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Zaključna naloga
(Final project paper)

Kromatsko število pandiagonalnih Latičnih kvadratnih grafov

(On the chromatic number of pandiagonal Latin square graphs)

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Izvleček:
Problem iskanja kromatičnega števila pandiagonalnih Latičnih grafov za $n \not\equiv \pm 1$ $(mod\ 6)$ sta v svojem članku o integralnih Cayleyevih grafih postavila Klotz in Sander. Ta članek bo postavil spodnje in zgornje meje, ter predstavil kromatično število za majhne vrednosti n. Ključni rezultati so torej sledeči: spodnja meja $PLSG(n) \geq n$ za lihe n in $PLSG(n) \geq n+2$ za sode n; zgornja meja $\chi(PLSG(n)) \leq n + c(log\ n)^4 n^{\frac{3}{4}}$; rekurzivna zgornja meja je $\chi(PLSG(n)) \leq \chi(PLSG(d)) \cdot \chi(PLSG(n/d))$, kjer je $1 < d < n$ delitelj n.

Key words documentation

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Abstract: The problem of finding chromatic numbers of pandiagonal Latin square graphs for $n \not\equiv \pm 1\ (mod\ 6)$ was posed by Klotz and Sander in their paper on integral Cayley graphs. This final project paper establishes lower and upper bounds and presents the chromatic numbers for small values of n. Namely, the main results are: a lower bound $PLSG(n) \geq n$ for odd n and $PLSG(n) \geq n + 2$ for even n; an upper bound $\chi(PLSG(n)) \leq n + c(log\ n)^4 n^{\frac{3}{4}}$; a recursive upper bound $\chi(PLSG(n)) \leq \chi(PLSG(d)) \cdot \chi(PLSG(n/d))$ where $1 < d < n$ is a divisor of n.

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Contents

List of Figures

1 Introduction

This final project contains my research work on the chromatic number of pandiagonal Latin square graphs. It is known that that $\chi(PLSG(n)) \geq n$, and equality holds if and only if $n \equiv \pm 1$ $(mod\ 6)$ [3]. The problem to find $\chi(PLSG(n))$ for $n \not\equiv \pm 1$ $(mod\ 6)$ was posed by Klotz and Sander in [5] as a problem.

In Section 2, we consider the so called *cyclic Latin square graphs* $LSG(n)$, defined as follows: the vertex set $V(LSG(n)) = \mathbb{Z}_n^2$, and vertices (r, c) and (r', c') are connected if any of the following conditions holds: $r = r'$, $c = c'$ or $r + c = r' + c'$. The graph $LSG(n)$ appears as a natural subgraph of $PLSG(n)$.

In section 3, we review the paper [2], where it is proved that

$$\chi(LSG(n)) = \begin{cases} n+2 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

In particular, we provide a detailed proof of this result for later use. As a consequence, we obtain the lower bound

$$\chi(PLSG(n)) \geq \begin{cases} n+2 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

In Section 4, we compute the chromatic number $\chi(PLSG(n))$ for $n \leq 8$ using computer, and also present an example of minimal proper coloring in each case. We used SageMath's function *sage.graphs.graph_coloring.vertex_coloring*, which converts the chromatic number problem into an integer linear optimization problem. We also find proper colorings of $PLSG(9)$ and $PLSG(10)$ with 12 and 13 colors, respectively, and hence obtain that $\chi(PLSG(9)) \leq 12$ and $\chi(PLSGL(10)) \leq 13$.

In Section 5, we represent the pandiagonal Latin square graphs as 4-uniform linear hypergraphs through the following procedure. The vertices of the hypergraph are the rows, columns and all types of diagonals of an $n \times n$ matrix. Every vertex of $PLSG(n)$ is then a 4-sized hyperedge and the hypergraph has degree n. This gives us the following asymptotic upper bound, described by Molloy and Reed in the paper [6]:

$$\chi(PLSG(n)) \leq n + c(log\ n)^4 n^{\frac{3}{4}}$$

for some constant c. This constant c is the same not just for all pandiagonal Latin square graphs, but for chromatic index of all 4-uniform linear hypergraphs, which we use to establish the lower bound $c \geq \frac{3}{2^{\frac{3}{4}}(ln\ 2)^4}$.

In Section 6, we show that pandiagonal Latin square graphs are isomorphic to certain Cayley graphs over the goup $\mathbb{Z}_n^2$. Based on this observation, we provide an alternative proof for the known result that $\chi(PLSG(n)) = n$ for $n \equiv \pm 1$ $(mod\ 6)$. This was previously proved by Hedayat [3] through the existence of Knut Vik designs of the given sizes.

Finally, in Section 7, we derive a recursive upper bound:

$$\chi(PLSG(n)) \leq \chi(PLSG(d)) \cdot \chi(PLSG(\frac{n}{d})),$$

where d is a divisor of n. The proof is based on finding a homomorphism φ from $PLSG(n)$ to $PLSG(d)$ such that for every vertex v of $PLSG(d)$, the induced subgraph of $PLSG(n)$ on the preimage $\varphi^{-1}(v)$ is isomorphic to $PLSG(\frac{n}{d})$. As an application, we obtain the following upper bound in the case when $n = 2^a \cdot 3^b \cdot p$:

$$\chi(PLSG(n)) \leq 2^{\lceil \frac{2a}{3} \rceil} \cdot 3^b \cdot \frac{11}{8}^{\lfloor \frac{a}{3} \rfloor} \cdot n.$$

2 Latin square graphs

We start with the definition of a Latin square graph arising from a finite group G.

Definition 2.1. For a finite group G of order n, the *Latin square graph* L_G has vertex set $V(L_G) = \{(r, c) | r, c \in G\}$ and vertices (r, c) and (r', c') are connected if any of the following conditions holds: $r = r'$, $c = c'$ or $r + c = r' + c'$.

In the special case when $G = \mathbb{Z}_n$, that is, the cyclic group of order n, the Latin square graph $L_{\mathbb{Z}_n}$ will be called *cyclic Latin square graph*, denoted by $LSG(n)$.

Definition 2.2. Let G be a group and S be a subset of G such that $1 \notin S$ and $g^{-1} \in S$ whenever $g \in S$. The *Cayley Graph* $\Gamma = \mathrm{Cay}(G, S)$ has vertex set $V(\Gamma) = G$ and $g \in G$ is connected to gs for all $s \in S$.

It is not hard to see that the cyclic Latin square graph $\mathrm{LSG}(n)$ is isomorphic to the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_n, S^n)$, where $S^n = S_1^n \cup S_2^n \cup S_3^n$ and

- $S_1^n = \langle (0, 1) \rangle \setminus \{(0, 0)\}$ (in the same row),
- $S_2^n = \langle (1, 0) \rangle \setminus \{(0, 0)\}$ (in the same column),
- $S_3^n = \langle (-1, 1) \rangle \setminus \{(0, 0)\}$ (in the same diagonal).

The chromatic number of the graph $\mathrm{LSG}(n)$ was determined in [2].
They give a lower bound for a general case Latin square graph.

Corollary 2.3.

- *$\chi(L_G) = |G|$ for every group G of odd order*
- *For every group G of order n, either $\chi(L_G) = n$ or $\chi(L_G) \geq n + 2$.*
- *Let G be an abelian group of order n, $\chi(L_G) \leq n+2$ if and only if G has a unique element of order 2*

And then show that this bound is exact for cyclic Latin square graphs.
We list a more detailed proof of the following here, as we will use it later.

Theorem 2.4.

$$\chi(\mathrm{LSG}(n)) = \begin{cases} n & \textit{if } n \textit{ is odd,} \\ n+2 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. We will consider the sets $T_i = \{(r, r+i) | r \in \mathbb{Z}_n\}$. We are going to find below the edges connecting vertices in T_i. It is sufficient to consider only the set T_0 and its vertex $v_0 = (0, 0)$.

Suppose that v_0 is connected to $v_1 = (a, a)$ for some $a \in \mathbb{Z}_n$ and $a \neq 0$. This means $v_1 = v_1 - v_0 \in S$. For $v_1 \in S_1$ we get $a = 0$, and for $v_1 \in S_2$ $a = 0$ as well, so they can not give an edge. For $v_1 \in S_3$, $(a, a) \in \langle(-1, 1)\rangle$ implying that $a = -a$ in $\mathbb{Z}_n$, so $2a = 0$ holds in $\mathbb{Z}_n$. This means that either $a = 0$ or $2a = n$.

Thus if n is odd, then T_i is an independent set of the graph. If we color vertices of each subset T_i with the same color, we get an n-coloring and by this the theorem is proved for n odd.

Now, we turn to the case when n is even. Each vertex $(a, a+i) \in T_i$ is connected to $(m+a, m+a+i) \in T_i$, where $n = 2m$.

Let us consider the subgraph Γ_i: $V(\Gamma_i) = T_i \cup T_{i+1}$. To find all the edges connecting a vertex from T_i with a vertex from T_{i+1}, it is sufficient to consider the vertex $v_0 = (0, 0) \in T_0$ and $v_2 = (a, a+1) \in T_1$, where $a \in \mathbb{Z}_n$. We find below a so that $\{v_0, v_2\}$ is an edge:
$v_2 \in S_1$: $a + 1 = 0$, hence $a = n - 1$.
$v_2 \in S_2$: $a = 0$.
$v_2 \in S_3$: $(a, a+1) \in \langle(-1, 1)\rangle$, hence $a + 1 = -a$, hence $2a = n - 1$. But n is even, so $n - 1$ is odd and there is no $a \in \mathbb{Z}_n$ satisfying $2a = n - 1$.

These edges $(x, x+i) \in T_i \sim (x, x+i+1) \in T_{i+1}$ and $(x, x+i) \in T_i \sim (x-1, x+i) \in T_{i+1}$ form a cycle of length $2n$. Let us label the vertices of this cycle as w_j where $j \in [0; 2n-1]$ in such a way that $w_0 = (0, i)$, w_j is connected to w_{j-1} and w_{j+1}, $w_j \in T_i$ for even j and $w_j \in T_{i+1}$ for odd j.

The fact that $(x, x+i) \in T_i$ is connected to $(x+m, x+m+i) \in T_i$ means w_j is connected to w_{j+n} for even j. The same can be said about T_{i+1}, so $w_j \sim w_{j+n}$ for odd j too.

The resulting graph Γ_i is known as a Möbius ladder. It is also isomorphic to the Circulant graph $Ci_{2n}(1, n)$, in which every vertex w_j is connected to $w_{j-1}, w_{j+1}, w_{j-n}, w_{j+n}$, where $w_{j-n} = w_{j+n}$.

If we remove w_j and $w_{j+n\pm1}$ for some j from this graph, it will become bipartite.

It is sufficient to show that Γ_i is bipartite when removing w_0 and $w_{m\pm1}$ as the result of removing w_a and $w_{a+m\pm1}$ is simply a rotation $j \mapsto j + a$.

Figure 1: $Ci_{16}(1, 8)$

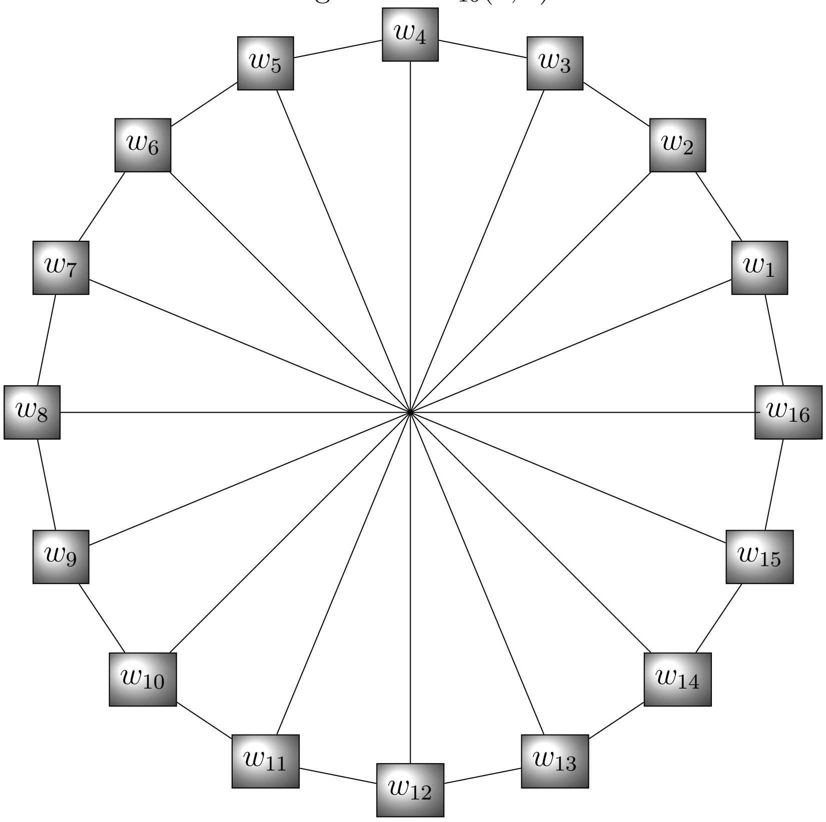


Figure 2: $Ci_{16}(1, 8)$ with w_0 and w_9 removed

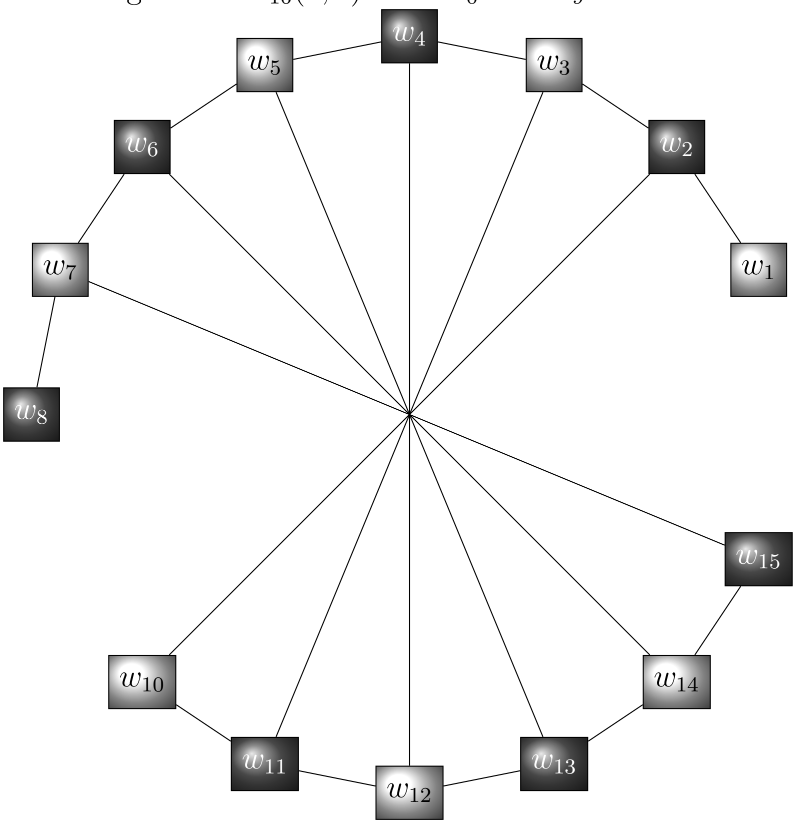
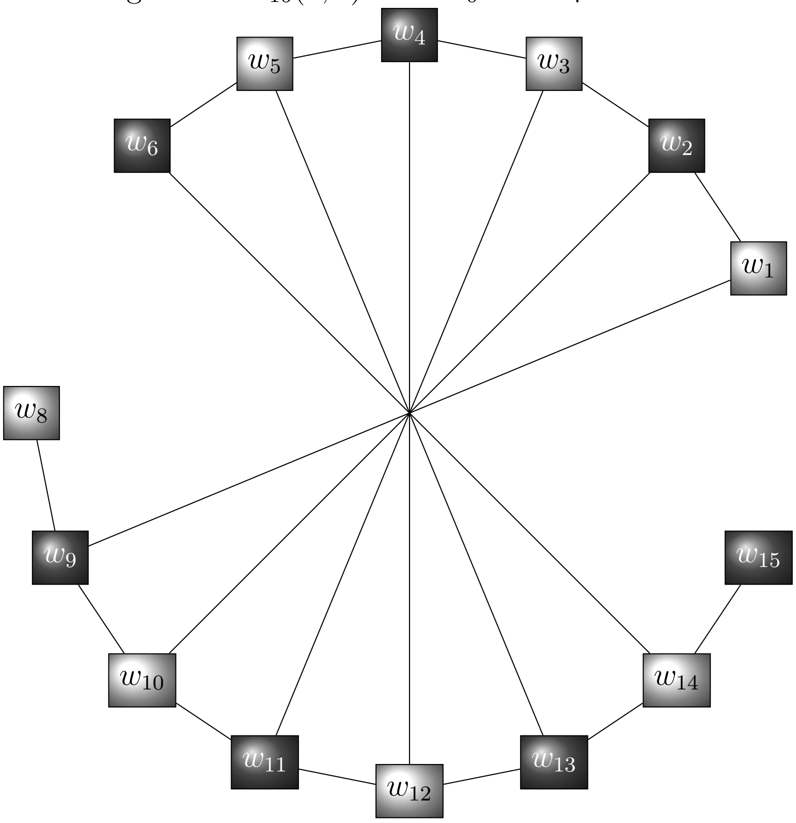


Figure 3: $Ci_{16}(1,8)$ with w_0 and w_7 removed



Let us assume that we removed w_0 and w_{m-1}. In case m is odd, we simply assign color 0 to w_j for even j and 1 for odd j. In case m is even, we assign color 0 to w_j for odd $j < m$ and even $j \geq m$, color 1 for even $j < m$ and even $j \geq m$.

Let us show that connected vertices are colored differently. This is clear for the edges $\{w_j, w_{j+1}\}$ with $j \in [0, m-2] \cup [m, n-1]$.
For edges $\{w_j, w_{j+m}\}$, if m is odd, then j and $j+m$ have different parity, so they are colored differently, for m even, j and $j+m$ are both even or odd, but are colored differently because they cannot be smaller or bigger than m at the same time.

The only case left is that there exists a 2−colorable subgraph Γ' such that for all even i, $V(\Gamma_i) \cap V(\Gamma') = \{w_j, w_{j+m\pm 1}\}$ for some j. If such subgraph does exist, then we can partition LSG(n) into a $m+1$ sized set of subgraphs $\{\Gamma_i - \Gamma' \mid \text{even } i\} \cup \{\Gamma'\}$, each of which can be colored with 2 colors, thus giving a coloring of LSG(n) with $2(m+1) = n+2$ colors.

Such subgraph Γ' can be constructed in the following way. Let $k = \lceil \frac{m}{2} \rceil$.

$$X = \{(j, 3j) | j \in [0, k-1]\}$$

$$Y = \{(j, 3j+2k) | j \in [0; m-k-1]\}$$

$$d = \begin{cases} (m, m+1), & \text{if } m \equiv 0 \ (mod\ 3) \\ (m-1, m), & \text{if } m \not\equiv 0 \ (mod\ 3) \end{cases}$$

$$X' = X + d$$

$$Y' = Y + d$$

$$V(\Gamma') = X \cup X' \cup Y \cup Y'$$

With a following lemma the proof is complete. □

Lemma 2.5. *The sets $X \cup X'$ and $Y \cup Y'$ are independent sets in* $\mathrm{LSG}(n)$.

Proof. First let us prove it for $X \cup X'$.

If for $j \in [1, k-1]$, $(0,0) \in X \sim (j, 3j) \in X$, then $j = 0$ or $3j = 0$ or $2j = 0$. But $j \leq k - 1 < \frac{n}{4}$ so $2j < 3j < n$. This means that X is an independent set.
$X' = X + d$, so it is an independent set as well. Now let us show that there are no edges between X and X'. Let $(a, 3a) \in X$, $(b, 3b) + d \in X'$, $a, b \in [0, k-1]$.

Let us consider the case when $m \equiv 0$ $(mod\ 3)$.
The chosen vertices are connected when $a = b+m$, $3a = 3b+m+1$, $2a = 2b+1$ in $\mathbb{Z}_n$.

1. $a \neq b + m$ as $a \in [0, k-1], b + m \in [m, m + k - 1]$
2. $3a \equiv 3b + m + 1 \ (mod\ n) \iff$
   $3(a - b) \equiv m + 1 \ (mod\ n) \iff$
   $3(a - b) = (2t + 1)m + 1$ for some t (due to $n = 2m$)
   $3(a - b) - (2t + 1)m \equiv 0 \ (mod\ 3),$ so it cannot be equal to 1
3. $2a \equiv 0 \ (mod\ 2)$ and $2b + 1 \equiv 1 \ (mod\ 2)$ while $2a, 2b + 1 < n$.

If $m \not\equiv 0$ $(mod\ 3)$, the vertices are connected if $a = b + m - 1$, $3a = 3b + m$, $2a + m = 2b + m + 1$.

1. Still $a \neq b + m - 1$ as $a, b \in [0, k - 1]$.
2. $3(a - b) \equiv m \iff$
   $3(a - b) = (2t + 1)m$ while $|a - b| \leq 2k - 2 \leq m$ so $t = 0$ or $t = -1$, which are equivalent.
   $3(a - b) = m$, but $m \not\equiv 0 \ (mod\ 3)$.
3. The third equation is unchanged.

A very similar discourse shows that $Y \cup Y'$ is independent.
For $j \in [1, m-k-1]$, $(0, 2k) \in Y \sim (j, 3j+2k) \in Y'$, hence $j = 0$, $3j + 2k = 2k$ or $2j + 2k = 2k$. $3j \leq 3m - 3k - 3 \leq n - k - 3$ so $3j \neq 0$. $2j < 3j$, so $2j \neq 0$.
$Y' = Y + d$, so it is also independent.

Showing that there are no edges between Y and Y' goes completely analogously as above for X and X'. Let $(a, 3a+2k) \in Y$, $(b, 3b+2k) + d \in Y'$, $a, b \in [0, m-k-1]$.

Let us consider $m \equiv 0$ $(mod\ 3)$. The chosen vertices are connected when $a = b+m$, $3a + 2k \neq 3b + 2k + m + 1$, $2a + 2k \neq 2b + 2k + 1$.

1. $a \neq b + m$, as $a \in [0, m-k-1]$, $b + m \in [m, n-k-1]$.

2. $3a \equiv 3b + m + 1$ $(mod\ n)$ $\iff$
   $3(a-b) \equiv m + 1$ $(mod\ n)$ $\iff$
   $3(a-b) = (2t+1)m + 1$ for some t $(n = 2m)$.
   $3(a-b) - (2t+1)m \equiv 0$ $(mod\ 3)$, so it cannot be equal to 1.

3. $2a + 2k \equiv 2b + 2k + 1$ $(mod\ n)$ $\iff$ $2(a-b) \equiv 1$ $(mod\ n)$.
   $n \equiv 0$ $(mod\ 2)$, hence $2(a-b) \not\equiv 1$ $(mod\ n)$

If $m \not\equiv 0$ $(mod\ 3)$, the chosen vertices are connected if $a = b + m - 1$,
$3a + 2k = 3b + 2k + m$ or $2a + 2k = 2b + 2k + 1$

1. $a \neq b + m - 1$ as $a, b \in [0, m-k-1]$.

2. $3a \equiv 0$ $(mod\ 3)$ while $3b + m \not\equiv 0$ $(mod\ 3)$.

3. $2a + 2k \equiv 0$ $(mod\ 2)$, $2b + 2k + 1 \equiv 1$ $(mod\ 2)$, $2b + 2k + 1 \leq n - 2k - 2 + 2k + 1 = n - 1 < n$.

$\square$

3 Pandiagonal latin square graphs

Definition 3.1. A *pandiagonal Latin square* is an $n \times n$-matrix with entries from $\{1, ..., n\}$ such that every number appears exactly once in every row, in every column, in the main diagonal and its broken parallels, as well as in the secondary diagonal and its broken parallels.

Pandiagonal Latin square graphs were introduced in [5].

Definition 3.2. The *pandiagonal Latin square graph* $\mathrm{PLSG}(n), n \geq 2$ is the graph whose vertes set is the n^2 positions of an $n \times n$ matrix, and two vertices are adjacent if they are in the same row, column, diagonal or broken diagonal (parallel to the main or secondary diagonal).

It can be easily checked that PLSG(2) and PLSG(3) are simply the complete graphs on 4 and 9 vertices, respectively.

It is also trivial to see that $\chi(PLSG(n)) \geq n$, as $PLSG(n)$ contains n-sized cliques (rows, columns and all types of diagonals).

Proposition 3.3.

$$\chi(\mathrm{PLSG}(n)) \geq \chi(\mathrm{LSG}(n)).$$

Proof. It is clear that any proper coloring of the pandiagonal Latin square graph is also a proper coloring of the cyclic Latin square graph, as $V(\mathrm{PLSG}(n)) = V(\mathrm{LSG}(n))$. Also, $E(\mathrm{PLSG}(n)) \subseteq E(\mathrm{LSG}(n))$ (hence $\mathrm{LSG}(n)$ is a subgraph of $\mathrm{PLSG}(n)$. Thus every proper coloring of $\mathrm{PLSG}(n)$ is also a proper coloring of $\mathrm{LSG}(n)$, hence $\chi(\mathrm{LSG}(n)) \leq \chi(\mathrm{PLSG}(n))$. □

Corollary 3.4. $\chi(\mathrm{PLSG}(n)) \geq n + 2$ *for even* n.

Proof. It follows directly from Theorem 2.4 and Proposition 3.3. □

It follows that $\mathrm{PLSG}(n)$ can be interpreted as a combination of two cyclic Latin square graphs. If we apply the mapping $\alpha : \mathrm{LSG}(n) \to \mathrm{PLSG}(n)$ defined by $\alpha((x, y)) = (-x, y)$, this induces a bijection from S_1 and S_2 onto themselves and from S_3 into S_4. This means that proper colorings of two Latin square graphs give rise to a proper coloring of the pandiagonal Latin square graph.
In general case, however, this does not give non-trivial results: best-case chromatic

number of a LSG(n) is n, so the constructed coloring of PLSG(n) uses up to n^2 colors (pairs of colors in the corresponding Latin square graphs).

4 Computational results

Finding the chromatic number of a graph was listed as one of the Karp's 21 NP-complete problems [4], so there is no general solution in polynomial time unless P=NP. I decided to use the function *sage.graphs.graph_coloring.vertex_coloring* from SageMath, which calculates the chromatic number by converting the problem into an integer linear optimization in the following way.

We have to fix the total amount of colors k (not all of which will be used); it is always possible to use one of the trivial upper bounds on the chromatic number to achieve that.
For every color i, we define a boolean (valued 0 or 1) variable c_i which is 1 exactly when at least one vertex has the color i. This allows us to calculate the amount of colors , which we will minimize in our linear problem, as

$$\sum_i c_i.$$

For every vertex v and color i, we define a boolean variable $d_{v,i}$, which is 1 exactly when v has color i. Every vertex has only one color, so we introduce a constraint

$$\sum_i d_{v,i} = 1.$$

Adjacent vertices cannot have the same color, so for every such pair v,w we introduce k constraints, one for every color i :

$$d_{v,i} + d_{w,i} \leq 1.$$

This produced the following results:

$$\chi(\mathrm{PLSG}(3)) = 9$$

Figure 4: Example of a proper coloring of PLSG(3) with 9 colors.

0	1	2
3	4	5
6	7	8

$$\chi(PLSG(4)) = 8$$

Figure 5: Example of a proper coloring of PLSG(4) with 8 colors.

7	0	6	4
6	4	1	0
3	7	5	2
5	2	3	1

$$\chi(\mathrm{PLSG}(5)) = 5$$

Figure 6: Example of a proper coloring of PLSG(5) with 5 colors.

3	4	1	0	2
1	0	2	3	4
2	3	4	1	0
4	1	0	2	3
0	2	3	4	1

$$\chi(\mathrm{PLSG}(6)) = 9$$

Figure 7: Example of a proper coloring of PLSG(6) with 9 colors.

2	5	3	7	1	6
3	7	1	2	0	5
6	0	5	8	7	4
5	8	4	3	6	1
4	2	0	1	8	3
7	6	8	4	2	0

$$\chi(PLSG(7)) = 7$$

Figure 8: Example of a proper coloring of PLSG(7) with 7 colors.

6	5	4	1	3	2	0
4	1	3	2	0	6	5
3	2	0	6	5	4	1
0	6	5	4	1	3	2
5	4	1	3	2	0	6
1	3	2	0	6	5	4
2	0	6	5	4	1	3

$$\chi(\text{PLSG}(8)) = 11$$

Figure 9: Example of a proper coloring of PLSG(8) with 11 colors.

9	6	5	0	4	10	7	3
5	0	3	9	2	8	4	6
4	2	6	8	3	9	5	7
6	1	4	5	7	0	3	10
0	3	2	1	10	4	6	8
7	9	0	3	8	1	10	2
10	5	1	2	9	7	8	0
2	8	10	7	1	5	9	4

As expected, these match the chromatic numbers known for $n = \pm 1 \ (mod\ 6)$ (see Theorem 5.2).

While the computation time grows extremely quickly and we were unable to compute $\chi(PLSG(9))$ and above this way, it is still possible to generate upper bounds and sub-optimal colorings by guessing the amount of colors.

$$\chi(\mathrm{PLSG}(9)) \leq 12$$

Figure 10: Example of a proper coloring of PLSG(9) with 12 colors.

0	3	8	4	10	11	7	1	6
5	2	1	7	6	9	10	3	4
3	10	4	9	1	8	2	5	11
9	11	2	8	4	5	1	6	7
6	1	5	3	11	7	4	9	8
7	4	6	0	8	1	11	2	5
2	0	9	10	7	4	5	8	3
11	7	3	2	5	6	9	10	0
10	6	0	1	3	2	8	11	9

$$\chi(\mathrm{PLSG}(10)) \leq 13$$

Figure 11: Example of a proper coloring of PLSG(10) with 13 colors.

0	4	11	7	6	1	9	2	5	12
8	10	0	5	2	12	4	3	11	6
2	3	7	8	4	6	11	10	12	1
12	1	10	6	9	8	3	4	0	5
4	6	3	0	12	2	10	9	8	11
9	11	8	10	5	3	6	12	4	2
10	12	2	4	11	7	1	5	9	0
5	0	6	1	10	4	2	8	3	7
6	8	5	3	7	9	12	11	2	10
7	2	12	9	8	10	5	0	1	4

5 An approach via hypergraphs

Definition 5.1. *Hypergraph* H is a pair (V, E), where V is a set of vertices and E is a set of non-empty subsets of V - hyperedges.

Definition 5.2. *Edge coloring* of a hypergraph H is a mapping $\phi : E \to C$ for some set C, such that $\forall e_1, e_2 \in E\ \exists v \in V : v \in e_1, v \in e_2 \Rightarrow \phi(e_1) \neq \phi(e_2)$.

Definition 5.3. *Chromatic index* of a hypergraph H is the smallest number $k = \chi'(H)$ such that there exists an edge coloring of H with k colors.

Definition 5.4. Hypergraph H is *k-uniform* if every hyperedge has exactly k vertices in it.

Definition 5.5. A vertice in hypergraph H has *degree* Δ when exactly Δ hyperedges from $E(H)$ contain it.

Definition 5.6. Hypergraph H *has degree* Δ if it's every vertice has degree at most Δ.

In their paper [6] Molloy and Reed prove the following:

Theorem 5.7. *For all k, there is a constant c_k depending on k such that any $k-$uniform linear hypergraph has list chromatic index at most $\Delta + c_k\Delta^{\frac{k-1}{k}}(log\Delta)^4$, where Δ is the degree of the hypergraph.*

The Theorem was proven using the Lovász Local Lemma, so it was not possible to get a coloring or value of c_k directly. However since then an algorithmic approach to the Local Lemma was developed [1], so it can be done directly. We, however, will consider a different approach.

Corollary 5.8. *There exists c such that for $n \geq 3$ $\chi(PLSG(n)) \leq n + c(log\ n)^4 n^{\frac{3}{4}} \Rightarrow$ For $n \to \infty$ $\chi(PLSG(n)) \leq n + o(n)$*

Proof. Let us construct a hypergraph H the following way: let rows, columns, diagonals and broken diagonals of $PLSG(n)$ be the vertices of H, so $|V(H)| = 4n$. For every vertex of $PLSG(n)$ we will have a hyperedge in H connecting a row, a column and 2 (broken) diagonals, which gives $|E(H)| = n^2$. This way H is a 4-uniform linear hypergraph with degree n, and chromatic index of H is equal to $\chi(PLSG(n))$. □

Knowing some values of $\chi(PLSG(n))$ we can find a lower bound for the constant c in 5.8:

$$c \geq \frac{\chi(PLSG(n)) - n}{n^{\frac{3}{4}}(logn)^4}$$

This gives us roughly $c \geq 0.38$ for $n = 4$, $c \geq 0.08$ for $n = 6$ and $c \geq 0.04$ for $n = 8$.

But the constant c is the same for any 4-uniform linear hypergraph H with degree Δ and chromatic index $\chi'(H)$.

$$c \geq \gamma(\chi, \Delta) = \frac{(\chi - \Delta)}{\Delta^{\frac{3}{4}}(log\Delta)^4}$$

Due to the way Theorem 5.7's proof was constructed, c has to be exactly the largest value of γ, as it is an optimal bound for some 4-uniform linear hypergraph.

Lemma 5.9. *Let H be a linear 4-uniform hypergraph of degree Δ. $\chi'(H) \leq 4\Delta - 3$.*

Proof. We reverse the operation we used to prove 5.8. Let's define graph G: $V(G) = E(H)$, For $v_1, v_2 \in V(G)$ $v_1 \sim v_2 \iff \exists a \in V(H) : a \in v_1, a \in v_2$. Any edge in H has 4 vertices, and each of those vertices has $\Delta - 1$ other edges connected to it. Thus G is a graph of degree at most $D = 4(\Delta - 1)$. This means $\chi(G) \leq D + 1 = 4\Delta - 3$. But because of the way we constructed G $\chi(G) = \chi'(H)$. □

Corollary 5.10.

$$c \geq \frac{3}{2^{\frac{3}{4}}(ln\ 2)^4} \approx 7.73$$

Proof. As we've shown, $\chi \in [\Delta; 4\Delta - 3]$. Ignoring if graphs of specific Δ and χ' exist, γ's maximum should be reached for $\chi' = 4\Delta - 3$

Let us consider a function

$$f = \frac{3(\Delta - 1)}{n^{\frac{3}{4}}log^4 n}$$

$$f' = \frac{\Delta^{\frac{3}{4}}log^4\ \Delta - (\Delta - 1)(\frac{3}{4}\Delta^{\frac{-1}{4}} + 4log^3\ \Delta\Delta^{\frac{-1}{4}}))}{\Delta^{\frac{6}{4}}log^8\ \Delta}$$

$$= \frac{log^3\ \Delta\Delta^{\frac{-1}{4}}}{\Delta^{\frac{6}{4}}log^8\ \Delta} \cdot (\frac{1}{4}\Delta log\ \Delta - 4\Delta + \frac{3}{4}log\ \Delta + 4)$$

$$= \frac{1}{\Delta^{\frac{7}{4}}log^5\ \Delta} \cdot (\frac{1}{4}\Delta log\ \Delta - 4\Delta + \frac{3}{4}log\ \Delta + 4)$$

f' has two roots in $[1; +\inf]$: $\Delta_1 = 1$ and a large one, roughly $\Delta_2 \approx 8 \cdot 10^6$. For $\Delta \in [\Delta_1; \Delta_2]$ $f'(\Delta) < 0$, for $\Delta > \Delta_2$ $f'(\Delta) > 0$.

Let's first consider $\Delta = 1$. Hyperedges don't have common vertices, $\chi' \in 0, 1$, so γ is at most 0.

For $\Delta = 2$ there exists a hypergraph H_2 with $\chi'(H_2) = 4\Delta - 3$: 5 hyperedges going through 10 vertices, each hyperedge intersecting all others. As this is the largest χ' for $\Delta = 2$ and $f' < 0$ $\gamma(5, 2) = \frac{3}{2^{\frac{3}{4}}(ln\ 2)^4} \approx 7.73$ is the largest value on $\Delta \in [\Delta_1; \Delta_2]$. □

For $\Delta > \Delta_2$ $f' > 0$, so $\gamma(\chi, \Delta)$ grows indefinitely. However, we can establish an upper bound on it's value.

$$\lim_{\Delta \to +\inf} \gamma(\Delta) \leq \lim_{\Delta \to +\inf} \frac{\chi - \Delta}{\Delta^{\frac{3}{4}} (log \Delta)^4}$$

So acquiring an exact bound on chromatic index of 4-uniform linear hypergraphs of large enough degree yields the actual value of c with this approach.

6 PLSG as a Cayley Graph

Observe that pandiagonal Latin square graphs can also be represented as Cayley graphs. The proof is straightforward.

Proposition 6.1. *The pandiagonal Latin square graph* PLSG(n) *is isomorphic to the Cayley graph* $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_n, S^n)$*, where* $S^n = S_1^n \cup S_2^n \cup S_3^n \cup S_4^n$*, and*

- $S_1^n = \langle(0,1)\rangle \setminus \{(0,0)\}$ *(in the same row),*
- $S_2^n = \langle(1,0)\rangle \setminus \{(0,0)\}$ *(in the same column),*
- $S_3^n = \langle(-1,1)\rangle \setminus \{(0,0)\}$ *(in the same diagonal),*
- $S_4^n = \langle(1,1)\rangle \setminus \{(0,0)\}$ *(in the same secondary diagonal).*

In his paper, Hedayat [4] has shown that pandiagonal Latin squares (he denotes them as Knut Vik designs) exist exactly for $n \equiv \pm 1$ $(mod\ 6)$. This is equivalent to saying that $\chi(\mathrm{PLSG}(n)) = n$ if and only if $n \equiv \pm 1$ $(mod\ 6)$. Here we provide an alternative proof for the sufficiency part of this fact.

Theorem 6.2. $\chi(\mathrm{PLSG}(n)) = n$ *for* $n \equiv \pm 1$ $(mod\ 6)$.

Proof. We will consider the following subsets of $V(\mathrm{PLSG}(n))$: $T_j = \{(r, 2r+j) \mid r \in \mathbb{Z}_n\}$ for $j \in \mathbb{Z}_n$. Let us determine the edges connecting vertices in T_j. It is sufficient to find those edges with one end-vertex equal to $(0, j)$, as the other ones will be translates of these. Vertices $(0, j)$ and $(a, 2a + j)$, where $a \in \mathbb{Z}_n$, $a \neq 0$, are connected if any of the following equalities holds:

$$2a + j \equiv j\ (mod\ n)\ \text{(in the same row)},$$

$$a + j \equiv j\ (mod\ n)\ \text{(in the same column)}$$

$$a + j \equiv j\ (mod\ n)\ \text{(in the same diagonal)}$$

$$3a + j \equiv j\ (mod\ n)\ \text{(in the same secondary diagonal)}.$$

These are independent of j, so the edges are going to be the same for all T_j.
We end up with 3 possibilities: $a \equiv 0$, $2a \equiv 0$, $3a \equiv 0$, all *mod* n.
The first is impossible because of the assumption $a \neq 0$. Since $n \equiv \pm 1$ $(mod\ 6)$, $2a \not\equiv 0$

and $3a \not\equiv 0 \ mod \ n$. This means that all T_j are independent subsets of PLSG(n). They also form an exact cover, as $T_{j_1} \cap T_{j_2} = \emptyset$, and $(x, y) \in V(\text{PLSG}(n))$ will be in T_j for $j = y - 2x$.
By coloring T_j with color j, we achieve a proper n-coloring of PLSG(n). □

Definition 6.3. *Eigenvalues* of an undirected graph G are the eigenvalues of an arbitrary adjacency matrix of G.

In Klotz's and Sander's paper [5] they provide the eigenvalues of $PLSG(n)$:

$$\text{Case n odd: } 4n - 4[1], n - 4[4n - 4], -4[n^2 - 4n + 3]$$

$$\text{Case n even: } 4n - 5[1], 2n - 5[1], n - 3[n], n - 5[3n - 6], -3\left[\frac{n^2}{2} - n\right], -5\left[\frac{n^2}{2} - 3n + 4\right]$$

However applying the Hoffman bound gives no non-trivial results:
$\chi(PLSG(n)) \geq 1 + \frac{4n-4}{4} = n$ for odd n and $\chi(PLSG(n)) \geq 1 + \frac{4n-5}{5} = \frac{4}{5}n$ for even n.

7 A recursive upper bound

In the following chapter we will prove this Theorem:

Theorem 7.1. *Let n be a non-prime number, d be a divisor of n, $1 < d < n$. Then*

$$\chi(\mathrm{PLSG}(n)) \leq \chi(\mathrm{PLSG}(d)) \cdot \chi(\mathrm{PLSG}(\frac{n}{d})).$$

Proof. Let us consider the mapping $\varphi : \mathbb{Z}_n^2 \to \mathbb{Z}_d^2$ defined by $\varphi((i,j)) = (x,y)$ such that $(x,y) \in \mathbb{Z}_d^2$ is the unique element satisfying

$$i \equiv x \ (mod\ d) \text{ and } j \equiv y \ (mod\ d).$$

Let $H = \langle (d,d) \rangle$, that is, $H = \{(id, jd) \,|\, i,j \in \mathbb{Z}_h\} \simeq \mathbb{Z}_h^2$. Then $\varphi^{-1}(v) = H + v$ for every $v = (x,y) \in \mathbb{Z}_d^2$. Let Γ_v denote the induced subgraph of $\Gamma = \mathrm{PLSG}(n)$ on the set $H + v$.

Claim 1. For every $v = (x,y) \in \mathbb{Z}_d^2$, Γ_v is isomorphic to $\mathrm{PLSG}(h)$.

$\varphi : Cay(\mathbb{Z}_n^2, S^n) \to Cay(\mathbb{Z}_d^2, S^d)$.
On vertices it acts as $\varphi : \mathbb{Z}_n^2 \to \mathbb{Z}_d^2$, so for $x = (a,b) \in PLSG(d)$: $\varphi^{-1}(x) = H + x'$, where $x` = (a,b)$ $(a,b \in [0, d-1])$;
$|V(H + x')| = h^2$, so the only thing left to show is that S^n reduced to this subset (let's denote it as $\mathbb{S}^n$) is equal to S^h.
$x' + (0,0), (0,1), ..., (0, h-1), (0, n-h) \subset x' + S_1^n$ forms $\mathbb{S}_1^n$, similar constructions form the rest of $\mathbb{S}^n$. This shows that $\mathbb{S}^n \simeq S^h$.

We can construct a graph Γ' in the following way: $V(\Gamma') = \mathbb{Z}_d^2$, and $v = (x,y)$ is connected to $v' = (x', y')$ when there are vertices in $\varphi^{-1}(v)$ and $\varphi^{-1}(v')$, respectively, which are adjacent in $\mathrm{PLSG}(n)$.

Caim 2. $\Gamma' = \mathrm{PLSG}(d)$.

We have to show that v is connected to v' in Γ' exactly when v is connected to v' in $\mathrm{PLSG}(d)$.

First let us prove that x connected to y in $\mathrm{PLSG}(d)$ means there exists an edge $\{x', y\}$ in $PLSG(n)$, $x' \in \varphi^{-1}(x)$, $y' \in \varphi^{-1}(y)$.
$x = (x_1, x_2)$ $y = (y_1, y_2)$, $x' = (x_1 + i \cdot d, x_2 + j \cdot d)$, $y' = (y_1 + k \cdot h, y_2 + l \cdot h)$ for any i, j, k, l.

Let $y - x \in S_1^d$, then $x_2 \equiv y_2 \ (mod\ d)$. But $x_2, y_2 \in [0; d-1]$, so $x_2 \equiv y_2 \ (mod\ n)$ and for $i = j = k = l = 0$ $x'_2 = y'_2$.

For $y - x \in S_2^d$ similarly we can take $i = j = k = l = 0$ and have $x`_1 = x_1 \equiv y_1 = y`_1 \ (mod\ n)$.

Let $y - x \in S_3^d$, $x_1 + x_2 \equiv y_1 + y_2 \ (mod\ d)$.
First possibility is that $x_1 + x_2 = y_1 + y_2 \ (mod\ n)$, in which case for $i, j, k, l = 0$ we have y' connected to x'.
The other one is $x_1 + x_2 = y_1 + y_2 + d \ (mod\ n)$ (symmetric to when $x_1 + x_2 + d = y_1 + y_2 \ (mod\ n)$). In this case we can take $i = 1$, $j = k = l = 0$, making $x'_1 + x'_2 = y'_1 + y'_2 \ (mod\ n)$.
Same construction takes place for $y - x \in S_4^d$.

Vice versa, if x' is connected to y' in $PLSG(n)$, $x = \varphi(x')$ is connected to $y = \varphi(y')$ in $PLSG(d)$.
If for $i \in \{1, 2\}$ $x'_i \equiv y'_i \ (mod\ n)$, then $x'_i \equiv y'_i \ (mod\ d)$, which means $x_i \equiv y_i \ (mod\ d)$ and they are connected.
Same for S_3^n and S_4^n: $x'_1 + x'_2 \equiv y'_1 + y'_2 \ (mod\ n) \Rightarrow x_1 + x_2 \equiv y_1 + y_2 \ (mod\ d)$, $x'_1 - x'_2 \equiv y'_1 - y'_2 \ (mod\ n) \Rightarrow x_1 - x_2 \equiv y_1 - y_2 \ (mod\ d)$.

Now, we can prove Theorem 7.1. By Claim 1, we divided PLSG(n) into d^2 subgraphs Γ_v, $v \in \mathbb{Z}_d^2$. If we color each of them with a different set of $\chi(\Gamma_v)$ colors, we will get a prooper coloring of Γ using

$$d^2 \cdot \chi(\Gamma_v) = d^2 \cdot \chi(\text{PLSG}(h))$$

colors.

However, if there is no edge in Γ between Γ_u and Γ_v, then we can use the same set of colors for them and still get a proper coloring of PLSG(n).
This is equivalent to coloring the graph Γ' that we have constructed above. By Claim 2, $\Gamma' = \text{PLSG}(d)$, and thus in this way we get a proper coloring using

$$\chi(\Gamma') \cdot \chi(\text{PLSG}(h)) = \chi(\text{PLSG}(d) \cdot \chi(\text{PLSG}(\frac{n}{d}))$$

colors. This completes the prof of the theorem. □

We can now use the Theorem 7.1 to construct various upper bounds. For example: we know that $\chi(PLSG(6)) = 9$ and for $n \equiv \pm 1 \ (mod\ 6)$ $\chi(PLSG(n)) = n$. Applying the Theorem gives: for $n \equiv 6 \ (mod\ 36)$ $\chi(PLSG(n)) \leq 9 \cdot \frac{n}{6} = \frac{3n}{2}$.

We can generalize this construction for any n. We can factor n into $n = 2^a \cdot 3^b \cdot m$, where m is coprime to 6. We can then use that $\chi(PLSG(2)) = 4$, $\chi(PLSG(3)) = 9$ and $\chi(PLSG(8)) = 11$.

$$\chi(PLSG(n)) \leq 9^b \cdot 11^{\lfloor \frac{a}{3} \rfloor} \cdot 4^{a - \lfloor \frac{a}{3} \rfloor} \cdot p$$

$$= 2^{\lceil \frac{2a}{3} \rceil} \cdot 3^b \cdot \frac{11}{8}^{\lfloor \frac{a}{3} \rfloor} \cdot n$$

These bounds can be improved significantly for most n by computing (or finding good upper bounds) some $\chi(PLSG(n))$ where $n = 2^a 3^b p$, either a or b being not zero.

8 Conclusion

The results of this final project paper could be improved.

It is possible to construct colorings for the upper bound introduced by Molloy and Reed by using an algorithmic proof of the Local Lemma by Beck. This will also produce the exact value of the constant c. Another approach is to find an exact upper bound on chromatic index of 4-uniform hypergraphs.

The general case bound we based on the recursive bound can be improved by computing $\chi(PLSG(n))$ for greater n.

It is also possible to apply the same construction we used to prove Theorems 2.4 6.2 for $n \not\equiv \pm 1 \ (mod\ 6)$, however applying it in a straightforward way yields graphs whose chromatic numbers are too high to yield non-trivial results.

9 Povzetek naloge v slovenskem jeziku

Kromatsko število $\chi(G)$ končnega enostavnega neusmerjenega grafa G je najmanjše število barv k, potrebnih za pravilno barvanje grafa G. Za celo število $n \geq 2$, ima *pandiagonalni latični kvadratni graf* PLSG(n) za točke n^2 elementov $n \times n$ matrike, dve točki sta si sosednji, če in samo če ležita v isti vrstici, stolpcu, zlomljeni diagonali ali sekundarni zlomljeni diagonali.

To diplomsko delo vsebinsko obsega moje raziskovalno delo na področju kromatskega števila pandiagonalnih latičnih kvadratnih grafov. Znano je da je $\chi(PLSG(n)) \geq n$ in da enakost drži, če in samo če je $n \equiv \pm 1$ $(mod\ 6)$ [3]. Problem iskanja $\chi(\mathrm{PLSG}(n))$ za $n \not\equiv \pm 1$ $(mod\ 6)$ je bil postavljen s strani Klotz-a and Sander-ja v [5] (natančneje gre za Problem 5).

V drugem poglavju bomo obravnavali tako imenovane *ciklične latične kvadratne grafe* LSG(n), definirane na sledeč način: množica točk je $V(\mathrm{LSG}(n)) = \mathbb{Z}_n^2$, točki (r, c) in (r', c') pa sta povezani, če drži kateri izmed pogojev: $r = r'$, $c = c'$ ali $r + c = r' + c'$. Graf LSG(n) se pojavlja kot naraven podgraf grafa PLSG(n). V tem poglavju bo podan tudi pregled članka [2], v katerem je dokazano, da

$$\chi(\mathrm{LSG}(n)) = \begin{cases} n+2 & \text{če je } n \text{ sod,} \\ n & \text{če je } n \text{ lih.} \end{cases}$$

Posebej se bomo osredotočili na natančen dokaz tega rezultata saj ga bomo kasneje še potrebovali. Kot posledico dobimo spodnjo mejo

$$\chi(\mathrm{PLSG}(n)) \geq \begin{cases} n+2 & \text{če je } n \text{ sod,} \\ n & \text{če je } n \text{ lih.} \end{cases}$$

V četrtem poglavju bomo, s pomočjo računalnika, izračunali kromatično število $\chi(\mathrm{PLSG}(n))$ za $n \leq 8$. Za vsak primer bomo tudi predstavili eno minimalno barvanje. Uporabili bomo funkcijo SageMath-a *sage.graphs.graph_coloring.vertex_coloring*, ki spremeni problem iskanja kromatičnega števila v problem celoštevilskega linearnega programiranja. Prav tako bomo najdli pravilno barvanje PLSG(9) z 12 in PLSG(10) s 13 barvami, psledično bomo prišli do neenakosti $\chi(PLSG(9)) \leq 12$ in $\chi(PLSGL(10)) \leq 13$.

V petem poglavju predstavimo pandiagonalni latični kvadratni graf kot 4-uniformni hipergraf s sledečim postopkom. Točke hipergrafa so vrstice, stolpci in vsi tipi diagonal $n \times n$ matrike. Vsaka točka PLSG(n) je potem hiperpovezava velikosti 4 in hipergraf je stopnje n. To nam da asimptotsko zgornjo mejo, ki sta jo Molloy in Reed opisala v članku [6]:

$$\chi(\mathrm{PLSG}(n)) \leq n + c(log\ n)^4 n^{\frac{3}{4}}$$

za neko konstanto c. Konstanta c je enaka ne le za vse pandiagonalne latične kvadatne grafe, ampak tudi za kromatične indekse vseh 4-uniformnih linearnih hipergrafov, to dejstvo uporabimo za postavitev spodnje meje $c \geq \frac{3}{2^{\frac{3}{4}}(ln\ 2)^4}$.

V šestem poglavju bomo pokazali da so pandiagonalni latični kvadratni grafi izomorfni določenim Cayleyevim grafom nad grupo $\mathbb{Z}_n^2$. Na podlagi te opazke bomo podali alternativni dokaz za znan rezultat, da je $\chi(PLSG(n)) = n$ za $n \equiv \pm 1$ $(mod\ 6)$. Ta enakost je Hedayat [3] prvotno dokazal z obstojem Knut Vik dizajnov dane velikosti.

Za konec bomo v sedmem poglavju izpeljali rekurzivno zgornjo mejo:

$$\chi(\mathrm{PLSG}(n)) \leq \chi(\mathrm{PLSG}(d)) \cdot \chi(\mathrm{PLSG}(\frac{n}{d})),$$

kjer je d delitelj n. Dokaz temelji na iskanju homomorfizma φ iz PLSG(n) v PLSG(d), tako da je za vsako točko v grafa PLSG(d), induciran podgraf grafa PLSG(n) na prasliki $\varphi^{-1}(v)$ izomorfen PLSG($\frac{n}{d}$). Kot primer uporabe izračunamo zgornjo mejo v primeru, ko je $n = 2^a \cdot 3^b \cdot p$:

$$\chi(\mathrm{PLSG}(n)) \leq 2^{\lceil \frac{2a}{3} \rceil} \cdot 3^b \cdot \frac{11}{8}^{\lfloor \frac{a}{3} \rfloor} \cdot n.$$

10 Bibliography

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