo PLT grafov (različnih sodih valenc), katerih red je potenca praštevila. Izkaže se, da so vsi Bouwerjevi grafi, kot tudi grafi iz člankov [5] in [31], tesno speti. A tudi če združimo vse tri omenjene družine, dobimo le majhen delež grafov iz družine vseh štirivalentnih tesno spetih PLT grafov (ki sta jih popolnoma klasificirala Marušič in Šparl [32, 53]). V doktorski disertaciji posplošimo družino Bouwerjevih grafov na mnogo večjo družino vozliščno in povezavno tranzitivnih grafov vseh mogočih sodih valenc večjih od 2. Posplošitev je zelo naravna, dobljena družina pa vsebuje skoraj vse štirivalentne tesno spete PLT grafe. To družino posplošenih Bouwerjevih grafov smo podrobno raziskali in podali popolno klasifikacijo članov družine, ki so PLT, ter izračunali njihove grupe avtomorfizmov.

7.2 Rezultati

7.2.1 Ločno tranzitivni zemljevidi, katerih temeljni grafi so rozetni grafi

Leta 2008 je Wilson [59] vpeljal družino štirivalentnih grafov, ki so danes znani kot rozetni grafi. Ta razred grafov je bil v zadnjih desetih letih precej proučevan in ga zdaj dokaj dobro razumemo (glej na primer [15, 27, 28]). V članku [59] je Wilson identificiral štiri posebne poddružine rozetnih grafov (definirane so spodaj) in dokazal, da so vsi pripadniki teh družin ločno tranzitivni grafi. Njegovo domnevo, da vsak povezavno tranzitiven rozetni graf (izkaže se, da je pri rozetnih grafih povezavna tranzitivnost ekvivalentna ločni tranzitivnosti) pripada eni od teh družin, so leta 2010 potrdili Kovács, Kutnar in Marušič [27].

Naj bo $n \geq 3$ celo število in naj celi števili r in a zadoščata pogojem $1 \leq r \leq n-1$, $r \neq n/2$ in $0 \leq a \leq n-1$. *Rozetni graf* $R_n(a, r)$ je graf z množico vozlišč $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$, množico povezav pa tvorijo povezave naslednjih štirih tipov:

- podmnožica vseh *obodnih povezav* $x_i x_{i+1}$, $i \in \mathbb{Z}_n$;
- podmnožica vseh *povezav pesta* $y_i y_{i+r}$, $i \in \mathbb{Z}_n$;
- podmnožica vseh *ravnih naper* $x_i y_i$, $i \in \mathbb{Z}_n$;
- podmnožica vseh *poševnih naper* $x_i y_{i-a}$, $i \in \mathbb{Z}_n$,

kjer vse indekse računamo po modulu n.

Naj bo dan zemljevid $\mathcal{M}$. Njegova grupa avtomorfizmov deluje na množici njegovih praporov (tj. trojic, ki jih tvorijo vozlišče, povezava in lice, ki so med seboj incidenčni). Naj bosta $\Phi = (v, e, f)$ in $\Phi' = (v', e', f')$ prapora zemljevida $\mathcal{M}$, kjer so $\{v, v'\}$, $\{e, e'\}$ in $\{f, f'\}$ po vrsti podmnožice njegove množice vozlišč, povezav in lic. Če je $\Phi \cap \Phi' = \{e, f\}$, $\Phi \cap \Phi' = \{v, f\}$ oziroma $\Phi \cap \Phi' = \{v, e\}$, pravimo, da sta prapora Φ' in Φ 0-, 1- oziroma 2-sosedna. Obravnavamo samo takšne zemljevide, kjer ima vsak prapor natanko en i-sosedni prapor za vsak $i \in \{0, 1, 2\}$ (*politopni zemljevidi*). Ni težko dokazati, da obstaja natanko 5 razredov ločno tranzitivnih zemljevidov glede na lokalno konfiguracijo praporov in orbit, ki jim ti prapori pripadajo. Zemljevidi iz treh izmed teh razredov, kjer so temeljni grafi teh zemljevidov rozetni grafi, so bili klasificirani v članku [28].

V disertaciji dokončamo klasifikacija ločno tranzitivnih zemljevidov, katerih temeljni grafi so rozetni grafi. Določimo namreč vse zemljevide razreda $2_{\{0,1\}}$, katerih temeljni grafi so rozetni grafi.

V disertaciji dokažemo naslednje štiri izreke:

Izrek 1. *Naj bo $\Gamma = R_n(2, 1)$ rozetni graf, kjer je $n \geq 3$. Graf Γ je temeljni graf zemljevida $\mathcal{M}$ iz razreda $2_{\{0,1\}}$ natanko tedaj, ko je $D(n, 6) \neq 1$. Nadalje, naj* *bo* $n_0 \in \{0, 2, 3, 4, 6, 8, 9, 10\}$ *ostanek števila* n *pri deljenju z* 12. *Veljajo naslednje trditve:*

(i) če je $n = 4$, *potem je* Γ *temeljni graf natanko enega zemljevida iz razreda* $2_{\{0,1\}}$, *njegova lica pa so dolžin* 4 *in* 8.

(ii) če je $n_0 \in \{3, 9\}$, *potem je* Γ *temeljni graf natanko enega zemljevida iz razreda* $2_{\{0,1\}}$, *njegova lica pa so dolžin* 4 *in* n.

(iii) če je $n_0 \in \{4, 8\}$, *potem je* Γ *temeljni graf dveh neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *pri enem so lica dolžin* 4 *in* n, *pri drugem pa so lica dolžin* 4 *in* $2n$.

(iv) če je $n_0 \in \{2, 10\}$, *potem je* Γ *temeljni graf treh neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *prvi ima lica dožin* 4 *in* n, *drugi ima lica dolžin* 4 *in* $2n$, *tretji pa ima lica dolžin* n *in* $2n$.

(v) če je $n_0 = 0$, *potem je* Γ *temeljni graf treh neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *dva imata lica dolžin* 4 *in* n, *eden pa ima lica dolžin* 4 *in* $2n$.

(vi) če je $n_0 = 6$, *potem je* Γ *temeljni graf štirih neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *dva imata lica dolžin* 4 *in* n, *tretji ima lica dolžin* 4 *in* $2n$, *četrti pa ima lica dolžin* n *in* $2n$.

Izrek 2. *Naj bo* $\Gamma = R_{2n}(n+2, n+1)$ *rozetni graf, kjer je* $n \geq 3$. *Graf* Γ *je temeljni graf zemljevida* $\mathcal{M}$ *iz razreda* $2_{\{0,1\}}$ *natanko tedaj, ko je* $D(n, 12) > 2$. *Nadalje, naj bo* $n_0 \in \{0, 3, 4, 6, 8, 9\}$ *ostanek števila* n *pri deljenju z* 12. *Veljajo naslednje trditve:*

(i) če je $n = 4$, *potem je* Γ *temeljni graf natanko enega zemljevida iz razreda* $2_{\{0,1\}}$, *njegova lica pa so dolžin* 4 *in* 8.

(ii) če je $n_0 \in \{3, 9\}$, *potem je* Γ *temeljni graf treh neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *prvi ima lica dolžin* 4 *in* n, *drugi ima lica dolžin* 4 *in* $2n$, *tretji pa ima lica dolžin* n *in* $2n$.

(iii) če je $n_0 \in \{4, 6, 8\}$, *potem je* Γ *temeljni graf dveh neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *pri enem so lica dolžin* 4 *in* n, *pri drugem pa so lica dolžin* 4 *in* $2n$.

(iv) če je $n_0 = 0$, *potem je* Γ *temeljni graf štirih neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$; *dva imata lica dolžin* 4 *in* n, *druga dva pa imata lica dolžin* 4 *in* $2n$.

Izrek 3. *Naj bo* $\Gamma = R_{2m}(2b, r)$, *kjer je* $b^2 \equiv \pm 1 \pmod{m}$ *in je bodisi* $r = 1$ *ali* $r = m - 1$, *pri čemer je* m *sodo število. Poleg tega naj graf* Γ *ne pripada nobeni od družin (i) in (ii) iz trditve 3.14. Graf* Γ *je temeljni graf zemljevida iz razreda* $2_{\{0,1\}}$ *natanko tedaj, ko je* $b^2 \equiv 1 \pmod{m}$. *V tem primeru obstajajo natanko trije takšni paroma neizomorfni zemljevidi.*

Izrek 4. *Naj bo* $\Gamma = R_{12m}(3d+2, 9d+1)$ *rozetni graf, kjer je* $d = m$ *ali pa* $d = 11m$. *Veljajo naslednje trditve:*

(i) če je $m \not\equiv 2 \pmod{4}$, *potem je* Γ *temeljni graf natanko treh paroma neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$.

(ii) *če je* $m \equiv 2 \pmod 4$, *potem je* Γ *temeljni graf natanko dveh neizomorfnih zemljevidov iz razreda* $2_{\{0,1\}}$.

7.2.2 Štirivalentni G-PLT grafi

V 4. poglavju doktorske disertacije vpeljemo nov parameter štirivalentnih G-PLT grafov, ki mu rečemo parameter alternirajočega skoka. Naj bo Γ štirivalenten G-PLT graf za neko podgrupo $G \leq \mathrm{Aut}(\Gamma)$. Izberimo si eno od dveh možnih orientacij povezav, ki ju porodi delovanje grupe G. Naj bo $r = \mathrm{rad}_G(\Gamma)$ in $a = \mathrm{att}_G(\Gamma)$. Naj bo $v \in V(\Gamma)$ in naj bosta $C = (u_0, u_1, \ldots, u_{2r-1})$ in $C' = (v_0, v_1, \ldots, v_{2r-1})$ G-alternirajoča cikla, ki vsebujeta v, pri čemer je $u_0 = v_0 = v$, vozlišče v pa je rep tistih dveh povezav na C, ki sta indicenčni temu vozlišču. Iz [32, Lemma 2.6] sledi, da je G-spoj $V(C) \cap V(C')$, ki vsebuje v in ga na kratko označimo s $C \cap C'$, enaka

$$C \cap C' = \{u_{i\ell} : 0 \leq i < a\} = \{v_{i\ell} : 0 \leq i < a\}, \tag{7.1}$$

kjer je $\ell = 2r/a$. Definirajmo

$$q_t(v) = \min\{q : v_{q\ell} \in \{u_\ell, u_{-\ell}\}\} \text{ in } q_h(v) = \min\{q : u_{q\ell} \in \{v_\ell, v_{-\ell}\}\},$$

kjer v primeru, ko je $a = 1$, to razumemo kot $q_t(v) = q_h(v) = 0$. Ker G na grafu Γ deluje vozliščno in povezavno tranzitivno, parametra $q_t(v)$ in $q_h(v)$ nista odvisna od izbire vozliča v. Zato lahko za vsak štirivalenten G-pol-ločno tranzitiven graf Γ definiramo $Q_G(\Gamma) = \{q_t, q_h\}$, kjer je $q_t = q_t(v)$ in $q_h = q_h(v)$ za neko vozlišče $v \in V(\Gamma)$ pri eni od obeh možnih orientacij povezav grafa Γ, ki ju inducira grupa G. Tako lahko za štirivalenten G-pol-ločno tranzitiven graf Γ definiramo parameter $\mathrm{jmp}_G(\Gamma) = \min(Q_G(\Gamma))$. Parametru $\mathrm{jmp}_G(\Gamma)$ pravimo *alternirajoči G-skok* grafa Γ. V primeru, ko je $G = \mathrm{Aut}(\Gamma)$, namesto $\mathrm{jmp}_{\mathrm{Aut}(\Gamma)}(\Gamma)$ pišemo $\mathrm{jmp}(\Gamma)$ in mu pravimo *alternirajoči skok* grafa Γ.

V doktorski disertaciji smo dokazali nekatere lastnosti tega novega parametra in jih uporabili kot orodje za določanje jedra naravnega delovanja grupe G na pripadajočem grafu $\mathrm{Alt}_G(\Gamma)$. Konkretneje, dokazali smo naslednji rezultat.

Izrek 5. *Naj bo* Γ *štirivalenten* G*-pol-ločno tranzitiven graf za neko grupo* $G \leq \mathrm{Aut}(\Gamma)$ *in naj bo* $r = \mathrm{rad}_G(\Gamma)$ *ter* $a = \mathrm{att}_G(\Gamma)$. *Naj bo* $K = K_G(\mathrm{Alt}_G(\Gamma))$ *jedro delovanja grupe* G *na grafu* $\mathrm{Alt}_G(\Gamma)$, *tj. na grafu* G*-alternirajočih ciklov grafa* Γ. *Potem velja ena od naslednjih trditev:*

(i) $a = 2r$ *in* $K = D_r$;

(ii) $a = r = 2$, *pri čemer je* Γ *izomorfen leksikografskemu produktu* $C_n[\overline{K_2}]$ *za neko celo število* n, *jedro* K *pa je izomorfno podgrupi elementarno abelske* 2*-grupe reda* 2^n;

(iii) $a = r > 2$, *pri čemer je jedro* K *izomorfno diedrski grupi reda* $2a$;

(iv) $a < r$, *kjer* $a \mid r$, *pri čemer je jedro* K *izomorfno ciklični grupi reda* a, *razen če je* $a = 2$ *(v tem primeru je jedro* K *lahko trivialno)*;

(v) $a < r$, *kjer* $a \nmid r$, *pri čemer je jedro* K *izomorfno ciklični grupi reda* $a/2$.

Izboljšali smo odgovor na vprašanje iz [49], če spojno število deli polmer pri vseh štirivalentnih PLT grafih. Dokazali smo sledeči izrek.

Izrek 6. *Naj bo Γ štirivalenten G-pol-ločno tranzitiven graf za neko podgrupo $G \leq \mathrm{Aut}(\Gamma)$ in naj bodo $r = \mathrm{rad}_G(\Gamma)$, $a = \mathrm{att}_G(\Gamma)$ in $q = \mathrm{jmp}_G(\Gamma)$. Denimo, da a ne deli r in da velja $4 < a < r$. Če velja $q = 1$ ali če je graf $\mathrm{Alt}_G(\Gamma)$ dvodelen, potem obstaja avtomorfizem ρ grafa Γ, ki ohranja vse G-alternirajoče cikle grafa Γ in vsaj na enem od njih deluje kot $2r/a$-koračna rotacija. Posledično je graf Γ ločno tranzitiven. Če torej velja $4 < a < r$, a ne deli r in $q \neq a/2 - 1$, potem je graf Γ ločno tranzitiven.*

7.2.3 Klasifikacija pol-ločno tranzitivnih posplošenih Bouwerjevih grafov

V 5.poglavju disertacije klasificiramo PLT posplošene Bouwerjeve grafe. V nadaljevanju $\mathbb{Z}_n$ in $\mathbb{Z}_n^*$ označujeta kolobar ostankov celih števil po modulu n in njegovo multiplikativno grupo obrnljivih elementov, kjer je n naravno število. V disertaciji je predstavljena sledeča konstrukcija.

Konstrukcija 1. *Naj bodo $m \geq 3$, $n \geq 2$ in $k \geq 2$ naravna števila in naj bosta $r \in \mathbb{Z}_n^*$ in $t \in \mathbb{Z}_n$ takšna, da velja*

$$r^m = 1,\ tr = t \text{ in } 1 + r + \cdots + r^{m-1} + kt = 0. \tag{7.2}$$

Množica vozlišč posplošenega Bouwerjevega grafa $\mathcal{GB}(m, n, k; r, t)$ je

$$V(\mathcal{GB}(m, n, k; r, t)) = \{(a; \boldsymbol{b}) \mid a \in \mathbb{Z}_m, \boldsymbol{b} \in \mathbb{Z}_n^{k-1}\},$$

sosednost pa je določena s sledečim pravilom:

$$(a; \boldsymbol{b}) \sim \begin{cases} (a+1; \boldsymbol{b}) & ;\ 0 \leq a < m-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (a+1; \boldsymbol{b} + r^a \boldsymbol{e_i}) & ;\ 0 \leq a < m-1,\ 1 \leq i \leq k-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (0; \boldsymbol{b} + t\mathbf{1}) & ;\ a = m-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1} \\ (0; \boldsymbol{b} + r^{m-1}\boldsymbol{e_i} + t\mathbf{1}) & ;\ a = m-1,\ 1 \leq i \leq k-1,\ \boldsymbol{b} \in \mathbb{Z}_n^{k-1}. \end{cases}$$

Pri tem $\boldsymbol{e_i} \in \mathbb{Z}_n^{k-1}$ predstavlja i-ti standardni vektor, $\mathbf{1} = \boldsymbol{e_1} + \boldsymbol{e_2} + \cdots + \boldsymbol{e_{k-1}}$ je vektor samih enic, $\lambda \boldsymbol{v}$ pa označuje običajni skalarni produkt v $\mathbb{Z}_n$-modulu $\mathbb{Z}_n^{k-1}$.

Graf $\mathcal{GB}(m, n, k; r, t)$ je regularen stopnje $2k$. Ta družina grafov je posplošitev Bouwerjevih grafov $\mathcal{B}(k, m, n)$ [8], ki sovpadajo z grafi $\mathcal{GB}(m, n, k; 2, 0)$, kjer je $2^m = 1$ v $\mathbb{Z}_n$ (iz zahteve $1 + 2 + 2^2 + \cdots + 2^{m-1} = 0$ sledi $2^m = 1$). Poleg tega je v primeru $r = 2$ pogoj $t(r-1) = 0$ enak pogoju $t = 0$, zato so Bouwerjevi grafi natanko posplošeni Bouwerjevi grafi $\mathcal{GB}(m, n, k; r, t)$ za $r = 2$. Klasifikacijo PLT Bouwerjevih grafov sta pred nedavnim našla Conder in Žitnik ter jo opisala v članku [11].

V doktorski disertaciji podamo popolno klasifikacijo pol-ločno tranzitivnih članov te nove družine grafov in dokažemo, da so vsi pol-ločno tranzitivni člani tesno speti grafi. Med drugim dokažemo naslednji izrek.

Izrek 7. *Naj bodo $m \geq 3$, $n \geq 2$ in $k \geq 2$ naravna števila in naj bosta $r \in \mathbb{Z}_n^*$ in $t \in \mathbb{Z}_n$ takšna, da velja $r^m = 1$, $tr = t$ in $1 + r + \cdots + r^{m-1} + kt = 0$. Potem je graf $\Gamma = \mathcal{GB}(m, n, k; r, t)$ Cayleyjev graf meta-abelske grupe, ki premore pol-ločno tranzitivno podgrupo grupe avtomorfizmov, glede na katero je graf tesno spet. Poleg tega je graf Γ pol-ločno tranzitiven, razen če velja:*

- $r^2 = 1$, *ali*
- $k = 2$ *in velja ena od sledečih možnosti:*
  - ◦ $r^2 = -1$;
  - ◦ $(m, n; r, t) \in \{(3, 7; 2, 0), (3, 7; 4, 0)\}$;
  - ◦ $(m, n) = (6, 7n_0)$ *za nek* $n_0 \geq 1$, *kjer* $7 \nmid n_0$, *in obstaja natanko ena rešitev* $r' \in \{r, -r, r^{-1}, -r^{-1}\}$ *enačbe* $2 - r' - r'^2 = 0$ *z* $r' \equiv 5 \pmod 7$ *in* $2+r'+t' = 0$, *kjer velja* $t' = t$ *v primeru* $r' \in \{r, r^{-1}\}$ *in* $t' = t+r+r^3+r^5$ *v primeru* $r' \in \{-r, -r^{-1}\}$.

Določili smo tudi polne grupe avtomorfizmov PLT $\mathcal{GB}(m, n, k; r, t)$ grafov in poiskali vse možne izomorfizme med njimi. Dokazali smo, da so stabilizatorji vozlišč v polni grupi avtomorfizmov izomorfni simetrični grupi S_k.

Stvarno kazalo

Kazalo

Declaration

I declare that this thesis does not contain any material previously published or written by another person except where due reference is made in the text.

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