UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA

(DOCTORAL THESIS)

NOVE KARAKTERIZACIJE V STRUKTURNI TEORIJI GRAFOV: 1-POPOLNO USMERLJIVI GRAFI, PRODUKTNI GRAFI IN CENA POVEZANOSTI
(NEW CHARACTERIZATIONS IN STRUCTURAL GRAPH THEORY: 1-PERFECTLY ORIENTABLE GRAPHS, GRAPH PRODUCTS, AND THE PRICE OF CONNECTIVITY)

TATIANA ROMINA HARTINGER

KOPER, 2017

MENTOR: IZR. PROF. DR. MARTIN MILANIČ

Acknowledgements

First and foremost I would like to thank my supervisor Martin Milanič for all his help, encouragement and thoughtful guidance during all these years.

I would like to thank Dragan Stevanović for giving me the oportunity of becoming a Young Researcher, and everyone at the Andrej Marušič Institute and the Faculty of Mathematics, Natural Sciences and Information Technologies for their help and support during my stay in Koper. Special thanks to Nina Chiarelli and Nastja Cepak for their help with the translation to Slovene, and to the co-authors from papers included in this thesis, Boštjan Brešar, Tim Kos, Matthew Johnson, and Daniel Paulusma, for working with me and allowing me to learn from them.

I would also like to express my gratitude to my husband Juan, without whom I would not have had the courage to embark on this journey in the first place, to my parents, Susana and Eduardo, my mother-in-law and sister-in-law, Viviana and Paz, and to the rest of my family and friends for all their love and support.

Abstract

This PhD thesis consists of three interrelated parts: 1-perfectly orientable graphs, graph products, and the price of connectivity. The central common theme among the three parts is the study of graph classes. In particular, three well known intersection graph classes will play a key role in many of our results: chordal graphs, interval graphs, and circular arc graphs.

Following the terminology of Kammer and Tholey, we say that an orientation of a graph is 1-perfect if the out-neighborhood of every vertex induces a tournament, and that a graph is 1-perfectly orientable if it has a 1-perfect orientation. This hereditary graph class forms a common generalization of the classes of chordal graphs and circular arc graphs. 1-perfectly orientable graphs are known to be polynomially recognizable, but a complete structural understanding of the class, a problem posed in 1982 by Skrien, is still an open question. In the thesis we obtain characterizations of 1-perfectly orientable graphs in various graph classes, including cographs, block-cactus graphs, cobipartite graphs, K_4-minor-free graphs, outerplanar graphs, and nontrivial product graphs for each of the four standard graph products (Cartesian, direct, strong, and lexicographic product).

As a consequence of the characterization of nontrivial 1-perfectly orientable product graphs, characterizations of when a nontrivial product of two graphs is chordal, interval, or circular arc, respectively, are derived.

For a family of graphs $\mathcal{F}$, an $\mathcal{F}$-transversal of a graph G is a subset $S \subseteq V(G)$ that intersects every subset of $V(G)$ that induces a subgraph isomorphic to a graph in $\mathcal{F}$. Given a connected graph G, we denote by $t_{\mathcal{F}}(G)$ the minimum size of an $\mathcal{F}$-transversal of G, and by $ct_{\mathcal{F}}(G)$ the minimum size of an $\mathcal{F}$-transversal of G that induces a connected graph. For a class of connected graphs, we say that the price of connectivity of $\mathcal{F}$-transversals is multiplicative if, for all G in the class, $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ is bounded by a constant, and additive if $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant. The price of connectivity is zero-additive if $t_{\mathcal{F}}(G)$ and $ct_{\mathcal{F}}(G)$ are always equal and unbounded if $ct_{\mathcal{F}}(G)$ cannot be bounded in terms of $t_{\mathcal{F}}(G)$. The price of connectivity is discussed in the context of hereditary graph classes defined by a single forbidden induced subgraph.

Math. Subj. Class (2010): 05C20, 05C40, 05C62, 05C69, 05C75, 05C76

Keywords: 1-perfectly orientable graph, structural characterization of families of graphs, chordal graph, interval graph, circular arc graph, cograph, block-cactus graph, cobipartite graph, K_4-minor-free graph, outerplanar graph, graph product, Cartesian product, lexicographic product, direct product, strong product, price of connectivity, cycle transversal, path transversal

Povzetek

Disertacija je sestavljena iz treh medsebojno povezanih delov: 1-popolno usmerljivi grafi, produktni grafi in cena povezanosti. Skupna točka vsem trem delom je proučevanje razredov grafov. Pomembno vlogo v naših raziskavah imajo trije znani presečni razredi grafov: tetivni grafi, intervalni grafi in grafi krožnih lokov.

Z uporabo terminologije Kammerja in Tholeyja pravimo, da je usmeritev grafa 1-popolna, če izhodna soseščina vsake točke inducira turnir, in da je graf 1-popolno usmerljiv, če premore 1-popolno orientacijo. Omenjeni razred grafov tvori skupno posplošitev razredov tetivnih grafov in grafov krožnih lokov. Kljub temu da lahko 1-popolno usmerljive grafe prepoznamo v polinomskem času, pa je razumevanje strukture tega razreda grafov, problem, ki ga je že leta 1982 podal Skrien, še vedno odprt problem. V disertaciji podamo karakterizacije 1-popolno usmerljivih grafov, ki pripadajo različnim grafovskim razredom, vključujoč kografe, bločno-kaktus grafe, ko-dvodelne grafe, grafov brez K_4 minorja, zunanje ravninske grafe in netrivialne produktne grafe za vsakega od štirih standardnih grafovskih produktov (kartezičnega, direktnega, krepkega in leksikografskega).

Kot posledica karakterizacije netrivialnih 1-popolno usmerljivih produktnih grafov so izpeljane tudi karakterizacije, kdaj je netrivialni produkt dveh grafov tetiven graf, intervalen graf ali graf krožnih lokov.

Za družino grafov $\mathcal{F}$ je $\mathcal{F}$-trasverzala grafa G poljubna podmnožica $S \subseteq V(G)$, ki ima neprazen presek z vsako podmnožico množice $V(G)$, ki inducira podgraf, izomorfen grafu iz $\mathcal{F}$. Naj bo $t_{\mathcal{F}}(G)$ minimalna velikost $\mathcal{F}$-transverzale grafa G in naj bo $ct_{\mathcal{F}}(G)$ minimalna velikost $\mathcal{F}$-transverzale grafa G, ki inducira povezan graf. Za razred povezanih grafov $\mathcal{G}$ rečemo, da je cena povezanosti $\mathcal{F}$-transverzal multiplikativna, če je za vse grafe $G \in \mathcal{G}$ razmerje $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ omejeno s konstanto, in aditivna, če je razlika $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ omejena s konstanto. Cena povezanosti je ničelno aditivna, če sta vrednosti $t_{\mathcal{F}}(G)$ in $ct_{\mathcal{F}}(G)$ vedno enaki, in neomejena, če je vrednost $ct_{\mathcal{F}}(G)$ neomejena glede na $t_{\mathcal{F}}(G)$. Ceno povezanosti preučimo v kontekstu hereditarnih grafovskih razredov definiranih s pomočjo enega prepovedanega induciranega podgrafa.

Math. Subj. Class (2010): 05C20, 05C40, 05C62, 05C69, 05C75, 05C76

Ključne besede: 1-popolno usmerljivi grafi, strukturna karakterizacija grafovskih razredov, tetivni grafi, intervalni grafi, grafi krožnih lokov, kografi, bločno-kaktus grafi, ko-dvodelni grafi, grafi brez K_4 minorja, zunanje ravninski grafi, produkt grafov, kartezični produkt, leksikografski produkt, direktni produkt, krepki produkt, cena povezanosti, transverzala ciklov, transverzala poti

Contents

Chapter 1

Introduction

The PhD thesis consists of three interrelated parts: 1-perfectly orientable graphs, graph products, and the price of connectivity. The central common theme among the three parts is the study of graph classes. In particular, three well known intersection graph classes will play a key role in many of our results: chordal graphs, interval graphs, and circular arc graphs. A graph G is said to be *chordal* if it is the intersection graph of subtrees of a tree (equivalently, if every cycle of length at least 4 in G has a chord), *interval* if it is the intersection graph of a family of closed intervals on the real line, and *circular arc* if it is the intersection graph of a set of closed arcs on a circle. The classes of chordal, interval, and circular arc graphs are well studied in the literature. Every interval graph is both a chordal and a circular arc graph; both inclusions are proper. Chordal graphs and interval graphs are subclasses of the class of perfect graphs. For more information on these graph classes we refer the reader to [10,25,37,41,58,59], for example.

It has been shown by Kammer and Tholey [55] that several geometric intersection graph classes are *k-perfectly orientable*, that is, they admit an orientation in which the out-neighborhood of every vertex can be covered with at most k cliques.

The first main topic of the PhD thesis will be that of 1-perfectly orientable graphs. A *tournament* is an orientation of a complete graph. We say that an orientation of a graph is 1-*perfect* if the out-neighborhood of every vertex induces a tournament and that a graph is 1-*perfectly orientable* (1-*p.o.* for short) if it admits a 1-perfect orientation.

The notion of 1-p.o. graphs was introduced by Skrien [81] in 1982 (under the name $\{B_2\}$-graphs), where the problem of characterizing this graph class was posed. By definition, 1-p.o. graphs are exactly the graphs that admit an orientation that is an out-tournament. A simple arc reversal argument shows that that 1-p.o. graphs are exactly the graphs that admit an orientation that is an in-tournament. Such orientations were called *fraternal orientations* in several papers [35,36,38,39,65,66,85].

1-p.o. graphs form a hereditary class of graphs that forms a common generalization of the two well studied classes of chordal graphs and circular arc graphs, as observed in [85] and in [81,85], respectively. While a structural understanding of 1-p.o. graphs is still an open question, partial results are known. Bang-Jensen et al. [4] (see also [70]) gave characterizations of 1-p.o. line graphs and of 1-p.o. triangle-free graphs and proved that every graph having a unique induced cycle of order at least 4 is 1-p.o..

We will develop several results about the structure of 1-perfectly orientable graphs. In particular, we will give a characterization of 1-perfectly orientable graphs in terms of edge clique covers, identify several graph transformations preserving the class of 1-perfectly orientable graphs, exhibit an infinite family of minimal forbidden induced minors for the class of 1-perfectly orientable graphs, and characterize the class of 1-perfectly orientable graphs within the classes of cographs and of cobipartite graphs. We will show that the class of 1-perfectly orientable cobipartite graphs coincides with the class of cobipartite circular arc graphs. As a side result we will define a new infinite family of bipartite graphs and show that their complements are 1-p.o.

Based on a reduction of the study of 1-perfectly orientable graphs to the biconnected case, we will characterize, both in terms of forbidden induced minors and in terms of composition theorems, the classes of 1-perfectly orientable block-cactus graphs, 1-perfectly orientable K_4-minor-free graphs and of 1-perfectly orientable outerplanar graphs. As part of our approach, we will introduce a class of graphs defined similarly as the class of 2-trees and relate the classes of graphs under consideration to two other graph classes closed under induced minors studied in the literature: cyclically orientable graphs and graphs of separability at most 2.

The second topic to be studied in this thesis is that of graph products. Product graphs within various graph classes have been considered in several papers; however, complete characterizations of graph theoretic properties within all four standard products (Cartesian, direct, strong, and lexicographic) are often difficult to obtain. Ravindra and Parthasarathy [74] characterized perfect Cartesian, direct, and lexicographic product graphs; the Cartesian case was also studied further by de Werra and Hertz [20]. There is no known characterization of perfect strong product graphs; partial characterizations and sufficient conditions were obtained by Ravindra [73] (see also [1]). Characterizations of line graphs and total graphs for various products were given by Rao [71] and by Rao and Vartak [72], of modulo m well covered lexicographic product graphs by Orlovich [69], and of uniquely pairable Cartesian product graphs by Che [14]. The results of this thesis will contribute to the knowledge of characterizations of graph classes within graphs decomposable with respect to one of the four standard graph products, by adding 1-perfectly orientable, chordal, interval, and circular arc graphs to the list.

The third topic that will be considered in the thesis is the so-called price of connectivity. For a family of graphs $\mathcal{F}$, an $\mathcal{F}$-transversal of a graph G is a subset $S \subseteq V(G)$ that intersects every subset of $V(G)$ that induces a subgraph isomorphic to a graph in $\mathcal{F}$. Let $t_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G, and $ct_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G that induces a connected graph. For a class of connected graphs $\mathcal{G}$, we say that the *price of connectivity* of $\mathcal{F}$-transversals is *multiplicative* if, for all $G \in \mathcal{G}$, the ratio $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ is bounded by a constant, and *additive* if the difference $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant. The price of connectivity is *zero-additive* if $t_{\mathcal{F}}(G)$ and $ct_{\mathcal{F}}(G)$ are always equal and *unbounded* if $ct_{\mathcal{F}}(G)$ cannot be bounded in terms of $t_{\mathcal{F}}(G)$. In certain cases, $\mathcal{F}$-transversals are well studied. For example, a vertex cover is a $\{P_2\}$-transversal and a feedback vertex set is an $\mathcal{F}$-transversal for the infinite family $\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$. As the examples suggest, it is natural to study minimum size $\mathcal{F}$-transversals.

We can put an additional constraint on an $\mathcal{F}$-transversal S of a connected graph G by requiring that the subgraph of G induced by S is connected. Minimum size connected $\mathcal{F}$-transversals of a graph have been investigated. In particular, minimum size connected vertex covers are well studied (see, for example, [8,11,12,21,27,33,79,89]) and minimum size connected feedback vertex sets have also received attention (see, for example, [6, 19, 42, 62, 80]).

In the PhD thesis we will consider the following question: *What is the effect of adding the connectivity constraint on the minimum size of an $\mathcal{F}$-transversal for a graph family $\mathcal{F}$?*

More precisely, we will study classes of graphs characterized by one forbidden induced subgraph H and $\mathcal{F}$-transversals where $\mathcal{F}$ contains an infinite number of cycles and, possibly, also one or more anticycles or short paths. We aim to determine exactly those classes of connected H-free graphs where the price of connectivity of these $\mathcal{F}$-transversals is unbounded, multiplicative, additive, or zero-additive. In particular, our tetrachotomies will extend known results for the case when $\mathcal{F}$ is the family of all cycles.

The thesis is structured as follows. In Chapter 2, we define the notions needed, fix the notation, and give some preliminary results. In Chapter 3, some initial structural results for the class of 1-perfectly orientable graphs are derived. In Chapter 4, characterizations for 1-perfectly orientable graphs within the classes of cobipartite graphs, cographs, block-cactus graphs, K_4-minor-free graphs, and outerplanar graphs are obtained. Chapters 5 and 6 deal with product graphs. More precisely, in Chapter 5 we characterize when a nontrivial product of two graphs is 1-perfectly orientable for each of the four standard graph products, namely the Cartesian product, the lexicographic product, the direct product, and the strong product, respectively, and in Chapter 6, we characterize nontrivial chordal, interval, and circular arc product graphs, respectively, for each of the four standard graph products. Chapter 7 deals with the price of connectivity. In this chapter we determine almost exactly those classes of connected H-free graphs where the price of connectivity of $\mathcal{F}$-transversals is unbounded, multiplicative, additive, or zero-additive, for families $\mathcal{F}$ containing an infinite number of cycles and, possibly, also one or more anticycles or short paths. Additivity remains an open problem in the case when family $\mathcal{F}$ consists of all holes and all even cycles. All other cases are characterized giving necessary and sufficient conditions. In particular, our tetrachotomies will extend known results for the case when $\mathcal{F}$ is the family of all cycles. Finally, in Chapter 8 we give some concluding remarks and pose some open problems.

Chapter 2

Definitions, notation, and preliminary results

In this chapter, we provide the basic notation and definitions, recall the basic properties of some graph classes relevant to our study, and prove some preliminary results.

2.1 General preliminaries on graphs

All graphs in this thesis are finite and simple, but may be either directed (in which case we will refer to them as *digraphs*) or undirected (in which case we will refer to them as *graphs*). We use standard graph and digraph terminology. In particular, the vertex and edge sets of a graph G will be denoted by $V(G)$ and $E(G)$, respectively, and the vertex and arc sets of a digraph D will be denoted by $V(D)$ and $A(D)$. In this section, we recall the definitions of some of the most used notions in this work. For further background on graphs, we refer to [22, 87], on graph classes, to [10, 41], and on digraphs, to [3].

An *orientation* of a graph $G = (V, E)$ is a digraph $D = (V, A)$ obtained by assigning a direction to each edge of G. An edge in a graph (resp., arc in a digraph) connecting vertices u and v will be denoted by uv or $\{u, v\}$ (resp., (u, v)). We will also use the notation $u \to v$ to denote the fact that an edge uv of a graph G is oriented from u to v in an orientation of G. The set of all vertices adjacent to a vertex v in a graph G will be denoted by $N_G(v)$ (or simply by $N(v)$ if the graph is clear from the context), and its cardinality, the *degree of v in G*, by $d_G(v)$ (or simply by $d(v)$). The *closed neighborhood* of v in G is the set $N_G(v) \cup \{v\}$, denoted by $N_G[v]$ (or simply by $N[v]$ if the graph is clear from the context). A *tournament* is an orientation of a complete graph. Given a digraph D, the *in-neighborhood* of a vertex v in D, denoted by $N_D^-(v)$, is the set of all vertices w such that $(w, v) \in A$. Similarly, the *out-neighborhood* of v in D is the set of all vertices w such that $(v, w) \in A$. The cardinalities of the in- and the out-neighborhood of v are the *in-degree* and the *out-degree of* v and are denoted by $d_D^-(v)$ and $d_D^+(v)$, respectively.

The *distance* between two vertices u and v in a connected graph G is denoted by $d_G(u, v)$ and defined as the minimum length (that is, number of edges) of a u,v-path in G. The maximum distance in G is called the *diameter* of G.

A *cut vertex* in a connected graph G is a vertex v such that the graph $G - v$ is disconnected. A graph G is *biconnected* if it is connected and has no cut vertices, and 2*-connected* if it is biconnected and has at least 3 vertices. A *block* of a graph G is a maximal biconnected subgraph of G. Every connected graph decomposes into a tree of blocks called the *block tree* of the graph. The vertex set of the block tree T of G is the set $\mathcal{B} \cup C$ where $\mathcal{B}$ is the set of blocks of G and C is the set of cut vertices of G; a block $B \in \mathcal{B}$ and a cut vertex $v \in C$ are connected by en edge in T if and only if $v \in V(B)$. Blocks of G that are leaves of T are called *end blocks* of G. Every leaf of the block tree T is a block of G, thus every graph with a cut vertex has at least two end blocks.

Given two graphs G and H, their *union* is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Their *disjoint union* is the graph $G + H$ with vertex set $V(G) \dot\cup V(H)$ (disjoint union) and edge set $E(G) \cup E(H)$ (if G and H are not vertex disjoint, we first replace one of them with a disjoint isomorphic copy). We denote the disjoint union of r copies of G by rG. The *join* of two graphs G and H is the graph denoted by $G * H$ and obtained from the disjoint union of G and H by adding to it all edges joining vertices of G with vertices of H. Given two graphs G and H and a vertex v of G, the *substitution* of v in G for H consists in replacing v with H and making each vertex of H adjacent to every vertex in $N_G(v)$ in the new graph.

Given a graph G and a subset S of its vertices, we denote by $G[S]$ the *subgraph of G induced by S*, that is, the graph with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$. By $G - S$ we denote the subgraph of G induced by $V(G) \setminus S$, and when $S = \{v\}$ for a vertex v, we also write $G-v$. The graph G/e obtained from G by *contracting* an edge $e = uv$ is defined as $G/e = (V, E)$ where $V = (V(G) \setminus \{u, v\}) \cup \{w\}$ with w a new vertex and $E = E(G - \{u, v\}) \cup \{wx \mid x \in N_G(u) \cup N_G(v)\}$.

A *clique* (resp., *independent set*) in a graph G is a set of pairwise adjacent (resp., non-adjacent) vertices of G. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set as G in which two distinct vertices are adjacent if and only if they are not adjacent in G. The fact that two graphs G and H are isomorphic to each other will be denoted by $G \cong H$. In this work we will often not distinguish between isomorphic graphs.

Two distinct vertices u and v in a graph G are said to be *true twins* if $N_G[u] = N_G[v]$. The operation of *true twin addition* to a graph G is defined as adding a new vertex w to G and making it adjacent to some vertex v of G and all its neighbors. We say that a graph G is *true-twin-free* if no pair of vertices of G are true twins.

The path, the cycle, and complete graph on n vertices will be denoted by P_n, C_n, and K_n, respectively, and the complete bipartite graph with parts of size m and n by $K_{m,n}$. The *claw* is the complete bipartite graph $K_{1,3}$. The *bull* is a graph with 5 vertices and 5 edges, consisting of a triangle with two disjoint pendant edges. In Fig. 2.1 some examples for P_n, C_n, and K_n are shown, together with the claw and the bull.

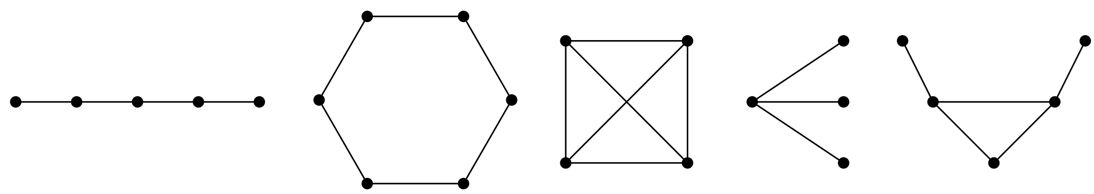


Figure 2.1: P_5, C_6, K_4, the claw ($K_{1,3}$), and the bull, respectively.

Induced subgraphs, minors and induced minors. A graph H is an *induced subgraph* of a graph G if it can be obtained from G by a sequence of vertex deletions. For graphs F and G, we write $F \subseteq_i G$ to denote that F is an induced subgraph of G. A graph H is said to be a *minor* of a graph G if H can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions. Equivalently, H is minor of G if there exists an *minor model of H in G*, that is, a collection $\{S_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ each inducing a connected subgraph such that for every two adjacent vertices u and v of H, we have $\{x, y\} \in E(G)$ for some $x \in S_u$ and $y \in S_v$. A graph H is said to be an *induced minor* of G if H can be obtained from G by a sequence of vertex deletions and edge contractions. Equivalently, H is an induced minor of G if there exists an *induced minor model of H in G*, that is, a collection $\{S_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ each inducing a connected subgraph such that for every two distinct vertices u and v of H, we have $\{u, v\} \in E(H)$ if and only if $\{x, y\} \in E(G)$ for some $x \in S_u$ and $y \in S_v$.

Note that for every graph H an induced minor model of H in G is also a minor model of it, and if H is a complete graph, then the converse holds as well. Moreover, if H is a graph of maximum degree at most three and G is any graph, then H is isomorphic to a minor of G if and only if G contains a subgraph isomorphic to a subdivision of H. A *subdivision* of a graph G is a graph resulting from a sequence of edge subdivisions in G. The subdivision of an edge e with endpoints $\{u, v\}$ yields a graph containing one new vertex w, and with edge set in which e is replaced by two new edges, $\{u, w\}$ and $\{w, v\}$.

Given a set $\mathcal{F}$ of graphs, a graph G is said to be:

- *$\mathcal{F}$-free* if no induced subgraph is isomorphic to a member of $\mathcal{F}$.
- *$\mathcal{F}$-induced-minor-free* if no induced minor of G is isomorphic to a member of $\mathcal{F}$.
- *$\mathcal{F}$-minor-free* if no minor of G is isomorphic to a member of $\mathcal{F}$.

Given two graphs G and H, we say that G is *H-free* (*H-minor-free*, resp., *H-induced-minor-free*) if no induced subgraph of G (no minor of G, resp., no induced minor of G) is isomorphic to H. A graph class is *hereditary* if it is closed under induced subgraphs, that is, every induced subgraph of a graph in the class is also in the class. The notions of *minor-closed* and *induced-minor-closed* graph classes are defined analogously. A *proper induced subgraph* (*proper minor*, resp., *proper induced minor*) of a graph G is any induced subgraph (*minor*, resp., *induced minor*) of G other than G itself.

Every minor-closed class $\mathcal{G}$ of graphs can be uniquely characterized in terms of *forbidden minors*. That is, there exists a unique set $\mathcal{F}$ of graphs such that: (i) a graph G is in $\mathcal{G}$ if and only if G is $\mathcal{F}$-minor-free, and (ii) every proper minor of every graph in $\mathcal{F}$ is in $\mathcal{G}$. The notions of *$\mathcal{F}$-induced-minor-free* graphs and of *forbidden induced minors* are defined analogously, with respect to the induced minor relation. For minor-closed graph classes, the sets of forbidden minors are always finite [77], while in the case of induced-minor-closed graph classes, the sets of forbidden induced minors can also be infinite (see, for example, [56]).

Some hereditary graph classes. A *hole* is a cycle of length at least 4. An *antihole* is the complement of a hole. A cycle, hole or antihole is *even* if it contains an even number of vertices; otherwise it is *odd*. A hole is *long* if it is of length at least 5, and a *long antihole* is the complement of a long hole. A graph is *odd-hole-free* or *odd-antihole-free* if it contains no induced odd holes or no induced odd antiholes, respectively. An *even-hole-free* graph is defined similarly. A graph is *weakly chordal* if it has no induced long hole and no induced long antihole. A graph is *perfect* if the chromatic number of every induced subgraph equals the size of a largest clique in that subgraph. By the Strong Perfect Graph Theorem [15], a graph is perfect if and only if it is odd-hole-free and odd-antihole-free. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. Split graphs coincide with the $\{2P_2, C_4, C_5\}$-free graphs [30]. A graph is *threshold* if it is $\{2P_2, P_4, C_4\}$-free, *trivially perfect* if it is $\{P_4, C_4\}$-free, *cotrivially perfect* if it is $\{2P_2, P_4\}$-free and a *cograph* if it is P_4-free. A graph is said to be *bipartite* if its vertex set can be partitioned into two independent sets. We say that a graph is *cobipartite* if it is the complement of a bipartite graph.

Given a family of sets, the *intersection graph* of the family is the graph that represents the pattern of intersections of sets in the family. More precisely, given a family of sets S_i, $i = 1, 2, \ldots, \ell$, its intersection graph is the undirected graph formed by creating one vertex v_i for each set S_i, and connecting two distinct vertices v_i and v_j by an edge whenever the corresponding two sets have a nonempty intersection, that is, $E(G) = \{\{v_i, v_j\} | S_i \cap S_j \neq \emptyset\}$. For more details, see [60].

A graph G is said to be *chordal* if it is C_k-free for all $k \geq 4$. Equivalently, a graph G is chordal if it is the vertex-intersection graph of subtrees of a tree [37]. A vertex in a graph G is *simplicial* if its neighborhood forms a clique. A *perfect elimination ordering* in a graph is a linear ordering of the vertices of the graph such that, for each vertex v, the neighbors of v that occur after v in the order form a clique. Fulkerson and Gross showed that graph is chordal if and only if it has a perfect elimination ordering [34]; equivalently, if it can be reduced to the one-vertex graph by a sequence of simplicial vertex removals. Note that the class of chordal graphs is closed both under vertex deletions and edge contractions, hence it is also closed under induced minors. Consequently, a graph G is chordal if and only if it is C_4-induced-minor-free. A well known subclass of the class of chordal graphs is the class of interval graph. A graph G is said to be *interval* if it is the intersection graph of closed intervals in the real line.

1-perfectly orientable graphs. Following the terminology of Kammer and Tholey [55], we say that a graph G is 1*-perfectly orientable* if it admits an orientation such that for every vertex $v \in V(G)$, the out-neighborhood of v in D induces a tournament (that is, it is a clique in G). The notion of 1-p.o. graphs was first introduced by Skrien [81] in 1982 (under the name $\{B_2\}$-graphs), where the problem of characterizing this graph class was posed.

The following theorem uses a reduction to 2-SAT to show that 1-p.o. graphs can be recognized in polynomial time.

Theorem 2.1 (Bang-Jensen, Huang and Prisner [4])**.** *1-perfectly orientable graphs can be recognized in polynomial time.*

While the complexity of recognizing 1-p.o. graphs is known, a structural understanding of this graph class remains an open problem.

A connected graph is said to be *unicyclic* if it has exactly one cycle. The following simple proposition will be used in some of our proofs.

Proposition 2.2. *Every unicyclic graph is 1-p.o.*

Proof. Every unicyclic graph G admits an orientation in which $d^+(v) \leq 1$ for all $v \in V(G)$. Any such orientation is 1-perfect. □

2.2 Preliminaries on circular arc graphs

A graph is *circular arc* if it is the intersection graph of a set of closed arcs on a circle. The class of circular arc graphs forms an important and well studied subclass of 1-p.o. graphs; see, e.g., [25, 59]. Given a circular arc graph G and a representation of G with arcs around a circle, a set of arcs whose union equals the entire circle is said to *cover the circle*. Notice that if the set of arcs in the representation does not cover the circle, the corresponding circular arc graph G is an interval graph. The following lemma characterizes when a disjoint union of two graphs is circular arc.

Lemma 2.3. *The disjoint union $G + H$ of two graphs G and H is a circular arc graph if and only if both G and H are interval graphs.*

Proof. Assume first that $G + H$ is circular arc. Both graphs G and H are circular arc, since they are induced subgraphs of $G+H$. Since there is no edge between $V(G)$ and $V(H)$ in $G+H$, then the sets of arcs F and F' corresponding to G and H respectively, cannot cover the circle, which implies that both G and H are interval graphs.

Conversely, if G and H are interval graphs, then so is $G + H$. Thus, $G + H$ is circular arc. □

The following fact is well known, see, e.g., [25].

Fact 2.4. *For every $n \geq 4$, every circular arc graph is $C_n + K_1$-free.*

While a characterization of the class of circular arc graphs by forbidden induced subgraphs remains an open problem, in a recent study the first forbidden structure characterization of circular arc graphs was obtained [32]. The class of cobipartite circular arc graphs, however, has been characterized in many ways (see, e.g., [59, Section 7] and [25]). In particular, we now state a characterization of cobipartite circular arc graphs due to Hell and Huang and a consequence of it, which we will use:

- in the characterization of 1-p.o. cobipartite graphs (Section 4.1),
- in order to prove that the family of complements of grid-walk graphs is an infinite family of 1-perfectly orientable graphs (Section 4.1.1), and
- in the characterization of the circular arc nontrivial lexicographic product graphs in Chapter 6.

Let G be a cobipartite graph with a bipartition $\{U, U'\}$ of its vertex set into two cliques. An edge of G connecting a vertex from U with a vertex of U' is said to be a *crossing edge* of G. A coloring of the crossing edges of G with colors red and blue is said to be *good* (with respect to $\{U, U'\}$) if for every induced C_4 in G, the two cross edges in it are of the opposite color. The following characterization of cobipartite circular arc graphs is a reformulation of [49, Corollary 2.3].

Theorem 2.5 (Hell and Huang [49])**.** *Let G be a cobipartite graph with a bipartition $\{U, U'\}$ of its vertex set into two cliques. Then G is a circular arc graph if and only if it has a good coloring.*

As a consequence of the previous theorem, we can obtain the following result.

Lemma 2.6. *The class of cobipartite circular arc graphs is closed under join.*

Proof. Let G and H be cobipartite circular arc graphs, with bipartitions of their vertex sets into two cliques U_1 and U_2, and V_1 and V_2, respectively. Then $F = G * H$ is cobipartite as well, with bipartition into two cliques $W_1 = U_1 \cup V_1$ and $W_2 = U_2 \cup V_2$. We will now show that F admits a good coloring. By Theorem 2.5 this will imply that the join of G and H is cobipartite.

By Theorem 2.5 there exists a good coloring of G and a good coloring of H. Every crossing edge of F is exactly of one of the following four types: a crossing edge of G, a crossing edge of H, a U_1, V_2-edge, or a U_2, V_1-edge. We construct a good coloring of F as follows: the crossing edges of G or of H are colored as in (some fixed) good colorings of G, resp. H, every U_1, V_2-edge is colored red, and every U_2, V_1-edge is colored blue. Since every induced C_4 in F either lies entirely in one of G and H, or it is formed by two non-adjacent vertices in G and two non-adjacent vertices in H, the so obtained coloring is indeed a good coloring of F. □

In the following lemma, combining the results of Lekkerkerker and Boland [58], and of Harary and Schwenk [46], we summarize the known characterizations of interval (resp., circular arc) forests. A *forest* is an acyclic graph. A *caterpillar* is a tree T such that the removal of all degree-one vertices yields a path. A *caterpillar forest* is a disjoint union of caterpillars. A *bipartite claw* is the graph obtained from the claw by subdividing each of its edges exactly once.

Lemma 2.7. *Let F be a forest. Then, the following are equivalent:*

1. *F is an interval graph,*

2. *F is a circular arc graph,*

3. *F is a caterpillar forest,*

4. *F contains no induced bipartite claw.*

Proof. Let F be a forest. Clearly, if F is interval then it is circular arc. Now, assume F is circular arc. Since F contains no cycle, the set of arcs in any circular arc representation of F cannot cover the circle, which implies that F is interval. The fact that F is interval if and only if it contains no induced bipartite claw follows from the characterization of interval graphs from [58]. The fact that F is a caterpillar forest if and only if F contains no induced bipartite claw was proved in [46]. □

2.3 Preliminaries on graphs of separability at most 2, cyclically orientable graphs, and outerplanar graphs

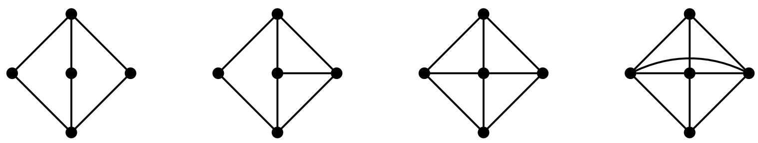
For a positive integer k, *graphs of separability at most k* were defined by Cicalese and Milanič in [16] as the graphs in which every two non-adjacent vertices are separated by a set of at most k other vertices. Several characterizations of graphs of separability at most 2 were given in [16]. In the next theorem we summarize those relevant to this work (Theorems 1 and 9 in [16]). We say that a graph G is obtained from two graphs G_1 and G_2 by *pasting along a k-clique*, and denote this by $G = G_1 \oplus_k G_2$, if for some $r \leq k$ there exist two r-cliques $K^{(1)} = \{x_1, \ldots, x_r\} \subseteq V(G_1)$ and $K^{(2)} = \{y_1, \ldots, y_r\} \subseteq V(G_2)$ such that G is isomorphic to the graph obtained from the disjoint union of G_1 and G_2 by identifying each x_i with y_i, for all $i = 1, \ldots, r$. In particular, if $k = 0$, then $G_1 \oplus_k G_2$ is the disjoint union of G_1 and G_2, and if $k = 1$, then the graph $G_1 \oplus_k G_2$ has a cut vertex.

Theorem 2.8 (Cicalese and Milanič [16]). *For every graph G, the following statements are equivalent.*

1. *G is of separability at most 2.*

2. *G is $\{K_{2,3}, K_{2,3}^{+1}, K_{2,3}^{+2}, K_{2,3}^{+3}\}$-induced-minor-free, where $K_{2,3}, K_{2,3}^{+1}, K_{2,3}^{+2}, K_{2,3}^{+3}$ are the four graphs depicted in Fig. 2.2.*

3. *G can be constructed from complete graphs and cycles by an iterative application of pasting along 2-cliques.*



$K_{2,3}$ $K_{2,3}^{+1}$ $K_{2,3}^{+2}$ $K_{2,3}^{+3}$

Figure 2.2: Forbidden induced minors for the class of graphs of separability at most 2

A related class of graphs is that of cyclically orientable graphs. A graph G is said to be *cyclically orientable* if it admits an orientation in which every chordless cycle is oriented cyclically. Motivated by applications of cyclically orientable graphs to cluster algebras, this family of graphs was introduced by Barot et al. [5] and studied further by Gurvich [44] and Zou [90]. The following theorem combines [44, Theorems 1 and 4] and [82, Theorem 1].

Theorem 2.9 (Combining results of Gurvich [44] and Speyer [82])**.** *For every graph G, the following statements are equivalent:*

1. *G is cyclically orientable.*
2. *G can be built from copies of K_1, K_2, and cycles by an iterative application of pasting along 2-cliques.*
3. *G is a K_4-free graph of separability at most 2.*

As a consequence, we obtain the following.

Corollary 2.10. *If G is biconnected and cyclically orientable, then G can be build from cycles by an iterative application of pasting along edges.*

The following is [44, Lemma 2].

Lemma 2.11 (Gurvich [44])**.** *Cyclically orientable graphs contain no subgraphs isomorphic to a subdivision of K_4.*

The above results imply the following characterization of cyclically orientable graphs in terms of forbidden induced minors.

Theorem 2.12. *For every graph G, the following statements are equivalent:*

1. *G is cyclically orientable.*
2. *G is $\{K_4, K_{2,3}\}$-induced-minor-free.*

Proof. We first argue that the class of cyclically orientable graphs is closed under induced minors. It is clearly closed under vertex deletions. To see that it is also closed under edge contractions, recall that by Theorem 2.9 G is cyclically orientable if and only if G is a K_4-free graph of separability at most 2. Since the class of graphs of separability at most 2 is closed under induced minors (cf. Theorem 2.8), contracting an edge of a cyclically orientable graph G results in a graph G' of separability at most 2. By Lemma 2.11, G does not contain any subdivision of K_4 (as a subgraph), which is equivalent to the fact that G does not contain K_4 as a minor. Since contracting an edge cannot produce a K_4 minor, graph G' has no K_4 minor, in particular, it is K_4-free. Thus, G' is cyclically orientable by Theorem 2.9.

Since the class of cyclically orientable graphs is closed under induced minors and the graphs K_4 and $K_{2,3}$ are not cyclically orientable, the implication $1 \Rightarrow 2$ follows. Suppose now that G is a $\{K_4, K_{2,3}\}$-induced-minor-free graph. Since each of the graphs in the set $\{K_{2,3}^{+1}, K_{2,3}^{+2}, K_{2,3}^{+3}\}$ (cf. Fig. 2.2) can be contracted to a K_4, the class of $\{K_4, K_{2,3}\}$-induced-minor-free graphs is a subclass of the class of $\{K_{2,3}, K_{2,3}^{+1}, K_{2,3}^{+2}, K_{2,3}^{+3}\}$-induced-minor-free graphs. It follows from Theorem 2.8 that every $\{K_4, K_{2,3}\}$-induced-minor-free graph is a K_4-free graph of separability at most 2. The implication $2 \Rightarrow 1$ now follows from Theorem 2.9. □

A graph G is *outerplanar* if it can be drawn in the plane without edge crossings and with all vertices incident with the outer face. Outerplanar graphs are exactly the $\{K_4, K_{2,3}\}$-minor-free graphs [13]. The following characterization in terms of forbidden induced minors is an immediate consequence of the characterization in terms of forbidden minors. We denote by $K_{2,3}^{+}$ the graph obtained from $K_{2,3}$ by adding an edge between the two vertices of degree 3.

Proposition 2.13. *For every graph G, the following statements are equivalent:*

1. *G is outerplanar.*

2. *G is $\{K_4, K_{2,3}, K_{2,3}^+\}$-induced-minor-free.*

Proof. Since none of the graphs K_4, $K_{2,3}$, $K_{2,3}^+$ is outerplanar (as it has either a K_4 or a $K_{2,3}$ as a minor), the class of outerplanar graphs is contained in the class of $\{K_4, K_{2,3}, K_{2,3}^+\}$-induced-minor-free graphs. Conversely, we will show that every $\{K_{2,3}, K_{2,3}^+, K_4\}$-induced-minor-free graph G is $\{K_4, K_{2,3}\}$-minor-free (and hence outerplanar). Indeed, suppose that G contains $H \in \{K_4, K_{2,3}\}$ as a minor. If $H = K_4$ then G contains K_4 as induced minor, which is impossible. So $H = K_{2,3}$. Consider a minor model $\mathcal{S} = \{S_v : v \in V(K_{2,3})\}$ of $K_{2,3}$ in G and let x and y be the two vertices of degree 3 in $K_{2,3}$. To avoid that $\mathcal{S}$ is an induced minor model of $K_{2,3}$ in G, we infer that G has an edge $\{x, y\}$ for some $x \in S_u$ and $y \in S_v$. This implies that either $K_{2,3}^+$ or K_4 is an induced minor of G, contrary to the assumption. □

2.4 Preliminaries on product graphs

We will now give the definitions and some basic facts about each of the four graph products studied in Chapters 5 and 6.

For each of the four considered products, we say that the product of two graphs is *nontrivial* if both factors have at least two vertices. For further details regarding product graphs and their properties, we refer to [45, 52].

Cartesian product graphs. The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) $u = u'$ and v is adjacent to v' in H, or

(b) $v = v'$ and u is adjacent to u' in G.

See Fig. 2.3 for an example.

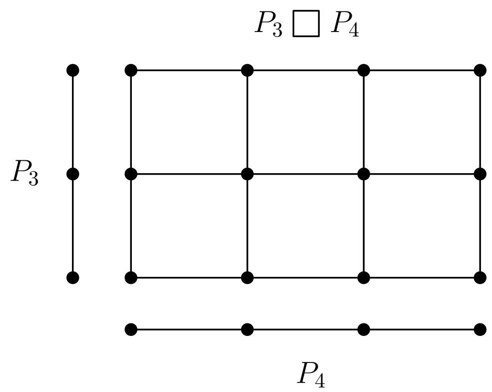


Figure 2.3: A small example for the Cartesian product.

The Cartesian product of two graphs is commutative, in the sense that $G \square H \cong H \square G$. The product $G \square H$ is connected if and only if both factors are connected (see [45, Corollary 5.3]). More precisely, if G has components $G_1, \ldots, G_k$ and H has components $H_1, \ldots, H_\ell$, then the components of $G \square H$ are exactly $G_i \square H_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$.

Lexicographic product graphs. Given two graphs G and H, the *lexicographic product* of G and H, denoted by $G[H]$ (sometimes also by $G \circ H$) is the graph with vertex set $V(G) \times V(H)$, in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) u is adjacent to u' in G, or

(b) $u = u'$ and v is adjacent to v' in H.

Fig. 2.4 shows an example.

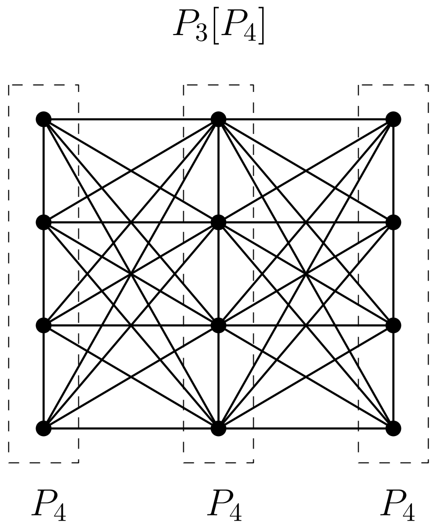


Figure 2.4: A small example for the lexicographic product.

Note that contrary to the other three products considered in this thesis, the lexicographic product is not commutative, that is, $G[H] \ncong H[G]$ in general. By [45, Corollary 5.14], the lexicographic product $G[H]$ of two nontrivial graphs is connected if and only if G is connected. In particular, if G has components $G_1, \ldots, G_k$, then the components of $G[H]$ are $G_1[H], \ldots, G_k[H]$.

Direct product graphs. The *direct product* $G \times H$ of two graphs G and H (sometimes also called *tensor product*, *categorical product*, or *Kronecker product*) is the graph with vertex set $V(G) \times V(H)$ in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) u is adjacent to u' in G, and

(b) v is adjacent to v' in H.

Fig. 2.5 gives an example.

The direct product of two graphs is commutative, in the sense that $G \times H \cong H \times G$. If the product $G \times H$ is connected, then both factors are connected, however the converse is generally not true. (For example, if $G \cong H \cong K_2$, then $G \times H \cong 2K_2$ is disconnected.) By [45, Corollary 5.10], the direct product of two connected nontrivial graphs is connected if and only if at most one of the factors is bipartite. If G has components $G_1, \ldots, G_k$ and H has components $H_1, \ldots, H_\ell$, then $G \times H$ is the disjoint union of the products of the components, $G_i \times H_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$.

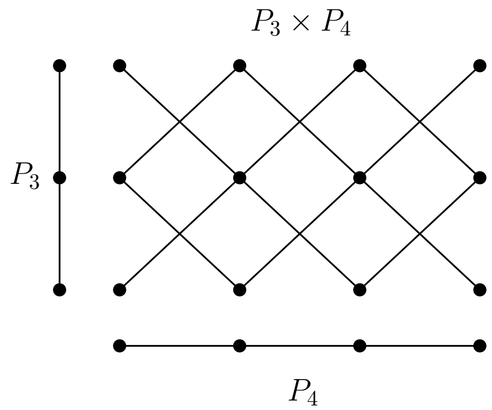


Figure 2.5: A small example for the direct product.

Strong product graphs. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which two distinct vertices (u, v) and (u', v') are adjacent if and only if

(a) u is adjacent to u' in G and $v = v'$, or

(b) $u = u'$ and v is adjacent to v' in H, or

(c) u is adjacent to u' in G and v is adjacent to v' in H.

It is easy to see that the fact that one of the conditions (a), (b) and (c) holds is equivalent to the pair of conditions $u' \in N_G[u]$ and $v' \in N_H[v]$, that is, that $(u', v') \in N_G[u] \times N_H[v]$. Consequently, for every two vertices $u \in V(G)$ and $v \in V(H)$, we have $N_{G \boxtimes H}[(u, v)] = N_G[u] \times N_H[v]$.

See Fig. 2.6 for an example.

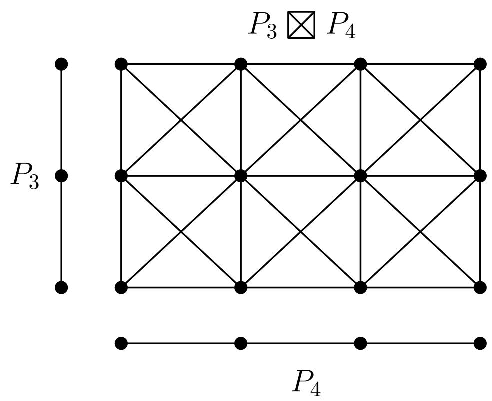


Figure 2.6: A small example for the strong product.

The strong product of two graphs is commutative, in the sense that $G \boxtimes H \cong H \boxtimes G$. By [45, Corollary 5.6], the strong product of two graphs G and H is connected if and only if both factors are connected. More precisely, if G has components $G_1, \ldots, G_k$ and H has components $H_1, \ldots, H_\ell$, then the components of $G \boxtimes H$ are exactly $G_i \boxtimes H_j$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$.

2.5 A few basic structural results

We now present a number of known results, that will be needed as lemmas in order to prove our results from Chapter 7.

A graph is a *linear forest* if it is the disjoint union of a set of paths.

Two vertex-disjoint subgraphs (or vertex subsets) F_1 and F_2 of a graph G are *adjacent* if there is at least one edge in G between a vertex in F_1 and a vertex in F_2. Similarly, a vertex u not in F_1 is *adjacent* to F_1 if $\{u\}$ and F_1 are adjacent. A set $D \subseteq V$ *dominates* G if every vertex $u \in V \setminus D$ is adjacent to D. We also say that $G[D]$ dominates G. If $D = \{u, v\}$ for two adjacent vertices u, v, then uv is called a *dominating edge* of G. A set $D \subseteq V$ *dominates* a set $S \subseteq V \setminus D$ if every vertex in S is adjacent to D.

We give four structural results (three known ones and one observation). The first result is well known (see, for example, [10]).

Lemma 2.14. *Every connected P_4-free graph on two or more vertices has a dominating edge.*

We will need the following result of Bacsó and Tuza [2] for the class of connected P_5-free graphs.

Lemma 2.15 (Bacsó and Tuza [2])**.** *Every connected P_5-free graph has a dominating P_3 or a dominating clique.*

We also need a lemma due to Duchet and Meyniel [24].

Lemma 2.16 (Duchet and Meyniel [24])**.** *Let G be a connected graph. Let β be the size of a minimum dominating set of G. Then G has a connected dominating set of size at most $3\beta - 2$.*

Lemma 2.17. *Let G be a connected graph with diameter d. Let A be a subgraph of G consisting of $r \geq 1$ components. Then G has a connected subgraph A' that contains A and that has less than $|V(A)| + (r-1)d$ vertices.*

Proof. Let the components of A be $D_1, \ldots, D_r$. We need to add less than d vertices to A in order to connect D_1 to each other D_i ($i \neq 1$). The resulting graph A' has size less than $|V(A)| + (r-1)d$. $\square$

Chapter 3

Basic results on 1-perfectly orientable graphs

In this chapter we prove various structural properties of 1-perfectly orientable graphs, which will be applied in Chapter 4 to derive characterizations of 1-perfectly orientable graphs within the classes of cobipartite graphs, cographs, block-cactus graphs, K_4-minor-free graphs, and outerplanar graphs, respectively.

Our results from this chapter can be summarized as follows:

1. We give a characterization of 1-p.o. graphs in terms of edge clique covers similar to a known characterization of squared graphs due to Mukhopadhyay.

2. We identify several graph transformations preserving the class of 1-p.o. graphs. In particular, we show that the class of 1-p.o. graphs is closed under taking induced minors. We also study the behavior of 1-p.o. graphs under the join operation, which motivates the study of 1-p.o. cobipartite graphs.

3. We identify several minimal forbidden induced minors for the class of 1-p.o. graphs, including 10 small specific graphs and two infinite families: the complements of even cycles of length at least 6 and the complements of the graphs obtained from odd cycles by adding a component consisting of a single edge.

4. We develop a reduction of the study of general 1-perfectly orientable graphs to the biconnected case.

5. We introduce a class of graphs defined similarly as the class of 2-trees, namely the class of hollowed 2-trees, and prove some structural results for these two graph classes. This graph class will prove to be useful in the study of 1-perfectly orientable K_4-minor-free and outerplanar graphs.

The results presented in this chapter are based on results from the following two papers.

- T. R. Hartinger and M. Milanič, *Partial Characterizations of 1-Perfectly Orientable Graphs.* J. Graph Theory. Vol. 85, 2, 2017, 378 – 394.

- B. Brešar, T. R. Hartinger, T. Kos, and M. Milanič (2016), 1-*perfectly orientable K_4-minor-free and outerplanar graphs.* Submitted. arXiv:1604.04598. An extended abstract appeared in Electronic Notes in Discrete Mathematics, Vol. 54, (2016), 199 – 204.

3.1 A characterization in terms of edge clique covers

A graph G is said to *have a square root* if there exists a graph H with $V(H) = V(G)$ such that for all $u, v \in V(G)$, we have $uv \in E(G)$ if and only if the distance in H between u and v is either 1 or 2. An *edge clique cover* in a graph G is a collection of cliques $\{C_1, \ldots, C_k\}$ in G such that every edge of G belongs to some clique C_i. In this section, we characterize 1-p.o. graphs in terms of edge clique covers, in a spirit similar to the well known Mukhopadhyay's characterization of graphs that admit a square root, which we now recall.

Theorem 3.1 (Mukhopadhyay [64]). *A graph G with $V(G) = \{v_1, \ldots, v_n\}$ has a square root if and only if G has an edge clique cover $\{C_1, \ldots, C_n\}$ such that the following two conditions hold:*

(a) $v_i \in C_i$ for all i,

(b) for every edge $v_iv_j \in E(G)$, we have $v_i \in C_j$ if and only if $v_j \in C_i$.

In the original statement of the theorem, the second condition is required for all $i \neq j$, but since $v_iv_j \notin E(G)$ clearly implies $v_i \notin C_j$ and $v_j \notin C_i$, the equivalence in condition 2 automatically holds for all non-adjacent vertex pairs.

Theorem 3.2. *For every graph G with $V(G) = \{v_1, \ldots, v_n\}$, the following conditions are equivalent:*

1. *G is 1-perfectly orientable.*
2. *G has an edge clique cover $\{C_1, \ldots, C_n\}$ such that the following two conditions hold:*
   - *(a) $v_i \in C_i$ for all i,*
   - *(b) for every edge $v_iv_j \in E(G)$, we have $v_i \in C_j$ or $v_j \in C_i$, but not both.*
3. *G has an edge clique cover $\{C_1, \ldots, C_n\}$ such that the following two conditions hold:*
   - *(a) $v_i \in C_i$ for all i,*
   - *(b) for every edge $v_iv_j \in E(G)$, we have $v_i \in C_j$ or $v_j \in C_i$.*

Before proving Theorem 3.2, we give two remarks. First, note that the difference between Theorem 3.1 and the equivalence of conditions 1 and 3 in Theorem 3.2 consists in replacing the equivalence in condition (b) of Theorem 3.1 with disjunction. This seemingly minor difference is in sharp contrast with the fact that recognizing graphs admitting a square root is NP-complete [63], while 1-p.o. graphs can be recognized in polynomial time 2.1. Second, a *pointed set* is a pair (S, v) where S is a nonempty set and $v \in S$. To every family $\mathcal{S} = \{(S_1, v_1), \ldots, (S_n, v_n)\}$ of pointed sets, one can associate a graph, the so called *catch graph* of $\mathcal{S}$ by setting $V(G) = \{v_1, \ldots, v_n\}$ and joining two distinct vertices v_i and v_j if and only if $v_i \in S_j$ or $v_j \in S_i$ (see, e.g. [60]). The equivalence between conditions 1 and 3 in the above theorem gives another proof of the fact that every 1-p.o. graph is the catch graph of a family of pointed sets, which also follows from the characterization of 1-p.o. graphs due to Urrutia and Gavril (stating that a graph is 1-p.o. if and only if it is the vertex-intersection graph of a family of mutually graftable subtrees in a graph) [85].

Proof of Theorem 3.2. First, we show the implication $1 \Rightarrow 2$. Given a 1-perfect orientation D of a 1-p.o. graph G with $V(G) = \{v_1, \ldots, v_n\}$, we define an edge clique cover $\{C_1, \ldots, C_n\}$ of G by setting $C_i = \{v_i\} \cup N_D^+(v_i)$. By definition, each C_i contains v_i, and, since D is 1-perfect, is a clique in G. Note that for all $i \neq j$, we have $v_j \in C_i$ if and only if $(v_i, v_j) \in A(D)$. In particular

$v_j \in C_i$ and $v_i \in C_j$ cannot happen simultaneously. Since for every edge $v_iv_j \in E(G)$, we have either $(v_i, v_j) \in A(D)$ or $(v_j, v_i) \in A(D)$ but not both, condition 2(b) follows.

The implication $2 \Rightarrow 3$ is trivial.

Finally, we show the implication $3 \Rightarrow 1$. Suppose that G has an edge clique cover $\{C_1, \ldots, C_n\}$ such that $v_i \in C_i$ for all i, and for every edge $v_iv_j \in E(G)$, $v_i \in C_j$ or $v_j \in C_i$. Define an orientation D of G as follows: for $1 \leq i < j \leq n$ such that $v_iv_j \in E(G)$, set $(v_i, v_j) \in A(D)$ if $v_j \in C_i$, and $(v_j, v_i) \in A(D)$, otherwise. By definition, for every vertex $v_i \in V(G)$ we have

$$\begin{aligned} N_D^+(v_i) &= \{v_j \mid j < i \ \wedge \ v_i \notin C_j\} \cup \{v_j \mid j > i \ \wedge \ v_j \in C_i\} \\ &\subseteq \{v_j \mid j < i \ \wedge \ v_j \in C_i\} \cup \{v_j \mid j > i \ \wedge \ v_j \in C_i\} \ \subseteq \ C_i\,, \end{aligned}$$

where the first inclusion relation holds due to condition 3(b). Hence, D is a 1-perfect orientation of G, and G is 1-p.o. □

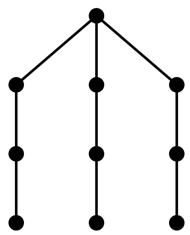
For later use, we also record the following immediate consequences of Theorem 3.2.

Corollary 3.3. *For every graph G with $V(G) = \{v_1, \ldots, v_n\}$, the following conditions are equivalent:*

1. *$\overline{G}$ is 1-perfectly orientable.*
2. *G has a collection of independent sets $\{I_1, \ldots, I_n\}$ such that the following two conditions hold:*
   (a) $v_i \in I_i$ for all i,
   (b) for every non-adjacent vertex pair $v_iv_j \in E(\overline{G})$, we have $v_i \in I_j$ or $v_j \in I_i$.

Corollary 3.4. *The edges of every 1-perfectly orientable graph with n vertices can be covered by n cliques.*

Note that the converse of Corollary 3.4 does not hold. For example, the complement of the 10-vertex graph G_1 (see Fig. 3.1 below) is not 1-p.o. (see Theorem 3.9), but can be edge-covered with (at most) 9 cliques. (Determining if the edges of a given n-vertex graph can be covered by n cliques is NP-complete [67]; see also [26].)



G_1

Figure 3.1: A graph on 10 vertices whose complement is not 1-p.o. and can be edge-covered with 9 cliques.

3.2 Operations preserving 1-perfectly orientable graphs

In this subsection, we identify several operations preserving 1-p.o. graphs and characterize when the join of two graphs is 1-p.o. Two distinct vertices u and v in a graph G are said to be *true* *twins* if $N_G[u] = N_G[v]$. (Similarly, u and v are said to be *false twins* if $N_G(u) = N_G(v)$.) Recall that a vertex v is *simplicial* if its neighborhood forms a clique, and *universal* if it is adjacent to every other vertex of the graph. The operations of adding a true twin, a simplicial vertex or a universal vertex to a given graph are defined in the obvious way. The operation of *duplicating a 2-branch in the complement* of a graph G is defined as follows. A *2-branch* in a graph G is a path (a, b, c) such that $d_G(b) = 2$ and $d_G(c) = 1$. We say that such a 2-branch is *rooted at a*. *Duplicating a 2-branch* G results in a graph H where (a, b, c) is a 2-branch in G, $V(H) = V(G) \cup \{b', c'\}$, where b' and c' are new vertices, $H - \{b', c'\} = G$, and (a, b', c') is a 2-branch in H. Finally, the result of *duplicating a 2-branch in the complement* of a graph G is the complement of a graph obtained by duplicating a 2-branch in $\overline{G}$.

Theorem 3.5. *The class of 1-perfectly orientable graphs is closed under each of the following operations:*

1. *Disjoint union.*
2. *Adding a universal vertex (that is, join with K_1).*
3. *Adding a true twin.*
4. *Adding a simplicial vertex.*
5. *Duplicating a 2-branch in the complement.*
6. *Vertex deletion.*
7. *Edge contraction.*

Proof. For a 1-p.o. graph G, let us denote by $D(G)$ an arbitrary (but fixed) 1-perfect orientation of G.

1. If $G = G_1 + G_2$ is the disjoint union of two 1-p.o. graphs G_1 and G_2, then the disjoint union of $D(G_1)$ and $D(G_2)$ is a 1-perfect orientation of G.

2. Suppose we have a 1-p.o. graph G with orientation $D(G)$ and we add a universal vertex v to G, thus obtaining a graph G'. A 1-perfect orientation D' of G' can be obtained by orienting an edge $xy \in E(G)$ from x to y if the edge is oriented from x to y in $D(G)$, and orienting the edges of the form uv from u to v. It is easy to see that D' is indeed 1-perfect.

3. Let w be a vertex in a 1-p.o. graph G, and let G' be the graph obtained from G by adding to it a true twin of w, say v. We obtain a 1-perfect orientation D' of G' by maintaining the same orientation as in $D(G)$ for the edges in G and orienting the new edges (incident with v) as $v \to u$ if $u \in N^+_{D(G)}(w)$, and $u \to v$ if $u \in N^-_{D(G)}(w)$. We also orient the edge between w and v as $w \to v$. It is a matter of routine verification to check that the so obtained orientation of G' is 1-perfect.

4. If we add a simplicial vertex v to a 1-p.o. graph G, then extending $D(G)$ by orienting all edges incident with v away from v results in an orientation D' of the new graph, say G', such that $N^+_{D'}(v)$ is a clique in G'. The other out-neighborhoods were not changed, so they are cliques in G' as well.

5. Let $V(G) = \{v_1, \ldots, v_n\}$. If G is 1-p.o., then Corollary 3.3 applies to $\overline{G}$. Hence, $\overline{G}$ has a collection of independent sets $\{I_1, \ldots, I_n\}$ such that $v_i \in I_i$ for all i, and for every edge $v_iv_j \in E(G)$, we have $v_i \in I_j$ or $v_j \in I_i$. Let H be the graph resulting from duplicating a 2-branch (a, b, c) in $\overline{G}$; without loss of generality, we may assume that $(a, b, c) = (v_1, v_2, v_3)$;

furthermore, let the two new vertices b' and c' be labeled as v_{n+1} and v_{n+2}, respectively. It suffices to prove that H has a collection of independent sets $\{J_1, \ldots, J_{n+2}\}$ such that $v_k \in J_k$ for all k, and for every edge $v_i v_k \in E(\overline{H})$, we have $v_i \in J_k$ or $v_k \in J_i$. We may assume without loss of generality that the sets I_j are maximal independent sets in $\overline{G}$, which in particular implies that each I_j contains exactly one of the vertices b and c. We define the sets J_k for $k \in \{1, \ldots, n+2\}$ with the following rule:

- For all $v_k \in V(G)$, set
$$J_k = \begin{cases} I_k \cup \{b'\}, & \text{if } b \in I_k; \\ I_k \cup \{c'\}, & \text{if } c \in I_k. \end{cases}$$
- For $k = n+1$ (that is, $v_k = b'$), set $J_k = (I_2 \setminus \{b\}) \cup \{b', c\}$.
- For $k = n+2$ (that is, $v_k = c'$), set $J_k = (I_3 \setminus \{a, c\}) \cup \{b, c'\}$.

Clearly, each J_k is an independent set in H. Let $v_i v_k \in E(\overline{H})$. Since $b'c' \notin E(\overline{H})$, we may assume that $v_i \in V(G)$. We analyze three cases according to where is v_k.

If $v_k \in V(G)$, then $v_i v_k \in E(G)$ and hence $v_i \in I_k$ or $v_k \in I_i$, implying $v_i \in J_k$ or $v_k \in J_i$.

If $v_k = b'$, then either $v_i \in J_k$ (in which case we are done), or $v_i \notin J_k = (I_2 \setminus \{b\}) \cup \{b', c\}$, in which case either $v_i = b$ or $v_i \notin I_2$. In the former case, we have $i = 2$ and $v_k = b' \in J_2$, while in the latter case, we have $b = v_2 \in I_i$, which implies $v_k = b' \in J_i$.

If $v_k = c'$, then either $v_i \in J_k$ (in which case we are done), or $v_i \notin J_k = (I_3 \setminus \{a, c\}) \cup \{b, c'\}$, in which case either $v_i \in \{a, c\}$ or $v_i \notin I_3$. In the former case, we have $c \in I_i$ (if $v_i = a$ this follows from the maximality of I_i) and consequently $v_k = c' \in J_i$. In the latter case, we have $c = v_3 \in I_i$, which implies $v_k = c' \in J_i$.

We have shown that $v_k \in J_k$ for all k, and for every edge $v_i v_k \in E(\overline{H})$, we have $v_i \in J_k$ or $v_k \in J_i$. By Corollary 3.3, H is the complement of a 1-p.o. graph, which establishes item 5.

6. Closure under vertex deletions follows immediately from the fact that the class of complete graphs is closed under vertex deletions.

7. Let $e = uv$ be an edge of a 1-p.o. graph G, and let D be a 1-perfect orientation of G, with (without loss of generality) $u \to v$. Let $G' = G/e$ be the graph obtained by contracting the edge e, and let w be the vertex replacing u and v.

Set

$$\begin{aligned} X &= N_G(u) \setminus N_G(v)\,, \\ Y &= \{x \in N_G(u) \cap N_G(v) \mid (x, v) \in A(D)\}\,, \\ U &= \{x \in N_G(u) \cap N_G(v) \mid (v, x) \in A(D)\}\,, \\ W &= \{x \in N_G(v) \setminus N_G(u) \mid (x, v) \in A(D)\}\,, \\ Z &= \{x \in N_G(v) \setminus N_G(u) \mid (v, x) \in A(D)\}\,, \\ R &= V(G) \setminus (X \cup Y \cup U \cup W \cup Z \cup \{u, v\})\,. \end{aligned}$$

Let D' be an orientation of G' defined as follows:

(i) For all edges $e \in E(G')$ whose endpoints are not incident with w, orient e the same way as it is oriented in D.

(ii) For all $x \in X$, orient the edge xw as $x \to w$.

(iii) For all $x \in Y$, orient the edge xw as $x \to w$.

(iv) For all $x \in U$, orient the edge xw as $w \to x$.

(v) For all $x \in W$, orient the edge xw as $x \to w$.

(vi) For all $x \in Z$, orient the edge xw as $w \to x$.

We complete the proof by showing that D' is a 1-perfect orientation of G'. We do this by directly verifying the defining condition that for every vertex x of $V(G')$, the set $N^+_{D'}(x)$ is a clique in G'. Note that $X \cup Y \cup U \cup W \cup Z \cup \{w\} \cup R$ is a partition of $V(G')$. We consider seven cases depending on to which part of this partition x belongs.

(1) $x \in X$. In this case, $N^+_{D'}(x) = (N^+_D(x) \setminus \{u\}) \cup \{w\}$. Note that since $(u,v) \in A(D)$ and D is a 1-perfect orientation of G, we have $u \in N^+_D(x)$. Consequently, since $N^+_D(x)$ is a clique in G containing u, it contains no vertex from $R \cup Z$, and thus $N^+_{D'}(x) = (N^+_D(x) \setminus \{u\}) \cup \{w\}$ is a clique in G'.

(2) $x \in W$. In this case, $v \in N^+_D(x)$, and a similar reasoning as above shows that $N^+_{D'}(x) = (N^+_D(x) \setminus \{v\}) \cup \{w\}$ is a clique in G'.

(3) $x \in Z$. In this case, $N^+_{D'}(x) = N^+_D(x)$ and this set is a clique in G and hence in G'.

(4) $x \in Y$. In this case, we have two possibilities, either $u \in N^+_D(x)$ or not. In the former case, we have $N^+_{D'}(x) = (N^+_D(x) \setminus \{u,v\}) \cup \{w\}$ which is a clique in G', since $N^+_D(x)$ is a clique in G containing u and v, and every neighbor of w in G' is a neighbor of either u or of v in G. In the latter case, we have $N^+_{D'}(x) = (N^+_D(x) \setminus \{v\}) \cup \{w\}$, which is again a clique in G' by a similar argument.

(5) $x \in U$. Now, $N^+_{D'}(x) = N^+_D(x) \setminus \{u\}$, which is a clique in G not containing u or v, and hence a clique in G'.

(6) $x \in R$. Since the edges with endpoints in R have no endpoint in $\{u,v\}$, the edges which have x as an endpoint will maintain the same orientation as in D. Therefore, $N^+_{D'}(x) = N^+_D(x)$ is a clique in G'.

(7) $x = w$. In this case, we have $N^+_{D'}(x) = N^+_D(v)$, therefore $N^+_{D'}(x)$ forms a clique in G'. □

In the study of 1-p.o. graphs we may restrict our attention to connected graphs. It is a natural question whether we may also assume that G is co-connected, that is, that its complement is connected, or, equivalently, that G is not the join of two smaller graphs. The join operation does not generally preserve the class of 1-p.o. graphs: the graphs $2K_1$ and $3K_1$ are trivially 1-p.o., but their join, $K_{2,3}$, is not (as can be easily verified; see also Theorem 3.9). In the next theorem we characterize when the join of two graphs is 1-p.o. Recall that a graph is said to be *cobipartite* of its complement is bipartite.

Theorem 3.6. *Suppose that a graph G is the join of two graphs G_1 and G_2. Then, G is 1-perfectly orientable if and only if one of the following conditions hold:*

1. *G_1 is complete and G_2 is 1-p.o., or vice versa.*
2. *Each of G_1 and G_2 is a cobipartite 1-p.o. graph.*

In particular, the class of cobipartite 1-p.o. graphs is closed under join.

Proof. Suppose first that G is 1-p.o. Clearly, both G_1 and G_2 are 1-p.o. graphs. If one of G_1 or G_2 is complete or both are cobipartite, we are done. So suppose that neither of them is complete and G_1, say, is not cobipartite. Then, G_1 contains the complement of an odd cycle, $\overline{C_{2k+1}}$ for some $k \geq 1$, as induced subgraph. Since G_2 is not complete, it contains $2K_1$ as induced subgraph. Consequently, G contains the join of $\overline{C_{2k+1}}$ and $2K_1$ as induced subgraph.

As this graph is isomorphic to the complement of $C_{2k+1} + K_2$, it is not 1-p.o. (see Theorem 3.9), and hence neither is G, a contradiction.

For the converse direction, suppose first that G_1 is complete and G_2 is 1-p.o., or vice versa. In this case G is 1-p.o., since it can be obtained from a 1-p.o. graph by a sequence of universal vertex additions, and Theorem 3.5 applies. Suppose now that G_1 and G_2 are two cobipartite 1-p.o. graphs with bipartitions of their respective vertex sets into cliques $\{A_1, B_1\}$ and $\{A_2, B_2\}$, respectively (one of the two cliques in each graph can be empty). Fixing a 1-perfect orientation D_i of each G_i (for $i = 1, 2$), we can construct a 1-perfect orientation, say D, of $G = G_1 * G_2$, as follows. Every edge of G that is an edge of some G_i is oriented as in D_i. Orient the remaining edges of the join from A_1 to A_2, from B_1 to B_2, from A_2 to B_1 and from B_2 to A_1. Let us verify that the out-neighborhood of a vertex $x \in A_1$ with respect to D forms a clique in G (the other cases follow by symmetry). We have $N_D^+(x) = N_{D_1}^+(x) \cup A_2$, and since $N_{D_1}^+(x)$ is a clique in G_1, A_2 is a clique in G and there are all edges between G_1 and A_2, the set $N_D^+(x)$ is indeed a clique in G. This shows that G is 1-p.o.

Since the class of bipartite graphs is closed under disjoint union, the class of cobipartite graphs is closed under join. Consequently, the set of cobipartite 1-p.o. graphs is closed under join. □

3.3 A family of minimal forbidden induced minors

Theorem 3.5 implies that the class of 1-p.o. graphs is closed under vertex deletions and edge contractions. Hence, it is also closed under taking induced minors. Recall that a graph H is said to be an *induced minor* of a graph G if H can be obtained from G by a series of vertex deletions or edge contractions. Graph classes closed under induced minors include all the minor-closed graph classes, as well as many others (see, e.g., [16,53,56,57,84]). Since the class of 1-p.o. graphs is closed under induced minors, it can be characterized in terms of *minimal forbidden induced minors*. That is, there exists a unique minimal set of graphs $\tilde{\mathcal{F}}$ such that (i) a graph G is 1-p.o. if and only if G is *$\tilde{\mathcal{F}}$-induced-minor-free* (that is, no induced minor of G is isomorphic to a member of $\tilde{\mathcal{F}}$), and (ii) every proper induced minor of every graph in $\tilde{\mathcal{F}}$ is 1-p.o. In this section we identify an infinite subfamily $\mathcal{F} \subseteq \tilde{\mathcal{F}}$ of minimal forbidden induced minors for the class of 1-p.o. graphs.

We start with two preliminary observations. The fact that every circular arc graph is 1-p.o. implies the following.

Proposition 3.7. *The complement of every odd cycle is* 1*-perfectly orientable.*

Proof. Recall that the *k-th power* of a graph G is the graph with the same vertex set as G in which two distinct vertices are adjacent if and only if their distance in G is at most k. It is easy to see (and also follows from the fact that the class of circular arc graphs is closed under taking powers [75]) that all powers of cycles are circular arc graphs. Therefore, the fact that the complement of every odd cycle is 1-p.o. follows from two facts: (i) that the complement of C_3 is 1-p.o., and (ii) for every $k \geq 2$, the complement of the odd cycle C_{2k+1} is isomorphic to a power of a cycle, namely to C_{2k+1}^{k-1}. □

Since every disjoint union of paths is an induced subgraph of a sufficiently large odd cycle, Proposition 3.7 and Theorem 3.5 yield the following. Recall that a linear forest is the disjoint union of a set of paths.

Corollary 3.8. *The complement of every linear forest is* 1*-perfectly orientable.*

The following theorem describes a set of minimal forbidden induced minors for the class of 1-p.o. graphs.

Theorem 3.9. *Let $\mathcal{F} = \{F_1, F_2, F_5, \ldots, F_{12}\} \cup \mathcal{F}_3 \cup \mathcal{F}_4$, where:*

- *graphs F_1, F_2 are depicted in Fig. 3.2, and*
- *$\mathcal{F}_3 = \{\overline{C_{2k}} \mid k \geq 3\}$, the set of complements of even cycles of length at least 6,*
- *$\mathcal{F}_4 = \{\overline{K_2 + C_{2k+1}} \mid k \geq 1\}$, the set of complements of the graphs obtained as the disjoint union of K_2 and an odd cycle,*
- *for $i \in \{5, \ldots, 12\}$, graph F_i is the complement of the graph G_{i-4}, depicted in Fig. 3.2.*

Then, every graph in set $\mathcal{F}$ is a minimal forbidden induced minor for the class of 1-perfectly orientable graphs.

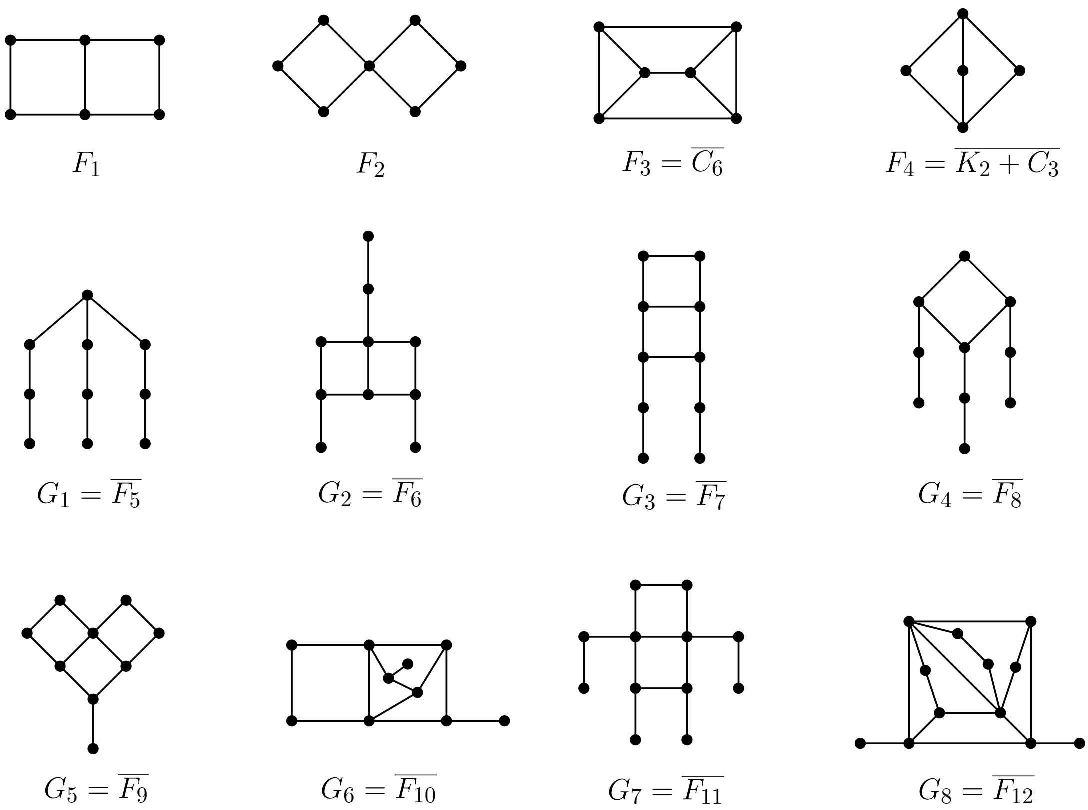


Figure 3.2: Four non-1-p.o. graphs and 8 complements of non-1-p.o. graphs. Graphs F_3 and F_4 are the smallest members of families $\mathcal{F}_3$ and $\mathcal{F}_4$, respectively.

Proof. We need to show that each $F \in \mathcal{F}$ is not 1-p.o., but every proper induced minor of F is. We first show that no graph in $\mathcal{F}$ is 1-p.o., and will argue minimality for all $F \in \mathcal{F}$ in the second part of the proof.

No graph is in $\mathcal{F}$ is 1-p.o.

First consider the graphs F_1 and F_2. Since they are both triangle-free, every edge clique cover of F_i (for $i \in \{1, 2\}$) contains all edges of F_i and hence has at least $|E(F_i)| > |V(F_i)|$ members. Hence, Corollary 3.4 implies that F_1 and F_2 are not 1-p.o.

The family $\mathcal{F}_3$ consists precisely of complements of even cycles of length at least 6. In particular, every $F \in \mathcal{F}_3$ is cobipartite. By Theorem 4.2, F is 1-p.o. if and only if F is circular arc. Since the family $\mathcal{F}_3$ is one of the six infinite families of minimal forbidden induced subgraphs for the class of circular arc cobipartite graphs [83], we infer that F is not 1-p.o.

Now let $F \in \mathcal{F}_4$, that is, $F = \overline{K_2 + C_{2k+1}}$ for some $k \geq 1$. We will prove that F is not 1-p.o. using Lemma 4.1. Let the vertices of the cycle component of $\overline{F}$ be named $u_1, \ldots, u_{2k+1}$, according to a cyclic order of C_{2k+1}. Also, let the two vertices of the K_2 component of $\overline{F}$ be named v_1 and v_2. Suppose that F admits a 1-perfect orientation D. For every two consecutive vertices from the cycle we have an induced C_4 given by these two vertices together with v_1 and v_2. By Lemma 4.1, every such C_4 must be oriented cyclically. Consider the C_4 induced by vertices $\{u_1, u_2, v_1, v_2\}$. Without loss of generality we may assume that it is oriented as $v_1 \to u_1 \to v_2 \to u_2 \to v_1$. This determines the orientation of the C_4 induced by $\{u_2, u_3, v_1, v_2\}$. Since the edge $\{v_1, u_2\}$ is oriented as $u_2 \to v_1$, the edge $\{v_1, u_3\}$ must be oriented as $v_1 \to u_3$. Proceeding along the cycle, we infer that $v_1 \to u_i$ for odd i and $u_i \to v_1$ for even i. However, this implies that $v_1 \to u_1$ and $v_1 \to u_{2k+1}$, contrary to the fact that D is a 1-perfect orientation of F. Therefore, F is not 1-p.o.

Each of the remaining graphs, F_5–F_{12}, belongs to the list of minimal forbidden induced subgraphs for the class of circular arc cobipartite graphs [83]. By Theorem 4.2, none of these graphs is 1-p.o.

It remains to show minimality, that is, that *every proper induced minor of every graph in* $\mathcal{F}$ *is* 1-*p.o.*

First consider the graphs F_1 and F_2. Deleting any vertex of either F_1 or F_2 results in either a chordal graph or in a unicyclic graph, hence in a 1-p.o. graph (cf. Proposition 2.2). Contracting any edge of F_1 results in a graph that is either chordal, or is obtained from a cycle by adding to it a simplicial vertex, hence in either case a 1-p.o. graph. Contracting any edge of F_2 results in a graph that can be reduced to a cycle by removing true twins and simplicial vertices, hence this graph is also 1-p.o.

We are left with graphs that are defined in terms of their complements. To argue minimality for them, it will be convenient to understand the effect of performing the operation of edge contraction on a given graph on its complement. It can be seen that if G is the graph obtained from a graph H by contracting an edge uv, then $\overline{G}$ is the graph obtained from $\overline{H}$ identifying a pair of non-adjacent vertices (namely, u and v) and making the new vertex adjacent exactly to the common neighbors in $\overline{H}$ of u and v. We will refer to this operation as *co-contracting a non-edge*.

Let $F \in \mathcal{F}_3$, that is, $F = \overline{C_{2k}}$ for some $k \geq 3$. Deleting a vertex from F results in the complement of a path, which is 1-p.o. by Corollary 3.8. Similarly, one can verify that co-contracting a non-edge of $\overline{F}$ results in a disjoint union of paths. Thus, every proper induced minor of F is 1-p.o.

Let $F \in \mathcal{F}_4$, that is, $F = \overline{C_{2k+1} + K_2}$ for some $k \geq 1$. Deleting a vertex in the cycle component of $\overline{F}$ from F results in the complement of a disjoint union of path, which is 1-p.o. by Corollary 3.8. Deleting a vertex in the K_2 component of $\overline{F}$ from F results in the graph that consists of the join of K_1 and the complement of an odd cycle, which is 1-p.o. by Theorem 3.5 and Proposition 3.7. Furthermore, co-contracting a non-edge of $\overline{F}$ results in a disjoint union of paths, and Corollary 3.8 applies again. Thus, every proper induced minor of F is 1-p.o.

We recall that each of the remaining graphs, F_5–F_{12}, is a minimal forbidden induced subgraph for the class of circular arc cobipartite graphs. Deleting a vertex from any of them results in a circular arc cobipartite graph, hence in a 1-p.o. graph. Note that F_9 has 9 vertices, each of $F_5, F_6, F_7, F_8, F_{10}$ has 10 vertices, and F_{11} and F_{12} have 12 vertices. Also note that since cobipartite graphs are closed under edge contractions, in order to show that every graph obtained from one of the graphs F_5–F_{12} by contracting an edge is 1-p.o., it suffices to argue that it is circular arc, which (since it is cobipartite) is equivalent to verifying that it does not contain any of the minimal forbidden induced subgraphs for the class of circular arc cobipartite graphs [83]. The only graphs with at most 10 vertices on this list are $\overline{C_6}$, $\overline{C_8}$, $\overline{C_{10}}$, and graphs F_5–F_{10}. The list also contains a unique graph of order 11; let G_9 denote its complement. Let $G \in \{\overline{F_5}, \ldots, \overline{F_{12}}\} = \{G_1, \ldots, G_8\}$. A direct inspection of the possible graphs resulting from co-contracting a non-edge of G shows that every such graph has either at most 10 vertices, in which case its complement is $\{C_6, C_8, C_{10}, G_1, \ldots, G_6\}$-free, or it has 11 vertices, in which case its complement either has an isolated vertex and the rest is $\{C_6, C_8, C_{10}, G_1, \ldots, G_6\}$-free, or it is connected, of order 11, and $\{C_6, C_8, C_{10}, G_1, \ldots, G_6, G_9\}$-free. Thus, in all cases contracting an edge of a graph in $\{F_5, \ldots, F_{12}\}$ results in a circular arc graph, hence in a 1-p.o. graph. This completes the proof. □

The previous result implies that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is the set of minimal forbidden induced minors for the class of 1-p.o. graphs. However, the complete set $\tilde{\mathcal{F}}$ remains unknown. It is conceivable that one can obtain further graphs in $\tilde{\mathcal{F}}$ by computing the minimal elements with respect to the induced minor relation of the list of forbidden induced subgraphs for the class of circular arc cobipartite graphs due to Trotter and Moore [83]. Besides the three small graphs F_5, F_6, F_7 and the family $\mathcal{F}_3$ of complements of even cycles of length at least 6, the list contains five other infinite families, the smallest members of which are graphs $F_8, \ldots, F_{12}$, respectively.

3.4 Reduction to the biconnected case

Since a graph is 1-p.o. if and only if each component of G is 1-p.o., we may restrict our attention to connected graphs. In this subsection, we analyze to what extent the study of 1-perfectly orientable graphs can be reduced to the biconnected case. It turns out that biconnectivity comes at a price: the study of slightly more general structures is required, namely of pairs (G, v) where G is a biconnected 1-perfectly orientable graph having a 1-perfect orientation D such that v is a sink in D. A *sink* in a directed graph is a vertex of out-degree zero. A directed graph is said to be *sink-free* if it has no sinks.

The reduction to the biconnected case presented in this section will be used in Sections 4.3, 4.4, and 4.5 for the characterizations of 1-perfectly orientable block-cactus, K_4-minor-free, and outerplanar graphs, respectively.

Definition 3.10. A *rooted graph* is a pair (G, v), denoted also by G^v, such that G is a graph and $v \in V(G)$. A rooted graph G^v is said to be *connected* (resp., biconnected) if G is connected (resp., biconnected), and 1-*perfectly orientable* if G has a 1-perfect orientation in which v is a sink.

Before stating the main theorem of this subsection we write the following two lemmas, one regarding sinks in 1-perfect orientations of connected graphs and one characterizing 1-perfect orientations of trees.

Lemma 3.11. *Every* 1*-perfect orientation of a connected graph has at most one sink.*

Proof. Let D be a 1-perfect orientation of a connected graph G with two sinks x and y. Suppose for a contradiction that $x \neq y$. Let $P = (x = v_0, v_1, \ldots, v_k = y)$ be a shortest x,y-path in G. Since $v_0 = x$ is a sink, the edge $\{v_0, v_1\}$ is oriented as $v_1 \to v_0$ in D. This implies that there is a unique maximum index $j \in \{1, \ldots, k\}$ such that (v_j, v_{j-1}) is an arc of D. Since y is a sink, the edge $\{v_{k-1}, v_k\}$ is oriented as $v_{k-1} \to v_k$ in D, which implies $j < k$. The definition of j implies that (v_{j+1}, v_j) is not an arc of D. Hence (v_j, v_{j+1}) is an arc of D. Since the out-neighborhood of v_j in D is a clique in G, vertices v_{j-1} and v_{j+1} are adjacent, contradicting the minimality of P. □

An *in-tree* is a directed graph D that has a vertex r called the *root* such that for every vertex $v \in V(D)$, there is exactly one directed path from v to r. Equivalently, an in-tree is a directed rooted tree in which all arcs point towards the root (that is, for every edge $\{x, y\}$ of the underlying undirected tree T, we have that (x, y) is an arc of D if and only if $d_T(x, r) < d_T(y, r)$). It is easy to see in every in-tree D, every vertex $v \in V(D)$ satisfies $d_D^+(v) = 1$, except for the root r, which is a sink.

Lemma 3.12. *Let T be a tree and let D be an orientation of T. Then, D is 1-perfect if and only if D is an in-tree. Moreover, for every vertex $r \in V(T)$ there exists a 1-perfect orientation D of T such that r is the root of the in-tree D.*

Proof. If D is an in-tree then the fact that D is a 1-perfect orientation of T follows from the fact that every vertex $v \in V(T)$ satisfies $d_D^+(v) \leq 1$.

Suppose now that D is a 1-perfect orientation of T. Then $d_D^+(v) \leq 1$ for all vertices $v \in V(T)$. By Lemma 3.11, D has at most one sink. If D does not have any sink, then for every vertex v of T, we have $d_D^+(v) = 1$, which implies that the total number of arcs in D equals $|V(D)| = |V(T)|$, contrary to the fact that T is acyclic. It follows that D has a unique sink, say r. We claim that D is an in-tree with root r, that is, for every vertex $v \in V(D)$ there is exactly one directed path from v to r. Since D is an orientation of a tree, for every vertex $v \in V(T)$ there is at most one directed path from v to r. Clearly, for $v = r$ there is a unique v,r-directed path. Since every vertex v that is not the root has a unique out-neighbor and D has no directed cycles, any maximal path from v ends in the root.

The last statement of the lemma is immediate since, given a vertex $r \in V(T)$, orienting all edges of T towards r results in an in-tree with root r. □

We summarize the reduction to biconnected rooted graphs in the following theorem. Given a tree T and a vertex $r \in V(T)$, the 1-perfect orientation of T in which r is the unique sink (cf. Lemma 3.12) will be referred to as the *r-rooted orientation of T*.

Theorem 3.13. *Let G be a connected graph with a cut vertex, let $\mathcal{B}$ and C be the sets of blocks and cut vertices of G, respectively, and let T be the block tree of G. Then, G is 1-perfectly orientable if and only if one of the following conditions holds:*

1. *There exists a block B_r of G such that B_r is 1-perfectly orientable and for every arc $(B, v) \in \mathcal{B} \times C$ of the B_r-rooted orientation of T, the rooted graph B^v is 1-perfectly orientable.*

2. *There exists a cut vertex v_r of G such that for every arc $(B, v) \in \mathcal{B} \times C$ of the v_r-rooted orientation of T, the rooted graph B^v is 1-perfectly orientable.*

Proof. Necessity. Suppose first that G is 1-perfectly orientable, and let D be a 1-perfect orientation of G. Consider the orientation T_D of the block tree T defined by orienting any edge $\{v, B\}$ of T (with $v \in C$ and $B \in \mathcal{B}$) as $B \to v$ if and only if v is a sink in the subgraph of D induced by $V(B)$.

We claim that for every node x of the block tree T, we have $d_{T_D}^+(x) \leq 1$. If $x = B$ is a block of G, then the inequality $d_{T_D}^+(B) \leq 1$ follows from Lemma 3.11. So let $x = v$ be a cut vertex of G and suppose for a contradiction that $d_{D_T}^+(v) \geq 2$. Then there exist two blocks B and B' of G containing v such that v is not a sink in the subgraph of D induced by X for any $X \in \{V(B), V(B')\}$. It follows that the out-neighborhood of v in D contains a vertex from $V(B) \setminus \{v\}$ and a vertex from $V(B') \setminus \{v\}$. As these two vertices are not adjacent in G, this contradicts the fact that D is 1-perfect.

Since every node of T_D is of out-degree at most 1, T_D is a 1-perfect orientation of T. Lemma 3.12 implies that orientation T_D is an in-tree. As there are two types of nodes in T, the blocks of G and the cut vertices of G, the unique sink of T_D can be either a block of G or a cut vertex. Suppose first that the unique sink of T_D is a block of G, say B_r. Then T_D is the B_r-rooted orientation of T. Since the subgraph of D induced by $V(B_r)$ is a 1-perfect orientation of B_r, it follows that B_r is 1-perfectly orientable. Moreover, for every arc $(B, v) \in \mathcal{B} \times C$ of T_D, the subgraph of D induced by $V(B)$ is 1-perfect orientation of B in which $v \in V(B)$ is a sink, which implies that B^v is 1-perfectly orientable. Thus, condition 1 holds in this case. A similar argument shows that condition 2 holds if the unique sink of T_D is a cut vertex of G.

Sufficiency. We now show that each of the two conditions is sufficient for G to be 1-perfectly orientable.

First, suppose that condition 1 holds, that is, there exists a block B_r of G such that B_r is 1-perfectly orientable and for every arc $(B, v) \in \mathcal{B} \times C$ of the B_r-rooted orientation of T, the rooted graph B^v is 1-perfectly orientable. Fix a 1-perfect orientation D_{B_r} of B_r and, for every arc $(B, v) \in \mathcal{B} \times C$ of the B_r-rooted orientation of T, fix a 1-perfect orientation D_B of B in which v is a sink. Note that since every block $B \neq B_r$ is of out-degree 1 in the B_r-rooted orientation of T, each of the blocks of G is oriented by exactly one of the above $|\mathcal{B}|$ orientations. Since each edge of G lies in a unique block of G, combining the above orientations defines a unique orientation of G, say D. We claim that D is a 1-perfect orientation of G. Every vertex $v \in V(G)$ that is not a cut vertex belongs to a unique block, say B, and therefore $N_D^+(v) = N_{D_B}^+(v)$. Since D_B is a 1-perfect orientation of B, the set $N_{D_B}^+(v)$ is a clique in B, and hence also a clique in G. If $v \in V(G)$ is a cut vertex, then there is a unique block B of G such that (v, B) is an arc of the B_r-rooted orientation of T, which means that for every block B' containing v other than B, the vertex v is a sink in $D_{B'}$. Again we obtain that $N_D^+(v) = N_{D_B}^+(v)$, hence this set is a clique in G. This shows that D is a 1-perfect orientation of G, showing that G is 1-perfectly orientable.

The proof in the case when condition 2 holds is very similar. Suppose that there is a cut vertex v_r of G such that for every arc $(B, v) \in \mathcal{B} \times C$ of the v_r-rooted orientation of T, the rooted graph B^v is 1-perfectly orientable. For every such arc (B, v), fix a 1-perfect orientation D_B of B in which v is a sink. In this case, every block B of G is of out-degree 1 in the v_r-rooted orientation of T and combining the above $|\mathcal{B}|$ orientations defines a unique orientation of G, say D. Arguments analogous to those in the above paragraph show that D is a 1-perfect orientation of G, hence G is 1-perfectly orientable in this case too. This completes the proof. □

We also prove a lemma on chordal graphs for later use.

Lemma 3.14. *Every rooted extension of a chordal graph is* 1*-perfectly orientable.*

Proof. Let G be a chordal graph and $v \in V(G)$. Since G is chordal, it has a perfect elimination ordering, that is, a linear ordering $\sigma = (v_1, \ldots, v_n)$ of the vertices of G such that for all $i \in \{1, \ldots, n\}$, vertex v_i is a simplicial vertex in the subgraph of G induced by $\{v_1, \ldots, v_i\}$. Moreover, the perfect elimination orderings of G are exactly the sequences of the form (σ', v_n) where v_n is a simplicial vertex of G and σ' is a perfect elimination ordering of $G - v_n$.

We claim that G has a perfect elimination ordering $\sigma = (v_1, \ldots, v_n)$ such that $v = v_1$. As observed already by Dirac [23], every minimal separator in a chordal graph is a clique, which implies that every chordal graph is either complete or has a pair of non-adjacent simplicial vertices. It follows that every chordal graph with at least two vertices has a pair of perfect elimination orderings $\sigma = (u_1, \ldots, u_n)$ and $\sigma' = (u'_1, \ldots, u'_n)$ such that $u_n \neq u'_n$. In particular, one can construct a perfect elimination ordering $(v_1, \ldots, v_n)$ of G by iteratively deleting simplicial vertices (and at the end reversing the order of deleted vertices) so that vertex v is deleted only at the very end, that is, so that $v = v_1$, as claimed.

Let $\sigma = (v_1, \ldots, v_n)$ be a perfect elimination ordering of G such that $v = v_1$. Orienting the edges of G as $v_i \to v_j$ if and only if $i > j$ result in a 1-perfect orientation of G in which v is a sink, showing that G^v is 1-perfectly orientable. □

3.5 Hollowed 2-trees and their relation to 1-perfectly orientable graphs

In this section we introduce the graph class of hollowed 2-trees, which is defined similarly as the class of 2-trees, and prove some structural results for these two graph classes. The classes of 2-trees and hollowed 2-trees will play an important role in our characterizations of 1-perfectly orientable K_4-minor-free and outerplanar graphs, obtained in Sections 4.4 and 4.5, respectively.

It is well known that trees can be constructed recursively as follows: (i) K_1 is a tree, (ii) a graph obtained from a tree by adding to it a vertex of degree 1 is a tree, and (iii) there are no other trees. The class of *2-trees* is defined in a similar way: (i) K_2 is a 2-tree, (ii) a graph obtained from a 2-tree by adding to it a simplicial vertex of degree 2 is a 2-tree, and (iii) there are no other 2-trees. We now consider the following extension of the notion of 2-trees.

Definition 3.15. A *hollowed 2-tree* is defined as follows: (i) any cycle of length at least four is a hollowed 2-tree, (ii) a graph obtained from a hollowed 2-tree by adding to it a simplicial vertex of degree 2 is a hollowed 2-tree, and (iii) there are no other hollowed 2-trees.

The name of this graph class relates to the fact that a *hole* in a graph G often refers to an induced cycle of length at least four in G. Every hollowed 2-tree has a unique hole (and, in particular, is not a 2-tree).

Note that all 2-trees and all hollowed 2-trees are biconnected. They will play an important role in our characterization of 1-perfectly orientable K_4-minor-free graphs (Theorem 4.12) and in its reduction to the biconnected case.

We first note some properties of 1-perfect orientations of 2-trees and of hollowed 2-trees.

Lemma 3.16. *All 2-trees and their rooted extensions are 1-perfectly orientable. Every hollowed 2-tree is 1-perfectly orientable, however, all its 1-perfect orientations are sink-free. (That is, no rooted extension of a hollowed 2-tree is 1-perfectly orientable.)*

Proof. Since 2-trees are chordal, Lemma 3.14 implies that all their rooted extensions are 1-perfectly orientable. In particular, every 2-tree is 1-perfectly orientable.

Now, let G be a hollowed 2-tree. We prove by induction on $|V(G)|$ that G is 1-perfectly orientable, having only sink-free 1-perfect orientations. If G is a cycle of length at least 4, then this holds by Lemma 4.1. Otherwise, G is obtained from a hollowed 2-tree G' by adding to it a simplicial vertex, say v, of degree 2. Extending a 1-perfect orientation of G' by orienting the two edges incident with v away from v yields a 1-perfect orientation of G, hence G is 1-perfectly orientable. Suppose for a contradiction that G has a 1-perfect orientation D with a sink s. If $s \neq v$, then the subgraph of D induced by $V(G')$ would be a 1-perfect orientation of G' with a sink, contrary to the inductive hypothesis. Therefore $s = v$. Let x and y be the two neighbors of v and suppose without loss of generality that $x \to y$ in D. Since D is a 1-perfect orientation of G, we infer that y is a sink in D', the subgraph of D induced by $V(G')$. However, this implies that D' is a 1-perfect orientation of G' with a sink, contrary to the inductive hypothesis. □

In the rest of the section, we prove four interrelated lemmas: one regarding K_4-minor-free biconnected graphs, one showing that 2-trees and hollowed 2-trees are the only biconnected $\{K_4, K_{2,3}, F_1\}$-induced-minor-free graphs, one showing that every connected $\{K_4, K_{2,3}, F_1, F_2\}$-induced-minor-free graph has at most one hole, and, finally, one characterizing the $\{K_{2,3}, F_1, F_2\}$-induced-minor-free graphs within the class of connected K_4-minor-free graphs.

Lemma 3.17. *Let G be a biconnected K_4-minor-free graph with at least two vertices. Then G is chordal if and only if G is a 2-tree.*

Proof. It follows immediately from the definition of 2-trees that every 2-tree is chordal.

Conversely, suppose that G is a biconnected chordal K_4-minor-free graph with at least two vertices. The fact that G is a 2-tree can be proved by induction on the number of vertices. If G has exactly 2 vertices, then $G = K_2$ is a 2-tree. Suppose that $|V(G)| > 2$. Since G is chordal, it has a simplicial vertex, say v. Since G is K_4-free, v is of degree at most 2. Since G is biconnected, v is of degree at least 2. Therefore, v is of degree exactly 2. It is easy to see that the graph $G - v$ is a biconnected chordal K_4-minor-free graph with at least two vertices. Therefore, by the inductive hypothesis, $G - v$ is a 2-tree. It follows that G is also a 2-tree. □

Lemma 3.18. *Let G be a biconnected K_4-minor-free graph. Then, G is $\{K_{2,3}, F_1\}$-induced-minor-free if and only if G is either K_1, a 2-tree, or a hollowed 2-tree.*

Proof. If G is either K_1, a 2-tree or a hollowed 2-tree, then G has at most one hole, which immediately implies that neither F_1 nor $K_{2,3}$ is an induced minor of G.

Suppose now that G is $\{K_{2,3}, F_1\}$-induced-minor-free. If G is chordal, then, since G is K_4-minor-free, it follows from Lemma 3.17 that G is a 2-tree. Therefore we may assume that G is non-chordal. We will show that in this case G is a hollowed 2-tree. It follows from Theorem 2.12 that G is cyclically orientable. By Corollary 2.10, G can be constructed from cycles by an iterative application of pasting along an edge. Assume that we are in step $k > 1$ of this construction procedure, and assume inductively that the graph G' constructed right before step k is a hollowed 2-tree. Now, in step k we will paste a cycle C along some edge $e = xy$ of G'. If C is of length 3, the graph will remain a hollowed 2-tree after the last operation. So we can assume that C is of length at least 4. Let C' be the unique hole in G'. Since G is biconnected, it has a pair P, Q of vertex-disjoint paths between x and C' and between y and C', respectively. Let P and Q be chosen so that their common length $|E(P)|+|E(Q)|$ is minimized. Let x' and y' be the endpoints of P and Q on C', respectively. Note that G' contains three pairwise internally vertex-disjoint x', y'-paths (two along C' and one more through $P \cup Q$); in particular, vertices x' and y' they cannot be separated by a set of less than 3 other vertices. Since G' is cyclically orientable, it is of separability at most 2 (by Theorem 2.9). Therefore, x' and y' are adjacent. Let z be the neighbor of x on C other than y, and similarly, z' be the neighbor of x' on C' other than y'. Now, the sets $V(P)$, $V(Q)$, $\{z\}$, $\{z'\}$, $V(C) \setminus \{x, y, z\}$, $V(C') \setminus \{x', y', z'\}$, form an induced minor model of F_1 in G, contrary to the fact that G is F_1-induced-minor-free. □

Lemma 3.19. *Let G be a connected $\{K_4, K_{2,3}, F_1, F_2\}$-induced-minor-free graph. Then G has at most one hole.*

Proof. Let G be a biconnected $\{K_4, K_{2,3}, F_1, F_2\}$-induced-minor-free graph. If G has a hole, then G is not chordal and in this case G is a hollowed 2-tree (by Lemma 3.18). Therefore, G has at most one hole.

Therefore we may assume that G is not biconnected. Since each block of G is $\{K_4, K_{2,3}, F_1, F_2\}$-induced-minor-free, each block of G can contain at most one hole. Suppose that G contains two holes, say C and C'. Then C and C' belong to different blocks, say B and B', respectively. Let $P = v_1, \ldots, v_n$ be a shortest path between C and C'. If $n = 1$ then F_2 appears as induced minor, a contradiction. If $n = 2$, then we consider the adjacencies between v_1 and C'. If v_1 has exactly one neighbor or exactly two neighbors in C' which are consecutive then we get F_2 as induced minor, if it has exactly two neighbors in C' which are not consecutive, we get either F_2 or $K_{2,3}$ as induced minor, and if it has 3 or more neighbors in C' we get K_4 as induced minor. If $n \geq 3$, then by minimality of the path we cannot have adjacencies between the vertices of the two cycles and internal vertices of the path, and thus we may contract $n-2$ edges of P to reduce it to the previous case. □

Lemma 3.20. *Let G be a connected K_4-minor-free graph with a cut vertex. Then, G is $\{K_{2,3}, F_1, F_2\}$-induced-minor-free if and only if every block of G is a 2-tree, except possibly one, which is a hollowed 2-tree.*

Proof. Let G be a connected K_4-minor-free graph with a cut vertex.

Suppose first that G is $\{K_{2,3}, F_1, F_2\}$-induced-minor-free. Since every block of G is $\{K_{2,3}, F_1\}$-induced-minor-free, Lemma 3.18 implies that every block of G is either a 2-tree or a hollowed 2-tree. Suppose for a contradiction that G has two distinct blocks, say B and B', that are not 2-trees. Each of these two blocks is a biconnected K_4-minor-free graph with at least two vertices. Therefore, by Lemma 3.17, neither of B and B' is chordal. By Lemma 3.19, G contains at most one hole, and therefore such a pair of blocks B and B' cannot exist.

Suppose now that every block is a 2-tree, except possibly one, which is a hollowed 2-tree. Then G has at most one hole. By Lemma 3.18, every block is $\{K_{2,3}, F_1\}$-induced-minor-free. Since any induced minor $K_{2,3}$ or F_1 can only belong to a single block, G is $\{K_{2,3}, F_1\}$-induced-minor-free. It remains to show that G is F_2-induced-minor-free. Assume by contradiction that G contains F_2 as an induced minor. Fix an induced minor model of F_2 in G, say $S_{v_1}, \ldots, S_{v_7}$, minimizing the size of the union of the S_{v_i}'s. Suppose that the two four-cycles of F_2 are induced by vertex sets $\{v_1, v_2, v_3, v_4\}$ and $\{v_4, v_5, v_6, v_7\}$. By the minimality of the model, the set $S_{v_1} \cup S_{v_2} \cup S_{v_3}$ together with a path within S_{v_4} forms a hole in G. Similarly, the sets $S_{v_5} \cup S_{v_6} \cup S_{v_7}$ together with a path within S_{v_4} form a hole in G. However, since these two holes are distinct, this contradicts the fact that G has at most one hole. □

Chapter 4

Characterizations of 1-perfectly orientable graphs within five induced-minor-closed graph classes

In this chapter we will give characterizations of 1-perfectly orientable graphs within the classes of cobipartite graphs, cographs, block-cactus graphs, K_4-minor-free graphs, and outerplanar graphs, respectively.

We show that within the class of cobipartite graphs, 1-p.o. graphs coincide with circular arc graphs. This adds to the list of the many characterizations of cobipartite circular arc graphs.

We characterize 1-p.o. cographs, obtaining characterizations both in terms of forbidden induced subgraphs and in terms of a composition theorem.

Finally, based on a reduction of the study of 1-perfectly orientable graphs to the biconnected case, we characterize, both in terms of forbidden induced minors and in terms of composition theorems, the classes of 1-perfectly orientable block-cactus graphs, 1-perfectly orientable K_4-minor-free graphs and of 1-perfectly orientable outerplanar graphs. As part of our approach, we relate the classes of graphs under consideration to the classes of 2-trees and hollowed 2-trees and two other graph classes closed under induced minors studied in the literature, namely the classes of cyclically orientable graphs and graphs of separability at most 2.

This chapter contains the main results from the following two papers.

- T. R. Hartinger and M. Milanič, *Partial Characterizations of 1-Perfectly Orientable Graphs.* J. Graph Theory. Vol. 85, 2, 2017, 378 – 394.
- B. Brešar, T. R. Hartinger, T. Kos, and M. Milanič (2016), *1-perfectly orientable K_4-minor-free and outerplanar graphs.* Submitted. arXiv:1604.04598. An extended abstract appeared in Electronic Notes in Discrete Mathematics, Vol. 54, (2016), 199 – 204.

4.1 1-perfectly orientable cobipartite graphs

The behavior of 1-p.o. graphs under the join operation motivates the study of 1-p.o. cobipartite graphs. In this section we show that a cobipartite graph is 1-p.o. if and only if it is circular arc. This equivalence will be derived using two ingredients: a necessary condition for the 1-p.o. property, which holds in general, and a characterization of cobipartite circular arc graphs due to Hell and Huang (Theorem 2.5).

We say that a chordless cycle C in a graph G is oriented *cyclically* in an orientation D of G if every vertex of the cycle has exactly one out-neighbor on the cycle (see [44, 90] for results on orientations defined in terms of this property).

Lemma 4.1. *In every 1-perfect orientation D of a 1-p.o. graph G, every chordless cycle of length at least four is oriented cyclically.*

Proof. Suppose that a chordless cycle C in G is not oriented cyclically in some 1-perfect orientation D of G. Let C' be the orientation of C induced by D. By assumption, C contains a vertex v with $d^+_{C'}(v) \neq 1$. Since $\sum_{u\in V(C)} d^+_{C'}(u) = |A(C')| = |E(C)| = |V(C)|$, it is not possible that $d^+_{C'}(u) \leq 1$ for all $u \in V(C)$, as this would imply $d^+_{C'}(v) = 0$ and consequently $\sum_{u\in V(C)} d^+_{C'}(u) < |V(C)|$. Thus, C contains a vertex v with $d^+_{C'}(v) = 2$. Since C is of length at least 4 and chordless, the out-neighborhood of v in C', and hence in D, is not a clique in G, contradicting the fact that D is a 1-perfect orientation of G. □

Recall the characterization of cobipartite circular arc graphs due to Hell and Huang described in Theorem 2.5 from Chapter 2. The characterization states that, given a graph G and a bipartition of its vertex set into cliques $\{U, U'\}$, G is circular arc if and only if it has a good coloring with respect to the bipartition. Recall that a good coloring of the crossing edges (those with one endpoint in U and one in U') with two colors is good with respect to a bipartition whenever for every induced 4-cycle of G its two crossing edges have different colors.

Theorem 4.2. *The following statements are equivalent for a cobipartite graph G:*

1. *G is 1-perfectly orientable.*
2. *G has an orientation in which every induced 4-cycle is oriented cyclically.*
3. *G is circular arc.*

Proof. As shown by Skrien [81], implication 3 ⇒ 1 holds for general (not necessarily cobipartite) graphs. Similarly, implication 1 ⇒ 2 holds in general as follows from Lemma 4.1.

It remains to prove that if G is cobipartite, then condition 2 implies condition 3. Let D be an orientation of G in which every induced 4-cycle of G is oriented cyclically. Fix a partition $\{U, U'\}$ of $V(G)$ into two cliques. We will now show that G admits a good coloring (with respect to $\{U, U'\}$), and Theorem 2.5 will imply that G is circular arc. We obtain a good coloring of G as follows: for every crossing edge e of G, we color e red if the arc of D corresponding to e goes from U to U', and blue if it goes from U' to U. To see that this is indeed a good coloring, let C be an arbitrary induced 4-cycle of G. Since C is oriented cyclically in D, out of the two crossing edges of C exactly one is oriented from U to U' in D. This implies that the two crossing edges of C will have different colors in the above coloring. It follows that the obtained coloring is a good coloring, as claimed. □

Note that Theorems 4.2 and 3.6 yield an alternative proof of Lemma 2.7.

Many characterizations of circular arc cobipartite graphs are known, including a characterization in terms of forbidden induced subgraphs due to Trotter and Moore [83] and several (at least five) others, see, e.g., [25, 59]. By Theorem 4.2, each of these yields a characterization of 1-p.o. cobipartite graphs. Theorem 4.2 can also be seen as providing further characterizations of cobipartite circular arc graphs.

The forbidden induced subgraph characterization of 1-perfectly orientable graphs within the class of complements of forests was given in [47, Theorem 15]. The characterization states that the complement of a forest is 1-perfectly orientable if and only if it is G_1-free, where G_1 is displayed in Fig. 3.1. Note that this characterization follows from Theorem 4.2 along with the characterization of cobipartite circular arc graphs in terms of forbidden induced subgraphs due to Trotter and Moore [83] (see also [28]).

4.1.1 Grid-walk graphs

In this subsection we define a family of bipartite graphs, which we name grid-walk graphs. We show that the complement of every member of this family is a circular arc graph, a result that, by Theorem 4.2, implies that the complement of each member of this family is 1-p.o..

In her PhD thesis, R. Zhang [88] introduced the notion of (k,n)-ladders, which for $k=4$ coincides with the notion of grid-walk graphs. She gave a formula for the number of spanning trees of (k,n)-ladders, generalizing the corresponding formulas for fans and ladders, and shown some problems to be NP-complete for (k,n)-ladder graphs.

A *grid-walk graph* is a graph G that can be written as the union of $n \geq 1$ four-cycles $C^1,\dots,C^n$ such that for every $i \in \{2,\dots,n\}$, cycle C^i intersects the graph $\cup_{j=1}^{i-1} C^j$ in a single edge, say e_{i-1}, of C^{i-1}. See Fig. 4.1 for an example with $n=20$.

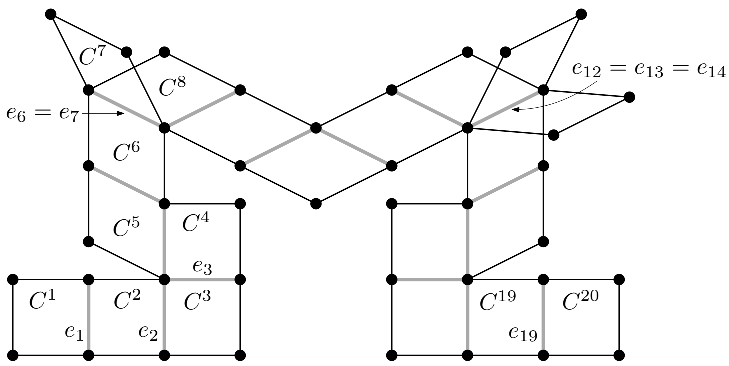


Figure 4.1: A grid-walk graph composed of twenty four-cycles. The edges $e_1,\dots,e_{19}$ are depicted grey.

The name "grid-walk graph" is motivated by the fact that every grid-walk graph can be represented by a finite walk in the infinite square grid graph. The *infinite square grid* is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{(i,j)(k,\ell) : i,j,k,\ell \in \mathbb{Z} \text{ and } |i-j|+|k-\ell| = 1\}$. And conversely, every such walk gives rise to a unique (up to isomorphism) grid-walk graph. A walk starting in the origin representing the graph depicted in Fig. 4.1 is shown in Fig. 4.2. The correspondence follows from the fact that, assuming the usual planar embedding of the grid, vertices of the square grid graph correspond bijectively to faces (all of which are four-cycles) of the dual grid, and that edges of the square grid are in a bijective correspondence with edges of the dual grid. The edges corresponding to the edges of the walk are precisely the edges $e_1,\dots,e_{n-1}$ of the corresponding grid-walk graph (as in the definition of grid-walk graphs).

To prove the following theorem, we will make use of the characterization of cobipartite circular arc graphs due to Hell and Huang (Theorem 2.5).

Theorem 4.3. *Let G be a grid-walk graph. Then, $\overline{G}$ is a cobipartite circular arc graph.*

Proof. Let G be a grid-walk graph. By definition, G can be written as the union of $n \geq 1$ four-cycles $C^1,\dots,C^n$ such that for every $i \in \{2,\dots,n\}$, cycle C^i intersects the graph $\cup_{j=1}^{i-1} C^j$ in a single edge, say e_{i-1}, of C^{i-1}. For $1 \leq i \leq n$, let G_i denote the graph $\cup_{j=1}^{i} C^j$.

A straightforward inductive argument on n shows that G is bipartite, which implies that $\overline{G}$ is cobipartite. Fix a bipartition $\{U,U'\}$ of its vertex set into two cliques. By Theorem 2.5 it suffices to prove that $\overline{G}$ has a good coloring with respect to $\{U,U'\}$. We will prove a slightly stronger statement. For every $i \in \{2,\dots,n\}$, let u_i, u_i' be the two (adjacent) vertices of cycle C^i that are not in G_{i-1} and such that $u_i \in U$ and $u_i' \in U'$ (in particular, $V(C^i) = e_{i-1} \cup \{u_i, u_i'\}$). Then, we can write $U = \{u_0,u_1,\dots,u_n\}$ and $U' = \{u_0',u_1',\dots,u_n'\}$ so that for every $i \in \{0,1,\dots,n\}$,

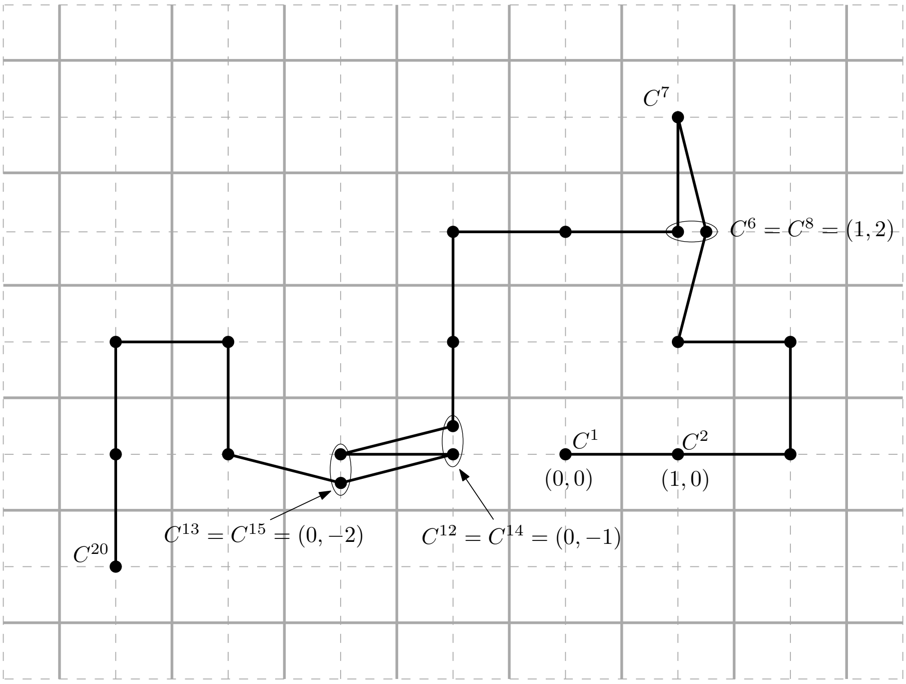


Figure 4.2: A walk in the square grid corresponding to the grid-walk graph depicted in Fig. 4.1. For reasons of clarity, pairs of points that repeat in the walk are shown with displacement.

vertices u_i and u_i' are adjacent in G_i and the graph $\overline{G_i}$ is cobipartite, with a bipartition $\{U_i, U_i'\}$ of its vertex set into two cliques, where $U_i = \{u_0, u_1, \ldots, u_i\}$ and $U_i' = \{u_0', u_1', \ldots, u_i'\}$. The statement that we will prove is the following:

For every $i \in \{1, \ldots, n\}$, the graph $\overline{G_i}$ has a good coloring with respect to $\{U_i, U_i'\}$ such that

($*$) *a crossing edge $u_j u_k'$ of $\overline{G_i}$ (with $0 \leq j, k \leq i$ and $j \neq k$) is colored red if $j < k$ and blue if $j > k$.*

We prove the above claim by induction on i. If $i = 1$, then $\overline{G_i} \cong 2K_2$, and coloring the edge $u_0 u_1'$ red and the edge $u_1 u_0'$ blue yields the desired coloring. Now, let $i > 1$ and let c_{i-1} be a good coloring of $\overline{G_{i-1}}$ with respect to $\{U_{i-1}, U_{i-1}'\}$ satisfying condition ($*$) (with $i-1$ in place of i). Since $U_i = U_{i-1} \cup \{u_i\}$ and $U_i' = U_{i-1}' \cup \{u_i'\}$, we can extend this coloring to a coloring c_i of the crossing edges of $\overline{G_i}$ (with respect to $\{U_i, U_i'\}$) by setting, for every crossing edge $u_j u_k'$ of $\overline{G_i}$:

$$c_i(u_j u_k') = \begin{cases} c_{i-1}(u_j u_k'), & \text{if } j \neq i \text{ and } k \neq i; \\ \text{blue}, & \text{if } j = i; \\ \text{red}, & \text{if } k = i. \end{cases}$$

Note that the mapping c_i is well-defined since vertices u_i and u_i' are non-adjacent in $\overline{G_i}$. Clearly, since c_{i-1} satisfies condition ($*$), so does c_i. Therefore, it remains to show that c_i is a good coloring, that is, that for every induced 4-cycle in $\overline{G_i}$, the two crossing edges in it are of the opposite color.

Suppose for a contradiction that there is an induced 4-cycle, say C, of $\overline{G_i}$ in which the two crossing edges are of the same color. Since the coloring c_{i-1} was good in $\overline{G_{i-1}}$, cycle C must contain at least one of the vertices u_i and u_i'. Without loss of generality, we may assume that it contains u_i. Moreover, since all the crossing edges incident with u_i are colored blue by c_i, we infer that the two crossing edges of C are colored blue. Since all the crossing edges incident with u_i' are colored red by c_i, this implies that $u_i' \notin V(C)$. Writing $V(C) = \{u_i, u_j, u_k', u_\ell'\}$ where

$E(C) = \{u_iu_j, u_ju'_k, u'_ku'_\ell, u'_\ell u_i\}$, we thus have $0 \leq j, k, \ell \leq i-1$. Vertex u'_k is non-adjacent to u_i in C (and thus also not in $\overline{G_i}$). Since in graph G_i, vertex u_i has a unique neighbor other than u'_i, this unique neighbor must be u'_k. It follows that u'_k is a common vertex of cycles C^{i-1} and C^i; in particular, u'_k is adjacent to u_{i-1} in the graph G_i. Since the edge $u_ju'_k$ is a crossing edge of C, it is colored blue, which implies that $j > k$. If $j = i-1$, then vertex $u_j = u_{i-1}$ belongs to cycle C^{i-1} and is therefore adjacent to $u'_k \in V(C^{i-1})$ in G_i, contrary to the fact that u_j and u'_k are adjacent in $\overline{G_i}$. Therefore $k < j < i-1$. Since $j \notin \{k, i-1\}$ and vertices u'_k and u'_{i-1} are the only neighbors of u_{i-1} in the graph G_{i-1}, vertex u_{i-1} is adjacent to vertex u'_j in the graph $\overline{G_{i-1}}$; moreover, since $j < i-1$, the edge $u_{i-1}u'_j$ is colored blue by c_i and therefore by c_{i-1}. It follows that the subgraph of $\overline{G_{i-1}}$ induced by $\{u_{i-1}, u_j, u'_k, u'_j\}$ is a 4-cycle in which the two crossing edges are of the same color, contrary to the inductive hypothesis. This completes the proof. □

4.2 1-perfectly orientable cographs

We now derive a characterization of 1-p.o. cographs, obtaining characterizations both in terms of forbidden induced subgraphs and in terms of structural properties. Recall that cographs can be characterized in terms of forbidden induced subgraphs by a single obstruction, namely the 4-vertex path P_4. Alternatively, the class of *cographs* can be defined recursively by stating that K_1 is a cograph, the disjoint union of two cographs is a cograph, the join of two cographs is a cograph, and there are no other cographs.

Theorem 4.4. *For every cograph G, the following conditions are equivalent:*

1. *G is 1-perfectly orientable.*
2. *G is $K_{2,3}$-free.*
3. *One of the following conditions holds:*
   - *$G \cong K_1$.*
   - *$G \cong \overline{mK_2}$ for some $m \geq 2$.*
   - *G is the disjoint union of two smaller 1-p.o. cographs.*
   - *G is obtained from a 1-p.o. cograph by adding to it a universal vertex.*
   - *G is obtained from a 1-p.o. cograph by adding to it a true twin.*

Proof. The implication $1 \Rightarrow 2$ follows from Theorems 3.5 and 3.9.

To show the implication $2 \Rightarrow 3$, suppose that G is a $K_{2,3}$-free cograph on at least two vertices that is not disconnected and does not have a universal vertex or a pair of true twins. We want to show that $G = \overline{mK_2}$. Since G is not disconnected and $G \neq K_1$, its complement $\overline{G}$ is disconnected. Let $m \geq 2$ denote the number of *co-components* of G (subgraphs of G induced by the vertex sets of components of $\overline{G}$). If one of the co-components has exactly one vertex, then that vertex is universal in G, which is a contradiction. Therefore, each co-components has at least two vertices. The recursive structure of cographs implies that each co-component of G is disconnected. In particular, it has independence number at least 2. On the other hand, since G is $K_{2,3}$-free, each co-component of G has independence number at most 2. This implies that each co-component is the disjoint union of two complete graphs. Since G has no true twins, each co-component is isomorphic to $2K_1$, that is, $G \cong \overline{mK_2}$ for some $m \geq 2$, as claimed.

Finally, we show the implication $3 \Rightarrow 1$. Suppose that G is a cograph such that one of the five conditions in item 3 holds. An inductive argument shows that G is 1-p.o., using Theorem 3.5

and the fact that K_1 and all graphs of the form $\overline{mK_2}$ are 1-p.o. (which follows, e.g., from Corollary 3.8). $\square$

As a consequence from Theorem 4.4, we obtain the following result.

Corollary 4.5. *Let G be a graph. Then, the following conditions are equivalent.*

(a) G is a 1-perfectly orientable cograph.

(b) G is $\{K_{2,3}, P_4\}$-induced-minor-free.

(c) G is $\{K_{2,3}, P_4\}$-free.

Proof. The implication $(b) \Rightarrow (c)$ is trivial, and the fact that (c) implies (a) follows from Theorem 4.4. Implication $(a) \Rightarrow (b)$ follows from the fact 1-p.o. graphs and cographs are closed under induced minors, $K_{2,3}$ is not 1-p.o. and P_4 is not a cograph. $\square$

4.3 1-perfectly orientable block-cactus graphs

In this section we derive a characterization of 1-p.o. graphs within the class of block-cactus graphs. A *block-cactus graph* is a graph such that all its blocks are either cycles or complete graphs. It is not difficult to see that the class or block-cactus graphs is induced-minor-closed, and thus, can be characterized in terms of minimal forbidden induced minors. Theorem 4.6 states such a characterization. The *diamond* graph consists of the complete graph K_4 minus an edge.

Very recently, Kamiński and Raymond [54] characterized the connected graphs that cannot be contracted to a diamond as exactly the connected block-cactus graphs (which the authors refer to as connected clique-cactus graphs). The reverse implication of the following equivalence could also be derived from their characterization, but we here give an independent and shorter proof.

Theorem 4.6. *A graph G is a block-cactus graph if and only if G is diamond-induced-minor-free.*

Proof. Let G be a block-cactus graph. Since block-cactus graphs are induced-minor-free, and the diamond is not a block-cactus graph (it is 2-connected and is neither a complete graph nor an induced cycle), G must be diamond-induced-minor-free.

Suppose now that G is a diamond-induced-minor-free graph, and suppose that G has a block B that is not complete and not an induced cycle. Then, there exist two vertices $x, y \in B$ such that $\{x, y\} \notin E(G)$. Since x and y belong to the same block, there exist two vertex-disjoint x, y-paths. Let us consider the two such paths minimizing their total length, say P^1 and P^2. If any vertex of P^1 is adjacent to any vertex of P^2, we would immediately obtain a diamond as induced minor, a contradiction. So we may assume that x and y together with P^1 and P^2 form an induced cycle in B, say C.

Since B is not a cycle, there must exist some vertex z in $B \setminus V(C)$. Suppose that z has exactly two neighbors in C, say v_1 and v_2. If $\{v_1, v_2\} \in E(G)$, then, since C contains at least two more vertices, we obtain a diamond as induced minor by a suitable contraction of edges from the v_1, v_2-path which is not the edge v_1, v_2. If $\{v_1, v_2\} \notin E(G)$, then the two v_1, v_2-paths in C each contain at least one more vertex. Thus, a suitable contraction of edges in each of the paths would result in a diamond as induced minor. Suppose now that z has at least three neighbors in C, and denote them by $v_1, v_2, v_3, \ldots, v_\ell$ ($\ell \geq 3$). Since C has at least 4 vertices, there exist two neighbors of z in C that are non-adjacent. We may assume without loss of generality that $\{v_1, v_3\} \notin E(G)$ and denote by v_2 the neighbor of z between them in C. But then, by contracting some edges in the v_1, v_3-path in C that passes through v_2 and deleting the other vertices of C, we obtain a diamond as induced minor.

Thus, we may assume that every vertex outside of C has at most one neighbor in C. Consider then a vertex z outside of C with its neighbor v in C minimizing the length of a shortest path P between z and C not going through v. Note that we may assume that v has no other neighbor in P, since we could then select another vertex instead of z which would give a shorter path than P. Let v' be the vertex in C such that P is a z, v'-path, and z' its neighbor in P. Using an analogous argument we may assume that v' has no other neighbor in P. Note that the minimality of P also implies that the internal vertices of P have no neighbors in C.

Therefore, we may contract some edges in P, and, possibly, in the two vertex disjoint v, v'-paths in C to obtain a diamond as induced minor.

We showed then that every block of G is either complete or a cycle, and so G is a block-cactus graph. □

As a consequence of the previous result and Theorem 3.13, we obtain the following characterization of 1-perfectly orientable block-cactus graphs. A *rooted extension of a graph G* is a rooted graph G^v for any $v \in V(G)$.

Proposition 4.7. *Let G be a connected block-cactus graph. Then, the following statements are equivalent:*

1. *G is 1-perfectly orientable.*
2. *At most one block of G is not complete.*
3. *G is F_2-induced-minor-free (see Fig. 3.2).*

Proof. (1. $\Leftrightarrow$ 2.) If G is biconnected, then G has only one block and both conditions 1. and 2. can be seen to hold. Suppose now that G has a cut-vertex. Graph G has blocks of two types: blocks that are complete – for which every rooted extension is 1-perfectly orientable (by Lemma 3.14) – and blocks that are not complete – which are cycles of length at least four, and for which, by Lemma 4.1, no rooted extension is 1-perfectly orientable. The two conditions from Theorem 3.13 are now easily seen to be equivalent to the following two conditions, respectively: (i) there exists a block B of G such that all blocks of G other than B are complete, and (ii) all blocks of G are complete. By Theorem 3.13 G is 1-p.o. if and only if one of conditions (i) and (ii) holds, which is in turn equivalent to condition 2.

(1. $\Rightarrow$ 3.) Since G is 1-p.o., Theorem 3.9 implies that G is F_2-induced-minor-free.

(3. $\Rightarrow$ 2.) Let G be an F_2-induced-minor-free graph. Since G is block-cactus, its blocks are complete graphs or cycles. Suppose by contradiction that there exist two blocks of G, say B and B', that are not complete (and thus, are cycles of length at least 4). By contracting each of these cycles to a C_4, contracting the edges in a shortest path between B and B', and deleting all vertices not in the path or in either of B and B', we would obtain F_2 as an induced minor, a contradiction. □

The following corollary is an immediate consequence of Theorem 4.6 and Proposition 4.7.

Corollary 4.8. *A graph G is a 1-perfectly orientable block-cactus graph if and only if it is $\{$diamond$, F_2\}$-induced-minor-free.*

4.4 1-perfectly orientable K_4-minor-free graphs

In this section we develop a structural characterization of 1-perfectly orientable graphs within the class of K_4-minor-free graphs. Since the class of K_4-minor-free graphs contains the class of outerplanar graphs, this will imply a structural characterization of 1-perfectly orientable outerplanar graphs (developed in Section 4.5).

We first characterize the biconnected case and then apply Theorem 3.13 to characterize the general case.

4.4.1 The biconnected case

To apply Theorem 3.13, we need to understand both biconnected 1-perfectly orientable K_4-minor-free graphs and biconnected 1-perfectly orientable K_4-minor-free rooted graphs. Both characterizations are easy to obtain using results from Chapter 3.

Lemma 4.9. *For a biconnected K_4-minor-free graph G, the following statements are equivalent:*

1. *G is 1-perfectly orientable.*
2. *G is $\{K_{2,3}, F_1\}$-induced-minor-free.*
3. *G is either K_1, a 2-tree, or a hollowed 2-tree.*
4. *G is either K_1, K_2, or can be constructed from a cycle by a sequence of additions of simplicial vertices of degree 2.*
5. *G is either K_1, K_2, or has a sink-free 1-perfect orientation.*

Proof. The implication $1 \Rightarrow 2$ follows from Theorem 3.9, Lemma 3.18 yields the equivalence $2 \Leftrightarrow 3$. The equivalence between statements 3 and 4 follows immediately from the definitions of 2-trees and hollowed 2-trees. The implication $5 \Rightarrow 1$ is clear.

To complete the proof, we show the implication $3 \Rightarrow 5$. Suppose that G is either K_1, a 2-tree, or a hollowed 2-tree. If G is K_1 or K_2, then there is nothing to prove. Therefore, G is either a cycle or is obtained from a 2-connected possibly hollowed) 2-tree by adding to it a simplicial vertex of degree 2. We prove that G has a sink-free 1-perfect orientation by induction on $|V(G)|$. If G is a cycle, then G has a sink-free 1-perfect orientation. If G is obtained from a 2-connected (possibly hollowed) 2-tree G' by adding to it a simplicial vertex, say v, of degree 2, then the inductive hypothesis implies that G' has a sink-free 1-perfect orientation, say D'. Extending D' by orienting the two arcs incident with v away from v yields a sink-free 1-perfect orientation of G, as claimed. □

Lemma 4.10. *For a biconnected K_4-minor-free graph G, the following statements are equivalent:*

1. *Some rooted extension of G is 1-perfectly orientable.*
2. *All rooted extensions of G are 1-perfectly orientable.*
3. *G is chordal.*
4. *G is either K_1 or a 2-tree.*

Proof. First, we show the implication 1 ⇒ 4. Suppose that some rooted extension of a biconnected K_4-minor-free graph G is 1-perfectly orientable. In particular, G is 1-perfectly orientable, and hence $\{K_{2,3}, F_1\}$-induced-minor-free by Theorem 3.9. By Lemma 3.18, G is either K_1, a 2-tree, or a hollowed 2-tree. By Lemma 3.16, G cannot be a hollowed 2-tree, and hence condition 4 holds.

Implication 4 ⇒ 3 is clear, implication 3 ⇒ 2 follows from Lemma 3.14, and implication 2 ⇒ 1 is trivial. □

Corollary 4.11. *For a biconnected K_4-minor-free graph G and $v \in V(G)$, the rooted graph G^v is 1-perfectly orientable if and only if G is either K_1 or a 2-tree.*

4.4.2 The general case

Now we have all the ingredients ready to complete the characterization of 1-perfectly orientable K_4-minor-free graphs. In Theorem 4.12 we will use the following two operations:

- (A_1): attach a simplicial vertex of degree 1.
- (A_2): attach a simplicial vertex of degree 2 (that is, add a new vertex and connect it by an edge to exactly two vertices of the graph, which are adjacent to each other).

Fig. 4.3 shows an example of a graph constructed starting from C_6 and using a sequence of operations (A_1) and (A_2).

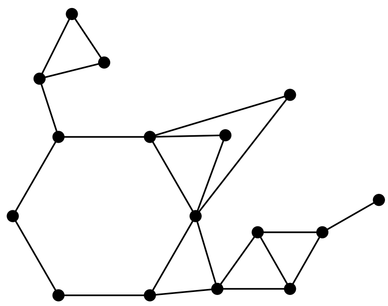


Figure 4.3

Theorem 4.12. *Let G be a connected K_4-minor-free graph. Then the following statements are equivalent:*

1. *G is 1-perfectly orientable.*
2. *G is $\{K_{2,3}, F_1, F_2\}$-induced-minor-free.*
3. *Every block of G is a 2-tree, except possibly one, which is either K_1 or a hollowed 2-tree.*
4. *G can be constructed from either K_1 or a cycle by a sequence of operations (A_1) and (A_2).*

Proof. Suppose first that G is biconnected. By Theorem 3.9, condition 1 implies condition 2. By Lemma 4.9, condition 2 implies condition 3, and condition 3 implies condition 4. Suppose now that G can be constructed from either K_1 or a cycle by a sequence of operations (A_1) and (A_2). Since G is biconnected, we may assume that operation (A_1) was never used in the sequence, unless G is isomorphic to K_2. Therefore, G is either K_1, K_2, or can be constructed from a cycle by a sequence of operations (A_2). By Lemma 4.9, this implies condition 1.

We are left with the case when G has a cut vertex. In this case, we first establish the equivalence of conditions 1 and 3. Let T be the block tree of G. By Theorem 3.13, G is 1-perfectly orientable if and only if one of the following conditions holds:

- There exists a block B_r of G such that B_r is 1-perfectly orientable and for every arc $(B, v) \in \mathcal{B} \times C$ of the B_r-rooted orientation of T, the rooted graph B^v is 1-perfectly orientable.

- There exists a cut vertex v_r of G such that for every arc $(B, v) \in \mathcal{B} \times C$ of the v_r-rooted orientation of T, the rooted graph B^v is 1-perfectly orientable.

Since each block of G is a biconnected K_4-minor-free graph, Lemma 4.9 and Corollary 4.11 imply that the above two conditions are equivalent, respectively, to the following two:

- There exists a block B_r of G such that B_r is either a 2-tree or a hollowed 2-tree, and for every arc $(B, v) \in \mathcal{B} \times C$ of the B_r-rooted orientation of T, the graph B is a 2-tree.

- There exists a cut vertex v_r of G such that for every arc $(B, v) \in \mathcal{B} \times C$ of the v_r-rooted orientation of T, the graph B is a 2-tree.

Since the only sink of a w-rooted orientation of a tree T' (with $w \in V(T')$) is w, the two conditions can be further simplified as follows:

- There exists a block B_r of G such that B_r is either a 2-tree or a hollowed 2-tree, and every other block $B \neq B_r$ is a 2-tree.

- All blocks of G are 2-trees.

Clearly, one of these two conditions holds if and only if condition 3 holds. This establishes the equivalence of conditions 1 and 3.

The equivalence of conditions 2 and 3 follows from Lemma 3.20. The implication $3 \Rightarrow 4$ can be proved by induction on $|V(G)|$, as follows. If $|V(G)| = 1$, then G is isomorphic to K_1 and we are done. Otherwise, G has an end block B that is not a hollowed 2-tree. Let v be the cut vertex of G contained in B. By induction, the graph $G' = G - (V(B) \setminus \{v\})$ can be constructed from either K_1 or a cycle by a sequence of operations (A_1) and (A_2). Such a sequence can be extended with an operation of the form (A_1) (resulting in a simplicial vertex w with a unique neighbor v) to create a new block corresponding to B and then with a sequence of operations of the form (A_2) to grow B out of the edge $\{v, w\}$. The implication $4 \Rightarrow 1$ can also be proved by induction on the number of vertices, using the fact that K_1 and cycles are 1-perfectly orientable and that a 1-perfect orientation of a graph G can be extended to a 1-perfect orientation of a graph obtained from G by adding to it a simplicial vertex v by orienting the edges incident with v away from v. □

Recall that a graph is said to be cyclically orientable if it admits an orientation in which every chordless cycle is oriented cyclically. As a consequence of Theorem 4.12 we obtain the following result.

Corollary 4.13. *For every graph G, the following statements are equivalent:*

1. *G is 1-perfectly orientable and K_4-minor-free.*

2. *G is 1-perfectly orientable and cyclically orientable.*

3. *G is $\{K_4, K_{2,3}, F_1, F_2\}$-induced-minor-free.*

Proof. Since each of the three properties are closed under taking components and disjoint union, we may assume that G is connected. The equivalence $1 \Leftrightarrow 3$ is then an immediate consequence of Theorem 4.12. The implication $2 \Rightarrow 3$ follows from Theorems 3.9 and 2.12. The implication $3 \Rightarrow 2$ follows from Theorems 4.12 and 3.9. □

4.5 1-perfectly orientable outerplanar graphs

Since every outerplanar graph is K_4-minor-free, we can derive from Theorem 4.12 a characterization of 1-perfectly orientable outerplanar graphs. In Theorem 4.14 we will use the following two operations:

- (A_1) attach a simplicial vertex of degree 1.
- (A_2') attach a simplicial vertex of degree 2 to adjacent vertices v and w where the edge vw lies in at most one induced cycle.

Note that the example in Fig. 4.3 is not constructed starting from C_6 using a sequence of operations (A_1) and $(A_2)'$ since there exists an edge which lies in the starting induced 6-cycle and two other induced 3-cycles, which means that at some point we must have attached a simplicial vertex of degree 2 to adjacent vertices v and w where the edge vw lies in more than one induced cycle.

Theorem 4.14. *For a connected outerplanar graph G, the following statements are equivalent:*

1. *G is 1-perfectly orientable.*
2. *G is $\{K_{2,3}, F_1, F_2\}$-induced-minor-free.*
3. *Every block of G is a 2-tree, except possibly one, which is either K_1 or a hollowed 2-tree.*
4. *G can be constructed from either K_1 or a cycle by a sequence of operations (A_1) and (A_2').*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ as well as the implication $4 \Rightarrow 1$ follow from Theorem 4.12. From Theorem 4.12 we also know that if one of conditions 1, 2, or 3 holds, then G can be constructed from either K_1 or a cycle by a sequence of operations (A_1) and (A_2). Suppose that, when using the operation (A_2) to add a simplicial vertex u with neighbors v and w, the edge vw already lies in two (distinct) induced cycles, say C and C'. First, we claim that C and C' intersect in a path (which contains the edge vw). Suppose that this is not the case. Then the intersection of C and C' consist of at least two components, each of which is a path. Let x be an endvertex of one of these path components, and let P be the path in C' with x as an endvertex, whose internal vertices and edges are not in C, and the other endvertex is $y \in V(C) \cap V(C')$. Now, it is easy to see that the subgraph induced by $V(C) \cup V(P)$ contains $K_{2,3}$ as a minor; this implies that the graph is not outerplanar, and since this property is preserved in further steps of the procedure, this contradicts the assumption that G is outerplanar. Thus C and C' intersect in a path, which contains vw. If this path contains other vertices than v and w, then one can again easily derive that $K_{2,3}$ appears as a minor, contradicting outerplanarity of G. Hence C and C' intersect exactly in the subgraph K_2 formed by v and w. Then, after applying the operation (A_2) of adding the vertex u as the neighbor of v and w, we infer that the sets $\{u\}$, $\{v\}$, $\{w\}$, $V(C) \setminus \{v, w\}$, and $V(C') \setminus \{v, w\}$ form an induced minor model of $K_{2,3}^+$. Hence, in this case the obtained graph would not be outerplanar, and it would remain non-outerplanar until the end of the procedure. Therefore, we deduce that in each step of the construction that uses (A_2), in fact an operation is of the form (A_2'). This proves the implication $1 \Rightarrow 4$. □

As a consequence of Proposition 2.13 and Theorem 4.14 we obtain the following result.

Corollary 4.15. *For every graph G, the following statements are equivalent:*

1. *G is 1-perfectly orientable and outerplanar.*
2. *G is $\{K_4, K_{2,3}, K_{2,3}^+, F_1, F_2\}$-induced-minor-free.*

Proof. Let G be a 1-perfectly orientable outerplanar graph. Then G is $\{K_4, K_{2,3}, K_{2,3}^+\}$-induced-minor-free since G is outerplanar, and $\{F_1, F_2\}$-induced-minor-free since it is 1-perfectly orientable. Conversely, if G is $\{K_4, K_{2,3}, K_{2,3}^+, F_1, F_2\}$-induced-minor-free then G is outerplanar by Proposition 2.13. By Theorem 4.14, G is also 1-perfectly orientable. $\square$

Chapter 5

1-perfectly orientable graphs and graph products

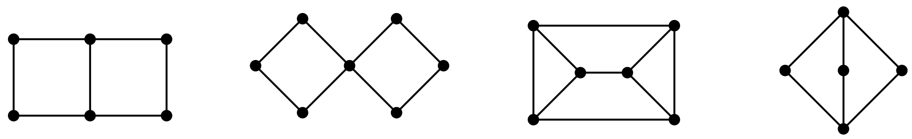
In this chapter we consider the four standard graph products: the Cartesian product, the strong product, the direct product, and the lexicographic product. For each of these four products, we completely characterize when a nontrivial product of two graphs G and H is 1-p.o. While the results for the Cartesian and the lexicographic products turn out to be rather straightforward, the characterizations for the cases of the direct and the strong product are more involved. Some common features of the structure of the factors involved in the characterization can be described as follows. In the cases of the Cartesian and the direct product the factors turn out to be very sparse and very restricted, always having components with at most one cycle. In the case of the lexicographic and of the strong product the factors can be dense. More specifically, cobipartite 1-p.o. graphs, including co-chain graphs in the case of strong products, play an important role in these characterizations. The case of the strong product also leads to a new infinite family of 1-p.o. graphs.

This chapter contains results from the following paper.

- T. R. Hartinger, and M. Milanič, 1-*perfectly orientable graphs and graph products.* Discrete Mathematics, 340 (2017), 1727 – 1737.

In the previous two chapters, several results about 1-p.o. graphs were proved. We now restate some of them for later use.

Proposition 5.1. *No graph in the set* $\{F_1, F_2, F_3, F_4\}$ *(see Fig 5.1) is* 1*-perfectly orientable.*



F_1 F_2 $F_3 = \overline{C_6}$ $F_4 = K_{2,3}$

Figure 5.1: Four small non-1-p.o. graphs.

Note that as a consequence from Theorem 3.5, we obtain the next result.

Proposition 5.2. *If* G *is* 1*-p.o. and* H *is an induced minor of* G*, then* H *is* 1*-p.o.*

Theorem 3.5 and Proposition 5.2 imply the following.

Corollary 5.3. *A graph G is* 1*-p.o. if and only if each component of G is* 1*-p.o.*

And from Propositions 5.1 and 5.2 we obtain the following.

Corollary 5.4. *Let G be a graph such that some graph F_i (with $1 \leq i \leq 4$, see Figure 5.1) is an induced minor of it. Then G is not* 1*-p.o.*

5.1 The Cartesian product

We start with a characterization of nontrivial Cartesian product graphs that are 1-p.o.

The product $G \Box H$ is connected if and only if both factors are connected (see [45, Corollary 5.3]). More precisely, if G has components $G_1, \ldots, G_k$ and H has components $H_1, \ldots, H_\ell$, then the components of $G \Box H$ are exactly $G_i \Box H_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$. Therefore, since the study of 1-p.o. graphs reduces to the connected case (by Corollary 5.3), no generality is lost in characterizing nontrivial Cartesian product graphs that are 1-p.o. only among the connected graphs (equivalently, only among the products having connected factors). For the proof, let us note that $P_3 \Box K_2$ is isomorphic to the *domino* (graph F_1 in Figure 5.1), and $K_3 \Box K_2$ is isomorphic to $\overline{C_6}$ (graph F_3 in Figure 5.1).

Theorem 5.5. *A nontrivial Cartesian product, $G \Box H$, of two connected graphs G and H is* 1*-p.o. if and only if $G \cong H \cong K_2$.*

Proof. If each of G and H is isomorphic to K_2, then $G \Box H$ is isomorphic to C_4 and thus 1-p.o. (the cyclic orientation of the C_4 is 1-perfect).

Conversely, suppose that $G \Box H$ is 1-p.o. and that one of G and H, say G, is not isomorphic to K_2. Since both G and H are induced subgraphs of $G \Box H$, they are both 1-p.o. (by Corollary 5.2). Since G and H are connected graphs on at least two vertices, each contains an edge. Moreover, G contains P_3 as a (not necessarily induced) subgraph. If G contains an induced P_3, then $G \Box H$ contains an induced domino, and is therefore not 1-p.o. by Corollary 5.4. Similarly, if G contains an induced K_3, then $G \Box H$ contains an induced copy of $K_3 \Box K_2 \cong \overline{C_6}$, and is therefore not 1-p.o., again by Corollary 5.4. In either case, we reach a contradiction. □

We now state the theorem for the general case, the proof of which follows easily from the connected case. For a positive integer k, we say that a *k-linear forest* is a disjoint union of paths each having at most k vertices. In particular, 1-linear forest are exactly the edgeless graphs, and 2-linear forests are exactly the graphs consisting only of isolated vertices and isolated edges.

Theorem 5.6. *A nontrivial Cartesian product, $G \Box H$, of two graphs G and H is* 1*-perfectly orientable if and only if one of the following conditions holds:*

(i) G is edgeless and H is 1*-perfectly orientable, or vice versa.*

(ii) G and H are 2*-linear forests.*

Proof. If $G \Box H$ is 1-p.o., then each of G and H is 1-p.o. as they are induced subgraphs of the product (by Corollary 5.2). Suppose that neither of G and H is edgeless. Then they both contain an induced K_2 and, as a consequence of Theorem 5.5, each component of either of them is either a K_1 or a K_2. Hence both are 2-linear forests.

Now, if G is edgeless and H is 1-p.o., the product $G \Box H$ consists of the disjoint union of copies of H, which is 1-p.o. due to Corollary 5.3. If both G and H are 2-linear forests, it follows from Theorem 5.5 that each component of $G \Box H$ is 1-p.o., and applying Corollary 5.3 we conclude that $G \Box H$ is 1-p.o. □

5.2 The lexicographic product

In this section, we characterize nontrivial lexicographic product graphs that are 1-p.o.

By [45, Corollary 5.14], the lexicographic product $G[H]$ of two nontrivial graphs is connected if and only if G is connected. In particular, if G has components $G_1, \ldots, G_k$, then the components of $G[H]$ are $G_1[H], \ldots, G_k[H]$. Therefore, when characterizing nontrivial lexicographic product graphs that are 1-p.o., we may without loss of generality restrict our attention to the case of nontrivial products $G[H]$ such that G is connected. The following theorem states the corresponding characterization.

Theorem 5.7. *A nontrivial lexicographic product, $G[H]$, of two graphs G and H such that G is connected is 1-p.o. if and only if one of the following conditions holds:*

(i) G is 1-p.o. and H is complete.

(ii) G is complete and H is a cobipartite 1-p.o. graph.

Proof. Suppose first that $G[H]$ is 1-p.o. Then, both G and H are 1-p.o. since they are induced subgraphs of $G[H]$. Suppose for a contradiction that none of conditions (i) and (ii) holds. Since G is a nontrivial connected graph, it has an edge, and, since (i) fails, H is not complete. We infer that $K_2[H]$ is an induced subgraph of $G[H]$ isomorphic to the join of two copies of H. Consequently, $H * H$ is 1-p.o. By Theorem 3.6 we obtain that H is cobipartite. Therefore, since we assume that (ii) fails, G is not complete. In particular, there exists an induced P_3 in G. Since H contains an induced $2K_1$ and $P_3[2K_1] \cong K_{2,4}$, we obtain that $G[H]$ contains $K_{2,3}$ as an induced subgraph, and by Corollary 5.4 it cannot be 1-p.o., a contradiction.

For the converse direction, we will show that in any of the two cases (i) and (ii), the graph $G[H]$ is 1-p.o. If G is 1-p.o. and H is complete, then the product $G[H]$ is isomorphic to the graph obtained by repeatedly substituting a vertex of G with a complete graph. Substituting a vertex v with a complete graph is the same as adding a sequence of true twins to vertex v, which by Theorem 3.5 results in a 1-p.o. graph. It follows that $G[H]$ is 1-p.o. If G is complete and H is a cobipartite 1-p.o. graph, then an inductive argument on the order of G together with the fact that cobipartite 1-p.o. graphs are closed under join (Theorem 3.6) shows that $G[H]$ is 1-p.o. □

The characterization for the general case follows from the previous theorem.

Theorem 5.8. *A nontrivial lexicographic product, $G[H]$, of two graphs G and H is 1-perfectly orientable if and only if one of the following conditions holds:*

(i) G is edgeless and H is 1-perfectly orientable.

(ii) G is 1-perfectly orientable and H is complete.

(iii) Every component of G is complete and H is a cobipartite 1-p.o. graph.

Proof. Suppose first that $G[H]$ is 1-p.o. Since each of G and H is an induced subgraph of the product, they are 1-p.o. (by Corollary 5.2). Suppose now that G is not edgeless and H is not complete (otherwise we are in cases (i) or (ii)). Let G_i be an arbitrary component of G. By Theorem 5.7 applied to the graph $G_i[H]$, which is 1-p.o., since H is not complete, we infer that G_i is complete and H is a cobipartite 1-p.o. graph. Thus, each component of G is complete and we are in case (iii).

Let us now prove the converse implication. If G is edgeless and H is 1-p.o., the product $G[H]$ consists of the disjoint union of copies of H, and is 1-p.o. due to Corollary 5.3. If G is 1-p.o. and

H is complete, it follows from Theorem 5.7 and Corollary 5.3 that $G[H]$ is 1-p.o. Finally, if every component of G is complete and H is a cobipartite 1-p.o. graph, applying Theorem 5.7 and Corollary 5.3 we conclude that $G[H]$ is 1-p.o. $\square$

5.3 The direct product

In this section, we characterize nontrivial direct product graphs that are 1-p.o.

If the direct product $G \times H$ is connected, then both factors are connected, however the converse is generally not true. (For example, if $G \cong H \cong K_2$, then $G \times H \cong 2K_2$ is disconnected.) By [45, Corollary 5.10], the direct product of two connected nontrivial graphs is connected if and only if at most one of the factors is bipartite. If G has components $G_1, \ldots, G_k$ and H has components $H_1, \ldots, H_\ell$, then $G \times H$ is the disjoint union of the products of the components, $G_i \times H_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$. It follows that, in order to characterize nontrivial direct product graphs that are 1-p.o., we may without loss of generality restrict our attention to the case of nontrivial products in which both factors are connected.

We start with some necessary conditions for the direct product of two graphs to be 1-p.o. We say that a graph is *triangle-free* if it is C_3-free.

Lemma 5.9. *Suppose that the direct product of two connected graphs G and H is* 1*-p.o. Then:*

1. *If one of G and H contains an induced P_3 or C_3, then the other one is $\{claw, C_3, C_4, C_5, P_5\}$-free.*

2. *At least one of G and H is triangle-free.*

3. *At least one of G and H is P_4-free.*

Proof. As we can see in Figures 5.2, 5.3, and 5.4 below, each of $P_3 \times claw$, $P_3 \times C_4$, and $C_3 \times claw$ contains an induced $K_{2,3}$, each of $P_3 \times C_3$, $P_3 \times C_5$, and $P_3 \times P_5$ contains an induced F_2, the graph $C_3 \times C_3$ contains an induced $F_3 = \overline{C_6}$, and $P_4 \times P_4$ contains an induced domino (F_1).

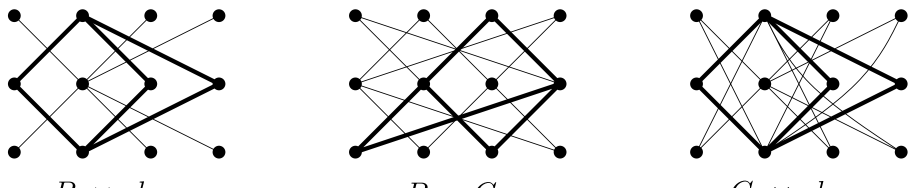


Figure 5.2: $K_{2,3}$ as induced subgraph of $P_3 \times claw$, $P_3 \times C_4$, and $C_3 \times claw$.

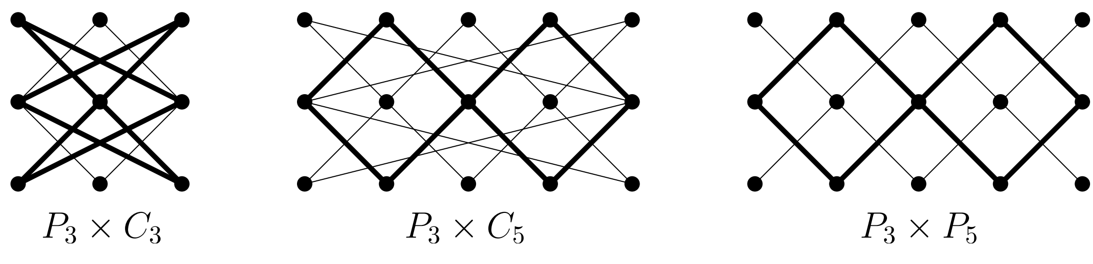


Figure 5.3: F_2 as induced subgraph of $P_3 \times C_3$, $P_3 \times C_5$, and $P_3 \times P_5$.

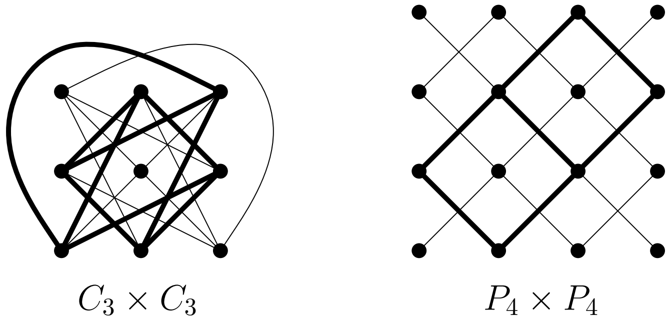


Figure 5.4: The complement of C_6 as induced subgraph of $C_3 \times C_3$ and the domino as induced subgraph of $P_4 \times P_4$.

Each of $C_3 \times C_4$, $C_3 \times C_5$, and $C_3 \times P_5$ contains an induced $C_3 \times P_3 \cong P_3 \times C_3$, and therefore also an induced F_2.

The lemma now follows from the above observations and Corollary 5.4. □

The characterization of 1-perfectly orientable direct products of two nontrivial connected graphs is given in the following theorem. Before stating the result, we define some concepts that will be necessary for the proof of Theorem 5.10. We say that an undirected graph is a *pseudoforest* if each component of it contains at most one cycle and a *pseudotree* if it is a connected pseudoforest. Recall that a graph is *unicyclic* if it contains exactly one cycle.

Theorem 5.10. *A nontrivial direct product, $G \times H$, of two connected graphs G and H is 1-p.o. if and only if one of the following conditions holds:*

(i) One factor is isomorphic to K_2 and the other one is a pseudotree.

(ii) One factor is isomorphic to P_3 and the other one to P_3 or to P_4.

Proof. We first show that each of the conditions (i) and (ii) is sufficient for $G \times H$ to be 1-p.o. Recall that every chordal graph and every graph having a unique induced cycle of order at least 4 is 1-p.o. [4]. In particular, this implies that every pseudoforest is 1-p.o. Suppose first that $G \cong K_2$ and H is a pseudotree. If H is bipartite, then $K_2 \times H$ is isomorphic to the pseudoforest $2H$, which is 1-p.o. If H is non-bipartite, then it is unicyclic, in which case $K_2 \times H$ is again unicyclic and therefore 1-p.o. Finally, $P_3 \times P_4$ is isomorphic to $2F$ where F is a unicyclic graph, and is therefore a 1-p.o. graph. This also implies that $P_3 \times P_3$ is 1-p.o.

To show necessity, suppose that $G \times H$ is 1-p.o. We consider two cases depending on whether one of the two factors is isomorphic to K_2 or not. Suppose first that G is isomorphic to K_2. Then $K_2 \times H$ is triangle-free, and it follows from [4, Corollary 5.7] that $K_2 \times H$ is a pseudoforest. If H is bipartite, then the product $K_2 \times H$ is isomorphic to $2H$, therefore H is a connected 1-p.o. bipartite graph, and by [4, Corollary 5.7] H must be a pseudotree. Suppose now that H is non-bipartite. Then, $K_2 \times H$ is connected [45, Theorem 5.9] and hence a pseudotree. Let us observe that in this case H must be a unicyclic graph (and therefore a pseudotree). Indeed, if H has a cycle $(v_1, \ldots, v_k)$ (for some odd k) then $K_2 \times H$ has a cycle of length $2k$ formed by vertices $(u_1, v_1), (u_2, v_2), (u_1, v_3), \ldots, (u_1, v_k), (u_2, v_1), (u_1, v_2), (u_2, v_3), \ldots, (u_2, v_k)$, where u_1 and u_2 are the two vertices of the K_2. Therefore if H had more than one cycle, then so would $K_2 \times H$, and we know that this is not the case.

Now consider the case when both factors have at least three vertices. By Lemma 5.9, at least one of the two factors, say G, is triangle-free. Since G has at least three vertices, it contains an induced P_3. Applying Lemma 5.9 further, we infer that H is $\{claw, C_3, C_4, C_5, P_5\}$-free. Since H is $\{claw, C_3\}$-free, it is of maximum degree at most 2, thus a path or a cycle. Since H is also

$\{C_4, C_5, P_5\}$-free and connected, we conclude that H is a path with either 3 or 4 vertices. If $H \cong P_4$, then G is P_4-free by Lemma 5.9, and since it contains a P_3, we must have $G \cong P_3$. If $H \cong P_3$, then applying the same arguments as above we obtain that $G \cong P_3$ or $G \cong P_4$. This concludes the proof of the forward implication, and with it the proof of the theorem. □

As a consequence from the previous theorem we can obtain a characterization for the general case.

Theorem 5.11. *A nontrivial direct product, $G \times H$, of two graphs G and H is 1-perfectly orientable if and only if one of the following conditions holds:*

(i) G is a 1-linear forest and H is any graph, or vice versa.

(ii) G is a 2-linear forest and H is a pseudoforest, or vice versa.

(iii) G is a 3-linear forest and H is a 4-linear forest, or vice versa.

Proof. Suppose first that $G \times H$ is 1-p.o. If one of G and H is a 1-linear forest, we are in case (i). So we may assume that both G and H contain K_2 as an induced subgraph. It follows from Theorem 5.10 that each component of both G and H is a pseudotree, and thus G and H are pseudoforests. If G is a 2-linear forest, we are in case (ii). Let H_j denote an arbitrary component of H. If G is a 3-linear forest containing a component $G_i \cong P_3$, it follows from Theorem 5.10 (applied to $G_i \times H_j$, which is an induced subgraph of $G \times H$ and hence 1-p.o.) that H_j is a path on at most 4 vertices, and thus H is a 4-linear forest and we are in case (iii). If G is a 4-linear forest containing a component $G_i \cong P_4$, by Theorem 5.10 applied to $G_i \times H_j$, component H_j must be a path on at most 3 vertices, and therefore H is a 3-linear forest. Finally, if G has a component different from K_1, K_2, P_3, and P_4, Theorem 5.10 applied to $G_i \times H_j$ implies that H_j is either K_1 or K_2, and thus H is a 2-linear forest.

Let us now show that for each of the three cases (i), (ii), and (iii), the product $G \times H$ is 1-p.o. If G is a 1-linear forest and H is any graph, then $G \times H$ is edgeless and therefore 1-p.o. If G is a 2-linear forest and H is a pseudoforest, then $G \times H$ is the disjoint union of the direct products $G_i \times H_j$ where G_i is a component of G and H_j is a component of H. Each of those products $G_i \times H_j$ is either the direct product of a K_1 and a pseudotree, which is 1-p.o., or of a K_2 and a pseudotree, which is 1-p.o. by Theorem 5.10. Then, $G \times H$ is 1-p.o. by Corollary 5.3. Finally, if G is a 3-linear forest and H is a 4-linear forest, the products $G_i \times H_j$ are of the form $P_i \times P_j$ with $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$, which are 1-p.o. due to Theorem 5.10. Applying Corollary 5.3, we obtain that $G \times H$ is 1-p.o. □

5.4 The strong product

In this section, we characterize nontrivial strong product graphs that are 1-p.o.

By [45, Corollary 5.6], the strong product of two graphs G and H is connected if and only if both factors are connected. More precisely, if G has components $G_1, \ldots, G_k$ and H has components $H_1, \ldots, H_\ell$, then the components of $G \boxtimes H$ are exactly $G_i \boxtimes H_j$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$. Therefore, in order to characterize nontrivial strong product graphs that are 1-p.o., we may again restrict our attention to the case of nontrivial products $G \boxtimes H$ in which both factors are connected.

Our characterization will be proved in several steps. In Section 5.4.1, we state two preliminary lemmas on the strong product and give two necessary conditions for 1-p.o. strong product graphs. The necessary conditions motivate the development of a structural characterization of $\{P_5, C_4, C_5, \mathit{claw}, \mathit{bull}\}$-free graphs. This is done in Section 5.4.2, where connected

$\{P_5, C_4, C_5, claw, bull\}$-free graphs are shown to be precisely the connected co-chain graphs. Connected true-twin-free co-chain graphs are further characterized in Section 5.4.3, and form the basis of an infinite family of 1-p.o. strong product graphs described in Section 5.4.4. Building on these results, we prove our main result of the section, Theorem 5.23 in Section 5.4.5, which gives a complete characterization of 1-p.o. strong product graphs both factors of which are nontrivial and connected.

5.4.1 Three lemmas

Recall that a vertex v in a graph G is *simplicial* if its neighborhood forms a clique. In Section 5.4.4 we will need the following property of simplicial vertices in relation to the strong product.

Lemma 5.12. *Let G and H be graphs and let u and v be simplicial vertices in G and H, respectively. Then, vertex (u, v) is simplicial in the strong product $G \boxtimes H$.*

Proof. It suffices to show that the closed neighborhood $N_{G\boxtimes H}[(u, v)]$ is a clique in $G \boxtimes H$. Note that $N_{G\boxtimes H}[(u, v)] = N_G[u] \times N_H[v]$, the set $N_G[u]$ is a clique in G (since u is simplicial in G) and, similarly, the set $N_H[v]$ is a clique in H. The desired result now follows from the fact that the strong product of two complete graphs is a complete graph. □

Recall also that two distinct vertices u and v in a graph G form a pair of *true twins* if $N_G[u] = N_G[v]$. We say that a graph is *true-twin-free* if it contains no pair of true twins. The next lemma shows that it suffices to characterize 1-p.o. strong product graphs in which both factors are true-twin-free.

Lemma 5.13. *Let G, G', and H be graphs such that G' is obtained from G by adding a true twin. Then, $G \boxtimes H$ is 1-p.o. if and only if $G' \boxtimes H$ is 1-p.o.*

Proof. Note that $G \boxtimes H$ is an induced subgraph of $G' \boxtimes H$. Therefore, by Proposition 5.2, if $G' \boxtimes H$ is 1-p.o., then so is $G \boxtimes H$.

Suppose now that $G \boxtimes H$ is 1-p.o., and that G' was obtained from G by adding to it a true twin x' to a vertex x of G. Note that for every $v \in V(H)$, we have $N_{G'\boxtimes H}[(x, v)] = N_{G'}[x] \times N_H[v]$ and $N_{G'\boxtimes H}[(x', v)] = N_{G'}[x'] \times N_H[v]$. Since $N_{G'}[x] = N_{G'}[x']$, each vertex of the form (x', v) for $v \in V(H)$ is a true twin in $G' \boxtimes H$ of vertex (x, v). It follows that $G' \boxtimes H$ can be obtained from $G \boxtimes H$ by a sequence of true twin additions. By Proposition 3.5, $G' \boxtimes H$ is 1-p.o. □

A similar approach as for the direct product (Lemma 5.9) gives the following necessary conditions for the strong product of two graphs to be 1-p.o.

Lemma 5.14. *Suppose that the strong product of two graphs G and H is 1-p.o. Then:*

1. *If one of G and H contains an induced P_3, then the other one is $\{P_5, C_4, C_5, claw, bull\}$-free.*

2. *At least one of G and H is P_4-free.*

Proof. We can verify that each of the graphs $P_3 \boxtimes C_4$, $P_3 \boxtimes C_5$, $P_3 \boxtimes claw$, and $P_3 \boxtimes bull$ has $K_{2,3}$ (graph F_4 in Figure 3.2) as induced minor, that $P_3 \boxtimes P_5$ contains an induced copy of F_2, and that $P_4 \boxtimes P_4$ contains an induced copy of F_1. Therefore, by Corollary 5.4, none of these graphs is 1-p.o. We can observe models of such induced minors in Figure 5.5.

The lemma now follows from the above observations and Corollary 5.4. □

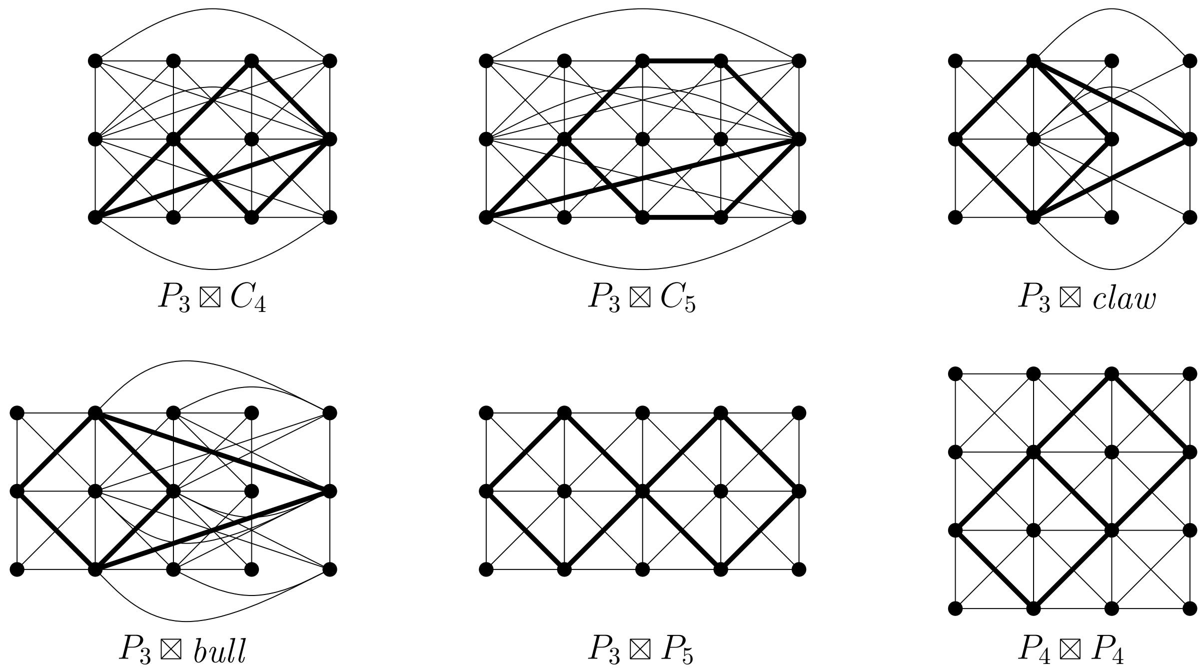


Figure 5.5: $K_{2,3}$ as induced minor of $P_3 \boxtimes C_4$, $P_3 \boxtimes C_5$, $P_3 \boxtimes claw$, $P_3 \boxtimes bull$, F_2 as induced subgraph of $P_3 \boxtimes P_5$, and the domino (F_1) as induced subgraph of $P_4 \boxtimes P_4$.

Lemma 5.14 motivates the development of structural characterizations of P_3-free graphs, of P_4-free graphs, and of $\{P_5, C_4, C_5, claw, bull\}$-free graphs. P_3-free graphs are precisely the disjoint unions of complete graphs. P_4-free graphs (also known as *cographs*) are also well understood: they are precisely the graphs that can be obtained from copies of K_1 by applying a sequence of the disjoint union and join operations [17]. The $\{P_5, C_4, C_5, claw, bull\}$-free graphs are characterized in the next section.

5.4.2 The structure of $\{P_5, C_4, C_5, claw, bull\}$-free graphs

Our characterization of $\{P_5, C_4, C_5, claw, bull\}$-free graphs will rely on the notion of co-chain graphs. A graph G is a *co-chain graph* if its vertex set can be partitioned into two cliques, say X and Y, such that the vertices in X can be ordered as $X = \{x_1, \ldots, x_{|X|}\}$ so that for all $1 \le i < j \le |X|$, we have $N[x_i] \subseteq N[x_j]$ (or, equivalently, $N(x_i) \cap Y \subseteq N(x_j) \cap Y$). The pair (X, Y) will be referred to as a *co-chain partition* of G. The following observation is an immediate consequence of the definitions.

Proposition 5.15. *The set of co-chain graphs is closed under true twin additions and universal vertex additions.*

The following structural characterization of connected $\{P_5, C_4, C_5, claw, bull\}$-free graphs can also be seen as a forbidden induced subgraph characterization of co-chain graphs within connected graphs.

Theorem 5.16. *A connected graph G is $\{P_5, C_4, C_5, claw, bull\}$-free if and only if it is co-chain.*

Proof. Sufficiency of the condition is easy to establish. The graphs P_5, C_5, the claw, and the bull, are not cobipartite and therefore not co-chain. The 4-cycle admits only one partition of its vertex set into two cliques, which however does not have the desired property.

Now we prove necessity. Let G be a connected $\{P_5, C_4, C_5, \mathit{claw}, \mathit{bull}\}$-free graph. We will show that G is $3K_1$-free. This will imply that G is co-chain due to the known characterization of co-chain graphs as exactly the graphs that are $\{3K_1, C_4, C_5\}$-free [48].

Suppose for a contradiction that G has an induced $3K_1$, with vertex set $\{x, y, z\}$, say. Since G is connected and P_5-free, every two vertices among $\{x, y, z\}$ are at distance 2 or 3.

Suppose first that $d(x, y) = d(x, z) = 2$. Let y' be a common neighbor of x and y, and let z' be a common neighbor of x and z. Since G is claw-free, $y'z \notin E(G)$ and similarly $yz' \notin E(G)$. In particular, $y' \neq z'$. Now, the vertex set $\{y, y', x, z', z\}$ induces either a P_5 (if y' and z' are non-adjacent), or a bull (otherwise), a contradiction.

Therefore, at least two out of the pairwise distances between x, y, and z are equal to 3. By symmetry, we may assume that $d(x, y) = d(x, z) = 3$. Note that the set of vertices at distance 2 from x form a clique, since otherwise we could apply the arguments from the previous paragraph to the triple $\{x, y', z'\}$ where $\{y', z'\}$ is a pair of non-adjacent vertices with $d(x, y') = d(x, z') = 2$.

Fix a pair of paths P and Q such that $P = (x = p_0, p_1, p_2, p_3 = y)$ is a shortest x-y path, $Q = (x = q_0, q_1, q_2, q_3 = z)$ is a shortest x-z path, and P and Q agree in their initial segments as much as possible, that is, the value of $k = k(P, Q) = \max\{j : p_i = q_i \text{ for all } 0 \leq i \leq j\}$ is maximized. Clearly, $k \in \{0, 1, 2\}$. If $k = 2$, then G contains a claw induced by $\{p_1, p_2, y, z\}$. Therefore $k \in \{0, 1\}$. If $k = 1$, then, recalling that p_2 is adjacent to q_2, we infer that G contains either a claw induced by $\{p_1, p_2, y, z\}$ (if p_2 is adjacent to z) or a bull induced by $V(Q) \cup \{p_2\}$. Therefore $k = 0$. By the minimality of (P, Q), we infer that $\{p_1q_2, p_2q_1, p_2z, yq_2\} \cap E(G) = \emptyset$. But now, G contains a claw induced by $\{p_1, p_2, y, q_2\}$. This contradiction completes the proof. □

5.4.3 Rafts and connected true-twin-free co-chain graphs

In Section 5.4.4, we will identify an infinite family of 1-p.o. strong product graphs. The family will be based on the following particular family of co-chain graphs. Given a non-negative integer $n \geq 0$, the *raft of order n* is the graph R_n consisting of two disjoint cliques on $n+1$ vertices each, say $X = \{x_0, x_1, \ldots, x_n\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ together with additional edges between X and Y such that for every $0 \leq i, j \leq n$, vertex x_i is adjacent to vertex y_j if and only if $i + j \geq n + 1$ [61]. Note that vertices x_0 and y_0 are simplicial in the raft. The cliques X and Y will be referred to as the *parts* of the raft. Figure 5.6 shows rafts of order n for $n \in \{1, 2, 3\}$.

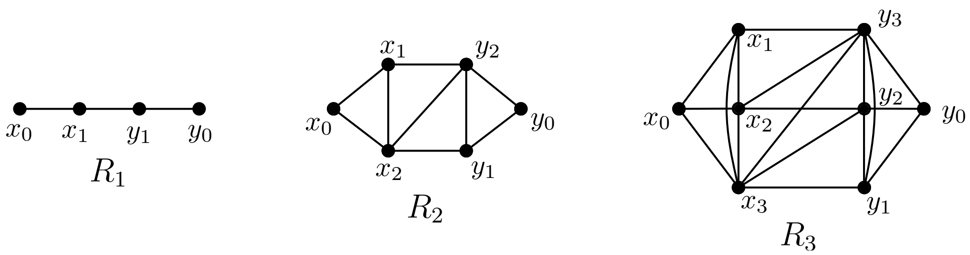


Figure 5.6: Three small rafts

It is an easy consequence of definitions that every raft is a co-chain graph. Moreover, as we show next, rafts play a crucial role in the classification of connected true-twin-free co-chain graphs.

Proposition 5.17. *Let G be a connected true-twin-free graph. Then, G is co-chain if and only if $G \in \{K_1\} \cup \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$. Moreover, if G is P_4-free, then G is co-chain if and only if $G \cong K_1$ or $G \cong P_3$.*

Proof. Sufficiency is immediate since every graph in $\{K_1\} \cup \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$ is co-chain.

Now, let G be a connected true-twin-free co-chain graph, with a co-chain partition (X, Y). Since G is true-twin-free, the closed neighborhoods of vertices in $X = \{x_1, \ldots, x_{|X|}\}$ are properly nested. Equivalently,

$$N(x_1) \cap Y \subset N(x_2) \cap Y \subset \cdots \subset N(x_{|X|}) \cap Y .$$

Since there are no pairs of true twins in Y, we have $|N(x_{i+1}) \cap Y| = |N(x_i) \cap Y| + 1$ for all $i \in \{1, \ldots, |X| - 1\}$. This implies an ordering of vertices in Y, say $Y = \{y_1, \ldots, y_{|Y|}\}$ such that $N(y_i) \cap X \subset N(y_{i+1}) \cap X$ and $|N(y_{i+1}) \cap X| = |N(y_i) \cap X| + 1$ for all $i \in \{1, \ldots, |Y| - 1\}$.

If $X = \emptyset$ or $Y = \emptyset$, then since both X and Y are cliques and G is true-twin-free, we infer that $G \cong K_1$.

Now, both X and Y are non-empty, and we analyze four cases depending on the smallest neighborhoods of vertices in the two parts. If $N(x_1) \cap Y = N(y_1) \cap X = \emptyset$, then since G is connected, we have $|X| = |Y| \geq 2$, and G is isomorphic to $R_{|X|-1}$. If $N(x_1) \cap Y = \emptyset$ and $N(y_1) \cap X \neq \emptyset$, then $|X| \geq 2$, and deleting the universal vertex $x_{|X|}$ from G leaves a graph isomorphic to $R_{|X|-2}$. Thus, $G \cong R_{|X|-2} * K_1$. The case when $N(x_1) \cap Y \neq \emptyset$ and $N(y_1) \cap X = \emptyset$ is symmetric to the previous one. Finally, if $N(x_1) \cap Y \neq \emptyset$ and $N(y_1) \cap X \neq \emptyset$, then vertices $x_{|X|}$ and $y_{|Y|}$ are both universal in G, contrary to the fact that G is true-twin-free.

Suppose now that G is also P_4-free but not isomorphic to either K_1 or P_3. Note that since $R_1 \cong P_4$, every raft of order at least 1 contains an induced P_4. It follows that G is isomorphic to a graph of the form $R_n * K_1$ for some $n \geq 0$. Since $R_0 * K_1 \cong P_3$, we have $n \geq 1$. But then $R_1 \cong P_4$ is an induced subgraph of G, a contradiction. □

5.4.4 An infinite family of 1-p.o. strong product graphs

The following observation is an immediate consequence of Lemma 5.12.

Observation 5.18. *Let G be a graph with a simplicial vertex v, and let $P_3 = (u_1, u_2, u_3)$ be the 3-vertex path, with leaves u_1 and u_3. Then, vertices (u_1, v) and (u_3, v) are simplicial in $P_3 \boxtimes G$.*

Proposition 5.19. *For every $n \geq 1$, the strong product $P_3 \boxtimes R_n$ is 1-p.o.*

Proof. First, notice that since R_n has two simplicial vertices, Observation 5.18 implies that the product $P_3 \boxtimes R_n$ has 4 simplicial vertices. Let G be the product $P_3 \boxtimes R_n$ minus these 4 simplicial vertices. Since 1-p.o. graphs are closed under simplicial vertex additions, it is enough to verify that G is 1-p.o. To prove this we will give an explicit orientation of G and show that it is a 1-perfect orientation.

Let $V(P_3) = \{u_1, u_2, u_3\}$ where u_1 and u_3 are the two leaves. Moreover, assuming the notation as in the definition of rafts, let $V(R_n) = X \cup Y$, where $X = \{x_0, x_1, \ldots, x_n\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ are the two parts of the raft. Vertices in G will be said to be *left*, resp. *right*, depending on whether their second coordinate is in X or in Y, respectively. A schematic representation of G is shown in Figure 5.7. We partition the graph's vertex set into 8 cliques: two singletons, $\{a\}$ and $\{b\}$, where $a = (u_2, x_0)$ and $b = (u_2, y_0)$, and 6 cliques of size n each, namely A_1, A_2, A_3, B_1, B_2, and B_3, defined as follows: for $i \in \{1, 2, 3\}$, we have $B_i = \{u_i\} \times (X \setminus \{x_0\})$ and $A_i = \{u_{4-i}\} \times (Y \setminus \{y_0\})$. Bold edges between certain pairs of sets mean that every possible edge between the two sets is present. If the corresponding edge is not bold, then only some of the edges between the two sets are present.

To describe such edges, we introduce the following ordering of the vertices within each of the 6 cliques of size n. Note that for every $1 \leq i < j \leq n$, we have that $N_{R_n}[x_i] \subset N_{R_n}[x_j]$ and

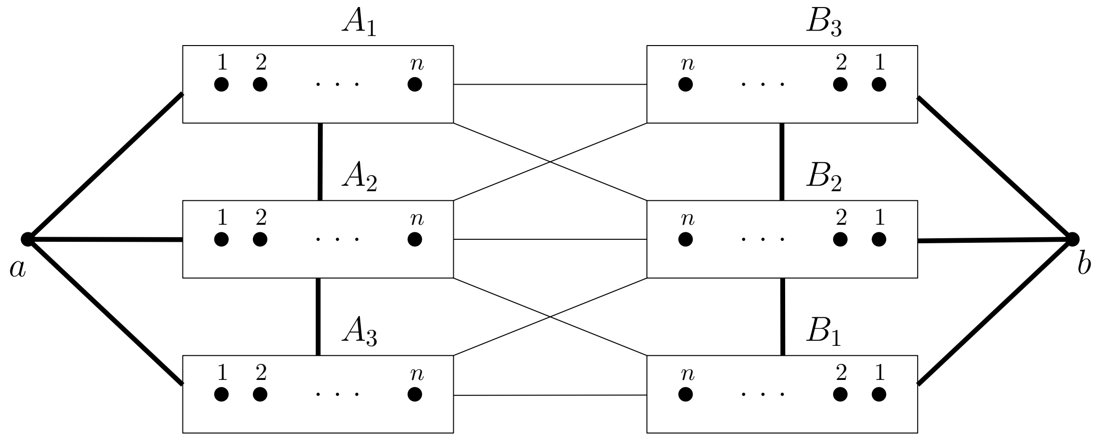


Figure 5.7: A schematic representation of graph G

$N_{R_n}[y_i] \subset N_{R_n}[y_j]$. We order the vertices in the 6 cliques accordingly, that is, for each clique of the form A_i, the linear ordering of its vertices is $(u_i, x_1), \ldots, (u_i, x_n)$; for each clique of the form B_i, the linear ordering of its vertices is $(u_{4-i}, y_1), \ldots, (u_{4-i}, y_n)$. To keep the notation light, we will slightly abuse the notation, speaking of "vertex i in clique C" (for $i \in \{1, \ldots, n\}$ and $C \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$) when referring to the i-th vertex in the linear ordering of C. We will also speak of "left" and of "right" cliques.

The edges of graph G can be now concisely described as follows. We will say that two cliques A_i and A_j (or B_i and B_j) are *adjacent* if $|i - j| \leq 1$. The neighborhood of a is $A_1 \cup A_2 \cup A_3$. The neighborhood of b is $B_1 \cup B_2 \cup B_3$. For each vertex i in a left clique, say A_j, its closed neighborhood consists of vertex a, all the vertices belonging to some left clique adjacent to A_j, and of vertices $\{n - i + 1, \ldots, n\}$ in each right clique adjacent to B_{4-j}. For each vertex i in a right clique, say B_j, its closed neighborhood consists of vertex b, all vertices belonging to some right clique adjacent to B_j, and of vertices $\{n - i + 1, \ldots, n\}$ in each left clique adjacent to A_{4-j}. Figure 5.8 shows a concrete example of G, namely for the case $n = 3$.

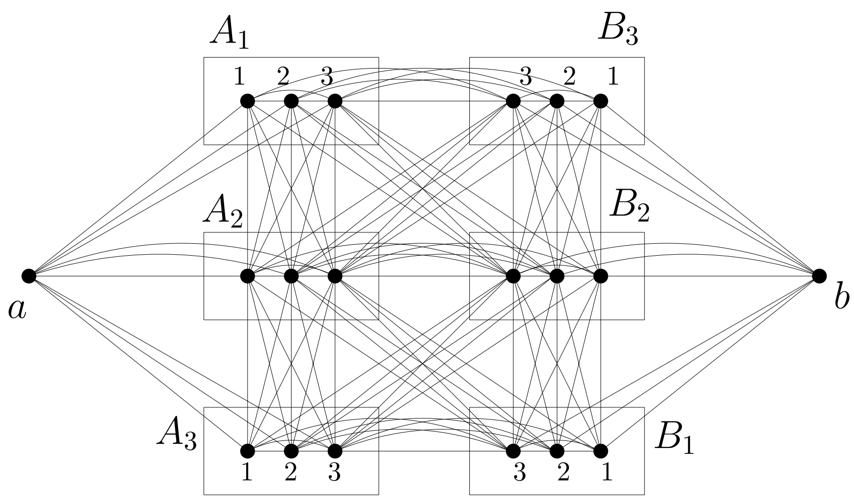


Figure 5.8: Graph G in the case $n = 3$

We now define an orientation of G, say D, as follows:

– Edges between vertex a and a vertex $i \in A_j$ are oriented from i to a for $j = 1$ and from a to i for $j \in \{2, 3\}$. Symmetrically, edges between vertex b and a vertex $i \in B_j$ are oriented from

i to b for $j = 1$ and from b to i for $j \in \{2, 3\}$.

– Edges within each clique are oriented from vertex i to vertex j (with $j \neq i$) if and only if $i < j$.

– All edges between vertices in A_1 and A_2 are oriented from A_1 to A_2. Symmetrically, all edges between vertices in B_1 and B_2 are oriented from B_1 to B_2.

– Edges between vertices in A_2 and A_3 are oriented as follows: For $i \in A_2$ and $j \in A_3$, from i to j if $i < j$, and from j to i, otherwise. Symmetrically, edges between B_2 and B_3 are oriented as follows: For $i \in B_2$ and $j \in B_3$, from i to j if $i < j$, and from j to i, otherwise.

– All edges between vertices in A_1 and B_3 are oriented from B_3 to A_1. Symmetrically, all edges between vertices in A_3 and B_1 are oriented from A_3 to B_1.

– All edges between vertices in A_1 and B_2 are oriented from B_2 to A_1. Symmetrically, all edges between vertices in A_2 and B_1 are oriented from A_2 to B_1.

– All edges between vertices in A_2 and B_1 are oriented from B_1 to A_2. Symmetrically, all edges between vertices in A_3 and B_2 are oriented from A_3 to B_2.

– Finally, all edges between vertices in A_2 and B_2 are oriented from A_2 to B_2.

To conclude the proof it remains to check that D is a 1-perfect orientation of G, that is, that for each vertex v in G, its out-neighborhood in D forms a clique in G. We consider several cases according to which part of the above vertex partition vertex v belongs to:

(i) $v \in \{a, b\}$. We have $N_D^+(a) = A_2 \cup A_3$, which forms a clique in G. Symmetrically, $N_D^+(b)$ forms a clique in G.

(ii) $v \in A_1 \cup B_1$. By symmetry, we may assume that $v \in A_1$, say $v = i$. Then, $N_D^+(i) = \{a\} \cup \{j \in A_1, j > i\} \cup A_2$, which forms a clique in G.

(iii) $v \in A_2$, say $v = i$. We have $N_D^+(i) = A \cup B$, where $A = \{j \in A_2 \cup A_3, j > i\}$ and $B = \{j \in B_2 \cup B_3, j > n - i\}$. Note that A and B are cliques in G. Moreover, if $j \in A$ and $k \in B$, then $j + k > i + (n - i) = n$, which implies that j and k are adjacent in G. Therefore, $N_D^+(i)$ is a clique in G.

(iv) $v \in A_3 \cup B_3$. By symmetry, we may assume that $v \in A_3$, say $v = i$. We have $N_D^+(i) = A \cup B$, where $A = \{j \in A_2, j \geq i\} \cup \{j \in A_3, j > i\}$ and $B = \{j \in B_2 \cup B_3, j > n - i\}$. Again, A and B are cliques in G. Moreover, if $j \in A$ and $k \in B$, then $j + k \geq i + (n - i + 1) = n + 1$, which implies that j and k are adjacent in G. Therefore, $N_D^+(i)$ is a clique in G.

(v) $v \in B_2$, say $v = i$. Then, $N_D^+(i) = A \cup B$, where $A = \{j \in A_1, j > n - i\}$ and $B = \{j \in B_2 \cup B_3, j > i\}$. Again, A and B are cliques in G such that if $j \in A$ and $k \in B$, then $j + k > (n - i) + i = n$, which implies that j and k are adjacent in G. Therefore, $N_D^+(i)$ is a clique in G.

This completes the proof that G is 1-p.o. □

Note that for every $n \geq 0$, the graph $R_n * K_1$ is isomorphic to an induced subgraph of R_{n+2}. Therefore, Proposition 5.19 and the fact that 1-p.o. graphs are closed under taking induced subgraphs implies the following.

Corollary 5.20. *For every $n \geq 0$, the strong product $P_3 \boxtimes (R_n * K_1)$ is 1-p.o.*

5.4.5 A characterization of connected nontrivial strong product graphs that are 1-p.o.

We now derive the main result of this section. We first show that Proposition 5.19 and Corollary 5.20 describe all nontrivial strong products of two true-twin-free connected graphs that are 1-p.o.

Lemma 5.21. *A nontrivial strong product, $G \boxtimes H$, of two true-twin-free connected graphs G and H is* 1*-p.o. if and only if one of them is isomorphic to P_3 and the other one belongs to $\{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$.*

Proof. If $G \cong P_3$ and $H \in \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$, the strong product $G \boxtimes H$ is 1-p.o. by Proposition 5.19 and Corollary 5.20.

Conversely, suppose that $G \boxtimes H$ is 1-p.o. If one of the factors is P_3-free, its connectedness would imply that the graph is complete and therefore contains a pair of true twins, which is a contradiction. Thus, both factors contain an induced P_3, and by the first part of Lemma 5.14, they are both $\{P_5, C_4, C_5, \mathit{claw}, \mathit{bull}\}$-free. In particular, by Theorem 5.16, they are both co-chain. Moreover, by Proposition 5.17, they both belong to the set $\{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$. By the second part of Lemma 5.14, at least one of G and H is P_4-free, and thus, by Proposition 5.17, isomorphic to P_3. □

To describe the main result of this section, the following notions will be convenient. We say that a graph is 2*-complete* if it is the union of two (not necessarily distinct) complete graphs sharing at least one vertex. Equivalently, a graph is 2-complete if and only if it can be obtained from either K_1 or P_3 by applying a sequence of true twin additions. Moreover, a *true-twin-reduction* of a graph G is any maximal induced subgraph of G that is true-twin-free. It is not difficult to observe that any two true-twin-reductions of a graph G are isomorphic to each other, thus we can speak of *the* true-twin-reduction of G.

Theorem 5.22. *A nontrivial strong product, $G \boxtimes H$, of two connected graphs G and H is* 1*-p.o. if and only if one of the following conditions holds:*

(i) One of the factors is 1*-p.o. and the other one complete.*

(ii) One of the factors is co-chain and the other one 2*-complete.*

Proof. Suppose first that given two nontrivial connected graphs G and H, the product $G \boxtimes H$ is 1-p.o. Then, G and H are both 1-p.o. We may assume that neither of the two factors is complete (since otherwise condition (i) holds). Let G' and H' be the true-twin-free reductions of G and H, respectively. Clearly, G' and H' are true-twin-free and $G' \boxtimes H'$ is 1-p.o. (since it is an induced subgraph of $G \boxtimes H$). Applying Lemma 5.21 to the product $G' \boxtimes H'$ (which is 1-p.o.), we infer that one of G' and H' is isomorphic to P_3 and the other one belongs to the set $\{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$. Without loss of generality, let $H' \cong P_3$. Then $G' \in \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$. In particular, by Proposition 5.17, G' is co-chain. Since G is obtained from G' by a sequence of true twin additions and G' is co-chain, Proposition 5.15 implies that G is co-chain. Since H', the true-twin-free reduction of H, is isomorphic to P_3, it follows that H is 2-complete, and hence condition (ii) holds.

Let us now prove that each of the two conditions is also sufficient. Suppose first that G is 1-p.o. and H is complete. Then, the product $G \boxtimes H$ can be obtained by applying a sequence of true twin additions to vertices of G. Applying Proposition 3.5, we infer that $G \boxtimes H$ is 1-p.o. in this case. In the other case, G is co-chain and H is 2-complete. Let G' and H' be the true-twin-free reductions of G and H, respectively. By Lemma 6.7, it suffices to show that $G' \boxtimes H'$ is 1-p.o. By Proposition 5.17, G' is isomorphic to a graph from the set $\{K_1\} \cup \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$.

Since H' is the true-twin-free reduction of a 2-complete graph, H' is isomorphic to either K_1 or to P_3. The fact that $G' \boxtimes H'$ is 1-p.o. now follows from Corollary 5.20 and the fact that 1-p.o. graphs are closed under taking induced subgraphs. $\square$

Finally, we state the theorem for the general case.

Theorem 5.23. *A nontrivial strong product, $G \boxtimes H$, of two graphs G and H is 1-perfectly orientable if and only if one of the following conditions holds:*

(i) Every component of G is complete and H is 1-perfectly orientable, or vice versa.

(ii) Every component of G is 2-complete and every component of H is co-chain, or vice versa.

Proof. Suppose first that $G \boxtimes H$ is 1-p.o. Then, each of G and H is 1-p.o., since they are induced subgraphs of the product. By Theorem 5.22, the components of the factors can either be complete, 2-complete, co-chain, or 1-p.o. Note that all complete graphs are 2-complete, all 2-complete graphs are co-chain, and all co-chain graphs are 1-p.o. If every component of G is complete, we are in case (i). Suppose that all components of G are 2-complete and G contains a component G_i that is not complete. Let H_j be an arbitrary component of H. Consider the product $G_i \boxtimes H_j$ (which is a component of $G \boxtimes H$ and hence 1-p.o.). By Theorem 5.22, H_j must be co-chain, and thus we are in case (ii). If all components of G are co-chain and G contains a component G_i that is not 2-complete, by Theorem 5.22 applied to $G_i \boxtimes H_j$, we infer that H_j is 2-complete, and thus we are in case (ii). Finally, if G contains a component G_i that is 1-p.o. but not co-chain, applying Theorem 5.22 to $G_i \boxtimes H_j$ we obtain that H_j is complete, and therefore we are in case (i).

For the converse implication, it follows from Theorem 5.22 that if every component of G is complete and H is 1-p.o., or every component of G is 2-complete and every component of H is co-chain, then every component of the product $G \boxtimes H$ is 1-p.o. We can then apply Corollary 5.3 to conclude that $G \boxtimes H$ is 1-p.o. $\square$

Chapter 6

Chordal, interval, and circular arc product graphs

In this chapter we consider the four standard graph products: the Cartesian product, the strong product, the direct product, and the lexicographic product. For each of these four products, we completely characterize when a nontrivial product of two graphs G and H is chordal, interval, or circular arc, respectively. While the characterizations for chordal and interval graphs are rather straightforward and can be proved directly, the characterizations of circular arc product graphs are more involved and are derived using characterizations of 1-perfectly orientable product graphs (for each of the four standard products) presented in the previous chapter. Recall that a graph is said to be 1-perfectly orientable if it admits an orientation such that the out-neighborhood of every vertex induces a tournament. As shown by Urrutia and Gavril [85] and by Skrien [81], respectively, the class of 1-perfectly orientable graphs generalizes both chordal graphs and circular arc graphs. Since every chordal, interval, or circular arc graph is 1-perfectly orientable, Theorems 5.6, 5.8, 5.11, and 5.23 give necessary conditions that every chordal, interval, resp. circular arc product graph must satisfy.

The results of this chapter contribute to the knowledge of characterizations of graph classes within graphs decomposable with respect to one the four standard graph products, by adding chordal, interval, and circular arc graphs to the list. This chapter is based on the following paper:

- T. R. Hartinger, *Chordal, interval, and circular-arc product graphs.* Applicable Analysis and Discrete Mathematics. Vol. 10, No 2 (2016), 532 – 551.

6.1 The Cartesian product

In the following theorem we characterize when a nontrivial Cartesian product of two graphs G and H is chordal, interval, or circular arc, respectively.

Theorem 6.1. *A nontrivial Cartesian product, $G \Box H$, of two graphs G and H is:*

- *chordal if and only if G is edgeless and H is chordal, or vice versa,*
- *interval if and only if G is edgeless and H is interval, or vice versa,*
- *circular arc if and only if one of the following conditions holds:*
  - *(i) G is edgeless and H is an interval graph, or vice versa,*
  - *(ii) $G \cong H \cong K_2$.*

Proof. First we characterize the chordal case. If G is edgeless and H is chordal, then $G\Box H$ is isomorphic to a disjoint union of $|V(G)|$ copies of H. Thus, since chordal graphs are closed under disjoint union, the stated condition is sufficient. To show necessity, assume now that $G\Box H$ is chordal. Both graphs G and H must be chordal since they are induced subgraphs of $G\Box H$. Suppose that none of G and H is edgeless. In that case, $G\Box H$ contains an induced $K_2\Box K_2 \cong C_4$ and is therefore not chordal, a contradiction. Thus, at least one of G and H is edgeless.

Suppose now that $G\Box H$ is interval. Both graphs, G and H, must be interval since they are induced subgraphs of $G\Box H$. Since $G\Box H$ is interval, it is chordal, and thus, by the above, one of G and H, say G, must be edgeless. Conversely, if G is edgeless and H is interval, the Cartesian product $G\Box H$ is isomorphic to a disjoint union of copies of H, and therefore interval.

For the circular arc case, it is clear that any of the conditions (i) and (ii) is sufficient for $G\Box H$ to be a circular arc graph. To prove necessity, suppose that $G\Box H$ is circular arc. If one of G and H, say G, is edgeless, then, since the product is nontrivial, it is isomorphic to the disjoint union of $|V(G)| \geq 2$ copies of H. By Lemma 2.3 and an inductive argument on the number of components of G, we infer that H is interval. Suppose now that both G and H have an edge. Since $G\Box H$ is circular arc, it is also 1-perfectly orientable. By Theorem 5.6, G and H are 2-linear forests. If one of G and H contains at least 2 edges, then $G\Box H$ contains $2C_4$ as an induced subgraph. This would imply the existence of an induced $C_4 + K_1$, contrary to Fact 2.4. A similar reasoning shows that each of G and H has a unique component, and thus each of them is isomorphic to K_2. $\Box$

Since the Cartesian product of a graph G with an n-vertex edgeless graph is isomorphic to the disjoint union of n copies of G, we obtain the following.

Corollary 6.2. *Let $\mathcal{C}_\Box$, $\mathcal{I}_\Box$, resp. $\mathcal{CA}_\Box$ denote the sets of (isomorphism classes of) nontrivial Cartesian product graphs that are chordal, interval, resp. circular arc. Then:*

$$\begin{aligned}
\mathcal{C}_\Box &= \{nG : G \textit{ chordal}, n \geq 2, |V(G)| \geq 2\}\,,\\
\mathcal{I}_\Box &= \{nG : G \textit{ interval}, n \geq 2, |V(G)| \geq 2\}\,,\\
\mathcal{CA}_\Box &= \{nG : G \textit{ interval}, n \geq 2, |V(G)| \geq 2\} \cup \{C_4\}\,.
\end{aligned}$$

6.2 The lexicographic product

The following theorem characterizes when a nontrivial lexicographic product of two graphs G and H is chordal, interval, or circular arc, respectively.

Theorem 6.3. *A nontrivial lexicographic product, $G[H]$, of two graphs G and H is:*

- *chordal if and only if one of the following conditions holds:*
  - *(i) G is edgeless and H is chordal,*
  - *(ii) G is chordal and H is complete,*
- *interval if and only if one of the following conditions holds:*
  - *(i) G is edgeless and H is interval,*
  - *(ii) G is interval and H is complete,*
- *circular arc if and only if one of the following conditions holds:*

(i) G is edgeless and H is interval,

(ii) G is circular arc and H is complete,

(iii) G is complete and H is cobipartite circular arc.

Proof. First, we characterize the chordal case. Suppose first that $G[H]$ is chordal. Then, both G and H are chordal since they are induced subgraphs of $G[H]$. If neither of conditions (i) or (ii) above holds, then G has an edge and H is not complete. This implies that the product $G[H]$ contains an induced subgraph isomorphic to $K_2[2K_1] \cong C_4$, contrary to the fact that it is chordal. For the converse direction, we will show that in both cases (i) and (ii), the product graph $G[H]$ is chordal. If G is edgeless and H is chordal, then the product $G[H]$ is isomorphic to the disjoint union of $|V(G)|$ copies of H, and therefore chordal. If G is chordal and H is complete, then the product $G[H]$ is isomorphic to the graph obtained by repeatedly substituting a vertex of G with a complete graph, and this operation is easily seen to preserve chordality.

Now we analyze the interval case. Assume that $G[H]$ is interval. Then, G and H are interval. Since $G[H]$ is interval, in particular $G[H]$ is chordal, and thus we obtain the desired result. Conversely, if G is edgeless and H interval, $G[H]$ is isomorphic to a disjoint union of copies of H, and if G is interval and H is complete, $G[H]$ can be obtained from a sequence of true twin additions to H. In both cases the lexicographic product $G[H]$ is interval.

Finally, we characterize the circular arc case. Suppose first that $G[H]$ is a circular arc graph. Then, both G and H are circular arc graphs, since they are induced subgraphs of $G[H]$. If G is edgeless, then the lexicographic product $G[H]$ is isomorphic to the Cartesian product $G\square H$ and by Theorem 6.1, conditions (i) holds. So we may assume that G has an edge. If H is complete then condition (ii) holds. Suppose now that G is not edgeless and that H is not complete. Since $G[H]$ is 1-perfectly orientable, one of conditions (i)–(iii) from Theorem 5.8 holds, and so we infer that every component of G is complete and H is cobipartite. Therefore, the product $G[H]$ contains an induced subgraph isomorphic to $K_2[2K_1] \cong C_4$, from which we infer that G is connected (that is, complete), since by Fact 2.4 $G[H]$ is $C_4 + K_1$-free. Therefore, condition (iii) holds. This completes the proof of the forward direction.

For the converse direction, we will show that in any of the three cases, the product graph $G[H]$ is circular arc. If G is edgeless and H interval, then the lexicographic product $G[H]$ is isomorphic to the disjoint union of $|V(G)|$ copies of H, and therefore circular arc. If G is circular arc and H is complete, then the product $G[H]$ is isomorphic to the graph obtained by repeatedly substituting a vertex of G with a complete graph. Substituting a vertex v with a complete graph is the same as adding a sequence of true twins to vertex v, an operation easily seen to preserve the property of being a circular arc graph. Finally, suppose that G is complete and H is a cobipartite circular arc graph. In this case, an inductive argument on the order of G together with the fact that the class of cobipartite circular arc graphs is closed under join (by Lemma 2.6) shows that $G[H]$ is a circular arc graph. □

Since the lexicographic product of an n-vertex edgeless graph with a graph G is isomorphic to the disjoint union of n copies of G, Theorem 6.3 has the following consequence.

Corollary 6.4. *Let $\mathcal{C}_{lex}$, $\mathcal{I}_{lex}$, resp. $\mathcal{CA}_{lex}$, denote the sets of (isomorphism classes of) nontrivial lexicographic product graphs that are chordal, interval, resp. circular arc. Then:*

$$\begin{aligned}
\mathcal{C}_{lex} &= \{nG : G \text{ chordal}, n \geq 2, |V(G)| \geq 2\} \cup \{G[K_n] : G \text{ chordal}, n \geq 2, |V(G)| \geq 2\}\,, \\
\mathcal{I}_{lex} &= \{nG : G \text{ interval},\ n \geq 2,\ |V(G)| \geq 2\} \cup \{G[K_n] : G \text{ interval},\ n \geq 2,\ |V(G)| \geq 2\}\,, \\
\mathcal{CA}_{lex} &= \{nG : G \text{ interval},\ n \geq 2,\ |V(G)| \geq 2\} \cup \{G[K_n] : G \text{ circular arc},\ n \geq 2,\ |V(G)| \geq 2\} \\
&\quad \cup \{K_n[G] : n \geq 2,\ G \text{ cobipartite circular arc},\ |V(G)| \geq 2\}\,.
\end{aligned}$$

6.3 The direct product

In the next theorem we characterize when a nontrivial direct product of two graphs G and H is chordal, interval, or circular arc, respectively. A *circular caterpillar* (resp. *odd circular caterpillar*) is a connected graph such that the removal of all degree-one vertices yields a cycle (resp. an odd cycle).

Theorem 6.5. *A nontrivial direct product, $G \times H$, of two graphs G and H is:*

- *chordal if and only if one of the following conditions holds:*
  - *(i) at least one of G and H is edgeless,*
  - *(ii) G is a 2-linear forest and H is a forest, or vice versa,*
- *interval if and only if one of the following conditions holds:*
  - *(i) at least one of G and H is edgeless,*
  - *(ii) G is a 2-linear forest and H is a caterpillar forest, or vice versa,*
- *circular arc graph if and only if one of the following conditions holds:*
  - *(i) at least one of G and H is edgeless,*
  - *(ii) G is a 2-linear forest and H is a caterpillar forest, or vice versa,*
  - *(iii) $G \cong K_2$ and H is an odd circular caterpillar, or vice versa.*

Proof. We prove the three equivalences in the order as stated in the theorem.

First suppose that $G \times H$ is chordal, and that both G and H contain an edge. We claim that G (and then, by symmetry, also H) is a forest. Indeed, if G contained a cycle, then $G \times H$ would contain an induced subgraph isomorphic to the direct product of K_2 with a cycle, which contains an induced cycle of length at least 4, contrary to the fact that $G \times H$ is chordal. It remains to show that at least one of G and H is a 2-linear forest. If this were not the case, then $G \times H$ would contain an induced copy of $P_3 \times P_3$, which contains an induced C_4 and therefore is not chordal, a contradiction.

For the converse direction, suppose that one of conditions (i) and (ii) holds. If condition (i) holds, then $G \times H$ is edgeless and hence chordal. Assume now that condition (ii) holds, say G is a 2-linear forest and H is a forest. In this case, for each component T of H, the graphs $K_1 \times T$ and $K_2 \times T$ are acyclic, and hence so is $G \times H$, which is the disjoint union of such graphs. It follows that $G \times H$ is chordal.

Assume now that $G \times H$ is interval. Since $G \times H$ is interval, it is chordal, thus one of the conditions for the chordal case holds. Therefore, necessity of the stated conditions is achieved, unless (without loss of generality) G is a 2-linear forest containing an edge and H is a forest that is not a caterpillar forest. By Lemma 2.7, H contains an induced bipartite claw, and consequently $G \times H$ contains an induced subgraph, say F, isomorphic to the direct product of K_2 with the bipartite claw. A direct inspection shows that F is isomorphic to the disjoint union of two copies of the bipartite claw, therefore by Lemma 2.7, F is not interval, and hence neither is $G \times H$. This establishes necessity. Let us now prove sufficiency. If one of G and H, say G, is edgeless, then $G \times H$ is edgeless and therefore interval. Now, if G is a 2-linear forest and H is a caterpillar forest, then each component of $G \times H$ is interval. This is because for each component K of H, the components of $G \times H$ are either $K_1 \times K$ or $K_2 \times K$, both caterpillar forests, and in particular interval graphs (Lemma 2.7). The result now follows from the fact that interval graphs are closed under disjoint union.

Finally, we consider the circular arc case. Suppose first that $G \times H$ is a circular arc graph. Then it is 1-perfectly orientable, in particular, one of the conditions (i)–(iii) from Theorem 5.11 holds. Condition (i) from that theorem coincides with condition (i) in Theorem 6.5, so we may assume that both G and H contain an edge.

Suppose that condition (ii) from Theorem 5.11 holds, say G is a 2-linear forest and H is a pseudoforest (the other case is symmetric). We consider two cases depending on whether H is acyclic or not.

Case 1: H is acyclic. We claim that in this case H is a caterpillar forest (and hence condition (ii) holds in this case). If this is not the case, then, by Lemma 2.7, H would contain an induced subgraph, say K, isomorphic to the bipartite claw, but then $G \times H$ would contain $K_2 \times K \cong 2K$ as induced subgraph, contradicting the fact that $2K$ is not is a circular arc graph (by Lemma 2.7). Hence, condition (ii) of the proposition holds in this case.

Case 2: H contains a component, say K, with a cycle. If K contains an even cycle (say of length $2k \geq 4$), then $G \times H$ contains $2C_{2k}$ as induced subgraph, contrary to the fact that it is a circular arc graph (by Fact 2.4, $C_{2k} + K_1$ is not circular arc and therefore neither is $2C_{2k}$). Hence, K contains a (unique) odd cycle, say C. If H has a vertex with no neighbors on C, then $G \times H$ contains an induced subgraph isomorphic to $C_{2k} + K_1$ where $k \geq 3$ is the length of C, contrary to the fact that $G \times H$ is a circular arc graph. It follows that every vertex not in C has a neighbor in C, and in particular, since H is a pseudoforest, that every vertex not in C has a unique neighbor in C and that $V(H) \setminus C$ is an independent set in H. Consequently, H is an odd circular caterpillar. If G were not isomorphic to K_2, the product $G \times H$ would contain an induced $C_{2k} + K_1$, contradicting Fact 2.4. We conclude that $G \cong K_2$ and hence condition (iii) applies in this case.

Finally, suppose that condition (iii) from Theorem 5.11 holds, say G is a 3-linear forest and H is a 4-linear forest. To avoid the already considered condition (ii) (from Theorem 5.11), we may assume that neither of G and H is a 2-linear forest. But then $G \times H$ contains an induced copy of $P_3 \times P_3$, which is not a circular arc graph (since it contains an induced $C_4 + K_1$), a contradiction.

For the converse direction, suppose that one of the conditions (i)–(iii) holds. If condition (i) holds, then $G \times H$ is edgeless and hence circular arc. Assume now that both G and H contain an edge and that condition (ii) holds, say G is a 2-linear forest and H is a caterpillar forest. In this case, for each component K of H, the graphs $K_1 \times K$ and $K_2 \times K$ are caterpillar forests, in particular, by Lemma 2.7, they are interval graphs. It follows that $G \times H$, which is the disjoint union of such graphs, is also interval, and hence circular arc. Finally, if condition (iii) holds, say $G \cong K_2$ and H is an odd circular caterpillar, then $G \times H$ is a circular caterpillar. It is easy to see that every circular caterpillar is circular arc: we can obtain a circular arc representation of it by covering the circle with arcs corresponding to vertices of the cycle, and placing a new arc corresponding to each leaf within the arc corresponding to its unique neighbor in the cycle without intersecting any other arc. □

Theorem 6.5 implies the following.

Corollary 6.6. *Let $\mathcal{C}_\times$, $\mathcal{I}_\times$, resp. $\mathcal{CA}_\times$, denote the sets of (isomorphism classes of) nontrivial* *direct product graphs that are chordal, interval, resp. circular arc. Then:*

$$
\begin{aligned}
\mathcal{C}_\times &= \{mnK_1 : m \geq 2,\ n \geq 2\} \\
&\quad \cup \{2mF + n|V(F)|K_1 : F \textit{ is a forest},\ m \geq 1,\ n \geq 0,\ |V(F)| \geq 2\}\,, \\
\mathcal{I}_\times &= \{mnK_1 : m \geq 2,\ n \geq 2\} \\
&\quad \cup \{2mF + n|V(F)|K_1 : F \textit{ is a caterpillar forest},\ m \geq 1,\ n \geq 0,\ |V(F)| \geq 2\}\,, \\
\mathcal{CA}_\times &= \{mnK_1 : m \geq 2,\ n \geq 2\} \\
&\quad \cup \{2mF + n|V(F)|K_1 : F \textit{ is a caterpillar forest},\ m \geq 1,\ n \geq 0,\ |V(F)| \geq 2\} \\
&\quad \cup \{G : G \textit{ is a circular caterpillar satisfying conditions } (*)\}\,,
\end{aligned}
$$

where conditions $(*)$ are the following:

- *the unique cycle C of G is of length $4k+2$ for some $k \geq 1$, and*
- *every two vertices at distance $2k+1$ on C are of the same degree in G.*

Proof. The statement of the corollary follows immediately from the characterizations given by Theorem 6.5 and the following facts:

– If $G \times H$ is a nontrivial direct product such that $m = |V(G)| \geq 2$, $n = |V(H)| \geq 2$, and at least one of the two factors is edgeless, then $G \times H$ is an edgeless graph of order mn.

– If H is a bipartite graph, then $K_2 \times H \cong 2G$ (see, e.g., [45, Exercise 8.14]). In particular, if H is a forest (resp. caterpillar forest), then $K_2 \times H \cong 2H$.

– The direct product is distributive (up to isomorphism) with respect to the disjoint union.

– Suppose that H is an odd circular caterpillar, with its unique cycle, say C, of length $2k+1$ for some $k \geq 1$. Then, $K_2 \times H$ is isomorphic to a circular caterpillar, say G, the unique cycle of which, say C', has length $2(2k+1) = 4k+2$. Moreover, every vertex v of C corresponds to a pair of vertices v', v'' of C' at distance $2k+1$ in C', such that $d_G(v') = d_G(v'') = d_H(v)$. □

6.4 The strong product

In this section we consider the strong product and we characterize when a nontrivial strong product of two graphs G and H is chordal, interval, or circular arc, respectively.

To prove the characterization of circular arc nontrivial strong product graphs, we need one further lemma.

Lemma 6.7. *Let G, G', and H be graphs such that G' is obtained from G by adding a true twin. Then, $G \boxtimes H$ is circular arc if and only if $G' \boxtimes H$ is circular arc.*

Proof. Note that $G \boxtimes H$ is an induced subgraph of $G' \boxtimes H$, therefore if $G' \boxtimes H$ is circular arc, then so is $G \boxtimes H$. Suppose now that $G \boxtimes H$ is circular arc, and that G' was obtained from G by adding to it a true twin x' to a vertex x of G. Note that for every $v \in V(H)$, we have $N_{G' \boxtimes H}[(x,v)] = N_{G'}[x] \times N_H[v]$ and $N_{G' \boxtimes H}[(x',v)] = N_{G'}[x'] \times N_H[v]$. Since $N_{G'}[x] = N_{G'}[x']$, each vertex of the form (x',v) for $v \in V(H)$ is a true twin in $G' \boxtimes H$ of vertex (x,v). It follows that $G' \boxtimes H$ can be obtained from $G \boxtimes H$ by a sequence of true twin additions. Since circular arc graphs are closed under true twin additions, $G' \boxtimes H$ is circular arc. □

We now state and prove the main result of this section. Recall that a graph G is said to be *co-chain* if its vertex set can be partitioned into two cliques, say X and Y, such that the vertices in X can be ordered as $X = \{x_1, \ldots, x_{|X|}\}$ so that for all $1 \leq i < j \leq |X|$, we have $N[x_i] \subseteq N[x_j]$, and 2-*complete* if G can be obtained from P_3 by applying a sequence of true twin additions.

Theorem 6.8. *A nontrivial strong product, $G \boxtimes H$, of two graphs G and H is:*

- *chordal if and only if every component of G is complete and H is chordal, or vice versa,*
- *interval if and only if every component of G is complete and H is interval, or vice versa,*
- *circular arc if and only if one of the following conditions holds:*
  - *(i) G is complete and H is a circular arc graph, or vice versa,*
  - *(ii) G is 2-complete and H is a connected co-chain graph, or vice versa,*
  - *(iii) each component of G is complete and H is interval, or vice versa.*

Proof. Again, we prove the three equivalences in the order as stated in the theorem.

Suppose first that $G \boxtimes H$ is chordal. Each graph G and H must also be chordal since they are induced subgraphs of $G \boxtimes H$. Suppose now that not all components of G are complete and no all components of H are complete. Therefore there is a component of G and a component of H each having an induced P_3. But $P_3 \boxtimes P_3$ contains an induced 4-cycle, and is therefore not chordal, a contradiction. Thus, all components of one of the factors must be complete.

To show sufficiency, let $G_1, \ldots, G_k$ be the components of G, let $H_1, \ldots, H_\ell$, be the components of H, and suppose that G_i is complete for $i = 1, \ldots, k$, and H is chordal. Note that the components of $G \boxtimes H$ are of the form $G_i \boxtimes H_j$ for $1 \leq i \leq k$, $1 \leq j \leq \ell$. Every component $G_i \boxtimes H_j$ of $G \boxtimes H$ is chordal since it is the result of applying a sequence of true twin additions to a chordal graph, namely H_j. (The operation of adding a true twin is easily seen to preserve chordality.) Since each component of $G \boxtimes H$ is chordal and chordal graphs are closed under disjoint union, we conclude that $G \boxtimes H$ is chordal.

Suppose now that $G \boxtimes H$ is interval. Again, G and H must be interval since they are induced subgraphs of the product. Necessity follows immediately from the chordal case. To conclude the proof for the interval case, assume that every component of G is complete and H is interval. In that case, the strong product $G \boxtimes H$ can be obtained as disjoint union of graphs each of which is the result of applying a sequence of true twin additions to the interval graph H. Since the operations of disjoint union and true twin addition preserve the class of interval graphs, we conclude that $G \boxtimes H$ is interval.

It remains to analyze the circular arc case.

Necessity. Suppose that $G \boxtimes H$ is circular arc. Then, G and H are induced subgraphs of $G \boxtimes H$ and therefore circular arc as well. Suppose that G and H are both connected. Since $G \boxtimes H$ is 1-perfectly orientable, by Theorem 5.23, either G is complete, or G is 2-complete and H is co-chain. So we are in cases (i) or (ii), respectively.

Now, if not both factors are connected, the product $G \boxtimes H$ is disconnected. Since $G \boxtimes H$ is circular arc, by Lemma 2.3 we know that all its components are interval. Moreover, since for every component G_i of G and every component H_j of H their product $G_i \boxtimes H_j$ is a component of $G \boxtimes H$ we infer that all components of G are interval, and similarly for H. Therefore, G and H are interval. Since $G \boxtimes H$ is a disjoint union of interval graphs, it is interval, and in particular chordal. Thus we can apply the already stablished characterization for the chordal case, and so one of G and H must be a disjoint union of complete graphs. This concludes the proof of the forward implication.

Sufficiency. We will show that if one of *(i), (ii)*, or *(iii)* holds, then $G \boxtimes H$ is circular arc.

If condition *(i)* holds, say G is complete and H is circular arc, then the product $G \boxtimes H$ is the result of applying a sequence of true twin additions to a circular arc graph, namely H, and so it is circular arc.

Suppose now that *(ii)* holds, say G is 2-complete and H is a connected co-chain graph. By Lemma 6.7, we may assume that both factors are true-twin-free. Therefore, $G \cong P_3$ and, by Lemma 5.21 $H \in \{K_1\} \cup \{R_n, n \geq 1\} \cup \{R_n * K_1, n \geq 0\}$. Notice first that $P_3 \boxtimes K_1 \cong P_3$ is circular arc. Since $R_n * K_1$ is an induced subgraph of R_{n+2}, it is enough to show that $P_3 \boxtimes R_n$ is circular arc for all $n \geq 1$.

Let $V(P_3) = \{u_1, u_2, u_3\}$ where u_1 and u_3 are the two leaves. Assuming the notation as in the definition of rafts, let $V(R_n) = X \cup Y$, where $X = \{x_0, x_1, \ldots, x_n\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ are the two parts of the raft. Vertices in $P_3 \boxtimes R_n$ will be said to be *left*, resp. *right*, depending on whether their second coordinate is in X or in Y, respectively.

Fig. 6.1 shows a schematic representation of $P_3 \boxtimes R_n$. We partition the vertex set of the graph in the following way: 6 singletons, namely $\{a_1\}, \{a_2\}, \{a_3\}, \{b_1\}, \{b_2\}, \{b_3\}$, where $a_i = (u_i, x_0)$ and $b_i = (u_{4-i}, y_0)$, and 6 cliques of size k each, namely A_1, A_2, A_3, B_1, B_2, and B_3, defined as follows: for $i \in \{1, 2, 3\}$, we have $A_i = \{u_i\} \times (X \setminus \{x_0\})$ and $B_i = \{u_{4-i}\} \times (Y \setminus \{y_0\})$. Bold lines between certain pairs of sets mean that every possible edge between the two sets is present. If the corresponding line is not bold, then only some of the edges between the two sets are present.

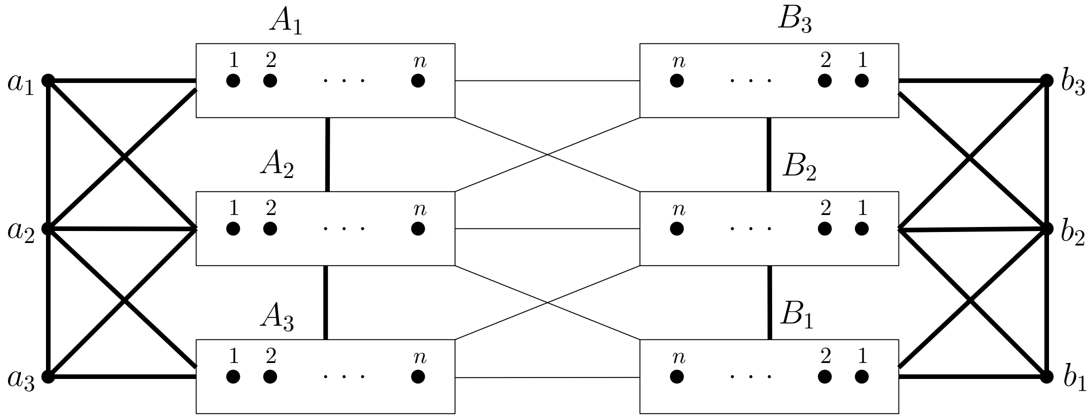


Figure 6.1: A schematic representation of $P_3 \boxtimes R_n$

To describe such edges, we introduce the following ordering of the vertices within each of the 6 cliques $A_1, A_2, A_3, B_1, B_2, B_3$ of size n. Note that for every $1 \leq i < j \leq n$, we have that $N_{R_n}[x_i] \subset N_{R_n}[x_j]$ and $N_{R_n}[y_i] \subset N_{R_n}[y_j]$. We order the vertices in the 6 cliques accordingly, that is, for each clique of the form A_i, the linear ordering of its vertices is $(u_i, x_1), \ldots, (u_i, x_n)$; for each clique of the form B_i, the linear ordering of its vertices is $(u_{4-i}, y_1), \ldots, (u_{4-i}, y_n)$. To keep the notation light, we will slightly abuse the notation, speaking of "vertex i in clique C" (for $i \in \{1, \ldots, n\}$ and $C \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$) when referring to the i-th vertex in the linear ordering of C.

The edges of graph G can be now concisely described as follows. We will say that two cliques K_i and K_j (where K_ℓ is either $A_\ell, B_\ell, \{a_\ell\}$, or $\{b_\ell\}$ for some ℓ) are *adjacent* if $|i - j| \leq 1$. The closed neighborhood of a_i is the union of the cliques A_j adjacent to a_i and $\{a_1\} \cup \{a_2\} \cup \{a_3\}$. The neighborhood of b_i is the union of the cliques B_j adjacent to b_i and $\{b_1\} \cup \{b_2\} \cup \{b_3\}$. For each vertex i in a left clique, say A_j, its closed neighborhood consists of the vertices a_i in its adjacent cliques $\{a_i\}$, all the vertices belonging to some left clique adjacent to A_j, and of vertices $\{n - i + 1, \ldots, n\}$ in each right clique adjacent to B_{4-j}. For each vertex i in a right clique, say B_j, its closed neighborhood consists of the vertices b_i in its adjacent cliques $\{b_i\}$, all vertices belonging to some right clique adjacent to B_j, and of vertices $\{n-i+1,\dots,n\}$ in each left clique adjacent to A_{4-j}.

For any two adjacent cliques A_i, B_j, the vertices of $A_i \cup B_j$ induce a special co-chain graph, called a semiraft. Given a non-negative integer $n \geq 0$, the *semiraft of order* n is the graph S_n consisting of two disjoint cliques on n vertices each, say $X = \{x_1,\dots,x_n\}$ and $Y = \{y_1,\dots,y_n\}$ together with additional edges between X and Y such that for every $1 \leq i, j \leq n$, vertex x_i is adjacent to vertex y_j if and only if $i + j \geq n$.

As shown by the interval representation given in Fig. 6.2, every semiraft is an interval graph.

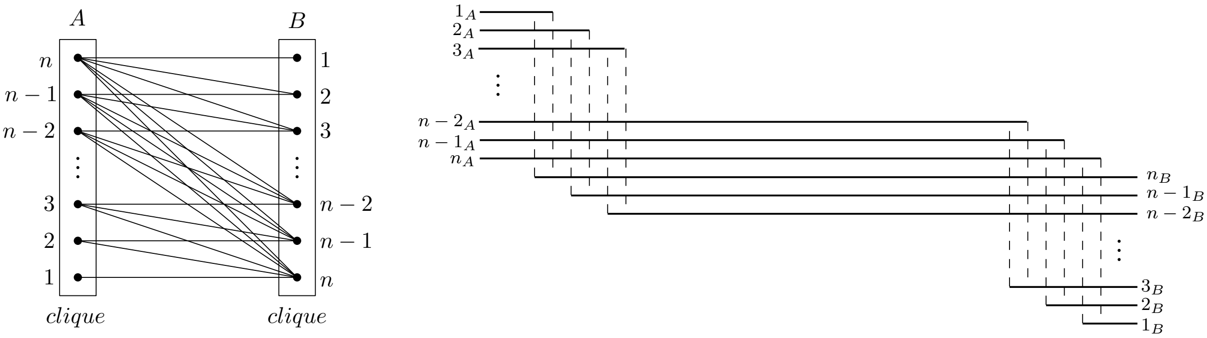


Figure 6.2: The semiraft S_n and its interval representation.

Suppose first that $n = 1$. A circular arc representation of $P_3 \boxtimes R_1$ is depicted in Fig. 6.3. (The rectangles P and Q also depicted in Fig. 6.3 are not part of the representation, they will be used later on in the proof.)

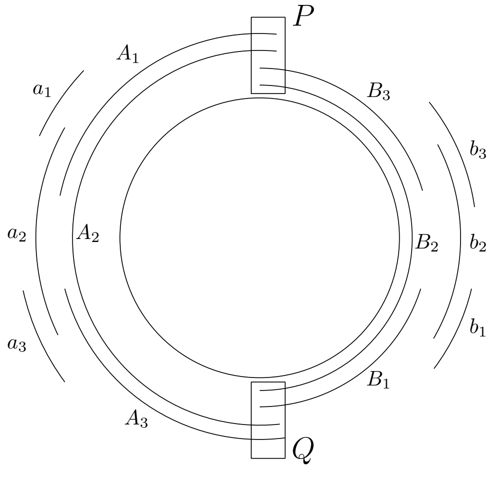


Figure 6.3: A circular arc representation of $P_3 \boxtimes R_1$.

Suppose now that $n > 1$. We will give a circular arc representation of $P_3 \boxtimes R_n$ similar to that of $P_3 \boxtimes R_1$ shown in Fig. 6.3, combined with the interval representations of semirafts represented by Fig. 6.2. The circular arc representation of $P_3 \boxtimes R_n$ is the same as in Fig. 6.3, but this time instead of each clique $C \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$ being represented by a single arc, it will consist of n arcs. If we were to "zoom in" at the rectangles marked as P and Q in Fig. 6.3 to see how the arcs representing the four cliques interact, then we would see the representations shown in Fig. 6.4 and 6.5 below.

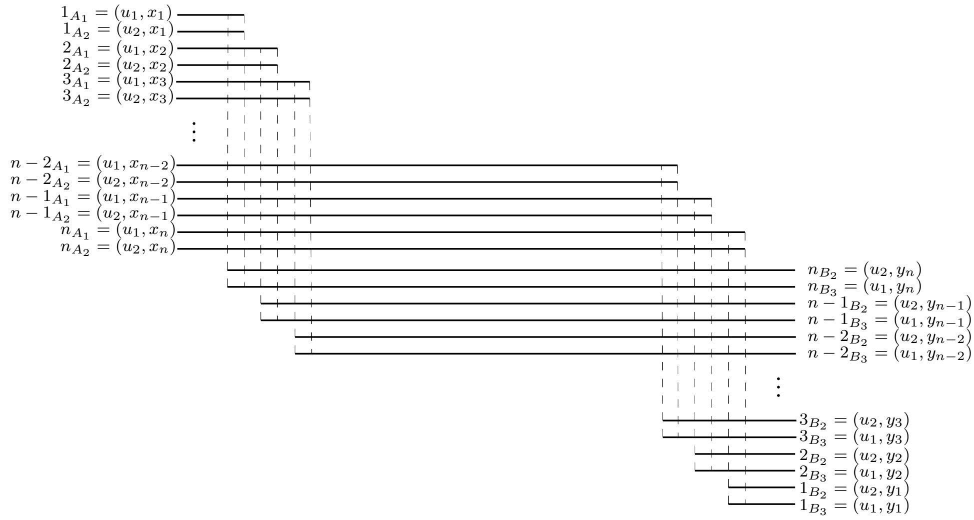


Figure 6.4: Intersection of cliques A_1, A_2, B_2, and B_3.

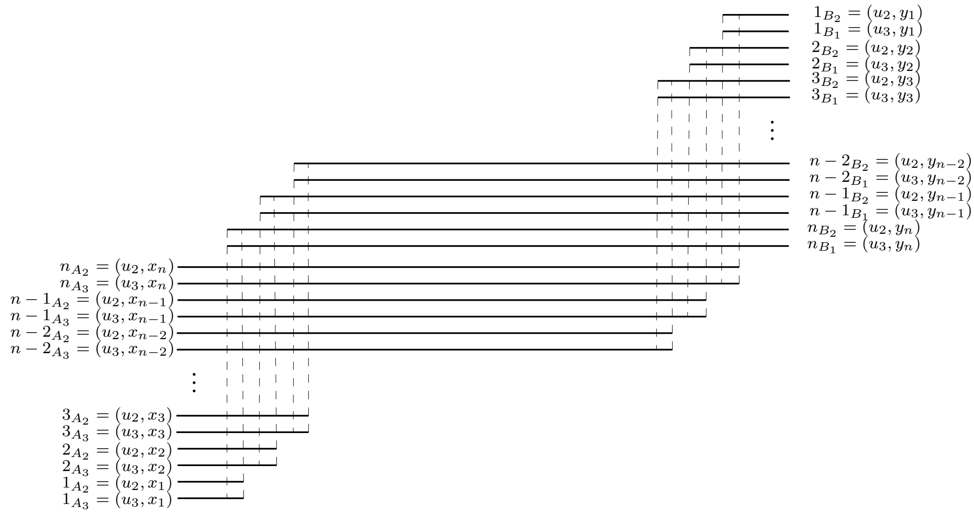


Figure 6.5: Intersection of cliques A_2, A_3, B_1, and B_2.

This gives a circular arc representation of the graph $P_3 \boxtimes R_n$. This implies that if G is 2-complete and H is co-chain, the strong product $G \boxtimes H$ is circular arc, concluding this part of the proof.

Finally, if condition *(iii)* holds, say each component of G is complete and H is interval, then each component of the strong product, $G_i \boxtimes H_j$, is interval, since it can be obtained by applying a sequence of true twin additions to an interval graph, H_j. It follows from Lemma 2.3 that $G \boxtimes H$ is circular arc. $\square$

It was shown in Lemma 5.19 that for each $n \geq 1$, the graph $P_3 \boxtimes R_n$ is 1-perfectly orientable. Theorem 6.8 (and its proof) imply that for each $n \geq 1$, the graph $P_3 \boxtimes R_n$ is circular arc. Since the class of circular arc graphs is a subclass of the class of 1-perfectly orientable graphs, this gives an alternative proof of the fact that graphs of the form $P_3 \boxtimes R_n$ are 1-perfectly orientable.

Since the strong product is distributive (up to isomorphism) with respect to the disjoint union, Theorem 6.8 implies the following.

Corollary 6.9. *Let $\mathcal{C}_{\boxtimes}$, $\mathcal{I}_{\boxtimes}$, resp. $\mathcal{CA}_{\boxtimes}$, denote the sets of (isomorphism classes of) nontrivial direct product graphs that are chordal, interval, resp. circular arc. Then:*

$$
\begin{aligned}
\mathcal{C}_{\boxtimes} &= \left\{ \bigplus_{i=1}^{k} (G \boxtimes K_{n_i}) : G \textit{ chordal}, \ |V(G)| \geq 2, \ k \geq 1, \ n_i \geq 1 \, \forall i = 1, \ldots, k, \ \sum_{i=1}^{k} n_i \geq 2 \right\}, \\
\mathcal{I}_{\boxtimes} &= \left\{ \bigplus_{i=1}^{k} (G \boxtimes K_{n_i}) : G \textit{ interval}, \ |V(G)| \geq 2, \ k \geq 1, \ n_i \geq 1 \, \forall i = 1, \ldots, k, \ \sum_{i=1}^{k} n_i \geq 2 \right\}, \\
\mathcal{CA}_{\boxtimes} &= \{ G \boxtimes K_n : G \textit{ circular arc}, \ n \geq 2, \ |V(G)| \geq 2 \} \\
&\quad \cup \{ G \boxtimes H : G \textit{ 2-complete}, \ H \textit{ connected and co-chain}, \ |V(H)| \geq 2 \} \\
&\quad \cup \left\{ \bigplus_{i=1}^{k} (G \boxtimes K_{n_i}) : G \textit{ circular arc}, \ |V(G)| \geq 2, \ k \geq 1, \ n_i \geq 1 \, \forall i = 1, \ldots, k, \ \sum_{i=1}^{k} n_i \geq 2 \right\}.
\end{aligned}
$$

Chapter 7

The price of connectivity for cycle transversals

Recall that an $\mathcal{F}$-transversal of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free; that is, S intersects every subset of V that induces a subgraph isomorphic to a graph in $\mathcal{F}$. For a connected graph G, $t_F(G)$ denotes the minimum size of an $\mathcal{F}$- transversal of G, and $ct_F(G)$ denotes the minimum size of a connected $\mathcal{F}$-transversal of G. Recall that for graphs H and G, we write $H \subseteq_i G$ to denote that H is an induced subgraph of G.

Our aim in this chapter is to find relationships between $ct_F(G)$ and $t_F(G)$; more particularly, we ask for a class of connected graphs $\mathcal{G}$, whether we can find a bound for $ct_F(G)$ in terms of $t_F(G)$ that holds for all $G \in \mathcal{G}$.

More precisely, we consider a number of families $\mathcal{F}$ that contain cycles, paths and complements of cycles. We study $\mathcal{F}$-transversals for graph classes characterized by one forbidden induced subgraph and ask whether the size of a minimum size connected $\mathcal{F}$-transversal can be bounded (and if so, to what extent) in terms of the size of a minimum size $\mathcal{F}$-transversal.

This chapter contains results from the following paper:

- T. R. Hartinger, M. Johnson, M. Milanič and D. Paulusma, *The price of connectivity for cycle transversals.* European Journal of Combinatorics. 58, (2016), 203-224. An extended abstract appeared in Mathematical Foundations of Computer Science 2015. Part II, volume 9234 of Lecture Notes in Computer Science, 395 – 406, Springer, 2015.

7.1 Our results

Table 7.1 summarizes our results together with related previous work. Results can be seen both according to the family $\mathcal{F}$ and the corresponding property of the graph $G - S$, where S is an $\mathcal{F}$-transversal of G. We note that when $\mathcal{F}$ is the family of even cycles or holes there is an open case. In all other cases, the stated conditions in Table 7.1 are both necessary and sufficient for $\mathcal{F}$-multiplicativity ($\mathcal{F}$-boundedness), $\mathcal{F}$-additivity, and $\mathcal{F}$-zero-additivity, respectively, in the class of connected H-free graphs.

Table 7.1 shows conditions on the graph H for the price of connectivity of $\mathcal{F}$-transversal for the class of H-free graphs to be multiplicative, additive or zero-additive, respectively, when $\mathcal{F}$ is a family of graphs that contains the specified infinite family of cycles and possibly some other small graphs. The results on cycles in the first row are due to Belmonte et al. [6] and the multiplicativity result on cycles and P_2 in the ninth row is due to Camby et al. [11]. All other results are new and presented in this work. All conditions are *necessary* and *sufficient* except for even cycles and holes, as in these two cases (marked by a † in the table) we do not know if H-free graphs are $\mathcal{F}$-additive for $H \subseteq_i P_3 + P_2 + sP_1$.

From Table 7.1 we can draw a number of conclusions. If a transversal that intersects (small) paths is wanted, we obtain multiplicative bounds for any class of H-free graphs. In all other cases, H may not contain a cycle or a claw (so is a linear forest). We also see that when we add a requirement that all triangles are intersected, there is always a jump from $H = P_4 + sP_1$ to $H = P_5 + sP_1$ for the additive bound. In general, it can be noticed that adding small graphs to $\mathcal{F}$ has differing effects. We say that a family of graphs $\mathcal{F}$ or a graph F *positively (negatively) influences* a family of graphs $\mathcal{F}'$ if the row in the table for their union contains more (fewer) bounded cases than the row for $\mathcal{F}'$. So, for example, $2P_2$ does not influence $\{C_4, C_5, C_6, \ldots\} \cup \{P_4\}$, and P_4 does not influence the family of long holes. Moreover, odd holes do not influence even holes, whereas even holes influence odd holes positively.

$\mathcal{F}$	Property of $G - S$	Condition for $\mathcal{F}$-multiplicativity (for $\mathcal{F}$-boundedness)	Condition for $\mathcal{F}$-additivity	Condition for $\mathcal{F}$-zero-additivity
cycles	forest	H is a linear forest [6]	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ [6]	$H \subseteq_i P_3$ [6]
odd cycles	bipartite	H is a linear forest	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$	$H \subseteq_i P_3$
even cycles† (equiv.: even holes)	even-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
holes†	chordal	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
odd holes	odd-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
odd holes and odd antiholes	perfect	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
long holes	long-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
long holes and long antiholes	weakly chordal	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
cycles and P_2 (equiv.: $\{P_2\}$)	edgeless	no restriction [11]	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$	$H \subseteq_i P_3$
holes and $2P_2$ (equiv.: $\{C_4, C_5, 2P_2\}$)	split	no restriction	$H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$	$H \subseteq_i P_3$
holes and $2P_2, P_4$ (equiv.: $\{C_4, 2P_2, P_4\}$)	threshold	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
holes and P_4 (equiv.: $\{C_4, P_4\}$)	trivially perfect	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
long holes and $2P_2$ (equiv.: $\{C_5, 2P_2\}$)	$\{C_5, 2P_2\}$-free	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ $H \subseteq_i P_2 + P_1$
long holes and $2P_2, P_4$ (equiv.: $\{2P_2, P_4\}$)	cotrivially perfect	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$
long holes and P_4 (equiv.: $\{P_4\}$)	cograph	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$

Table 7.1: Table showing conditions on the graph H for the price of connectivity of $\mathcal{F}$-transversal for the class of H-free graphs to be multiplicative, additive or zero-additive, respectively, for a given family $\mathcal{F}$.

In the remainder of the chapter, after presenting some known and new basic results, we present a number of general theorems, from which the results in Table 7.1 directly follow. We emphasize that all proofs of these theorems are algorithmic in nature, that is, they can be translated directly into polynomial-time algorithms that modify an $\mathcal{F}$-transversal into a connected $\mathcal{F}$-transversal of appropriate cardinality.

We provide a brief guide to the proof of Table 7.1. Theorem 7.9 implies the second row. Theorem 7.11 implies the third and fourth row, and Theorem 7.14 implies the next four rows. The ninth row follows from Theorem 7.22 and the tenth from Theorem 7.23. Theorem 7.24 implies the eleventh and twelfth rows. The final three rows follow from Theorems 7.25, 7.26 and 7.27, respectively.

7.1.1 Some results on the price of connectivity

We now give five results that are directly related to the concept of price of connectivity and that we will need in our later proofs. All results, except the first one, which follows from Lemma 2.14, can be found in the papers of Belmonte et al. [6, 7] or follow from results in these papers after a straightforward generalization (which we need).

Lemma 7.1. *For every family $\mathcal{F}$ of graphs, the class of connected P_4-free graphs is $\mathcal{F}$-additive.*

Proof. Let G be a connected P_4-free graph with two or more vertices, with a minimum $\mathcal{F}$-transversal S. By Lemma 2.14, G has a dominating edge, say uv. So $S \cup \{u, v\}$ is a connected $\mathcal{F}$-transversal of G, implying that $ct_{\mathcal{F}}(G) \leq t_{\mathcal{F}}(G)+2$. Since the above inequality trivially holds for the one-vertex graph, we conclude that the class of connected P_4-free graphs is $\mathcal{F}$-additive, with $d_{P_4} \leq 2$. □

The second result has been proven by Belmonte et al. [6] for the special case when the family $\mathcal{F}$ consists of all cycles.

Lemma 7.2. *For any family of graphs $\mathcal{F}$ with $K_r \in \mathcal{F}$ for some integer $r \geq 1$, the class of connected P_5-free graphs is $\mathcal{F}$-additive.*

Proof. Let G be a connected P_5-free graph. Let S be a minimum $\mathcal{F}$-transversal of G. By Lemma 2.15, G has a dominating set D that induces a P_3 or a complete graph. In the first case, $S \cup D$ is a connected $\mathcal{F}$-transversal of G of size at most $|S|+3$. In the second case, $|D \setminus S| \leq r-1$. So in this case $S \cup D$ is a connected $\mathcal{F}$-transversal of G of size at most $|S| + r - 1$. □

We also need to generalize a result that was proved by Belmonte et al. [6] for the graph $H = P_5$. The proof for the general case is the same and we state it here for completeness.

Lemma 7.3. *For a family of graphs $\mathcal{F}$ and a graph H, if the class of connected H-free graphs is $\mathcal{F}$-additive, then so is the class of connected $\{H + sP_1\}$-free graphs for all $s \geq 1$.*

Proof. Let G be a connected $\{H + sP_1\}$-free graph for some $s \geq 0$. We prove that $ct_{\mathcal{F}}(G) \leq t_{\mathcal{F}}(G) + d_{H+sP_1}$ for some constant d_{H+sP_1} by induction on s. If $s = 0$ the statement holds by assumption. Now let $s \geq 1$. If G is $\{H + (s-1)P_1\}$-free, then the statement holds by the induction hypothesis. Suppose G is not $\{H + (s-1)P_1\}$-free. Let F be an induced subgraph of G isomorphic to $H + (s-1)P_1$. Because G is $\{H + sP_1\}$-free, F dominates G. By Lemma 2.16 we find that G has a connected dominating set D of size at most $3|V(F)| - 2$. Let S be a minimum $\mathcal{F}$-transversal of G. Then $S \cup D$ is a connected $\mathcal{F}$-transversal of G of size at most $t_{\mathcal{F}}(G) + 3|V(F)| - 2$. Hence, we can take $d_{H+sP_1} = 3|V(H)| + 3s - 5$. □

Belmonte et al. [6] proved that the class of connected $\{P_2 + P_4, P_6\}$-free graphs is not $\mathcal{F}$-additive if $\mathcal{F}$ is the class of all cycles. To prove this result they showed that the family $\{L_k : k \geq 1\}$ of connected $\}P_2 + P_4, P_6\}$-free graphs displayed in Figure 7.1 is not $\mathcal{F}$-additive. Using the observation made in the caption of Figure 7.1 leads to the following more general result.

Lemma 7.4. *For any family of cycles $\mathcal{F}$ with $C_3 \in \mathcal{F}$, the class of connected $\{P_2+P_4, P_6\}$-free graphs is not $\mathcal{F}$-additive.*

As a consequence of Lemma 7.4, any class of connected graphs that contains all connected $\{P_2+P_4, P_6\}$-free graphs is not $\mathcal{F}$-additive either. More generally, if $\mathcal{G}$ and $\mathcal{G}'$ are two classes of connected graphs such that $\mathcal{G} \subseteq \mathcal{G}'$ and $\mathcal{G}$ is not $\mathcal{F}$-additive, then neither is $\mathcal{G}'$. We will use this fact implicitly throughout this chapter.

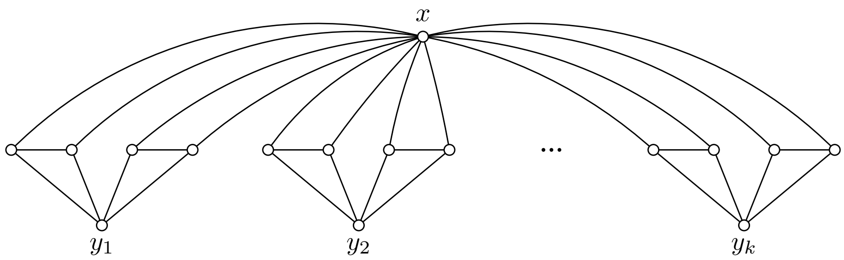


Figure 7.1: The graph L_k, defined by Belmonte et al. [6] for every $k \geq 1$; note that $\{y_1, \ldots, y_k, x\}$ is the unique minimum $\mathcal{F}$-transversal whenever $\mathcal{F}$ is any family of cycles with $C_3 \in \mathcal{F}$ and that any minimum connected $\mathcal{F}$-transversal has size $2k+1$.

Finally, the following technical lemma of Belmonte et al. [6] will also be useful for proving our results.

Lemma 7.5 (Belmonte et al. [6])**.** *Let $s \geq 1$ be an integer and let G be a connected sP_3-free graph with a subset $S \subseteq V(G)$ and an independent set $U \subseteq V(G) \setminus S$. If there exists a component Z of $G[S]$ that contains an induced copy of $(s-1)P_3$, then there exists a set S' with $S \subseteq S'$ of size at most $|S| + 2s - 2$ such that*

(i) $G[S']$ has a component Z' containing all vertices of $V(Z) \cup (S' \setminus S)$;

(ii) every vertex of $U' = U \setminus S'$ is adjacent to at most one component of $G[S']$ that is not equal to Z';

(iii) every component of $G[S']$ not equal to Z' is adjacent to at most one vertex of U'.

7.2 A new general theorem

The following theorem is used in all our tetrachotomies. The third part was shown by Belmonte et al. [6] for the case when $\mathcal{F}$ is the family of all cycles, and our proof for that part is a modification of theirs.

Theorem 7.6. *Let $\mathcal{F}$ be a family of graphs and let H be a graph. Then, the following three statements hold:*

(i) If $\mathcal{F}$ contains a linear forest, then the class of all connected graphs is $\mathcal{F}$-multiplicative.

(ii) If H is a linear forest, then the class of connected H-free graphs is $\mathcal{F}$-multiplicative.

(iii) If $\mathcal{F}$ contains an infinite number of cycles and no linear forests and H is not a linear forest, then the class of connected H-free graphs is $\mathcal{F}$-unbounded.

Proof. We start with (i). First suppose that $\mathcal{F}$ contains a linear forest F; that is, it is, say, the disjoint union of p paths. Let G be a connected graph, and let S be a minimum $\mathcal{F}$-transversal of G with components $D_1, \ldots, D_r$ for some integer $r \geq 1$. Because G is connected, we can connect the components of S by $r-1$ paths using vertices of $G-S$ only. Let S' be the resulting connected $\mathcal{F}$-transversal. Because $G-S$ is $\mathcal{F}$-free, $G-S$ is F-free. Let q be the length of a longest path in F. As the path $P_{p(q+2)}$ contains F as an induced subgraph and $G-S$ is F-free, $G-S$ is $P_{p(q+2)}$-free. Hence, each of the $r-1$ paths contains less than $p(q+2)$ vertices. Thus we find that $|S'| \leq |S| + rp(q+2) \leq |S| + |S|(p(q+2)) = (p(q+2)+1)|S|$, and we can take $c_{\mathcal{F}} = (p(q+2)+1)$.

Now we prove (ii). Suppose that H is a linear forest; that is, it is, say, the union of k paths, each of length at most ℓ. Let $G = (V, E)$ be a connected H-free graph. Then, as G is H-free, we find that G has diameter less than $k(\ell+2)$. Let $S \subseteq V$ be a minimum $\mathcal{F}$-transversal of G. Let $D_1, D_2, \ldots, D_r$ $(r \geq 1)$ be the components of $G[S]$. In order to make S connected we need to add less than $(r-1)k(\ell+2) \leq (|S|-1)k(\ell+2)$ vertices by Lemma 2.17. Hence we can take $c_H = k(\ell+2)$.

Finally, we prove (iii). Suppose that $\mathcal{F}$ contains an infinite number of cycles and no linear forests and that H is not a linear forest.

Let p' be an integer greater than the maximum length of a cycle in H; if H has no cycle, let $p' = 5$. Let p be an integer such that $p \geq p'$ and $C_p \in \mathcal{F}$ (such an integer p exists because $\mathcal{F}$ contains infinitely many cycles).

First suppose that H is C_3-free. We construct the following graph. Take two cycles $C = u_1 \cdots u_{p+1} u_1$ and $C' = u'_1 \cdots u'_{p+1} u'_1$. Connect u_1 and u'_1 via a path $u_1 v_1 \ldots v_k u'_1$ for some $k \geq 1$. Add the edges $u_2 u_{p+1}$ and $u'_2 u'_{p+1}$. Denote the resulting graph by G_k; see Figure 7.2 for an example. Note that G_k is connected and $K_{1,3}$-free and that it has four induced cycles, two of which have length p and two of which have length 3.

As H is not a linear forest, H either contains an induced $K_{1,3}$ or an induced cycle, which has length between 4 and $p-1$ by our choice of p and our assumption that H is C_3-free. Hence, every G_k is H-free. Let $S = \{u_2, u'_2\}$. As $G_k - S$ is a path and $\mathcal{F}$ contains no linear forests, S is an $\mathcal{F}$-transversal. Because G_k has two induced copies of C_p at distance more than k and $C_p \in \mathcal{F}$, the family $\{G_k\}$ is $\mathcal{F}$-unbounded.

Now suppose that H contains an induced C_3. Take two cycles $C = u_1 \cdots u_p u_1$ and $C' = u'_1 \cdots u'_p u'_1$. Connect u_1 and u'_1 via a path $u_1 v_1 \ldots v_k u'_1$ for some $k \geq 1$. The resulting graph G^*_k is connected and H-free, as it is C_3-free. We repeat the above arguments and find that the family $\{G^*_k\}$ is $\mathcal{F}$-unbounded. □

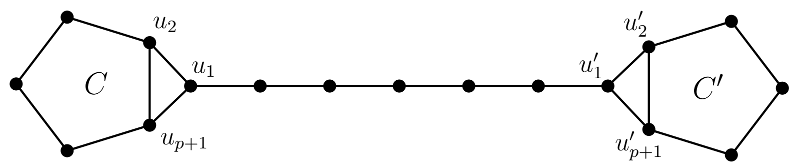


Figure 7.2: An example of the construction in the proof of Theorem 7.6 (iii) in the case when H is C_3-free, only contains cycles of length at most 4 and $C_5 \in \mathcal{F}$.

Parts *(ii)* and *(iii)* of Theorem 7.6 imply the following.

Corollary 7.7. *For any graph H and for any family of graphs $\mathcal{F}$ containing an infinite number of cycles and no linear forests, the class of connected H-free graphs is $\mathcal{F}$-multiplicative if and only if H is a linear forest.*

7.3 Cycle families with odd cycles

In this section we assume we are given a family $\mathcal{F}$ of graphs that contains all odd cycles, although we will show more general results whenever possible. We start with the following lemma, which generalizes the corresponding result of Belmonte et al. [6] when $\mathcal{F}$ is the family of all cycles. We use a similar approach as used in their proof but our arguments (which are based on bipartiteness instead of cycle-freeness) are different and this proof demonstrates some techniques used several times in obtaining our results.

Lemma 7.8. *For any family of graphs $\mathcal{F}$ containing either all odd cycles or P_2 and for any fixed $s \geq 1$, the class of connected sP_3-free graphs is $\mathcal{F}$-additive.*

Proof. The proof is by induction on s. Let $s = 1$. Then every connected sP_3-free graph G is complete. Hence, every minimum $\mathcal{F}$-transversal of G is connected.

Now let $s \geq 2$. Let G be a connected sP_3-free graph. We may assume by induction that G contains an induced copy Γ_0 of an $(s-1)P_3$. Let S be a minimum $\mathcal{F}$-transversal of G. Let Γ be a minimum connected induced subgraph of G that contains Γ_0. Because G is sP_3-free, G has diameter less than $4s$. Then, by Lemma 2.17, we find that Γ has size less than $3(s-1)+(s-2)4s = 4s^2-5s-3$. Let $S' = S \cup V(\Gamma)$. Then we have that $|S'| \leq |S|+4s^2-5s-3$.

If S' is connected then we take $d_{sP_3} = 4s^2 - 5s - 3$ as our desired constant and we are done. Suppose S' is not connected. Below we describe how to refine S'. During this process, we always use Z to denote the component of S' containing Γ, and we will never remove a vertex of Z from S'; in fact, one can think of the proof as "growing" Z and connecting it to the other vertices of S' until $Z = S'$.

Observe that the sP_3-freeness of G implies that every component of S' other than Z is complete. Throughout the proof, we let A denote the union of clique components of S', so $V(A) = S' \setminus V(Z) = S \setminus V(Z)$. We also note that the graph $G - S'$ is bipartite, as even its supergraph $G - S$ contains no odd cycles by the definition of S. Hence we can partition $G - S'$ into two (possibly empty) sets U_1 and U_2 so that U_1 and U_2 are independent sets.

We start with the following two claims, both of which follow from Lemma 7.5, which we apply twice, namely once with respect to U_1 and once with respect to U_2. By Lemma 7.5 this leads to a total increase in the size of S' by an additive factor of at most $2(2s-2) = 4s-4$.

Claim 1: Without loss of generality, we may assume that every vertex of $U_1 \cup U_2$ is adjacent to at most one component of A.

Claim 2: Without loss of generality, we may assume that every component of A is adjacent to at most one vertex of U_1 and to at most one vertex of U_2.

Using Claims 1 and 2 we prove the following crucial claim.

Claim 3: Without loss of generality, we may assume that every vertex of every component of A has exactly one neighbour in U_1 and exactly one neighbour in U_2.

We prove Claim 3 as follows. Let A^* be the union of components for which the statement of Claim 3 does not hold. Let D be a component of A^*. By Claim 2, D is adjacent to at most one vertex of U_1 and to at most one vertex of U_2. First suppose that D is non-adjacent to U_1 or to U_2, say D is not adjacent to U_1. Because G is connected, this means that D is adjacent to (exactly one) vertex $z \in U_2$, say $v \in D$ is adjacent to z. As D belongs to A^*, we find that D contains a vertex v' not adjacent to z. Hence, $vv'z$ is an induced P_3. Now suppose that D is adjacent to U_1 and to U_2, say D has vertices u, v (possibly $u = v$) so that u is adjacent to $x \in U_1$ and v is adjacent to $z \in U_2$. Then, as D is in A^*, there exists a vertex v' that is non-adjacent to at least one of x, z, say to z. Again, $vv'z$ is an induced P_3. As G is sP_3-free and no vertex in

$U_1 \cup U_2$ is adjacent to more than one component of A by Claim 1, we deduce that A^* contains at most $s-1$ components. Moreover, each vertex $z \in U_1 \cup U_2$ included in an induced P_3 as described above must be adjacent to Z (due to sP_3-freeness of G and the fact that Z contains an induced $(s-1)P_3$). Hence, we can add these vertices to Z increasing the size of Z, and thus the size of S', by at most $s-1$. The remaining components of A have the desired property. Moreover, Claims 1 and 2 are still valid. This completes the proof of Claim 3.

Due to Claim 3 we may assume without loss of generality that each vertex v in each component D of A has exactly two neighbours in $G - S'$, namely one neighbour in U_1 and one neighbour in U_2. By Claim 2, these neighbours are the same for all vertices in D. Hence, we may denote these two neighbours by s_D and t_D, respectively,

Consider a component D of A. If one of its neighbours in $U_1 \cup U_2$, say s_D, is adjacent to Z, then replacing S' with $(S' \cup \{s_D\}) \setminus \{v\}$ and Z with the connected component of S' containing $Z \cup \{s_D\}$ does not result in an odd cycle in $G - S'$. Moreover, such a swap does not increase the size of S' either. It does, however, reduce the number of vertices of S' that are not in Z (which is our goal). Consequently, we perform these swaps until, in the end, both the neighbours s_D and t_D of each component of A are not adjacent to Z. In particular this implies that s_D and t_D are adjacent, so $V_D \cup \{s_D, t_D\}$ is a clique. Then, due to Claims 1–3, the components in A together with their neighbours in $U_1 \cup U_2$ induce a union of complete graphs. This union is a disjoint union, as otherwise G would contain an induced P_3 not adjacent to Z and, as Z has an induced $(s-1)P_3$, we would obtain an induced sP_3 in G. Note that the swaps did not change the size of S'.

Let U_1' and U_2' denote the subsets of U_1 and U_2, respectively, that consist of vertices adjacent to no components of A. Let W_1 consist of all vertices s_D adjacent to U_2' and let W_2 consist of all vertices t_D adjacent to U_1'. Note that $W_1 \subseteq U_1 \setminus U_1'$ and that $W_2 \subseteq U_2 \setminus U_2'$. Because G is connected and no s_D or t_D is adjacent to Z or to some other component of A not equal to D, we find that $W_1 \cup W_2$ contain at least one of s_D, t_D for each component D of A.

We choose smallest sets U_1'' and U_2'' in U_1' and U_2', respectively, that dominate W_2 and W_1, respectively. By minimality, each vertex $u \in U_1''$ must have a "private" neighbour t_D in W_2, and hence together with t_D and s_D, corresponds to a "private" P_3. Consequently, as G is sP_3-free and $U_1'' \subseteq U_1$ is an independent set, U_1'' has size at most $s-1$. Similarly, U_2'' has size at most $s-1$. Moreover, each vertex in $U_1'' \cup U_2''$ is adjacent to Z (again due to the sP_3-freeness of G).

Figure 7.3 shows an example in which the components of A consist on three cliques (the first two of size two and the last one of size one) to illustrate the situation.

We now do as follows. First, for each component D of A we pick one of its vertices v and swap v with s_D if $s_D \in W_1$ and otherwise we swap v with t_D (note that $t_D \in W_2$ in that case). We also add all vertices of $U_1'' \cup U_2''$ to Z and thus to S'. The results of these swaps are as follows. First, $G[S']$ has become connected. Second, S' has increased in size at most by $2(s-1)$, which is allowed. Third, $G - S'$ is still bipartite (as swapping a vertex of a component D of A with s_D or t_D does not create any odd cycles). Consequently, we have found a connected $\mathcal{F}$-transversal of size at most $|S| + 4s^2 - 5s - 3 + 4s - 4 + (s-1) + 2(s-1) = |S| + 4s^2 + 2s - 10$, so we can take $d_{sP_3} = 4s^2 + 2s - 10$. □

We are now ready to prove the main result of this section.

Theorem 7.9. *For any graph H and for any family of cycles $\mathcal{F}$ containing all odd cycles, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 0$;*

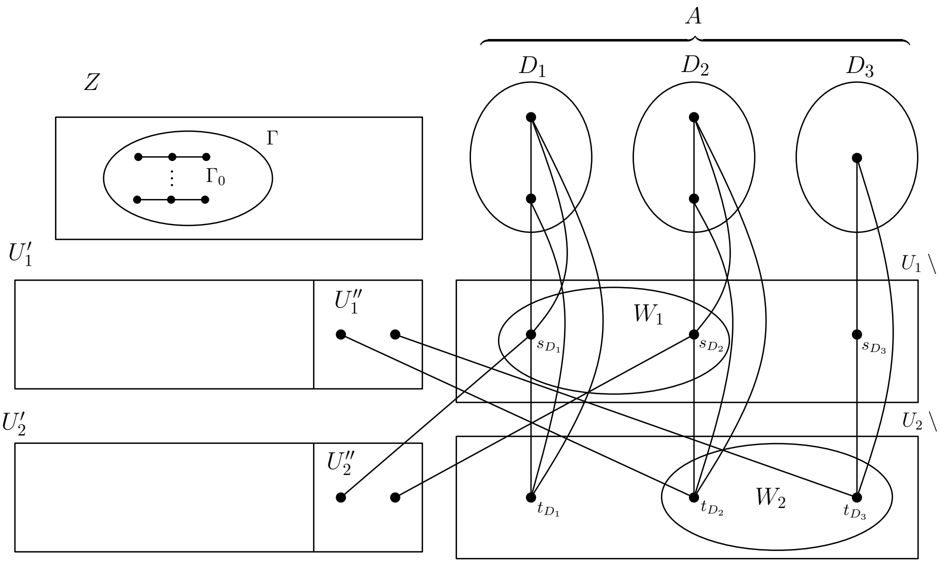


Figure 7.3: The situation in the proof of Lemma 7.8.

- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Corollary 7.7. We now prove the second claim. First suppose $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 0$. If $H \subseteq_i P_5 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.2 and 7.3. If $H \subseteq_i sP_3$ for some $s \geq 1$, the result follows from Lemma 7.8. Now suppose $H \not\subseteq_i P_5 + sP_1$ and $H \not\subseteq_i sP_3$ for any $s \geq 0$. By Theorem 7.6 (iii), we may assume that H is a linear forest. Then $P_6 \subseteq_i H$ or $P_2 + P_4 \subseteq_i H$, hence the class of connected H-free graphs is a superclass of the class of connected $\{P_2 + P_4, P_6\}$-free graphs and we can use Lemma 7.4.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. If $H \not\subseteq_i P_3$ then, by Theorem 7.6 (iii), we may assume that H is a linear forest. Hence, $3P_1 \subseteq_i H$ or $P_1 + P_2 \subseteq_i H$. Let $K_{2,2,2}$ be the graph on vertices $u_1, u_2, v_1, v_2, w_1, w_2$ and edges u_iw_j, u_iv_j and v_iw_j for $1 \leq i \leq j \leq 2$. Note that $K_{2,2,2}$ is $\{3P_1, P_1 + P_2\}$-free. Any minimum $\mathcal{F}$-transversal has size 2, whereas any minimum connected $\mathcal{F}$-transversal is of size 3.

□

7.4 Cycle families with 4-cycles but no 3-cycles

In this section we consider families of cycles $\mathcal{F}$ such that $C_3 \notin \mathcal{F}$ but $C_4 \in \mathcal{F}$. We need the following lemma.

Lemma 7.10. *For any family $\mathcal{F}$ of cycles with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$,*

- *the class of connected P_5-free graphs is not $\mathcal{F}$-additive;*
- *the class of connected $P_2 + P_4$-free graphs is not $\mathcal{F}$-additive;*
- *the class of connected $2P_3$-free graphs is not $\mathcal{F}$-additive;*
- *the class of connected $3P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We consider the four parts one at a time.

First, we describe a family of connected P_5-free graphs that is not $\mathcal{F}$-additive. Each graph G is a clique on k vertices, $k \geq 4$, and k copies of C_4. Each vertex in the clique is adjacent to every vertex in a distinct copy of C_4. Figure 7.4 gives an example with $k = 4$. Note that G is P_5-free: any induced path on at least four vertices can contain at most one vertex from each C_4, and thus at most two such vertices in total, and can only contain two vertices from the clique.

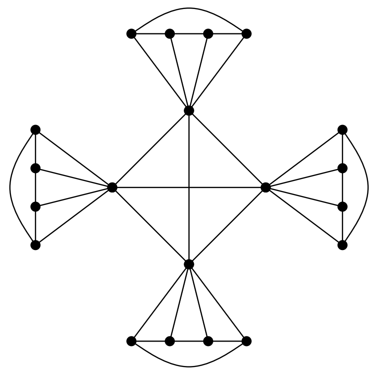


Figure 7.4: A graph in a family of P_5-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$.

We have $t_{\mathcal{F}}(G) \leq k$ since a set S containing one vertex from each copy of C_4 is an $\mathcal{F}$-transversal as $G - S$ is chordal. On the other hand, every connected $\mathcal{F}$-transversal of G contains, in addition to at least one vertex from each C_4, all the vertices of the clique. So $ct_{\mathcal{F}}(G) \geq 2k$.

Second, we describe a family of connected $P_2 + P_4$-free graphs that is not $\mathcal{F}$-additive. Each graph G consists of $k \geq 2$ copies of $K_{3,3}$, identified at a single vertex denoted v. Figure 7.5 shows the construction for $k = 4$.

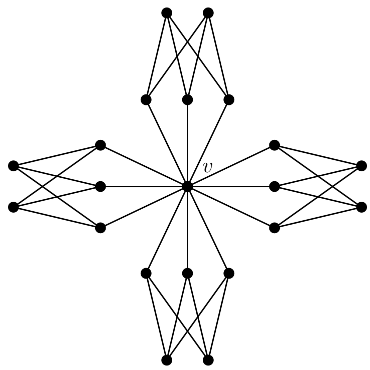


Figure 7.5: A graph in a family of $P_4 + P_2$-free graphs that is not $\mathcal{F}$-additive whenever $C_4 \in \mathcal{F}$.

Note that G is $P_4 + P_2$-free: every induced P_4 contains v, and deleting the vertices in such a P_4 and their neighbours results in an edgeless graph. We have $t_{\mathcal{F}}(G) \leq k + 1$ since a set S containing v and one vertex that is not adjacent to v from each $K_{3,3}$ is an $\mathcal{F}$-transversal as $G - S$ is a forest. On the other hand, every connected $\mathcal{F}$-transversal of G contains, in addition to v, at least two other vertices from each copy of $K_{3,3}$. So $ct_{\mathcal{F}}(G) \geq 2k + 1$.

Third, we describe a family of connected $2P_3$-free graphs that is not $\mathcal{F}$-additive. Each graph G consists of a complete graph K_{4k} for $k \geq 2$ denoted K, and a set M of $2k$ additional vertices forming an induced matching and each joined to two other vertices in K. Figure 7.6 shows the construction for $k = 3$. Note that G is $2P_3$-free: any induced P_3 contains a vertex from K, and deleting this vertex and all its neighbours results in a disjoint union of cliques, a P_3-free graph.

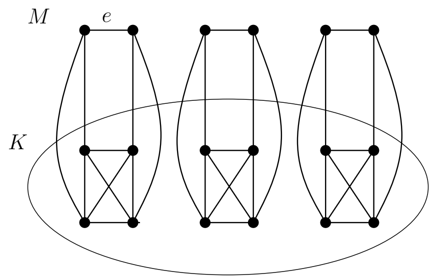


Figure 7.6: A graph in a family of $2P_3$-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \notin \mathcal{F}$, $C_4 \in \mathcal{F}$.

We have $t_{\mathcal{F}}(G) \leq k$, since a set S containing one vertex from each edge in M is an $\mathcal{F}$-transversal as $G - S$ is chordal. On the other hand, every connected $\mathcal{F}$-transversal of G contains at least two vertices from each subgraph consisting of an edge e in M and vertices in K adjacent to an endpoint of e. So $ct_{\mathcal{F}}(G) \geq 2k$.

Finally, we describe a family of connected $3P_2$-free graphs that is not $\mathcal{F}$-additive. Each graph G consists of three copies K, K' and K^* of a complete graph on $2k$ vertices for $k \geq 2$, and an independent set M of k vertices. Every vertex in K^* is joined to every vertex in K and K' and every vertex in M is joined to a distinct pair of vertices in K and K'. Figure 7.7 shows the construction for $k = 3$. Note that G is $3P_2$-free: when an induced P_2 and all its neighbours are deleted the resulting graph is either an independent set (if the P_2 is contained in K^*) or a graph in which every P_2 is incident with the same clique (if the P_2 intersects either K or K').

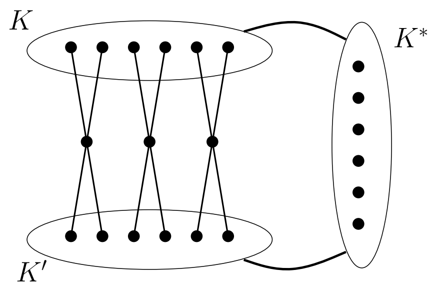


Figure 7.7: A graph in a family of $3P_2$-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \notin \mathcal{F}$, $C_4 \in \mathcal{F}$.

We have $t_{\mathcal{F}}(G) \leq k$, since M is an $\mathcal{F}$-transversal as $G - M$ is chordal. On the other hand, a connected $\mathcal{F}$-transversal of G either contains K^* or, for each vertex v of M, either v and one of its neighbours, or, if it does not contain v, two of its neighbours. So $ct_{\mathcal{F}}(G) \geq 2k$.

□

We now state our result for infinite families of cycles $\mathcal{F}$ with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$. It does not provide a complete characterization as we are unable to give necessary and sufficient conditions for the class of H-free graphs to be $\mathcal{F}$-additive. This would be possible if it could be shown that $\{P_3 + P_2 + sP_1\}$-free graphs are $\mathcal{F}$-additive for all $s \geq 0$. By Lemma 7.3, this is the case if and only if $\{P_3 + P_2\}$-free graphs are $\mathcal{F}$-additive, which we conjecture to be true.

Theorem 7.11. *For any graph H and for any infinite family of cycles $\mathcal{F}$ with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*

- *$\mathcal{F}$-additive if $H \subseteq_i P_4+sP_1$ for some $s \geq 0$, but not if $H \not\subseteq_i P_4+sP_1$ nor $H \not\subseteq_i P_3+P_2+sP_1$ for some $s \geq 0$;*

- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Corollary 7.7. We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from Lemmas 7.1 and 7.3. Now suppose $H \not\subseteq_i P_4 + sP_1$ and $H \not\subseteq_i P_3 + P_2 + sP_1$ for any $s \geq 0$. By Theorem 7.6 (iii), we may assume that H is a linear forest. Then $P_5 \subseteq_i H$, $P_2 + P_4 \subseteq_i H$, $2P_3 \subseteq_i H$, or $3P_2 \subseteq_i H$, and we can use Lemma 7.10.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. If $H \not\subseteq_i P_3$ then, by Theorem 7.6 (iii), we may assume that H is a linear forest. Hence, $3P_1 \subseteq_i H$ or $P_1 + P_2 \subseteq_i H$.

If $P_1 + P_2 \subseteq_i H$, then we have that the complete bipartite graph $G = K_{3,3}$ is a connected H-free graph (since it is $P_1 + P_2$-free). And $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$ so the class of connected H-free graphs is not $\mathcal{F}$-zero-additive.

Finally, suppose that $3P_1 \subseteq_i H$, and let G be the complement of the graph shown in Figure 7.8. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph. As every $\mathcal{F}$-transversal of G must intersect every induced $2P_2$ in $\overline{G}$, the minimum $\mathcal{F}$-transversals of G are in bijective correspondence with the four edges of the 4-cycle in $\overline{G}$. So $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$, and the class of connected H-free graphs is also not $\mathcal{F}$-zero-additive in this case.

□

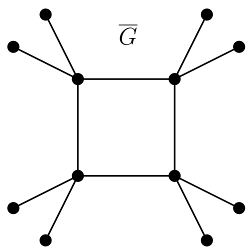


Figure 7.8: The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$.

7.5 Cycle families with 5-cycles but no 3- or 4-cycles

In this section we consider families of cycles $\mathcal{F}$ such that $C_3, C_4 \notin \mathcal{F}$ but $C_5 \in \mathcal{F}$. We first prove the following lemma; note that C_3 and C_4 are both induced subgraphs of $\overline{2P_4}$.

Lemma 7.12. *Let $\mathcal{F}$ be a family of graphs with $C_5 \in \mathcal{F}$ that contains no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$. Then the class of connected $2P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We describe a family of connected $2P_2$-free graphs that is not $\mathcal{F}$-additive, where $\mathcal{F}$ is any family of cycles as in the statement of the lemma. The graphs in the family are constructed from $k \geq 2$ copies $H_1, \ldots, H_k$ of the graph that is obtained from $2P_4$ by adding all possible edges between the vertices of one copy and the other one. For each H_i, there is a new vertex v_i adjacent to both endpoints of the two P_4s, and in addition there are all possible edges between vertices in different H_i's. Figure 7.9 shows an example for $k = 4$.

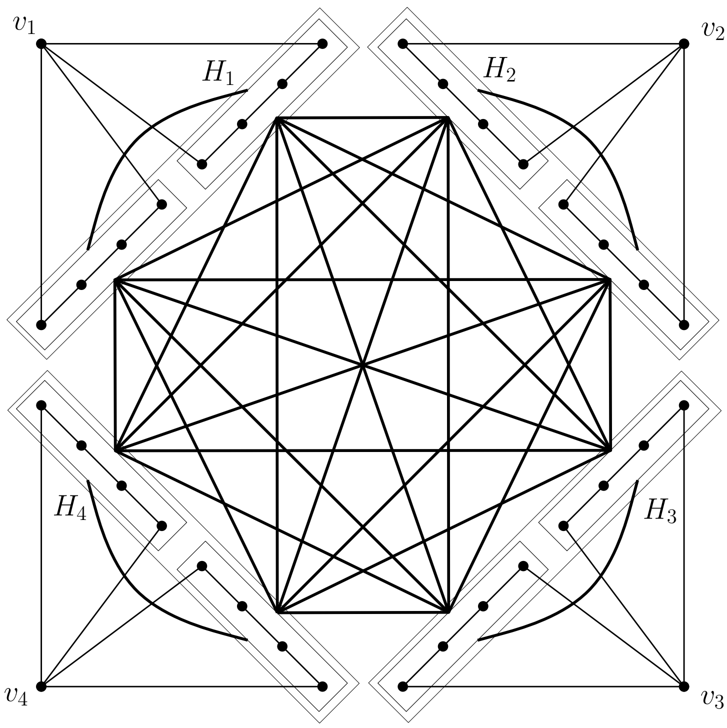


Figure 7.9: A member of a family of connected $2P_2$-free graphs that is not $\mathcal{F}$-additive whenever $C_5 \in \mathcal{F}$ and $\mathcal{F}$ contains no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$. A thick edge between two sets of vertices inducing a P_4 means the presence of all possible edges between the two sets.

We first show that every graph G in this family is $2P_2$-free. Every edge e of G has at least one endpoint in some H_i, say in H_1. Deleting the closed neighbourhood of e results in the subgraph induced by a subset of $\{v_1, \ldots, v_k\}$ (if $e \in E(H_1)$), or in the subgraph induced by $\{u, v_2, \ldots, v_k\}$ for some $u \in V(H_1)$ (otherwise). In either case, the resulting graph is edgeless. Therefore, G is $2P_2$-free.

Let G be a graph in this family, and let k be the number of H_i's. We have $t_{\mathcal{F}}(G) \leq k$ since deleting the vertices $v_1, \ldots, v_k$ results in a graph that is isomorphic to $\overline{2kP_4}$ and thus $\mathcal{F}$-free. On the other hand, every connected $\mathcal{F}$-transversal S of G must contain at least two vertices from each subgraph induced by $V(H_i) \cup \{v_i\}$, for every i (otherwise it either misses an induced C_5 or contains only v_i, making it isolated in $G[S]$). Therefore, $ct_{\mathcal{F}}(G) \geq 2k$, which establishes the non-$\mathcal{F}$-additivity of the family. □

We also need the following lemma.

Lemma 7.13. *Let $\mathcal{F}$ be a family of graphs that contains C_5 but no induced subgraph of $\overline{4P_4}$. Then the class of connected $3P_1$-free graphs is not $\mathcal{F}$-zero-additive.*

Proof. Let $\mathcal{F}$ be any family of cycles as in the statement of the lemma and let G be the complement of the graph depicted in Figure 7.10. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph.

Since $\overline{C_5} = C_5$, in the complement of G we need to cover all the C_5's. Therefore there is a unique minimum $\mathcal{F}$-transversal S of G, consisting of the two endpoints of the central edge of $\overline{G}$. Indeed $\overline{G} - S$ is isomorphic to $4P_4$, so the graph $G - S \cong \overline{4P_4}$ is $\mathcal{F}$-free. Since the graph $G[S]$ is not connected, we have $ct_{\mathcal{F}}(G) > t_{\mathcal{F}}(G)$. □

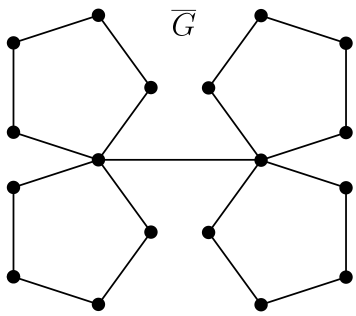


Figure 7.10: The complement of a graph that shows that the class of connected $3P_1$-free graphs is not $\mathcal{F}$-zero-additive whenever $C_5 \in \mathcal{F}$ and $\mathcal{F}$ contains no induced subgraphs of $\overline{4P_4}$.

Theorem 7.14. *For any graph H and for any graph family $\mathcal{F}$ which only contains graphs with an induced P_4, including C_5 and an infinite number of other cycles but no linear forests and no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_4$.*

Proof. The first claim follows immediately from Corollary 7.7. We now prove the second claim. First suppose that $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$. Then the class of connected H-free graphs is $\mathcal{F}$-additive due to Lemmas 7.1 and 7.3. Now suppose that $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. By Theorem 7.6 (iii), we may assume that H is a linear forest. Hence, $2P_2 \subseteq_i H$ and we can use Lemma 7.12. Finally, we show the third claim. Recall that if $H \subseteq_i P_4$ then any H-free graph is already $\mathcal{F}$-free. Suppose that $H \not\subseteq_i P_4$. If $2P_2 \subseteq_i H$ we use Lemma 7.12 again. Hence $3P_1 \subseteq_i H$. In that case we use Lemma 7.13. This completes the proof of Theorem 7.14. □

7.6 Families of short paths and cycles

In Section 7.6.2 we prove our results for families $\mathcal{F}$ of graphs that contain P_2, $2P_2$ or P_4, in particular for families $\mathcal{F}$ for which the graph minus an $\mathcal{F}$-transversal is a split graph, a threshold graph, a trivially perfect graph, or a cograph, respectively. In order to show these results we need a number of lemmas, which we will prove in Section 7.6.1. As before, lemmas and theorems are often stated in a more general form than needed.

7.6.1 Lemmas

Lemma 7.15. *For $\mathcal{F} = \{C_4, C_5, 2P_2\}$ and any fixed $s \geq 0$, the class of connected $\{P_3+sP_2\}$-free graphs is $\mathcal{F}$-additive.*

Proof. The proof is by induction on s. Let $s = 0$. Every connected P_3-free graph G is complete. Hence, every minimum $\mathcal{F}$-transversal of G is connected.

Now let $s \geq 1$. Let G be a connected $\{P_3 + sP_2\}$-free graph. We may assume by induction that G contains an induced copy Γ_0 of an $P_3 + (s-1)P_2$. Let S be a minimum $\mathcal{F}$-transversal of G. Let Γ be a minimum connected induced subgraph of G that contains Γ_0. Because G is $\{P_3 + sP_2\}$-free, G has diameter less than $3(s+1) - 1 = 3s - 2$. Then, by Lemma 2.17, we find that Γ has size less than $3(s-1) + (s-2)(3s-2) = 3s^2 - 3s + 1$. Let $S' = S \cup V(\Gamma)$. Then we have that $|S'| \leq |S| + 3s^2 - 3s + 1$.

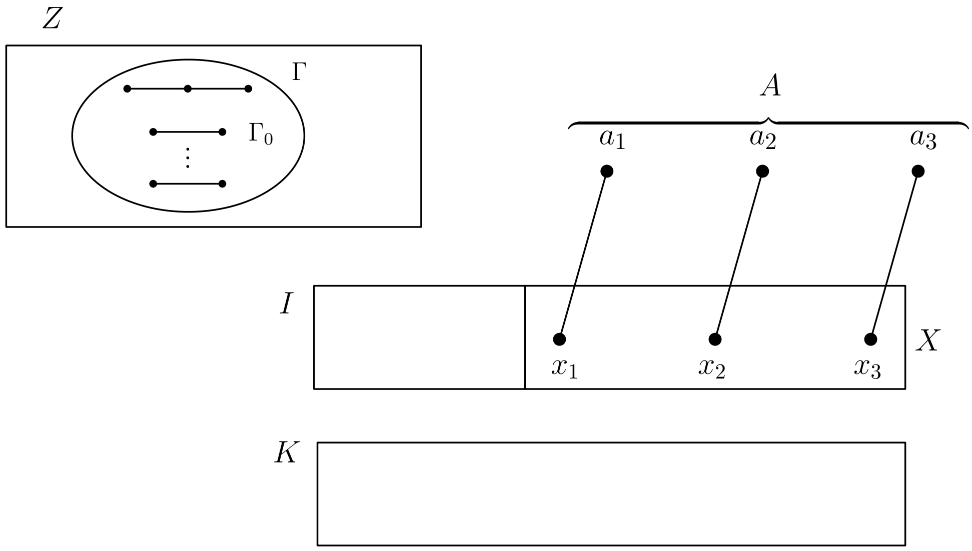


Figure 7.11: The decomposition of the graph G in the proof of Lemma 7.15.

If S' is connected then we take $d_{P_3+sP_2} = 3s^2 - 3s + 1$ as our desired constant and we are done. Suppose S' is not connected. Below we describe how to refine S'. During this process, we always use Z to denote the component of S' containing Γ, and we will never remove a vertex of Z from S'.

Observe that the $\{P_3 + sP_2\}$-freeness of G implies that every component of S' other than Z consists of a single vertex. We let A denote the union of these single vertices, so $A = S' \setminus V(Z) = S \setminus V(Z)$. We also note that the graph $G - S'$ is split, as even its supergraph $G - S$ is $\{C_4, C_5, 2P_2\}$-free by the definition of S. Hence we can partition $G - S'$ into two (possibly empty) sets: a clique K and an independent set I.

We start with the following two claims, both of which follow from Lemma 7.5. By Lemma 7.5, this leads to a total increase of S' by an additive factor of at most $2s - 2$.

Claim 1: Without loss of generality, we may assume that every vertex of I is adjacent to at most one vertex of A.

Claim 2: Without loss of generality, we may assume that every vertex of A is adjacent to at most one vertex of I.

We proceed as follows. If A contains a vertex u not adjacent to a vertex in I then we move u from A to I. Hence, we may assume without loss of generality that A has no such vertices. Then, by Claim 2, every vertex in A is adjacent to exactly one vertex of I. Let $A = \{a_1, \ldots, a_q\}$ for some integer $q \geq 1$ and let $X = \{x_1, \ldots, x_q\}$ be the subset of I in which x_i is the unique neighbour of a_i for $i = 1, \ldots, q$. By Claim 1, $G[A \cup X]$ is isomorphic to qP_2. See Figure 7.11 for an example.

Due to the $\{P_3 + sP_2\}$-freeness of G and the fact that Z contains an induced $P_3 + (s-1)P_2$, each x_i is adjacent to Z. We swap a_i and x_i, that is, we put a_i into I and x_i into A. Then, because a_i is not adjacent to any other vertex in I, we still have the property that $G - S'$ is split. However, we now also have that $Z = S'$, as desired. So we have found a connected $\mathcal{F}$-transversal S' of size at most $|S| + 3s^2 - 3s + 1 + 2s - 2 = |S| + 3s^2 - s - 1$ meaning we can take $d_{P_3+sP_2} = 3s^2 - s - 1$. This completes the proof of Lemma 7.15.

$\square$

Lemma 7.16. *Let $\mathcal{F}$ be a family of graphs with either $\mathcal{F} = \{P_2\}$, or $\mathcal{F} \cap \{P_4, 2P_2\} \neq \emptyset$ and $\mathcal{F} \setminus \{P_2, P_4, 2P_2\}$ a (possibly empty) set of holes. If H is not a linear forest then the class of connected H-free graphs is not $\mathcal{F}$-additive.*

Proof. Let H be a graph that is not a linear forest, so H contains a cycle or an induced $K_{1,3}$. Let us verify that the class of all paths is a class of H-free connected graphs that is not $\mathcal{F}$-additive.

If $\mathcal{F} = \{P_2\}$, then for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/2$ (since taking every other vertex on the path results in an $\mathcal{F}$-transversal), while $ct_{\mathcal{F}}(P_n) \geq n-2$ (since any $\mathcal{F}$-transversal contains a vertex u from the first 2 vertices of P_n and also a vertex v from the last 2 vertices, and these two need to be made connected by taking all the vertices of the path that lie in between).

If $\mathcal{F} \cap \{P_4, 2P_2\} = \{P_4\}$ then, similarly, for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/4$ while $ct_{\mathcal{F}}(P_n) \geq n-6$. If $\mathcal{F} \cap \{P_4, 2P_2\} = \{2P_2\}$ then for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/2$, while $ct_{\mathcal{F}}(P_n) \geq n-8$. Finally, if $\mathcal{F} \cap \{P_4, 2P_2\} = \{P_4, 2P_2\}$ then for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/2$, while $ct_{\mathcal{F}}(P_n) \geq n-6$. □

Lemma 7.17. *Let $\mathcal{F}$ be a family of graphs that contains C_4 but no induced subgraph of $K_{1,3}$. Then the class of $\{P_2 + P_1\}$-free graphs is not $\mathcal{F}$-zero-additive.*

Proof. The complete bipartite graph $K_{3,3}$ is $\{P_2 + P_1\}$-free. Removing a single vertex or two adjacent vertices does not make the graph C_4-free. If we remove two non-adjacent vertices then we obtain a claw, which is $\mathcal{F}$-free. Hence, a minimum $\mathcal{F}$-transversal has size 2 and a minimum connected $\mathcal{F}$-transversal has size at least 3.

□

Lemma 7.18. *Let $\mathcal{F}$ be a family of graphs that contains P_4 but no complete graph. Then the class of $2P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We construct a family of connected $2P_2$-free graphs $\{G_k\}$ as follows. Let G_k have a clique $K_k = \{u_1, \ldots, u_{2k}\}$ and an independent set $\{a_1, \ldots, a_k\}$. For $i = 1, \ldots, k$, add the edges $a_i u_{2i-1}$ and $a_i u_{2i}$. (See Figure 7.12.)

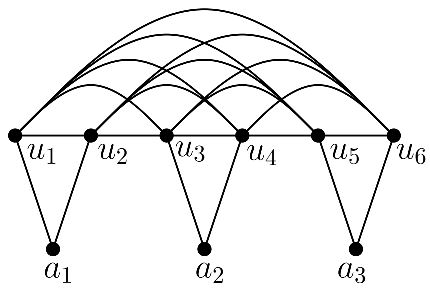


Figure 7.12: The graph G_k for $k = 3$ used in the proof of Lemma 7.18.

Note that G_k is $2P_2$-free, for all $k \geq 1$. Note that each set $\{a_i, a_j, u_i, u_{2i}, u_j, u_{2j}\}$ induces four different P_4's. On the one hand, the set $\{a_1, \ldots, a_k\}$ forms an $\mathcal{F}$-transversal of G of size k. On the other hand, as any two distinct a_i and a_j are non-adjacent and have no common neighbour, any connected $\mathcal{F}$-transversal of G contains at least two vertices from at least $k-1$ of the k pairwise disjoint sets $\{a_i, u_{2i-1}, u_{2i}\}$ and therefore has size at least $2(k-1)$. □

Lemma 7.19. *Let $\mathcal{F}$ be a family of graphs that contains P_4 but no disjoint union of two complete graphs. Then the class of $3P_1$-free graphs is not $\mathcal{F}$-zero-additive.*

Proof. Construct the following 14-vertex graph G^*. Take a set A of seven vertices a, a', b, b', c, d, d', add the edges making each of $A_1 = \{a, a', b, b'\}$ and $A_2 = \{d, d'\}$ a clique, and add the edges $bc, b'c, cd, cd'$. Take a set B of seven vertices s, s', t, t', u, v, v', add the edges making each of $B_1 = \{s, s', t, t'\}$ and $B_2 = \{v, v'\}$ a clique, and add the edges $tu, t'u, uv, uv'$. Add every edge between a vertex of A_1 and a vertex of B_1 (thus making $A_1 \cup B_1$ a clique), every edge between a vertex of B_1 and a vertex of B_2 (thus making $A_2 \cup B_2$ a clique), add edges from c to every vertex of $B \setminus \{u\}$, and add edges from u to every vertex of $A \setminus \{c\}$. See Figure 7.13 for a picture of G^*. Note that G^* is $3P_1$-free and that $\{u, c\}$ is the unique minimum $\mathcal{F}$-transversal, hence every minimum connected $\mathcal{F}$-transversal has size (at least) 3. $\square$

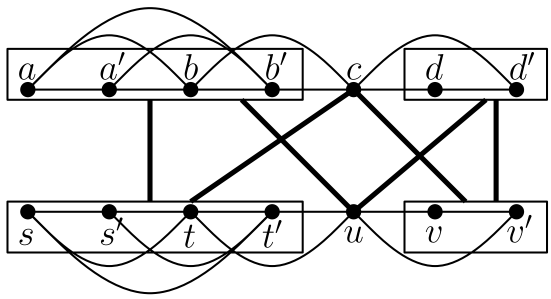


Figure 7.13: The graph G^* used in the Proof of Lemma 7.19. A thick edge between two sets of vertices means the presence of all possible edges between the two sets.

Let K_6^+ be the graph that consists of a clique on six vertices and another vertex made adjacent to three vertices of the clique.

Lemma 7.20. *Let $\mathcal{F}$ be a family of graphs that contains $2P_2$ and P_4 but no induced subgraph of K_6^+. Then the class of $3P_1$-free graphs is not $\mathcal{F}$-zero-additive.*

Proof. We construct the following graph G with ten vertices $a_1, a_2, b_1, b_2,\ u_1, u_2, u_3, v_1, v_2, v_3$ so that $\{a_1, a_2, u_1, u_2, u_3\}$, $\{b_1, b_2, v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ are three cliques. See Figure 7.14 for a picture of G. Note that G is $3P_1$-free, as the first two cliques partition $V(G)$. Then every minimum $\mathcal{F}$-transversal consists of three vertices, namely one of $\{a_1, a_2\}$ and two of $\{b_1, b_2\}$, or vice versa (as otherwise either an induced $2P_2$ is left or an induced P_4). Consequently, the size of a minimum connected $\mathcal{F}$-transversal is 4. $\square$

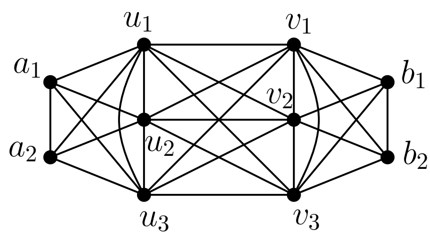


Figure 7.14: The graph G used in the proof of Lemma 7.20.

Lemma 7.21. *Let $\mathcal{F}$ be a family of graphs that contains $2P_2$ but no induced subgraph of $\overline{4P_3}$. Then the class of $3P_1$-free graphs is not $\mathcal{F}$-zero-additive.*

Proof. The proof mimics that of Lemma 7.13. Let G be the complement of the graph shown in Figure 7.15. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph. Since $\overline{2P_2} = C_4$, in the complement of G we need to cover all the C_4's. Therefore there is a unique minimum $\mathcal{F}$-transversal S of G, consisting of the two endpoints of the central edge of $\overline{G}$. Indeed $\overline{G} - S$ is isomorphic to $4P_3$, so the graph $G - S \cong \overline{4P_3}$ is $\mathcal{F}$-free. Since the graph $G[S]$ is not connected, we have $ct_{\mathcal{F}}(G) > t_{\mathcal{F}}(G)$. $\square$

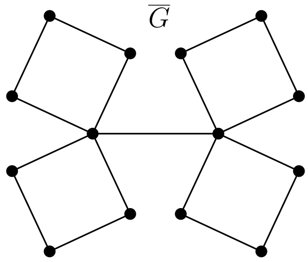


Figure 7.15: The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $2P_2 \in \mathcal{F}$ and no induced subgraph of $\overline{4P_3}$ is in $\mathcal{F}$.

7.6.2 Theorems

We are now ready to prove the following six theorems.

Theorem 7.22. *For any graph H and for $\mathcal{F} = \{P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 1$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Theorem 7.6 (i). We now prove the second claim. If $H \subseteq_i P_5 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.2 and 7.3. If $H \subseteq_i sP_3$ for some $s \geq 0$, the result follows from Lemma 7.8. Suppose that $H \not\subseteq_i P_5 + sP_1$ for any $s \geq 0$ and $H \not\subseteq_i sP_3$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 7.16. Hence we may assume that H is a linear forest. Then, since $H \not\subseteq_i P_5 + sP_1$ and $H \not\subseteq_i sP_3$ for any $s \geq 0$, we find that $P_4 + P_2 \subseteq_i H$ or $P_6 \subseteq_i H$. Consider the $\{P_4 + P_2, P_6\}$-free graph G_k obtained from k 4-cycles $a_ib_ic_id_ia_i$ for $i = 1, \ldots, k$ after identifying all $a_1, \ldots, a_k$ into a single vertex a (so G_k consists of disjoint P_3's, whose end-vertices are both adjacent to a). For every $k \geq 1$, a minimum $\mathcal{F}$-transversal has size $k + 1$ and a minimum connected $\mathcal{F}$-transversal has size $2k + 1$.

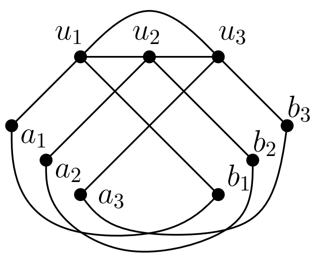
We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. Suppose $H \not\subseteq_i P_3$. By the previous claim we may assume that H is a linear forest. Thus, $H \not\subseteq_i C_4$ and the graph $G = C_4$ is an H-free graph with $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$. ☐

Theorem 7.23. *For any graph H and for $\mathcal{F} = \{C_4, C_5, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$ for some $s \geq 0$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Theorem 7.6 (i). We now prove the second claim. First suppose $H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$ for some $s \geq 0$. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.1 and 7.3. If $H \subseteq_i P_3 + sP_2$ for some $s \geq 0$, the result follows from Lemma 7.15. Now suppose $H \not\subseteq_i P_4 + sP_1$ and $H \not\subseteq_i P_3 + sP_2$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 7.16. Hence we may assume that H is a linear forest. Then $P_5 \subseteq_i H$ or $P_4 + P_2 \subseteq_i H$ or $2P_3 \subseteq_i H$.

First suppose that $P_5 \subseteq_i H$ or $2P_3 \subseteq_i H$. We construct a family of connected H-free graphs $\{G_k\}$ as follows. Let G_k have a clique $K_k = \{u_1, \ldots, u_k\}$ and two independent sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$. For $i = 1, \ldots, k$, add edges a_ib_i, a_iu_i and b_iu_i. See Figure 7.16 for an example.



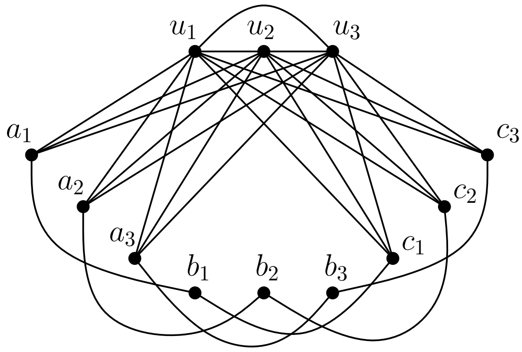


Figure 7.16: The graphs G_k (left) and G^*_k (right) for $k = 3$ used in the proof of Theorem 7.23.

Note that G_k is $\{2P_3, P_5\}$-free, and thus H-free, for all $k \geq 1$. Every minimum $\mathcal{F}$-transversal consists of exactly one vertex of each pair $\{a_i, b_i\}$, as we need to remove at least one vertex from at least $k-1$ pairs $\{a_i, b_i\}$ to remove induced $2P_2$'s and then another vertex from the remaining pair (which forms an induced $2P_2$ with a non-adjacent pair of clique vertices). On the other hand, every connected $\mathcal{F}$-transversal consists of at least $2k$ vertices.

Now suppose that $P_4 + P_2 \subseteq_i H$. We construct a family of connected H-free graphs $\{G^*_k\}$ as follows. Let G^*_k have a clique $K_k = \{u_1, \ldots, u_k\}$ and three independent sets $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$ and $\{c_1, \ldots, c_k\}$. For $i = 1, \ldots, k$, add edges a_ib_i and b_ic_i. Also add an edge between each a_i and each u_j, and an edge between each c_i and each u_j. See Figure 7.16 for an example. As each u_j is adjacent to all vertices of G^*_k except the mutually non-adjacent vertices $b_1, \ldots, b_k$, we find that G^*_k is $\{P_4 + P_2\}$-free for all $k \geq 1$. By the same arguments as in the previous case, we find that $\{b_1, \ldots, b_k\}$ is the unique minimum $\mathcal{F}$-transversal. On the other hand, every connected $\mathcal{F}$-transversal contains at least $2k + 1$ vertices.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. Now suppose $H \not\subseteq_i P_3$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$ for some integer $s \geq 0$.

Suppose that $3P_1 \subseteq_i H$, and let G be the complement of the graph shown in Figure 7.17.

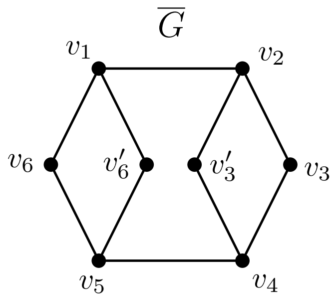


Figure 7.17: The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $\mathcal{F} = \{C_4, C_5, 2P_2\}$.

Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free (and hence H-free) graph. The set $S = \{v_1, v_2\}$ is an $\mathcal{F}$-transversal of G since $\overline{G} - S$ (and consequently $G - S$) is a split graph. On the other hand, deleting any pair of non-adjacent vertices from $\overline{G}$ leaves at least one subgraph isomorphic to $2P_2$ or C_4, which implies that $t_{\mathcal{F}}(G) = 2 < ct_{\mathcal{F}}(G)$.

Now suppose that $3P_1 \not\subseteq_i H$. If $P_2 + P_1 \subseteq_i H$ then we can apply Lemma 7.17. If H is $\{3P_1, P_2 + P_1\}$-free, then we conclude (since H is a linear forest) that $H \subseteq_i P_3$, a contradiction. □

Theorem 7.24. *For any graph H and for $\mathcal{F} = \{C_4, P_4\}$ or $\mathcal{F} = \{C_4, P_4, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Theorem 7.6 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.1 and 7.3. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 7.16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, we find that $2P_2 \subseteq_i H$ and we can use Lemma 7.18.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. Now suppose $H \not\subseteq_i P_3$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 0$. Hence it holds that $3P_1 \subseteq_i H$ or $P_2 + P_1 \subseteq_i H$.

We start with the case where $3P_1 \subseteq_i H$. If $2P_2 \in \mathcal{F}$ then we use Lemma 7.20. Suppose that $2P_2 \notin \mathcal{F}$. Then $\mathcal{F} = \{C_4, P_4\}$ and we can use Lemma 7.19. We now consider the case $P_2 + P_1 \subseteq_i H$. As $C_4 \in \mathcal{F}$ we apply Lemma 7.17. This completes the proof of Theorem 7.24. □

Theorem 7.25. *For any graph H and for $\mathcal{F} = \{C_5, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$.*

Proof. The first claim follows immediately from Theorem 7.6 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.1 and 7.3. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 7.16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, we find that $2P_2 \subseteq_i H$ and thus we can use Lemma 7.12.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, and if $H \subseteq_i P_1 + P_2$ then any connected H-free graph is $\mathcal{F}$-free. So in both cases the result follows directly. Now suppose that $H \not\subseteq_i P_3$ and $H \not\subseteq_i P_1 + P_2$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 0$. If $3P_1 \subseteq_i H$, then we can apply Lemma 7.21. If $3P_1 \not\subseteq_i H$, then $H = P_4$ and we can consider the 7-vertex graph G consisting of 6 vertices forming a $3P_2$ and one more vertex adjacent to all the other vertices. Graph G is a connected P_4-free graph with $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$. This completes the proof of Theorem 7.25. □

Theorem 7.26. *For any graph H and for $\mathcal{F} = \{P_4, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$.*

Proof. The first claim follows immediately from Theorem 7.6 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.1 and 7.3. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 7.16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, we find that $2P_2 \subseteq_i H$ and thus we can use Lemma 7.18.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, and if $H \subseteq_i P_1 + P_2$ then any connected H-free graph is $\mathcal{F}$-free. So in both cases the result follows directly. Now suppose that $H \not\subseteq_i P_3$ and $H \not\subseteq_i P_1 + P_2$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 0$. Hence it holds that $3P_1 \subseteq_i H$ and we can apply Lemma 7.20. This completes the proof of Theorem 7.26. □

Theorem 7.27. *For any graph H and for $\mathcal{F} = \{P_4\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_4$.*

Proof. The first claim follows immediately from Theorem 7.6 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7.1 and 7.3. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 7.16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, we find that $2P_2 \subseteq_i H$ and thus we can use Lemma 7.18.

We now prove the third claim. If $H \subseteq_i P_4$ then any connected H-free graph is $\mathcal{F}$-free, so the result follows directly. Now suppose $H \not\subseteq_i P_4$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 1$. Hence, $3P_1 \subseteq_i H$ and we can use Lemma 7.19.

□

Chapter 8

Conclusion

A number of graph theory research problems were considered in this PhD thesis. The results presented in the thesis will contribute to the expansion of knowledge in the field of structural graph theory.

Our approach in the research work was a combinatorial and structural one. We made use of various properties and characterizations of the graph classes under consideration, combining graph theoretic and combinatorial tools and proof techniques such as proof by contradiction, proof by minimal counterexample, inductive proofs, extremality, characterizations of graph classes by forbidden substructures (induced subgraphs, induced minors, minors), graph decompositions, etc. An important role in the study was played by various graph transformations and graph invariants. In this final chapter we summarize our findings and discuss open questions and possible directions for future research work.

The results and open problems given in this chapter can be found in the following papers.

- T. R. Hartinger and M. Milanič, *Partial Characterizations of 1-Perfectly Orientable Graphs.* J. Graph Theory. Vol. 85, 2, 2017, 378 – 394.
- B. Brešar, T. R. Hartinger, T. Kos, and M. Milanič (2016), 1*-perfectly orientable K_4-minor-free and outerplanar graphs.* Submitted. arXiv:1604.04598. An extended abstract appeared in Electronic Notes in Discrete Mathematics, Vol. 54, (2016), 199 – 204.
- T. R. Hartinger, M. Johnson, M. Milanič and D. Paulusma, *The price of connectivity for cycle transversals.* European Journal of Combinatorics. 58, (2016), 203-224. An extended abstract appeared in Mathematical Foundations of Computer Science 2015. Part II, volume 9234 of Lecture Notes in Computer Science, 395 – 406, Springer, 2015.

1-perfectly orientable graphs. In this PhD thesis, we developed several results on the structure of 1-perfectly orientable graphs, including the identification of several graph transformations preserving the class of 1-perfectly orientable graphs and an infinite family of minimal forbidden induced minors for the class of 1-p.o. graphs. We characterized the class of 1-perfectly orientable graphs within several induced-minor-closed graph classes, namely the classes of cographs, cobipartite graphs, K_4-minor-free graphs, outerplanar graphs, and block-cactus graphs.

Theorem 3.9 from Section 3 implies that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is the set of minimal forbidden induced minors for the class of 1-p.o. graphs. However, the complete set $\tilde{\mathcal{F}}$ is unknown. It is conceivable that one can obtain further graphs in $\tilde{\mathcal{F}}$ by computing the minimal elements with respect to the induced minor relation of the list of forbidden induced subgraphs for the class of circular arc cobipartite graphs due to Trotter and Moore [83]. Besides the three small graphs

F_5, F_6, F_7 and the family $\mathcal{F}_3$ of complements of even cycles of length at least 6, the list contains five other infinite families, the smallest members of which are graphs $F_8, \ldots, F_{12}$, respectively. (In [83], the lists represent the complementary property and are denoted by $\mathcal{T}_i$, $\mathcal{W}_i$, $\mathcal{D}_i$, $\mathcal{M}_i$, and $\mathcal{N}_i$, respectively.)

Open problem. *Determine the set of minimal forbidden induced minors for the class of 1-perfectly orientable graphs.*

Throughout the thesis, several graph classes proved to be important in the study of 1-perfectly orientable graphs. The known and new results on the relationships between the graph classes studied in this thesis are summarized in the Hasse diagram on Fig. 8.1. The results marked in grey have been proved in this PhD thesis.

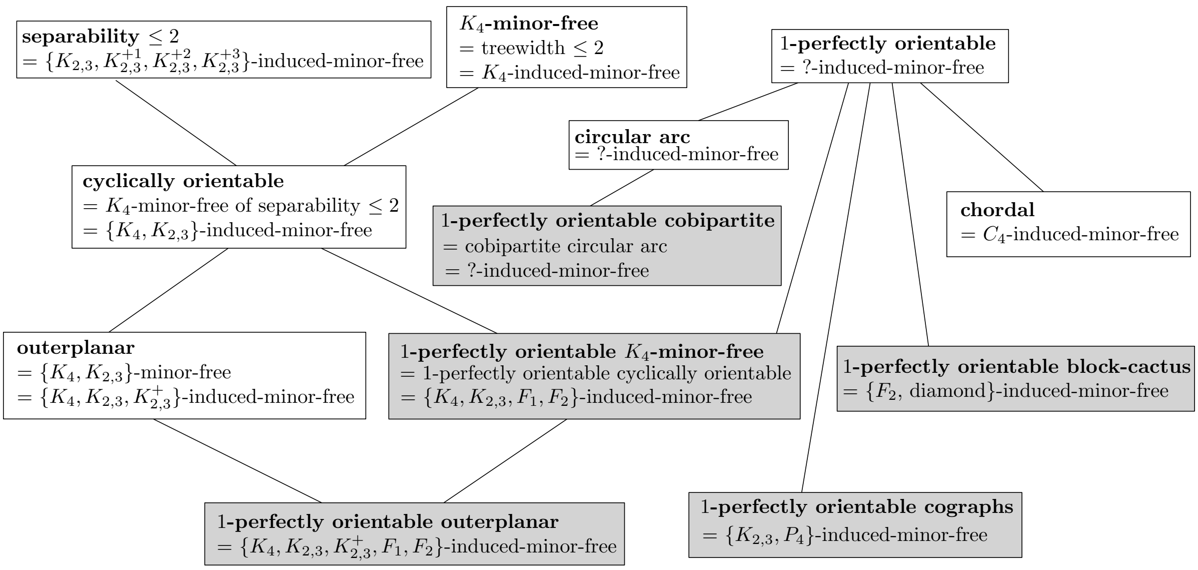


Figure 8.1: Hasse diagram of inclusion relations between induced-minor-closed graph classes considered, summarizing known results and results obtained in this thesis.

As a consequence of our results on K_4-minor-free and outerplanar 1-p.o. graphs (Chapter 4) we can observe the following. Since outerplanar and K_4-minor-free graphs are subclasses of the class of planar graphs (see, e.g., [10]), it is a natural question whether the characterizations of 1-perfectly orientable graphs within these two graph classes given by Theorems 4.12 and 4.14 could be generalized to the class of planar graphs. While no such characterizations are presently known, we observe below that known results on treewidth imply a partial result in this direction, namely that 1-perfectly orientable planar graphs are of bounded treewidth. We remind the reader that every outerplanar graph is of treewitdh at most 2 and, more generally, K_4-minor-free graphs are exactly the graphs of treewidth at most 2. For details on treewidth we refer the reader to [9].

A $k \times k$ grid is the graph with vertex set $\{1, \ldots, k\}^2$ and edge set $\{\{(i,j),(i',j')\} : 1 \leq i, j, i', j' \leq k, |i - i'| + |j - j'| = 1\}$. One of the results from the graph minor project due to Robertson and Seymour [76] states that for every positive integer k there is a positive integer N such that if G is a graph of treewidth at least N then the $k \times k$ grid is a minor of G. This result was further strengthened for planar graphs in several ways. For example, a result due to Gu and Tamaki [43] implies the following.

Proposition 8.1. *For every planar graph G, the treewidth of G is at most $4.5k - 1$ where k is the largest integer such that G contains a $k \times k$ grid as a minor.*

Proof. Let G be a planar graph, let k be as above, and let b and t denote the treewidth and the branchwitdh of G, respectively. Since G is planar, a result by Gu and Tamaki [43] implies that $b \leq 3k$. Moreover, we have $t \leq \max\{1.5b-1, 1\}$ by a general result relating the treewidth and the branchwidth due to Robertson and Seymour [78]. Consequently, $t \leq \max\{4.5k-1, 1\} = 4.5k-1$ since $k \geq 1$. □

Corollary 8.2. *For every $k > 1$, the treewidth of every planar graph having no $k \times k$ grid as minor is at most $4.5(k-1)-1$.*

Next, observe that a minor model of a 6×6 grid in a planar graph G can be used to obtain an induced minor model of F_1 in G (see Fig. 8.2).

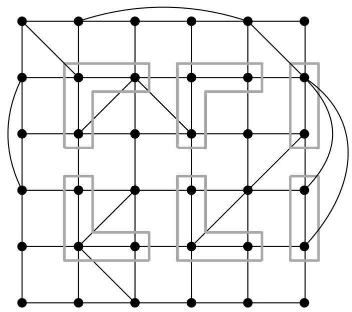


Figure 8.2: Obtaining F_1 as induced minor in a planar graph having the 6×6 grid as a minor.

Therefore, no 1-perfectly orientable planar graph can have a 6×6 grid as a minor and Corollary 8.2 implies the following.

Corollary 8.3. *The treewidth of every* 1*-perfectly orientable planar graph is at most* 21*.*

More generally, for every positive integer r, the treewidth is bounded in the class of 1-perfectly orientable K_r-minor-free graphs. This follows from the analogous statement in the more general setting, for K_r-minor-free graphs excluding any fixed planar graph as induced minor, which can be proved using arguments as in Case 2 of the proof of [86, Theorem 9]. (Theorem 9 from [86] was derived from results due to Fellows et al. [29] and Fomin et al. [31].) This observation has the following algorithmic consequence: since the defining property of 1-perfectly orientable graphs can be expressed in Monadic Second Order Logic with quantifiers over edges and edge subsets, Courcelle's Theorem [18] implies that 1-perfectly orientable graphs can be recognized in linear time in any class of graphs of bounded treewidth. In particular, by the above observation, this is the case for any class of graphs excluding some complete graph as a minor.

The next open question could lead to further insights on the structure of 1-perfectly orientable graphs; a positive answer would generalize Corollary 8.3.

A *maximal clique* is a clique that does not exist exclusively within the vertex set of a larger clique. The *clique number* of a graph G, denoted by $\omega(G)$, is the size of a largest clique or maximal clique of G.

Open problem. *Is it true that for every positive integer k there is a positive integer N such that every* 1*-perfectly orientable graph with clique number k is of treewidth at most N?*

Graph products. In this thesis we considered the four standard graph products, namely the Cartesian product, the lexicographic product, the direct product, and the strong product.

For each of them we characterized when a nontrivial product of two graphs is 1-p.o., chordal, interval, or circular arc, respectively (Chapters 5 and 6).

An open problem which could be considered is that of characterizing the studied graph classes in terms of other graph products, such as the conormal product [68], the modular product [51], the rooted product [40] or the homomorphic product [50], for instance.

The price of connectivity. In Chapter 7 we extended the tetrachotomy result of Belmonte et al. [6] for the family $\mathcal{F}$ of all cycles by giving tetrachotomy results for a number of natural families $\mathcal{F}$ containing cycles and anticycles (see Table 7.1). Let us recall that a tetrachotomy for the price of connectivity of $\mathcal{F}$-transversals when $\mathcal{F}$ is the family of even cycles or of all holes is still an open case. To settle it, it would suffice to show that the class of connected (P_3+P_2)-free graphs is $\mathcal{F}$-additive, which we conjecture to be true.

Conjecture. *The class of connected $(P_3 + P_2)$-free graphs is $\mathcal{F}$-additive if $\mathcal{F}$ consists of all even cycles or all holes.*

A final open problem worth mentioning regarding the price of connectivity if that of obtaining a tetrachotomy for infinite families $\mathcal{F}$ of cycles that contain C_3 but that miss some other odd cycle.

By Corollary 7.7 we know that the class of H-free graphs is $\mathcal{F}$-multiplicative if and only if H is a linear forest. We also know, due to Lemma 7.4, that the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive. Moreover, the class of connected H-free graphs is $\mathcal{F}$-zero-additive if and only if $H \subseteq_i P_3$, as we can use the example of $G = K_{2,2,2}$ from Theorem 7.9. Hence, using Lemmas 7.1–7.3, we see that what remains is to check, for every $s \geq 2$, whether the class of H-free graphs is $\mathcal{F}$-additive if $H = sP_3$.

Bibliography

[1] G. Alexe and E. Olaru. The strongly perfectness of normal product of t-perfect graphs. *Graphs Combin.*, 13(3):209–215, 1997.

[2] G. Bacsó and Z. Tuza. Dominating cliques in P_5-free graphs. *Period. Math. Hungar.*, 21(4):303–308, 1990.

[3] J. Bang-Jensen and G. Gutin. *Digraphs. Theory, algorithms and applications.* Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, second edition, 2009.

[4] J. Bang-Jensen, J. Huang, and E. Prisner. In-tournament digraphs. *J. Combin. Theory Ser. B*, 59(2):267–287, 1993.

[5] M. Barot, C. Geiss, and A. Zelevinsky. Cluster algebras of finite type and positive symmetrizable matrices. *J. London Math. Soc. (2)*, 73(3):545–564, 2006.

[6] R. Belmonte, P. van 't Hof, M. Kamiński, and D. Paulusma. The price of connectivity for feedback vertex set. In *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, volume 16 of *CRM Series*, pages 123–128. Ed. Norm., Pisa, 2013.

[7] R. Belmonte, P. van 't Hof, M. Kamiński, and D. Paulusma. Forbidden induced subgraphs and the price of connectivity for feedback vertex set. In *Mathematical Foundations of Computer Science 2014. Part II*, volume 8635 of *Lecture Notes in Comput. Sci.*, pages 57–68. Springer, Heidelberg, 2014.

[8] D. Binkele-Raible and H. Fernau. Parameterized measure & conquer for problems with no small kernels. *Algorithmica*, 64(1):189–212, 2012.

[9] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1):1 – 45, 1998.

[10] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[11] E. Camby, J. Cardinal, S. Fiorini, and O. Schaudt. The price of connectivity for vertex cover. *Discrete Math. Theor. Comput. Sci.*, 16(1):207–223, 2014.

[12] J. Cardinal and E. Levy. Connected vertex covers in dense graphs. *Theoret. Comput. Sci.*, 411(26-28):2581–2590, 2010.

[13] G. Chartrand and F. Harary. Planar permutation graphs. *Annales de l'I.H.P. Probabilités et statistiques*, 3(4):433–438, 1967.

[14] Z. Che. Uniquely pairable graphs. *Discrete Math.*, 308(24):6104–6110, 2008.

[15] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[16] F. Cicalese and M. Milanič. Graphs of separability at most 2. *Discrete Appl. Math.*, 160(6):685–696, 2012.

[17] D. G. Corneil, H. Lerchs, and L. S. Burlingham. Complement reducible graphs. *Discrete Appl. Math.*, 3(3):163–174, 1981.

[18] B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Inform. and Comput.*, 85(1):12–75, 1990.

[19] M. Cygan, M. Pilipczuk, M. Pilipczuk, and J. O. Wojtaszczyk. Kernelization hardness of connectivity problems in d-degenerate graphs. *Discrete Appl. Math.*, 160(15):2131–2141, 2012.

[20] D. de Werra and A. Hertz. On perfectness of sums of graphs. *Discrete Math.*, 195(1-3):93–101, 1999.

[21] F. Delbot, C. Laforest, and S. Rovedakis. Self-stabilizing algorithms for connected vertex cover and clique decomposition problems. In *Principles of Distributed Systems*, volume 8878 of *Lecture Notes in Comput. Sci.*, pages 307–322. Springer, Cham, 2014.

[22] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[23] G. A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, 25:71–76, 1961.

[24] P. Duchet and H. Meyniel. On Hadwiger's number and the stability number. In *Graph theory (Cambridge, 1981)*, volume 62 of *North-Holland Math. Stud.*, pages 71–73. North-Holland, Amsterdam-New York, 1982.

[25] G. Durán, L. N. Grippo, and M. D. Safe. Structural results on circular-arc graphs and circle graphs: A survey and the main open problems. *Discrete Appl. Math.*, 164(part 2):427–443, 2014.

[26] R. D. Dutton and R. C. Brigham. A characterization of competition graphs. *Discrete Appl. Math.*, 6(3):315–317, 1983.

[27] L. Fan, Z. Zhang, and W. Wang. PTAS for minimum weighted connected vertex cover problem with c-local condition in unit disk graphs. *J. Comb. Optim.*, 22(4):663–673, 2011.

[28] T. Feder, P. Hell, and J. Huang. List homomorphisms and circular arc graphs. *Combinatorica*, 19(4):487–505, 1999.

[29] M. R. Fellows, J. Kratochvíl, M. Middendorf, and F. Pfeiffer. The complexity of induced minors and related problems. *Algorithmica*, 13(3):266–282, 1995.

[30] S. Foldes and P. L. Hammer. Split graphs. In *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977)*, pages 311–315. Congressus Numerantium, No. XIX. Utilitas Math., Winnipeg, Man., 1977.

[31] F. V. Fomin, P. Golovach, and D. M. Thilikos. Contraction bidimensionality: the accurate picture. In *Algorithms—ESA 2009*, volume 5757 of *Lecture Notes in Comput. Sci.*, pages 706–717. Springer, Berlin, 2009.

[32] M. Francis, P. Hell, and J. Stacho. Forbidden structure characterization of circular-arc graphs and a certifying recognition algorithm. In *Proceedings of the Twenty-sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '15, pages 1708–1727, Philadelphia, PA, USA, 2015. Society for Industrial and Applied Mathematics.

[33] T. Fujito. On approximability of the independent/connected edge dominating set problems. *Inform. Process. Lett.*, 79(6):261–266, 2001.

[34] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. *Pacific J. Math.*, 15:835–855, 1965.

[35] H. Galeana-Sánchez. Normal fraternally orientable graphs satisfy the strong perfect graph conjecture. *Discrete Math.*, 122(1-3):167–177, 1993.

[36] H. Galeana-Sánchez. A characterization of normal fraternally orientable perfect graphs. *Discrete Math.*, 169(1-3):221–225, 1997.

[37] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. Combinatorial Theory Ser. B*, 16:47–56, 1974.

[38] F. Gavril. Intersection graphs of proper subtrees of unicyclic graphs. *J. Graph Theory*, 18(6):615–627, 1994.

[39] F. Gavril and J. Urrutia. Intersection graphs of concatenable subtrees of graphs. *Discrete Appl. Math.*, 52(2):195–209, 1994.

[40] C. Godsil and B. McKay. A new graph product and its spectrum. *BULL. AUSTRAL. MATH. SOC.*, 18:21–28, 1978.

[41] M. C. Golumbic. *Algorithmic Graph Theory and Pperfect Graphs*, volume 57 of *Annals of Discrete Mathematics*. Elsevier Science B.V., Amsterdam, second edition, 2004. With a foreword by Claude Berge.

[42] A. Grigoriev and R. Sitters. Connected feedback vertex set in planar graphs. In *Graph-theoretic concepts in computer science*, volume 5911 of *Lecture Notes in Comput. Sci.*, pages 143–153. Springer, Berlin, 2010.

[43] Q.-P. Gu and H. Tamaki. Improved bounds on the planar branchwidth with respect to the largest grid minor size. *Algorithmica*, 64(3):416–453, 2012.

[44] V. Gurvich. On cyclically orientable graphs. *Discrete Math.*, 308(1):129–135, 2008.

[45] R. H. Hammack, W. Imrich, and S. Klavžar. *Handbook of Product Graphs.* Discrete mathematics and its applications. CRC Press, Boca Raton, FL, 2011.

[46] F. Harary and A. Schwenk. Trees with hamiltonian square. *Mathematika*, 18:138–140, 1971.

[47] T. R. Hartinger and M. Milanič. On the structure of 1-perfectly orientable graphs. 2014, arXiv:1411.6663 [math.CO].

[48] P. Heggernes and D. Kratsch. Linear-time certifying recognition algorithms and forbidden induced subgraphs. *Nordic J. Comput.*, 14(1-2):87–108 (2008), 2007.

[49] P. Hell and J. Huang. Two remarks on circular arc graphs. *Graphs Combin.*, 13(1):65–72, 1997.

[50] P. Hell and J. Nešetřil. Graphs and homomorphisms. *Oxford University Press*, 2004.

[51] W. Imrich. Assoziative Produkte von Graphen. *Akad. Wiss. Math.-Natur. Kl. S-B.*, II(180):203–239, 1972.

[52] W. Imrich and S. Klavžar. *Product Graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[53] C. R. Johnson and T. A. McKee. Structural conditions for cycle completable graphs. *Discrete Math.*, 159(1-3):155–160, 1996.

[54] M. Kamiński, J.-F. Raymond, and T. Trunck. Well-quasi-ordering H-contraction-free graphs. *Discrete Applied Mathematics*, 2017.

[55] F. Kammer and T. Tholey. Approximation algorithms for intersection graphs. *Algorithmica*, 68(2):312–336, 2014.

[56] J. Kratochvíl. String graphs. I. The number of critical nonstring graphs is infinite. *J. Combin. Theory Ser. B*, 52(1):53–66, 1991.

[57] H.-J. Lai. Super-Eulerian graphs and excluded induced minors. *Discrete Math.*, 146(1-3):133–143, 1995.

[58] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962/1963.

[59] M. C. Lin and J. L. Szwarcfiter. Characterizations and recognition of circular-arc graphs and subclasses: a survey. *Discrete Math.*, 309(18):5618–5635, 2009.

[60] T. A. McKee and F. R. McMorris. *Topics in Intersection Graph Theory.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[61] M. Milanič. Hereditary efficiently dominatable graphs. *J. Graph Theory*, 73(4):400–424, 2013.

[62] N. Misra, G. Philip, V. Raman, S. Saurabh, and S. Sikdar. FPT algorithms for connected feedback vertex set. *J. Comb. Optim.*, 24(2):131–146, 2012.

[63] R. Motwani and M. Sudan. Computing roots of graphs is hard. *Discrete Appl. Math.*, 54(1):81–88, 1994.

[64] A. Mukhopadhyay. The square root of a graph. *J. Combinatorial Theory*, 2:290–295, 1967.

[65] J. Nešetřil and P. Ossona de Mendez. Fraternal augmentations of graphs, coloration and minors. In *6th Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications*, volume 28 of *Electron. Notes Discrete Math.*, pages 223–230.

[66] J. Nešetřil and P. Ossona de Mendez. Fraternal augmentations, arrangeability and linear Ramsey numbers. *European J. Combin.*, 30(7):1696–1703, 2009.

[67] R. J. Opsut. On the computation of the competition number of a graph. *SIAM J. Algebraic Discrete Methods*, 3(4):420–428, 1982.

[68] O. Ore. Theory of Graphs. *Amer. Math. Soci.*, 1962.

[69] Y. L. Orlovich. Characterizations and recognition of subclasses of modulo m well-covered graphs. *Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk*, (3):106–114, 128, 2003.

[70] E. Prisner. *Familien zusammenhängender Teilgraphen eines Graphen und ihre Durchschnittsgraphen.* Dissertation Hamburg, 1988.

[71] P. R. Rao. Characterization of line graphs and total graphs of some graph products. *Utilitas Math.*, 29:5–18, 1986.

[72] P. R. Rao and M. N. Vartak. On the characterization of the line graphs of some graph products. *Ars Combin.*, 11:107–121, 1981.

[73] G. Ravindra. Perfectness of normal products of graphs. *Discrete Math.*, 24(3):291–298, 1978.

[74] G. Ravindra and K. Parthasarathy. Perfect product graphs. *Discrete Mathematics*, 20:177 – 186, 1977.

[75] A. Raychaudhuri. On powers of strongly chordal and circular arc graphs. *Ars Combin.*, 34:147–160, 1992.

[76] N. Robertson and P. Seymour. Graph minors. III. Planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49 – 64, 1984.

[77] N. Robertson and P. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325 – 357, 2004.

[78] N. Robertson and P. D. Seymour. Graph minors. X. Obstructions to tree-decomposition. *J. Combin. Theory Ser. B*, 52(2):153–190, 1991.

[79] G. Salamon. Vulnerability bounds on the number of spanning tree leaves. *Ars Math. Contemp.*, 2(1):77–92, 2009.

[80] P. Schweitzer and P. Schweitzer. Connecting face hitting sets in planar graphs. *Inform. Process. Lett.*, 111(1):11–15, 2010.

[81] D. J. Skrien. A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular-arc graphs, and nested interval graphs. *J. Graph Theory*, 6(3):309–316, 1982.

[82] D. E. Speyer. Cyclically orientable graphs. preprint, 2005, arXiv:math/0511233 [math.CO].

[83] W. T. Trotter, Jr. and J. I. Moore, Jr. Characterization problems for graphs, partially ordered sets, lattices, and families of sets. *Discrete Math.*, 16(4):361–381, 1976.

[84] A. Tucker. Characterizing circular-arc graphs. *Bull. Amer. Math. Soc.*, 76:1257–1260, 1970.

[85] J. Urrutia and F. Gavril. An algorithm for fraternal orientation of graphs. *Inform. Process. Lett.*, 41(5):271–274, 1992.

[86] P. van 't Hof, M. Kamiński, D. Paulusma, S. Szeider, and D. M. Thilikos. On graph contractions and induced minors. *Discrete Appl. Math.*, 160(6):799–809, 2012.

[87] D. B. West. *Introduction to Graph Theory.* Prentice Hall, Upper Saddle River (N. J.), 2001.

[88] R. Zhang. Qadratic Bottleneck Problems: Algorithms, Complexity and Related Topics. *PhD Thesis. Simon Fraser Uiversity.*, 2011.

[89] F. Zou, X. Li, S. Gao, and W. Wu. Node-weighted Steiner tree approximation in unit disk graphs. *J. Comb. Optim.*, 18(4):342–349, 2009.

[90] T. Zou. Cyclical orientations of graphs. *Sichuan Daxue Xuebao*, 47(1):21–26, 2010.

Index

Chapter 9

Povzetek v slovenskem jeziku

V disertaciji obravnavamo tri glavne probleme.

Najprej se posvetimo problemu, kako pridobiti nekaj potrebnih ali zadostnih pogojev za razred 1-popolno usmerljivih grafov. Poudarek je na karakterizaciji 1-popolno usmerljivih grafov znotraj posebnih družin grafov, natančneje, znotraj kografov, ko-dvodelnih grafov, bločno-kaktus grafov, grafov brez K_4 minorja in zunanje ravninskih grafov.

Na drugem mestu proučujemo štiri standardne produkte grafov, torej kartezični produkt, leksikografski produkt, direktni produkt in krepki produkt, ter za vsakega od produktov karakteriziramo, kdaj je netrivialen produkt dveh grafov 1-popolno usmerljiv. Popolnoma karakteriziramo tudi tetivne grafe, intervalne grafe in grafe krožnih lokov, ki jih je moč razstaviti glede na poljubnega od štirih standardnih produktov.

Nazadnje obravnavamo, kako določiti ceno povezanosti za $\mathcal{F}$-tranzverzale grafov določenih družin grafov $\mathcal{F}$. Posvetimo se razredom grafov, karakteriziranih z enim prepovedanim induciranim podgrafom H, in $\mathcal{F}$-tranzverzalam, kjer $\mathcal{F}$ vsebuje neskončno mnogo ciklov in morda tudi enega ali več anticiklov ali kratkih poti. Določimo natanko tiste razrede povezanih H-prostih grafov, kjer je cena povezanosti teh $\mathcal{F}$-tranzverzal neomejena, multiplikativna, aditivna ali ničelno aditivna.

Rezultate disertacije lahko povzamemo s sledečim seznamom:

1. Karakterizacija 1-popolno usmerljivih grafov glede na pokritje povezav s klikami.
2. Identifikacija številnih transformacij grafov, ki ohranjajo razred 1-popolno usmerljivih grafov. Podan je dokaz dejstva, da je razred 1-popolno usmerljivih grafov zaprt za inducirane minorje.
3. Karakterizacija 1-popolno usmerljivih grafov znotraj razreda ko-dvodelnih grafov.
4. Identifikacija neskončne družine ko-dvodelnih 1-popolno usmerljivih grafov.
5. Identifikacija neskončne družine minimalnih prepovedanih induciranih minorjev za razred 1-popolno usmerljivih grafov.
6. Karakterizaciji 1-popolno usmerljivih kografov: s prepovedanimi induciranimi podgrafi in s kompozicijskim izrekom.
7. Karakterizacija 1-popolno usmerljivih bločno-kaktus grafov.
8. Karakterizacije 1-popolno usmerljivih grafov znotraj razreda grafov brez K_4 minorja in znotraj zunanje ravninskih grafov, tako glede na prepovedane inducirane minorje kot s kompozicijskim izrekom.

9. Karakterizacija, kdaj je netrivialen produkt dveh grafov G in H 1-popolno usmerljiv, za vsakega od štirih standardnih produktov.

10. Karakterizacija, kdaj je netrivialen produkt dveh grafov G in H tetiven graf, intervalen graf, oziroma graf krožnih lokov, za vsakega od štirih standardnih produktov.

11. Določitev natanko tistih razredov povezanih H-prostih grafov, kjer je cena povezanosti $\mathcal{F}$-transverzal neomejena, multiplikativna, aditivna ali ničelno aditivna, kjer $\mathcal{F}$ vsebuje neskončno število ciklov in morda enega ali več anticiklov ali kratkih poti.

Vsak izmed rezultatov pripomore k razširitvi znanja s področja strukturne teorije grafov. Večina rezultatov je vključena v naslednje znanstvene članke:

- T. R. Hartinger and M. Milanič, *Partial Characterizations of 1-Perfectly Orientable Graphs.* J. Graph Theory. Vol. 85, 2, 2017, 378–394.

- B. Brešar, T. R. Hartinger, T. Kos in M. Milanič (2016), *1-perfectly orientable K_4-minor-free and outerplanar graphs.* Poslan v objavo. arXiv:1604.04598. Razširjen povzetek je bil objavljen v Electronic Notes in Discrete Mathematics 54 (2016) 199–204.

- T. R. Hartinger in M. Milanič, *1-perfectly orientable graphs and graph products.* Discrete Mathematics 340 (2017) 1727–1737.

- T. R. Hartinger, (2016), *Chordal, interval, and circular-arc product graphs.* Applicable Analysis and Discrete Mathematics 10 (2016) 532–551.

- T. R. Hartinger, M. Johnson, M. Milanič in D. Paulusma, *The price of connectivity for cycle transversals.* European Journal of Combinatorics 58 (2016) 203–224. Razširjen povzetek je bil objavljen v Mathematical Foundations of Computer Science 2015. Part II, volume 9235 of Lecture Notes in Comput. Sci., strani 395–406. Springer, Heidelberg, 2014.

9.1 1-popolno usmerljivi grafi: operacije in karakterizacije v štirih razredih grafov, zaprtih za inducirane minorje

Prva obravnavana tema se nanaša na 1-popolno usmerljive grafe. *Turnir* je usmeritev polnega grafa. Z uporabo terminologije Kammerja and Tholeyja [55] pravimo, da je usmeritev grafa 1-*popolna*, če izhodna soseščina vsake točke inducira turnir, in da je graf 1-*popolno usmerljiv* (na kratko, 1-*p.u.*), če premore 1-popolno usmeritev.

Idejo 1-p.u. grafov je prvič predstavil Skrien [81] leta 1982 (pod imenom $\{B_2\}$-grafi), ko je tudi zastavil problem karakterizacije tega razreda grafov. Po definiciji so 1-p.u. grafi tisti grafi, ki premorejo usmeritev, ki je izhodni turnir. S preprostim argumentom zamenjave usmeritve povezav pokažemo, da so 1-p.u. grafi natanko grafi, ki premorejo usmeritev, ki je vhodni turnir. Taka usmeritev je v številnih člankih [35, 36, 38, 39, 65, 66, 85] poimenovana *bratska usmeritev.*

S pomočjo prevedbe na 2-SAT je bilo pokazano, da lahko družino 1-p.u. grafov prepoznamo v polinomskem času [4]. Čeprav je razumevanje strukture tega hereditarnega razreda grafov še vedno odprto vprašanje, so znani nekateri delni rezultati. Bang-Jensen idr. [4] (glej tudi [70]) so podali karakterizacijo 1-p.u. povezavnih grafov in 1-p.u. grafov brez trikotnikov in dokazali, da je vsak graf z enim samim induciranim ciklom reda vsaj 4 tudi 1-p.u. graf. V člankih [85] in [81, 85] je bilo pokazano, da sta razreda tetivnih grafov in grafov krožnih lokov podrazreda 1-p.u. grafov.

V disertaciji je predstavljenih kar nekaj rezultatov o strukturi 1-p.u. grafov. Podana je karakterizacija 1-p.u. grafov glede na pokritje povezav s klikami, identificirane so številne transformacije grafov, ki ohranjajo razred 1-p.u. grafov, prikazana je neskončna družina minimalnih prepovedanih induciranih minorjev za razred 1-p.u. grafov in karakteriziran je razred 1-p.u. grafov v razredih kografov in ko-dvodelnih grafov (tj. komplementov dvodelnih grafov). Pokažemo, da razred 1-p.u. ko-dvodelnih grafov sovpada z razredom ko-dvodelnih grafov krožnih lokov. Kot stranski rezultat definiramo novo neskončno družino dvodelnih grafov in dokažemo, da so njihovi komplementi 1-p.u.

Z uporabo prevedbe proučevanja razreda 1-p.u. grafov na 2-povezane grafe karakteriziramo, tako glede na prepovedane inducirane minorje kot glede na kompozicijske izreke, razreda 1-p.u. grafov brez K_4 minorja in 1-p.u. zunanje ravninskih grafov. Kot del našega pristopa uvedemo razred grafov, definiran podobno kot razred 2-dreves, ter povežemo obravnavani razred grafov z dvema drugima razredoma, zaprtima za inducirane minorje, ki sta že bila proučevana v literaturi: s ciklično usmerljivimi grafi in z grafi ločljivosti kvečjemu 2. Karakteriziramo tudi 1-p.u. bločne-kaktus grafe.

V nadaljevanju so natančneje predstavljeni glavni rezultati tega dela. Pravimo, da je graf H *induciran minor* grafa G, če lahko graf H dobimo iz grafa G z zaporednim odstranjevanjem točk in skrčevanjem povezav. Naši rezultati implicirajo, da je razred 1-p.u. grafov zaprt za inducirane minorje in ga torej lahko karakteriziramo glede na *minimalne prepovedane inducirane minorje.* Z drugimi besedami, obstaja taka minimalna množica grafov $\mathcal{F}$, da je graf G 1-p.u. graf, če in samo če ne vsebuje nobenega grafa iz množice $\mathcal{F}$ kot induciranega minorja. Taka množica je minimalna v smislu, da je vsak pravi induciran minor poljubnega grafa v $\mathcal{F}$ 1-p.u. graf. Naslednji rezultat bo opisal neskončno poddružino $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$ minimalnih prepovedanih induciranih minorjev za razred 1-p.u. grafov.

Izrek 9.1. *Naj bo $\mathcal{F} = \{F_1, F_2, F_5, \ldots, F_{12}\} \cup \mathcal{F}_3 \cup \mathcal{F}_4$ množica grafov, za katero velja:*

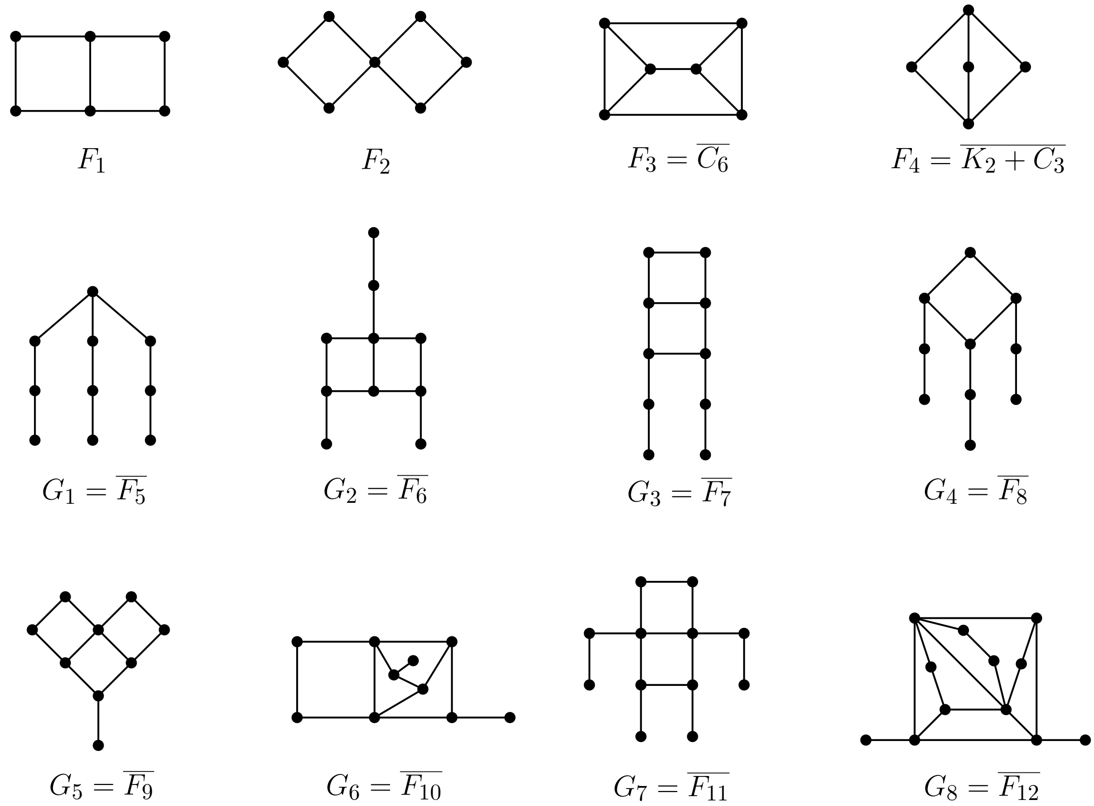
- *Grafa F_1 in F_2 sta prikazana na sliki 9.1,*
- *$\mathcal{F}_3 = \{\overline{C_{2k}} \mid k \geq 3\}$ je množica komplementov sodih ciklov dolžine vsaj 6,*
- *$\mathcal{F}_4 = \{\overline{K_2 + C_{2k+1}} \mid k \geq 1\}$ je množica komplementov grafov, ki jih dobimo kot disjunktno unijo grafa K_2 z nekim lihim ciklom,*
- *za $i \in \{5, \ldots, 12\}$ je graf F_i komplement grafa G_{i-4}, prikazanega na sliki 9.1.*

Potem je vsak graf iz množice $\mathcal{F}$ minimalen prepovedan induciran minor za razred 1-popolno usmerljivih grafov.

Sledijo karakterizacije 1-p.u. grafov znotraj naslednjih grafovskih razredov: ko-dvodelni grafi, kografi, bločno-kaktus grafi, grafi brez K_4 minorjev in zunanje ravninski grafi. Graf je *dvodelen*, če je njegovo množico točk moč razdeliti na dve neodvisni množici. Pravimo, da je graf *ko-dvodelen*, če je njegov komplement dvodelen. Dalje pravimo, da je cikel brez tetiv C v grafu G *usmerjen ciklično* v neki usmeritvi D grafa G, če ima vsaka točka cikla natanko enega izhodnega soseda, ki pripada ciklu.

Izrek 9.2. *Za vsak ko-dvodelen graf G so naslednje trditve ekvivalentne:*

1. *Graf G je 1-popolno usmerljiv.*
2. *Graf G premore usmeritev, v kateri je vsak induciran cikel dolžine 4 usmerjen ciklično.*
3. *G je graf krožnih lokov.*



Slika 9.1: Štirje ne-1-p.u. grafi in 8 komplementov ne-1-p.u. grafov. Grafa F_3 in F_4 sta najmanjša člana družin $\mathcal{F}_3$ in $\mathcal{F}_4$, v tem vrstnem redu.

Razred *kografov* je definiran rekurzivo, s pogoji, da je graf K_1 kograf, da je disjunktna unija dveh kografov kograf, da je spoj dveh kografov kograf in da so to vsi kografi. Kografi so natanko P_4-prosti grafi [10].

Izrek 9.3. *Za vsak kograf G so naslednje trditve ekvivalentne:*

1. *Graf G je* 1*-popolno usmerljiv.*
2. *Graf G je $K_{2,3}$-prost.*
3. *Velja ena od naslednjih trditev:*
   - $G \cong K_1$.
   - $G \cong \overline{mK_2}$ *za nek* $m \geq 2$.
   - *Graf G je disjunktna unija dveh manjših* 1*-p.u. kografov.*
   - *Graf G je rezultat dodajanja univerzalne točke nekemu manjšemu* 1*-p.u. kografu.*
   - *Graf G je rezultat dodajanja pravega dvojčka nekemu manjšemu* 1*-p.u. kografu.*

Blok grafa G je maksimalen povezan podgraf brez prereznih točk. Graf G je *bločno-kaktus*, če je vsak blok grafa G bodisi cikel ali pa poln graf.

Izrek 9.4. *Naj bo G povezan bločno-kaktus graf. Tedaj so naslednje trditve ekvivalentne:*

1. *G je* 1*-popolno usmerljiv.*
2. *Kvečjemu en blok grafa G ni poln.*

3. *G nima induciranega minorja, izomorfnega grafu F_2 (glej sliko 9.1).*

Razred 2-*dreves* je definiran na sledeč način: (i) graf K_2 je 2-drevo, (ii) graf, ki ga dobimo tako, da 2-drevesu dodamo simplicialno točko stopnje 2, je 2-drevo in (iii) drugih 2-dreves ni. Uvedli bomo razred netetivnih grafov, ki ga lahko dobimo s podobno induktivno konstrukcijo kot 2-drevesa. *Votlo* 2-*drevo* je definirano na sledeči način: (i) vsak cikel dolžine vsaj 4 je votlo 2-drevo, (ii) graf, ki ga dobimo tako, da votlemu 2-drevesu dodamo simplicialno točko stopnje 2, je votlo 2-drevo in (iii) drugih votlih 2-dreves ni.

V naslednjih rezultatih so uporabljene sledeče operacije:

- (A_1): dodajanje simplicialne točke stopnje 1.
- (A_2): dodajanje simplicialne točke stopnje 2 (tj. dodajanje točke, ki je povezana z natanko dvema sosednima točkama).
- (A_2'): dodajanje simplicialne točke stopnje 2 s sosednima točkama v in w, pri čemer je povezava vw vsebovana v največ enem induciranem ciklu.

Izrek 9.5. *Naj bo G povezan graf brez K_4 minorja. Potem so naslednje trditve ekvivalentne:*

1. *Graf G je* 1*-popolno usmerljiv.*
2. *Graf G je brez induciranih minorjev, izomorfnih $K_{2,3}, F_1$ ali F_2.*
3. *Vsak blok grafa G je* 2*-drevo, razen morda enega, ki je bodisi K_1 bodisi votlo* 2*-drevo.*
4. *Graf G lahko konstruiramo bodisi iz grafa K_1 bodisi iz cikla z nekim zaporedjem operacij (A_1) in (A_2).*

Graf G je *zunanje ravninski*, če ga lahko v ravnini narišemo brez presečišč povezav tako, da vse točke mejijo na zunanje lice. Zunanje ravninski grafi so natanko grafi brez minorjev, izomorfnih K_4 ali $K_{2,3}$ [13].

Izrek 9.6. *Naj bo G povezan zunanje ravninski graf. Potem so naslednje trditve ekvivalentne:*

1. *Graf G je* 1*-popolno usmerljiv.*
2. *Graf G je brez induciranih minorjev, izomorfnih $K_{2,3}, F_1$ ali F_2.*
3. *Vsak blok grafa G je* 2*-drevo, razen morda enega, ki je bodisi K_1 bodisi votlo* 2*-drevo.*
4. *Graf G lahko konstruiramo bodisi iz grafa K_1 bodisi iz cikla z nekim zaporedjem operacij (A_1) in (A_2').*

9.2 Karakterizacija glede na štiri standardne grafovske produkte

Druga tema, obravnavana v disertaciji, so produktni grafi. Produktni grafi znotraj raznih razredov grafov so bili obravnavani v številnih člankih; popolna karakterizacija dane grafovske lastnosti z vidika vseh štirih standardnih produktov (kartezični, direktni, krepki in leksikografski) je pogosto zahtevna. Ravindra in Parthasarathy [74] sta karakterizirala popolne kartezične, direktne in leksikografske produkte. Kartezični produkt sta nadalje proučevala tudi de Werra and Hertz [20]. Trenutno ni nobene poznane karakterizacije popolnih netrivialnih krepkih produktov; delno karakterizacijo in zadostne pogoje je opisal Ravindra [73] (glej tudi [1]). Karakterizacijo povezavnih grafov in totalnih grafov glede na razne produkte sta podala Rao [71] ter

Rao in Vartak [72], karalterizacijo modulo m dobro pokritih leksikografskih produktov je podal Orlovich [69], karakterizacijo enolično parnih kartezičnih produktov pa Che [14]. Rezultati te disertacije prispevajo k poznavanju karakterizacij razredov grafov znotraj grafov, ki jih je mogoče razstaviti glede na enega od štirih standardnih produktov grafov, ter na seznam dodajo 1-popolno usmerljive grafe, tetivne grafe, intervalne grafe ter grafe krožnih lokov.

9.2.1 1-popolno usmerljivi produkti grafov

V tem delu podamo karakterizacijo, kdaj je netrivialen produkt dveh grafov 1-p.u. graf za vsakega izmed štirih standardnih grafovskih produktov: za kartezični produkt, leksikografski produkt, direktni produkt in krepki produkt. Za poljubnega izmed štirih omenjenih produktov pravimo, da je *netrivialen*, če oba faktorja vsebujeta vsaj 2 točki. Za več informacij o grafovskih produktih in njihovih lastnostih bralca napotujemo na [45, 52].

Izrek 9.7. *Naj bosta G in H povezana grafa. Potem je netrivialen kartezični produkt $G \square H$ 1-p.u. graf če in samo če je $G \cong H \cong K_2$.*

Izrek 9.8. *Naj bosta G in H grafa in naj bo graf G povezan. Potem je netrivialen leksikografski produkt $G[H]$ 1-p.u. graf če in samo če je izpolnjen vsaj eden od naslednjih pogojev:*

(i) Graf G je 1-p.u. in je graf H poln graf.

(ii) Graf G je poln graf in je graf H ko-dvodelen 1-p.u. graf.

Psevdodrevo je povezan graf, ki vsebuje kvečjemu en cikel.

Izrek 9.9. *Naj bosta G in H povezana grafa. Potem je netrivialen direktni produkt $G \times H$ 1-p.u. graf če in samo če velja ena od naslednjih trditev:*

(i) Eden izmed faktorjev je izomorfen grafu K_2, drugi pa je psevdodrevo.

(ii) Eden izmed faktorjev je izomorfen grafu P_3, drugi pa grafu P_4.

(iii) Oba faktorja sta izomorfna grafu P_3.

Graf G je *ko-veriga*, če lahko njegovo množico točk razdelimo v taki dve kliki, recimo X in Y, da lahko točke v X uredimo tako, $X = \{x_1, \ldots, x_{|X|}\}$, da za vsak $1 \leq i < j \leq |X|$ velja $N[x_i] \subseteq N[x_j]$ (ali, ekvivalentno, da je $N(x_i) \cap Y \subseteq N(x_j) \cap Y$). Pravimo, da je graf 2-*poln*, če je unija dveh (ne nujno različnih) polnih grafov, ki imata skupno vsaj eno točko. Ekvivalentno je graf 2-poln če in samo če ga lahko dobimo bodisi iz grafa K_1 bodisi iz grafa P_3 z zaporednim dodajanjem pravih dvojčkov.

Izrek 9.10. *Naj bosta G in H povezana grafa. Potem je netrivialen krepki produkt $G \boxtimes H$ 1-p.u. graf če in samo če velja ena od naslednjih trditev:*

(i) Eden izmed faktorjev je 1-p.u. graf in drugi faktor je poln graf.

(ii) Eden izmed faktorjev je ko-veriga in drugi faktor je 2-poln.

9.2.2 Tetivni in intervalni produktni grafi ter produktni grafi krožnih lokov

V tem poglavju karakteriziramo netrivialne produktne grafe, ki so tetivni grafi, intervalni grafi in grafi krožnih lokov, za vsakega od štirih standardnih grafovskih produktov.

Izrek 9.11. *Netrivialen kartezični produkt $G \square H$ grafov G in H je:*

- *tetiven graf, če in samo če je G graf brez povezav in je H tetiven graf ali obratno,*
- *intervalen graf, če in samo če je G graf brez povezav in je H intervalen graf ali obratno,*
- *graf krožnih lokov, če in samo če velja vsaj ena od naslednjih trditev:*
  - *(i) je G graf brez povezav in je H graf krožnih lokov ali obratno,*
  - *(ii) $G \cong H \cong K_2$.*

Izrek 9.12. *Netrivialen leksikografski produkt $G[H]$ grafov G in H je:*

- *tetiven graf, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) G je graf brez povezav in H je tetiven graf,*
  - *(ii) G je tetiven graf in H je poln graf,*
- *intervalen graf, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) G je graf brez povezav in H je intervalen graf,*
  - *(ii) G je intervalen graf in H je poln graf,*
- *graf krožnih lokov, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) G je graf brez povezav in H je intervalen graf,*
  - *(ii) G je graf krožnih lokov in H je poln graf,*
  - *(iii) G je poln graf in H je ko-dvodelen graf krožnih lokov.*

2-*linearen gozd* je graf, ki sestoji samo iz izoliranih točk in izoliranih povezav. *Gosenica* je drevo, pri katerem z odstranitvijo vseh listov (točk stopnje 1) dobimo pot. *Gozd gosenic* je disjunktna unija gosenic. *Liha ciklična gosenica* je povezan graf, pri katerem z odstranitvijo vseh listov (točk stopnje 1) dobimo lih cikel.

Izrek 9.13. *Netrivialen direktni produkt $G \times H$ grafov G in H je:*

- *tetiven graf, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) vsaj en izmed faktorjev ne vsebuje povezave,*
  - *(ii) graf G je 2-linearen gozd in graf H je gozd, ali obratno,*
- *intervalen graf, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) vsaj en izmed faktorjev ne vsebuje povezave,*
  - *(ii) graf G je 2-linearen gozd in graf H je gozd gosenic, ali obratno,*
- *graf krožnih lokov, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) vsaj en izmed faktorjev ne vsebuje povezave,*
  - *(ii) graf G je 2-linearen gozd in graf H je gozd gosenic, ali obratno,*
  - *(iii) $G \cong K_2$ in H je liha ciklična gosenica, ali obratno.*

Izrek 9.14. *Netrivialen krepki produkt $G \boxtimes H$ grafov G in H je:*

- *tetiven graf, če in samo če je vsaka komponenta grafa G poln graf in je H tetiven graf, ali obratno,*

- *intervalen graf, če in samo če je vsaka komponenta grafa G poln graf in je H intervalen graf, ali obratno,*
- *graf krožnih lokov, če in samo če velja vsaj en od naslednjih pogojev:*
  - *(i) G je poln graf in H je graf krožnih lokov, ali obratno,*
  - *(ii) G je 2-poln graf in graf H je povezana ko-veriga, ali obratno,*
  - *(iii) vsaka komponenta grafa G je poln graf in H je intervalen graf, ali obratno.*

9.3 Cena povezanosti za transverzale ciklov

Tretja obravnavana tema v disertaciji je cena povezanosti. Za družino grafov $\mathcal{F}$ je *$\mathcal{F}$-transverzala* grafa G taka podmnožica $S \subseteq V(G)$, ki ima presek z vsako podmnožico množice $V(G)$, ki inducira podgraf, izomorfen grafu $\mathcal{F}$. Naj bo $t_{\mathcal{F}}(G)$ minimalna velikost $\mathcal{F}$-transverzale grafa G in naj bo $ct_{\mathcal{F}}(G)$ minimalna velikost $\mathcal{F}$-transverzale grafa G, ki inducira povezan graf. Za razred povezanih grafov $\mathcal{G}$ rečemo, da je *cena povezanosti* $\mathcal{F}$-transverzal *multiplikativna*, če je za vse grafe $G \in \mathcal{G}$, $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ omejena s konstanto, in *aditivna*, če je $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ omejena s konstanto. Cena povezanosti je *ničelno aditivna*, če sta vrednosti $t_{\mathcal{F}}(G)$ in $ct_{\mathcal{F}}(G)$ vedno enaki, in *neomejena*, če je vrednost $ct_{\mathcal{F}}(G)$ neomejena glede na $t_{\mathcal{F}}(G)$. Za nekatere premere so $\mathcal{F}$-transverzale dobro proučene. Na primer, točkovno pokritje je $\{P_2\}$-transverzala, množica povratnih točk pa je $\mathcal{F}$-transverzala za neskončno družino $\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$. Kot narekujeta ta dva primera, je smiselno proučevati $\mathcal{F}$-transverzale najmanjše velikosti.

Na $\mathcal{F}$-transverzale S povezanega grafa G lahko damo dodatno omejitev in zahtevamo, da je podgraf, ki ga inducira množica S, povezan. *Povezane* $\mathcal{F}$-transverzale minimalne velikosti grafa so bile v literaturi že študirane; raziskovalci so se med drugim posvetili najmanišim povezanim točkovnim pokritjem (glej npr. [8, 11, 12, 21, 27, 33, 79, 89]) in najmanišim povezanim množicam povratnih točk (glej npr. [6, 19, 42, 62, 80]).

V disertaciji obravnavamo sledeče vprašanje: *Kakšen vpliv ima omejitev povezanosti na minimalno velikosti $\mathcal{F}$-transverzal za družino grafov $\mathcal{F}$?*

Natančneje, proučujemo razrede grafov, karakterizirane z enim prepovedanim induciranim podgrafom H, in $\mathcal{F}$-transverzale, kjer $\mathcal{F}$ vsebuje neskončno število ciklov in morda enega ali več anticiklov ali kratkih poti. Cilj je določiti natanko tiste razrede povezanih grafov brez induciranega podgrafa H, kjer je cena povezanosti teh $\mathcal{F}$-transverzal neomejena, multiplikativna, aditivna ali ničelno aditivna. Naša tetrahotomija med drugim razširi do sedaj znane rezultate za primer, ko je $\mathcal{F}$ družina vseh ciklov.

Tabela, predstavljena v nadaljevanju, povzema rezultate tega dela in nekatere do sedaj znane rezultate v zvezi s ceno povezanosti. Rezultate je mogoče interpretirati tako glede na družino $\mathcal{F}$ kot glede na ustrezno lastnost grafa $G - S$, kjer je S neka $\mathcal{F}$-transverzala grafa G. Tabela 9.1 predstavlja pogoje za graf H, pod katerimi je cena povezanosti $\mathcal{F}$-transverzal za H-proste grafe multiplikativna, aditivna ali ničelno aditivna, v tem vrstnem redu, ko je $\mathcal{F}$ množica grafov, ki vsebuje podano neskončno družino ciklov in morda še nekaj drugih majhnih grafov.

Rezultati za cikle, podani v prvi vrsti tabele, so delo Belmonteja idr. [6]. Rezultat, podan v deveti vrsti tabele, v zvezi z multiplikativnostjo za cikle in P_2, je delo Cambyjeve idr. [11]. Vsi ostali rezultati, ki jih predstavimo v disertaciji, so novi. Vsi podani pogoji so tako potrebni kot zadostni, z izjemo rezultatov za sode cikle in luknje, v teh primerih namreč ne vemo, ali so H-prosti grafi $\mathcal{F}$-aditivni za $H \subseteq_i P_3 + P_2 + sP_1$. V vseh ostalih primerih so podani pogoji v tabeli 9.1 tako potrebni kot zadostni za $\mathcal{F}$-multiplikativnost ($\mathcal{F}$-omejenost), $\mathcal{F}$-aditivnost in $\mathcal{F}$-ničelno aditivnost, v tem vrstnem redu, v razredu povezanih H-prostih grafov.

Luknja je cikel dolžine vsaj štiri; *dolga luknja* je cikel dolžine vsaj pet. *(Dolga) antiluknja* je komplement (dolge) luknje. Disjunktno unijo dveh kopij grafa G označimo z $2G$.

$\mathcal{F}$	Lastnost grafa $G - S$	Pogoj za $\mathcal{F}$-multiplikativnost (za $\mathcal{F}$-omejenost)	Pogoj za $\mathcal{F}$-aditivnost	Pogoj za $\mathcal{F}$-ničelna aditivnost
cikli	gozd	H je linearen gozd [6]	$H \subseteq_i P_5 + sP_1$ ali $H \subseteq_i sP_3$ [6]	$H \subseteq_i P_3$ [6]
lihi cikli	dvodelen graf	H je linearen gozd	$H \subseteq_i P_5 + sP_1$ ali $H \subseteq_i sP_3$	$H \subseteq_i P_3$
sodi cikli (ekviv.: sode luknje)	graf brez sodih lukenj	H je linearen gozd	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
liknje	tetiven	H je linearen gozd	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
lihe luknje	graf brez lihih lukenj	H je linearen gozd	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
lihe luknje in lihe antiluknje	popoln graf	H je linearen gozd	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
dolge luknje	graf brez dolgih lukenj	H je linearen gozd	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
dolge luknje in dolge antiluknje	šibko tetiven graf	H je linearen gozd	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
cikli in P_2 (ekviv.: $\{P_2\}$)	graf brez povezav	ni omejitev [11]	$H \subseteq_i P_5 + sP_1$ ali $H \subseteq_i sP_3$	$H \subseteq_i P_3$
luknje in $2P_2$ (ekviv.: $\{C_4, C_5, 2P_2\}$)	razcepljen graf	ni omejitev	$H \subseteq_i P_4 + sP_1$ ali $H \subseteq_i P_3 + sP_2$	$H \subseteq_i P_3$
luknje in $2P_2, P_4$ (ekviv.: $\{C_4, 2P_2, P_4\}$)	pragoven graf	ni omejitev	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
luknje in P_4 (ekviv.: $\{C_4, P_4\}$)	trivialno popoln graf	ni omejitev	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
dolge luknje in $2P_2$ (ekviv.: $\{C_5, 2P_2\}$)	$(C_5, 2P_2)$-prost graf	ni omejitev	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ $H \subseteq_i P_2 + P_1$
dolge luknje in $2P_2, P_4$ (ekviv.: $\{2P_2, P_4\}$)	ko-trivialno popoln graf	ni omejitev	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ ali $H \subseteq_i P_2 + P_1$
dolge luknje in P_4 (ekviv.: $\{P_4\}$)	kograf	ni omejitev	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$

Tabela 9.1: V tabeli so povzeti pogoji na graf H, glede na katere je cena povezanosti $\mathcal{F}$-transverzal v razredu H-prostih grafov multiplikativna, aditivna oz. ničelno aditivna, za različne družine grafov $\mathcal{F}$.

Kazalo vsebine

Stvarno kazalo