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FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

Zaključna naloga
(Final project paper)

Krepko-regularni graf brez trikotnikov, Higman-Sims graf

(Strongly regular graphs without triangles, Higman-Sims graph)

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Izvleček:
V prvem delu naloge definiramo osnovne definicije o grupama i grafih. Potem definiramo krepko regularen graf in pogoje kdaj obstaja. Naslednja stvar definiramo dva krepko regularna grafa s parametrima $(5, 0, 2)$ i $(10, 0, 2)$ in na koncu konstruiramo Higman-Sims graf. Skozi celo delo se raztegujejo štiri naloge z rešitvami katere smo mi podali.

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Abstract: In first part of a project we define basic definitions about groups and graphs. Later we define strongly regular graphs and their feasibility conditions for existing. Also we define two types of strongly regular graphs with $(5, 0, 2)$ and $(10, 0, 2)$ parameters. At the end we construct Higman-Sims graph. In the paper there are four problems with our solutions.

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1 Introduction

In the final project we discuss strongly regular graphs. We recall that, by a strongly regular graph with parameters (k, α, γ) we mean a k-regular graph with the property that any two adjacent vertices have eaxctly α, while any two non-adjacent vertices have exactly γ common neighbours. We pay our attention to strongly regular graphs whose parameter $\alpha = 0$; or equivalently, to strongly regular graphs without triangles.

In the first part of our thesis we solve two project probelms from the book [5]. Our main tool by these will be the so called feasibility conditions for strongly regular graphs. We apply this conditions to derive more restrictions on the possible parameters in the case when the strongly regular graphs are without triangles. In particular, we show that the number of such graphs is finte when $\gamma \neq 2, 4, 6$ (see Problem 3.2.1). In fact, this is the task of Problem 4.7.1 from [BW]. Then we will solve Problem 4.7.2 from [BW], and show that for $\gamma = 2$, the degree k must be in the following form (see Problem 3.2.2):

$$k = t^2 + 1, \text{ where } t \not\equiv 0\,(mod\,4).$$

In the second part of our thesis we turn to strongly regular graphs with parameters $(5, 0, 2)$ and $(10, 0, 2)$. Note that, these parameters arise when we substitute $t = 2$ and $t = 3$ above. We first solve Problem 4.7.3 from [BW] and by this discover a nice relation with biplanes (see Problem 4.1.1). A biplane is symmetric 2-design with $\lambda = 2$. After that we solve Problem 4.7.4 from [BW], and show that any strongly regular graph with parameters $(5, 0, 2)$ has a rank 3 group of automorphisms (see Problem 4.2.1). Finally, we will solve Problem 4.7.5 from [BW], and construct a strongly regular graph with parameters $(10, 0, 2)$ as a subgraph of the so called Higman-Sims graph (see Problem 4.3.5).

Let us give an overview of the topics to be presented: In chapter 2 we recall some basic definitions and theorems about groups, graphs, designs and projective planes which are going to be frequently used in the upcoming chapters. In chapter 3 we define strongly regular graphs and present feasibility conditions for their parameters. In Chapter 4 we discuss strongly regular graphs with parameters $(5, 0, 2)$ and $(10, 0, 2)$.

2 Preliminaries

2.1 Groups

In this section we will mention some definitions, theorems and lemmas about groups.

Definition 2.1.1. A **group** $(G, *)$ is a nonempty set G with a binary operation $*$ that satisfy the following axioms:

1. (*closure*) For $\forall a, b \in G$, the result $a * b$ is also in G.
2. (*associativity*) For $\forall a, b, c \in G$, $(a * b) * c = a * (b * c)$.
3. (*identity element*) There exists an element e in G such that $\forall a \in G$, $e * a = a * e = a$.
4. (*inverse element*) For $\forall a \in G$, there exists $b \in G$ such that $a * b = b * a = e$, where e is an identity element.

If a group G also satisfies: $\forall a, b \in G$, $a * b = b * a$ (commutativity), then G is called **Abelian group**.

Definition 2.1.2. A group G is said to be **finite** if the set G has a finite number of elements. In this case, the number of elements is called the **order** of G, denoted by $|G|$

Definition 2.1.3. Let G be a group, and let H be a subset of G. Then H is called a **subgroup** of G if H itself is a group, under the operation of G.

An equivalent definition is as follows:

Definition 2.1.4. Let G be a group with identity element e, and let H be a subset of G. Then H is a **subgroup** of G if and only if the following conditions hold:

1. $\forall a, b \in H, ab \in H$.
2. The identity element $e \in H$.
3. $\forall a \in H, a^{-1} \in H$.

We will write $H \leq G$ when H is a subgroup of G.

Definition 2.1.5. Let N be a subgroup of a group G. We say that N is **normal** if its left and right cosets are equal, in other words, if $gH = Hg$ for all $g \in G$.

Definition 2.1.6. A group G is called **simple** if it is nontrivial, and its only normal subgroups are the trivial group and the group G itself.

2.2 Group actions

Definition 2.2.1. The **symmetric group** S_X or $Sym(X)$ of a nonempty set X is a group of all permutations of the set X with the operation being the usual composition of mappings.

In the special case wen the set $X = \{1, 2, 3, \ldots, n\}$, we write S_n for $Sym(X)$. It is well-known that $|S_n| = n!$.

Definition 2.2.2. By a **permutation group** G of degree n we mean a subgroup $G \leq S_X$ for any n-set X.

Now, after we have defined symmetric and permutation groups, we can tell what is an action of a group G on a set X.

Definition 2.2.3. An **action** of a group G on a set X is a mapping $\mu : X \times G \to X$ which satisfies the following properties:

1. $\forall x \in X, x^1 = x$, where 1 is identity of the group G.
2. $\forall x \in X$ and $\forall g_1, g_2 \in G$, $x^{g_1 g_2} = (x^{g_1})^{g_2}$.

Here by x^g we denote the image $\mu((x, g))$ of the element $(x, g) \in X \times G$.

The cardinality $|X|$ of the set X in the above definition is called the **degree** of the action.

Definition 2.2.4. Let G act on a set X and let $x \in X$. The **orbit** of x under G is the subset

$$Orb(x) = \{x^g : g \in G\}.$$

It is well-known that given two elements $x, y \in X$, either $Orb(x) = Orb(y)$ or $Orb(x) \cap Orb(y) = \emptyset$. Also, the orbits $Orb(x)$, $x \in X$, form a partition of X.

Definition 2.2.5. Let G act on a set X. We say that G acts **transitively** on X (or G is **transitive** on X) if $\forall x, y \in X$, there exists some $g \in G$ such that $y = x^g$.

Notice that, an equivalent way to say that G is transitive on X is to say that $\forall x \in X, Orb(x) = X$, i.e., there is only one orbit in the action of G on the set X.

Definition 2.2.6. Let G act on a set X and let $x \in X$. The **stabilizer** of x in the group G is the subset

$$G_x = \{g \in G : x^g = x\}.$$

It is well-known that the stabilizer G_x is a subgroup of G. In the case when both the order $|G|$ and the degree $|X|$ are finite, there is a nice relation between the stabilizer G_x and the orbit $Orb(x)$ given in the following lemma, also known as the **Orbit-stabilizer lemma**.

Lemma 2.2.7. Let G be a finite group acting on a finte set X, and let $x \in X$. Then

$$|G| = |Orb(x)| \cdot |G_x|.$$

Let G be a group acting on a set X. The **canonical action** of G on the Cartesian product $X \times X$ is defined by the following rule:

$$\forall (x, y) \in X \times X \text{ and } \forall g \in G, (x, y)^g = (x^g, y^g).$$

Definition 2.2.8. Let G be a finite group acting on a finte set X. The **rank** of G is equal to the number of orbits in the canonical action of G on the Cartesian product $X \times X$.

Our final lemma in this section will play an important rule later.

Lemma 2.2.9. Let $G \leq S_X$ be a transitive permutation group, and suppose that G has r orbits $D_0, D_1, \ldots, D_{r-1}$ on $X \times X$. Then r is equal to the rank of G.

2.3 Graphs

In this section we recall the basic definitions about graphs.

Definition 2.3.1. A **graph** Γ is a triple consisting of a vertex set $V(\Gamma)$, edge set $E(\Gamma)$, and a relation that associates with each edge two (not necessarily distinct) vertices, called its endpoints.

Wh say that two vertices are **adjacent** if there is an edge between them. The **neighbourhood** of vertex v in a graph G is the set of all vertices of G adjacent to v.

Definition 2.3.2. A **subgraph** of a graph Γ is a graph H, such that $V(H) \subseteq V(\Gamma)$ and $E(H) \subseteq E(\Gamma)$, and the assignment of endpoints to edges in H is the same as in Γ.

We write $H \subseteq \Gamma$ and say that "Γ contains H"

Definition 2.3.3. Let $\Gamma = (V, E)$ be a graph and S be a subset of V. The **induced subgraph** $\Gamma[S]$ is the subgraph of Γ with vertex set S and edge set consisting of all edges from E that have endpoints in S.

First we introduce regular graphs, to be easier to understand what is a strongly regular graph.

Definition 2.3.4. A graph is **regular** if each of its vertices has the same number of neighbours.

This means that every vertex has the same **degree** or **valency**, i.e., the number of edges which touch the given vertex. Now, after we have defined regular graphs, we can say what is a strongly regular graph. Except the number of vertices and edges, strongly regular graphs are described by two parameters.

Definition 2.3.5. A **strongly regular graph** is a k-regular graph which is not a complete graph or its complement with v vertices and e edges with parameters α and γ which satisfy the following conditions:

- Every two adjacent vertices have α common neighbours.
- Every two non-adjacent vertices have γ common neighbours.

Notation: $G = (k, \alpha, \gamma)$.

Definition 2.3.6. A **walk** in a graph is a list of $v_0, e_1, v_1, ..., e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.

A (u, v)**-walk** has first vertex u and last vertex v. The **length of a walk**, is the number of its edges. In a simple graph, a walk is completely specified by its ordered list of vertices.

Definition 2.3.7. An **automorphism** of a graph G is a permutation α in $S_{V(G)}$ such that:

$$\forall v_1, v_2 \in V(G),\ \{v_1, v_2\} \in E(G) \iff \{v_1^g, v_2^g\} \in E(G).$$

The set of all automorphisms forms a permutation group, a subgroup of $S_{V(G)}$, and this is denoted by $Aut(G)$.

Definition 2.3.8. A graph G is vertex-transitive if $Aut(G)$ acts transitively on $V(G)$.

We observe that, if a graph is vertex-transitive, then it is also regular.

2.4 Designs

Definition 2.4.1. A **design** with parameters $t-(v,k,\lambda)$ is a pair (X,B) such that

- X is a v-set.
- B is a collection of k-subsets of X;
- Each t-subset of X is contained in exactly λ members of B.

The elements of X are called **points** and the elements of B are called **blocks**. We shall assume that all the parameters are positive integers and that $v > k \geq t$ (to avoid trivial cases). Also, the members of B must be distinct: repeated blocks are not allowed. We also use the terms $t-(v,k,\lambda)$ **design** or t**-design**.

Besides the parameters v, k and λ, we have two more parameters b and r, where b denotes the cardinality $|B|$ and r denotes the number of blocks that contain a given point $x \in X$. It follows from the definition of a design that the latter number does not depend on the choice of the point x. We summarized the basic parameters of a $t-(v,k,\lambda)$ design in the table below:

v	number of points
b	number of blocks
r	number of blocks containing a given point
k	number of points in a block
λ	number of blocks containing any t specific points

Table 1: Parameters of a $t-(v,k,\lambda)$ design.

The parameters are not independent as shown in the following lemma.

Lemma 2.4.2. Let (X,B) be a $t-(v,k,\lambda)$ design with additional parameters b and r. Then

$$b = \frac{v(v-1)\lambda}{k(k-1)},$$

$$r = \frac{\lambda(v-1)}{k-1}.$$

Definition 2.4.3. A $t-(v,k,\lambda)$ design is **symmetric** if it is incomplete, $t \geq 2$ and $b = v$.

A $2-(v,k,1)$ design with $k \geq 3$ is also called a projective plane. We discuss several properties of projective planes in the next section.

2.5 Projective planes

Projective geometry is the study of geometric properties that are invariant with respect to projective transformations. More about projective geometry you can find in the book [3].

Definition 2.5.1. A **projective plane** consists of a set of **lines**, a set of **points**, and a relation between points and lines called **incidence**, having the following properties:

- Given any two distinct points, there is exactly one line incident with both of them.
- Given any two distinct lines, there is exactly one point incident with both of them.
- There are four points such that no line is incident with more than two of them.

The second condition means that there are no parallel lines. The term "incidence" is used to say that "point P is incident with line l", and it is used instead of either "P is on l" or "l passes through P".

Definition 2.5.2. A projective plane is **finite** if its point set has a finite number of elements. A projective plane has **order** n if it is finite and any of its lines passes through exactly $n+1$ points.

It is well-known that a projective plane of order n has n^2+n+1 points.

For a better understanding of the above definition we take a look at a finite projective plane of order 2, called the **Fano plane**. It has 7 points and 7 lines with 3 points on every line and 3 lines through every point.

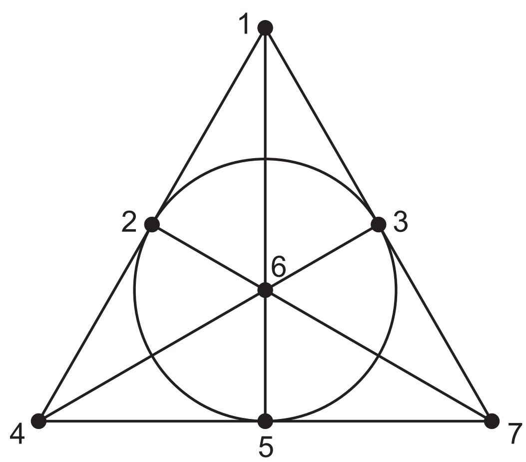

Figure 1: Fano plane

Definition 2.5.3. In a projective plane a set Ω of points is called an **oval** if:

- Any line l meets Ω in at most two points.
- For any point $P \in \Omega$ there exists exactly one tangent line t through P, i.e., $t \cap \Omega = \{P\}$.

When $|l \cap \Omega| = 0$, the line l is called an **exterior line** (or **passant**), if $|l \cap \Omega| = 1$, then a **tangent line**, and if $|l \cap \Omega| = 2$, then a **secant line**.

In a finite projective plane of order n a set Ω of points is well-known to be an oval if and only if $|\Omega| = n + 1$, and no three points are collinear (on a common line).

Definition 2.5.4. In a projective plane a set Ω of points is called an **hyperoval** if any line meets Ω in 0 or 2 points.

3 Feasibility conditions for strongly regular graphs without triangles

Given a triple (k, α, γ) of positive integers, it is natural to ask if these can be parameters of a strongly regular graph. We answer this question in this chapter. As we mentioned in the introduction we will derive very powerful feasibility conditions that we use to solve interesting problems.

3.1 Feasibility conditions

Definition 3.1.1. Let Γ be a graph with n vertices. The **adjacency matrix** of Γ is the $n \times n$ matrix A whose rows and columns are labelled by the vertices of Γ, and whose entries are given by:

$$(A)_{u,v} = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent in } G; \\ 0 & \text{otherwise.} \end{cases}$$

For an example let us take the **Petersen graph** shown in Fig. 5.

An adjacency matrix of the Petersen graph is given below.

Lemma 3.1.2. Suppose that A is the adjacency matrix of a graph Γ, and A^l denotes the l-th power of A $(l \geq 0)$. Then $(A^l)_{uv}$ is equal to the number of walks of length l in Γ, beginning at u and ending at v.

Proof. We use mathematical induction on l to prove this lemma.

Base case: for $l = 0$ and $l = 1$. $l = 0 => A^0 = I$, and when $l = 1 => A^1 = A$. This statement is clearly true.

Inductive step: if it is true for $l = k$ then also holds for $l = k+1$. This can be done as follows. Our idea is to decompose a walk from vertex u to vertex v into two walks. One walk of length k is from u to new vertex w, and from w to v a walk of length one (edge). By the induction hypothesis the number of walks of length k from u to w is

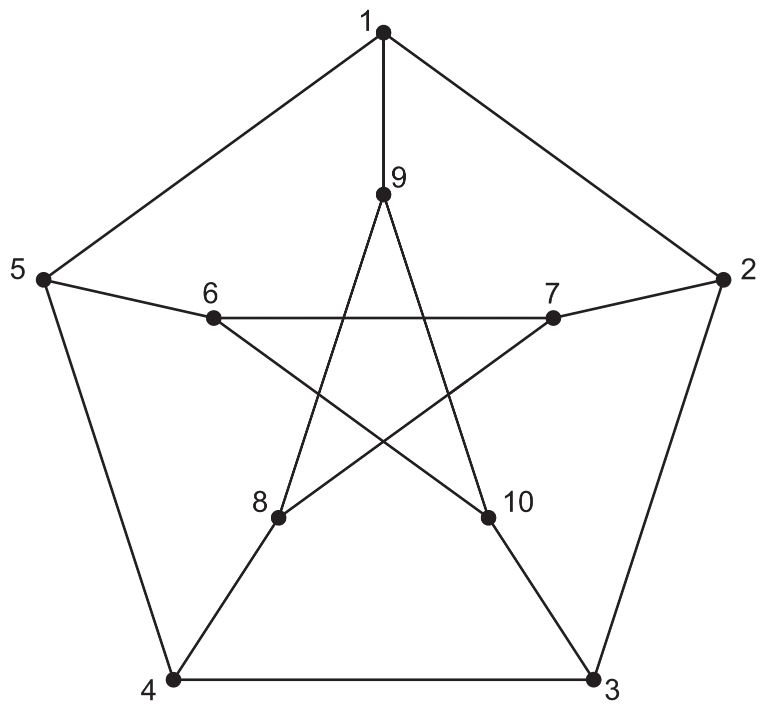

Figure 2: Petersen graph

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	1	0	0	1	0	0	0	1	0
2	1	0	1	0	0	0	1	0	0	0
3	0	1	0	1	0	0	0	0	0	1
4	0	0	1	0	1	0	0	1	0	0
5	1	0	0	1	0	1	0	0	0	0
6	0	0	0	0	1	0	1	0	0	1
7	0	1	0	0	0	1	0	1	0	0
8	0	0	0	1	0	0	1	0	1	0
9	1	0	0	0	0	0	0	1	0	1
10	0	0	1	0	0	1	0	0	1	0

Table 2: Adjacency matrix of the Petersen graph.

$(A^k)_{uw}$. Now, the number of walks of length k from u to v is

$$\sum_{w,v\in E}(A^k)_{uw} = \sum_{w\in V}(A^k)_{uw}(A)_{wv} = (A^{k+1})_{uv}.$$

This completes the proof of the lemma. □

Theorem 3.1.3. Let Γ be a strongly regular graph with parameters (k,α,γ) and let A be an adjacency matrix of Γ. Then A satisfies the equation

$$A^2 + (\gamma + \alpha)A + (\gamma - k)I = \gamma J.$$

Proof. The entry $(A^i)_{uv}$ means the number of walks of length i from u to v. From the above equation we conclude that there are three cases for i, i.e., $i = 2, 1, 0$. Also, we can see that for each i we have three possibilities:

- $u = v$;
- u and v are adjacent;
- u and v are distinct and not adjacent.

First, we consider the case when $i = 2$. $(A^2)_{uv}$ is the number of walks of length 2 from u to v. Now, we will consider our three possibilities.

- When $u = v$, it is obvious that we have k such walks (u, x, u) where x is any one of the k vertices adjacent to u.
- When u and v are adjacent, we should look at our second parameter α (the number of vertices adjacent to any two adjacent vertices of Γ). This is our solution because there are α such walks (u, y, v), where y is any one of the α vertices adjacent to both u and v.
- Similarly to the previous case, we should look at the parameter γ (the number of vertices adjacent to any two distinct non-adjacent vertices of Γ). Now γ is our solution, and there are γ such walks (u, z, v), where z is any one of the γ vertices adjacent to both u and v.

We can proceed similarly for $i = 1$ and $i = 0$. The results are summarized in the table below.

	$(A^2)_{uv}$	$(A)_{uv}$	$(I)_{uv}$
$u = v$	k	0	1
$\{u, v\} \in E(\Gamma)$	α	1	0
$\{u, v\} \notin E(\Gamma)$	γ	0	0

Table 3: The entries $(A^i)_{uv}$, $i = 2, 1, 0$.

We proved for all cases that $[A^2 + (\gamma - \alpha)A + (\gamma - k)I]_{uv} = \gamma$. □

In next theorem we will derive numerical conditions, involving only the parameters (k, α, γ).

Theorem 3.1.4. If a strongly regular graph Γ with parameters (k, α, γ) exists, then either

1. $k = 2\gamma$ and $\alpha = \gamma - 1$, or
2. $(\alpha - \gamma)^2 + 4(k - \gamma)$ is a perfect square, say s^2, and the expression

$$m = \frac{k}{2\gamma s}((k - 1 + \gamma - \alpha)(s + \gamma - \alpha) - 2\gamma)$$

   is a positive integer.

Before we prove this theorem, we recall some facts from linear algebra.

Definition 3.1.5. Let A be an $n \times n$ matrix. The number k is an **eigenvalue** of A if there exists a non-zero vector j such that

$$Aj = kj.$$

In this case, the vector j is called an **eigenvector** of A corresponding to k.

Let A be an adjacency matrix of the strongly regular graph Γ given in Theorem 3.1.4, and let j be the column vector whose entries are equal to 1. We know that A has k ones in each row, and thus we can say that

$$Aj = kj.$$

We conclude that j is an eigenvector of A and k is an eigenvalue.

Proof of Theorem 3.1.4. Let n be the number of vertices of strongly regular graph Γ. Denote by A an adjacency matrix of size $n \times n$. Let j denote the column vector whose entries are all equal to 1. Since A has k ones in each row, $Aj = kj$; that is, j is an eigenvector of A associated with the eigenvalue k. Now we recall a definition before the proof about eigenvectors and eigenvalues. Suppose vector e is any eigenvector of A with associated eigenvalue λ. Then the equation

$$A^2 + (\gamma - \alpha)A + (\gamma - k)I = \gamma J$$

Shows that e is also an eigenvector of J, and the associated eigenvalue is

$$\lambda^2 + (\gamma - \alpha)\lambda + (\gamma - k) = \gamma n \text{ or } 0.$$

Recall now definitions of rank and kernel. The rank of γJ is 1, so that its kernel has dimension $n - 1$ and there is an $(n - 1)$-dimensional space of eigenvectors associated with the eigenvalue 0. Thus every eigenvalue $\lambda \neq k$ of A corresponds to the zero eigenvalue of γJ and satisfies

$$\lambda^2 + (\gamma - \alpha)\lambda + (\gamma - k) = 0.$$

Therefore, we have shown that the eigenvalues of A are k, λ_1, λ_2, where λ_1, λ_2 are the roots of the quadratic equation $\frac{1}{2}(\alpha - \gamma \pm \sqrt{d})$, where $d = (\alpha - \gamma)^2 + 4(k - \gamma)$. The multiplicity of k is 1, and if m_i, $i = 1, 2$, denotes the multiplicity of λ_i we have the equations

$$m_1 + m_2 = n - 1,$$

$$(m_1 - m_2)\sqrt{d} = (m_1 + m_2)(\gamma - \alpha) - 2k.$$

The second equation follows from the fact that the trace of A is zero, so that $k + m_1\lambda_1 + m_2\lambda_2 = 0$

If $\sqrt{d}$ is irrational, then we deduce that $m_1 = m_2 = \frac{k}{\gamma-\alpha}$. Thus we must have $\gamma - \alpha \geq 1$; and if $\gamma - \alpha \geq 2$ then $m_1 = m_2 \leq k/2$, so that $n - 1 \leq k$ and we have the absurdity $l \leq 0$. Hence, in this case $\gamma - \alpha = 1$, which leads to $k = 2\gamma$ and case 1 of the theorem.

If $\sqrt{d}$ is an integer s, then eliminating m_2 from the two equations leads to the stated expression for $m = m_1$, and this must be an integer since it represents a multiplicity. Thus we have case 2 of the theorem. □

3.2 The case $\alpha = 0$

Before we start with the project problem, we try to explain a theorem on one example, what will be useful to solve our problem.

Now we take a look at the parameters $(k, 0, 1)$. The parameter k says that we have regular graph with valency (degree) k in which there are now 3-circuits and no 4-circuits, because $\alpha = 0$ and $\gamma = 1$. Then the number of vertices $n = 1 + k^2$. That is the smallest number of a possible k-valent graph whose shortest circuit has length 5. Now we use Theorem 3.1.4, and we get two possibilities:

- $k = 2$, or
- $4k - 3 = s^2$ and
$$m = \frac{k}{2s}(ks + k - 2)$$
is a positive integer.

In the second case after eliminating k and writing the result as a polynomial equation in s, we get

$$s^5 + s^4 + 6s^3 - 2s^2 + (9 - 32m)s - 15 = 0.$$

We have to find the values for s. Then s must be a divisor of 15, and the corresponding non-trivial values of $k = (s^2+3)/4$ are $3, 7$ or 57. Our graph can exist only if $k = 2, 3, 7$ or 57.

- $k = 2$, the graph is the pentagon;
- $k = 3$, the graph is the Petersen graph;
- $k = 7$, the graph is unique and it admits a rank 3 group of automorphisms;
- $k = 57$, it is not known whether or not a graph exists, although it has been shown that there cannot be a corresponding rank 3 group.

Problem 3.2.1. Use Theorem 3.1.4 to show that if the parameters $(k, 0, \gamma)$, $\gamma \geq 2$, are feasible then $\gamma^2 - 4\gamma + 4k$ is the square of an integer s, and

$$m = \frac{k}{2\gamma s}((k-1+\gamma)(s+\gamma) - 2\gamma)$$

is also an integer. Eliminate k and express the resulting identity as a polynomial equation in s. Deduce that for each value of $\gamma \neq 2, 4, 6$ there are only finitely many corresponding values of k.

Solution. We have a HINT in this problem, what we should use for solving this problem. Now, we use Theorem 3.1.4 as before.

$$(\alpha - \gamma)^2 + 4(k - \gamma) = s^2;$$
$$\alpha = 0$$
$$(0 - \gamma)^2 + 4(k - \gamma) = s^2$$
$$\gamma^2 - 4\gamma + 4k = s^2$$

We do the same for our expression:

$$m = \frac{k}{2\gamma s}((k-1+\gamma-\alpha)(s+\gamma-\alpha) - 2\gamma)$$
$$\alpha = 0$$
$$m = \frac{k}{2\gamma s}((k-1+\gamma-0)(s+\gamma-0) - 2\gamma)$$
$$m = \frac{k}{2\gamma s}((k-1+\gamma)(s+\gamma) - 2\gamma)$$

The next step is to eliminate k and express the above formula in a polynomial equation in s. From first equation we express k:

$$\gamma^2 - 4\gamma + 4k = s^2.$$

$$4k = s^2 - \gamma^2 + 4\gamma.$$
$$k = \frac{s^2 - \gamma^2 + 4\gamma}{4}.$$

Then we plug this in the second equation:

$$m = \frac{k}{2\gamma s}((k - 1 + \gamma)(s + \gamma) - 2\gamma).$$
$$m = \frac{s^2 - \gamma^2 + 4\gamma}{8\gamma s}((\frac{s^2 - \gamma^2 + 4\gamma}{4} - 1 + \gamma)(s + \gamma) - 2\gamma).$$
$$m = \frac{s^2 - \gamma^2 + 4\gamma}{8\gamma s}((\frac{s^2 - \gamma^2 + 4\gamma - 4 + 4\gamma}{4})(s + \gamma) - 2\gamma).$$
$$m = \frac{s^2 - \gamma^2 + 4\gamma}{8\gamma s}\frac{s^3 + s^2\gamma - \gamma^2 s - \gamma^3 + 4\gamma s + 4\gamma^2 - 4s - 4\gamma + 4\gamma s + 4\gamma^2 - 8\gamma}{4}.$$
$$m = \frac{s^2 - \gamma^2 + 4\gamma}{8\gamma s}\frac{s^3 + s^2\gamma - \gamma^2 s - \gamma^3 + 8\gamma s + 8\gamma^2 - 4s - 12\gamma}{4}.$$
$$m = \frac{s^5 + s^4\gamma - s^3\gamma^2 - s^2\gamma^3 + 8s^3\gamma + 8s^2\gamma^2 - 4s^3 - 12s^2\gamma - s^3\gamma^2 - s^2\gamma^3 + s\gamma^4 + \gamma^5}{32\gamma s}$$
$$\frac{-8s\gamma^3 - 8\gamma^4 + 4s\gamma^2 + 12\gamma^3 + 4s^3\gamma + 4s^2\gamma^2 - 4s\gamma^3 - 4\gamma^4 + 32s\gamma^2 + 32\gamma^3 - 16s\gamma - 48\gamma^2}{32\gamma s}$$
$$m = \frac{s^5 + s^4\gamma + s^3(8\gamma - \gamma^2 - 4 - \gamma^2 + 4\gamma) + s^2(8\gamma^2 - \gamma^3 - 12\gamma - \gamma^3 + 4\gamma^2)}{32\gamma s}$$
$$\frac{+s(\gamma^4 - 8\gamma^3 + 4\gamma^2 - 4\gamma^3 + 32\gamma^2 - 16\gamma) + \gamma^5 - 8\gamma^4 + 12\gamma^3 - 4\gamma^4 - 48\gamma^2 + 32\gamma^3}{32\gamma s}$$
$$m = \frac{s^5 + s^4\gamma + s^3(-2\gamma^2 + 12\gamma - 4) + s^2(-2\gamma^3 + 12\gamma^2 - 12\gamma)}{32\gamma s}$$
$$+\frac{s(\gamma^4 - 12\gamma^3 + 36\gamma^2 - 16\gamma) + (\gamma^5 - 12\gamma^4 + 44\gamma^3 - 48\gamma^2)}{32\gamma s}.$$

$$(s^5 + s^4\gamma + s^3(-2\gamma^2 + 12\gamma - 4) + s^2(-2\gamma^3 + 12\gamma^2 - 12\gamma)$$
$$+s(\gamma^4 - 12\gamma^3 + 36\gamma^2 - 16\gamma) + (\gamma^5 - 12\gamma^4 + 44\gamma^3 - 48\gamma^2) - 32\gamma sm = 0.$$

$$s^5 + s^4\gamma + s^3(-2\gamma^2 + 12\gamma - 4) + s^2(-2\gamma^3 + 12\gamma^2 - 12\gamma)$$
$$+s(\gamma^4 - 12\gamma^3 + 36\gamma^2 - 16\gamma - 32\gamma m) + (\gamma^5 - 12\gamma^4 + 44\gamma^3 - 48\gamma^2) = 0.$$

Now, we take the constant term of the equation and factorize it:

$$\gamma^5 - 12\gamma^4 + 44\gamma^3 - 48\gamma^2 = \gamma^2(\gamma - 2)(\gamma - 4)(\gamma - 6).$$

The zeros of this polynomial are $0, 2, 4, 6$. This means that for this numbers the constant term vanishes in our equation. As in our first example we see that, if $\gamma \neq$

$0, 2, 4, 6$, then s must be a divisor a non-zero number. This means that there are finitely many values of s; and since $k = \frac{s^2-\gamma^2+4\gamma}{4}$, we conclude that there are finitely many corresponding values of k as well. $\square$

Problem 3.2.2. Show that the parameters $(k, 0, 2)$ are feasible if and only if $k = t^2 + 1$, where t is an integer not congruent to $0\ (mod\, 4)$.

Solution. Like in the previous solution, we will use Theorem 3.1.4.

$$(\alpha - \gamma)^2 + 4(k - \gamma) = s^2$$

For α and γ we plug the numbers 0 and 2.

$$(0 - 2)^2 + 4(k - 2) = s^2.$$

$$4k - 4 = s^2.$$
$$k = \frac{s^2 + 4}{4}.$$
$$k = \frac{s^2}{4} + 1.$$
$$k = (\frac{s}{2})^2 + 1.$$
$$\frac{s}{2} = t.$$
$$k = t^2 + 1.$$

Now, let us see when the expression is a positive integer.

$$m = \frac{k}{2\gamma s}((k - 1 + \gamma - \alpha)(s + \gamma - \alpha) - 2\gamma.)$$

$$\alpha = 0; \gamma = 2; s = 2t; k = t^2 + 1$$

$$m = (\frac{t^2 + 1}{8t})((t^2 - 1 + 2 - 0)(2t + 2 - 0) - 4).$$

$$m = (\frac{t^2 + 1}{8t})(2t^3 + 2t^2 + 4t).$$

$$m = \frac{2t^5 + 2t^4 + 4t^3 + 2t^3 + 2t^2 + 4t}{8t}.$$

$$m = \frac{t^4 + t^3 + 3t^2 + t + 2}{4}.$$

We have to prove that m is positive integer when t is an integer not congruent to $0\ (mod\, 4)$. Now, if $t \equiv 0\ (mod\, 4)$, then we can write $t = 4j, j$ is a positive integer.

$$m = \frac{(4j)^4 + (4j)^3 + 3(4j)^2 + (4j) + 2}{4}.$$

We conclude m is not a integer, because every element is divisible by 4 except 2.

It remains to check that m becomes an integer for $t = 4j + 1, t = 4j + 2$ and $t = 4j + 3$. Then

1.
$$m = \frac{(4j+1)^4 + (4j+1)^3 + 3(4j+1)^2 + (4j+1) + 2}{4}.$$

2.
$$m = \frac{(4j+2)^4 + (4j+2)^3 + 3(4j+2)^2 + (4j+2) + 2}{4}.$$

3.
$$m = \frac{(4j+3)^4 + (4j+3)^3 + 3(4j+3)^2 + (4j+3) + 2}{4}.$$

It is enough to check only free members, because the other ones are divisible by 4.

1. m is integer because $1 + 1 + 3 * 1 + 1 + 2 = 8$ is divisible by 4.

2. m is integer because $2^4 + 2^3 + 3 * 2^2 + 2 + 2 = 40$ is divisible by 4.

3. m is integer because $3^4 + 3^3 + 3 * 3^2 + 3 + 2 = 140$ is divisible by 4.

We checked all possibilities for t and this completes the proof. □

4 Strongly graphs with parameters $(5, 0, 2)$ and $(10, 0, 2)$

In this chapter we prove a relation between strongly graphs with parameters $(k, 0, 2)$ and biplanes. We recall that a biplane is a symmetric 2-design with $\lambda = 2$. Then we turn to strongly regular graphs with parameters $(5, 0, 2)$ and $(10, 0, 2)$ and derive some of their interesting properties.

4.1 Strongly regular graphs and biplanes

Problem 4.1.1. Let Γ be a strongly regular graph with parameters $(k, 0, 2)$, and let X be its vertex set. Define

- $\beta_x = \{y \in X \mid y = x \text{or } y \text{ is adjacent to } x\}$.
- $B = \{\beta_x \mid x \in X\}$.

Show that (X, B) is a symmetric $2-(\frac{1}{2}(k^2+k+2), k+1, 2)$ design (such a design is called a biplane.) By Problem 3.2.2, biplanes could exist for $k = t^2+1, t = 1, 2, 3, 5 \ldots$. Find the graph and the corresponding biplane for $t = 1$.

Solution. We use Definition 2.4.1 to check that (X, B) is indeed a $2-(\frac{1}{2}(k^2+k+2), k+1, 2)$ design. First we find $|X|$. This is the number of vertices of the graph Γ, and we can write

$$|X| = m_1 + m_2 + 1.$$

$$(m_1 - m_2)s = (m_1 + m_2)2 - 2k.$$

We also know that

$$m_1 = m = \frac{1}{4s}((k+1)(s+2) - 4).$$

From these we find

$$m_2 = \frac{m_1(s-2) + 2k}{s+2}$$

and thus

$$m_1 + m_2 = \frac{m_1(s-2) + 2k + m_1(s+2)}{s+2} =$$

$$\frac{\frac{k}{2}(k+1)(s+2)-2k+2k}{s+2}=\frac{1}{2}(k^2+k).$$

This shows that $|X| = \frac{1}{2}(k^2+k+2)$.

Since the graph Γ is regular and it has degree k, every block in B has exactly $k+1$ points.

Finally, we have to show that any 2 points in X are contained in 2 blocks. Let these points be x and y. Then we have two possibilities:

1. x and y are adjacent in Γ.
2. x and y are not adjacent in Γ.

In case 1 it follows from the definition of blocks that the blocks β_x and β_y contain $\{x, y\}$. Also, since $\alpha = 0$, no other block can contain $\{x, y\}$. In case 2 there are $\gamma = 2$ vertices that are adjacent to both x and y. Let these vertices be denoted by z_1 and z_2. Then we see that in this case the blocks β_{z_1} and β_{z_2} contain $\{x, y\}$, and no other block does. By all these we proved that (X, B) is indeed a $2-(\frac{1}{2}(k^2+k+2), k+1, 2)$ design. It is clearly symmetric because $|B| = |X|$, that is, $b = x$ (see Definition 2.4.3).

Next, we have to find the graph and the corresponding biplane for $t = 1$. We recall that a biplane is a symmetric 2-design with parameter $\lambda = 2$, that is, every set of two points is contained in two blocks, while any two lines intersect in two points. Therefore, for

$$t = 1 \Rightarrow k = 2$$

, and the corresponding biplane $2-(\frac{1}{2}(2^2+2+2), 2+1, 2) = 2-(4, 3, 2)$ has 4 points. It is the complete design with $v = 4$ and $k = 3$ and geometrically the points are the vertices and the blocks are the faces of a tetrahedron, it is shown in Fig. 3.

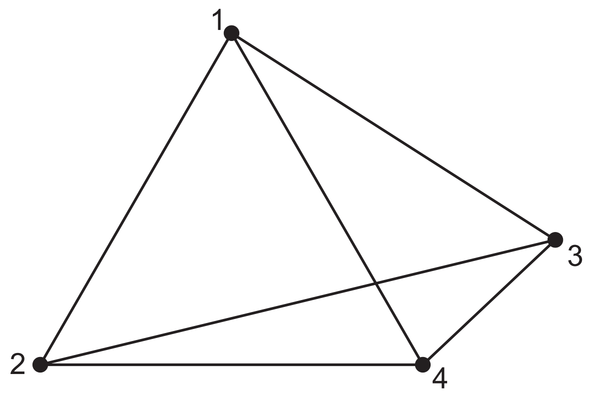

Figure 3: The tetrahedron.

4.2 The strongly regular graph $(5, 0, 2)$

Problem 4.2.1. Suppose Γ_5 is strongly regular graph with parameters $(5, 0, 2)$. Let $*$ be any vertex and label the adjacent vertices $1, 2, 3, 4, 5$. Show that the 10 vertices not adjacent to $*$ must be labelled $12, 13, \ldots, 45$ in canonical way, and that all the edges of the graph are determined uniquely. Show that Γ_5 admits a transitive, rank 3, group of automorphisms, so that we have in fact constructed a strongly regular graph.

Solution. Our graph has $\frac{1}{2}(5^2 + 5 + 2) = 16$ vertices. The parameter $k = 5$ says that the graph Γ_5 is 5-regular. In other words each vertex has 5 neighbours. We label the vertices with letters from A to P, see Fig. 4.2.

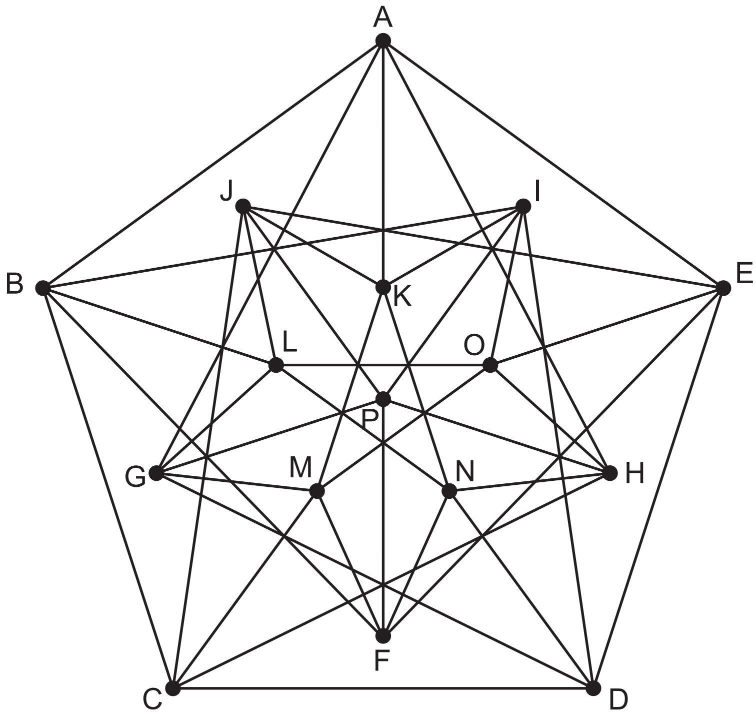

Figure 4: Clebsch graph

Next we change the above notation. With $*$ we denote vertex on the top. Adjacent vertices to $*$ we denote by $1, 2, 3, 4, 5$. Then it is easy to denote other vertices as follows: if two vertices have adjacent vertex to them denote by the name of these two vertices. For example, if we have two vertices labelled by 1 and 2, then the vertex adjacent to both will be 12.

From our parameters we can conclude that every two non-adjacent vertices in the graph have two adjacent vertex. Because we start from the top of a graph where we already denote one adjacent vertex of two others, it follows that we have just one more adjacent vertex to these. And then for this labelling we can say that it is uniquely determined. The obtained graph is shown in Fig. 4.3.

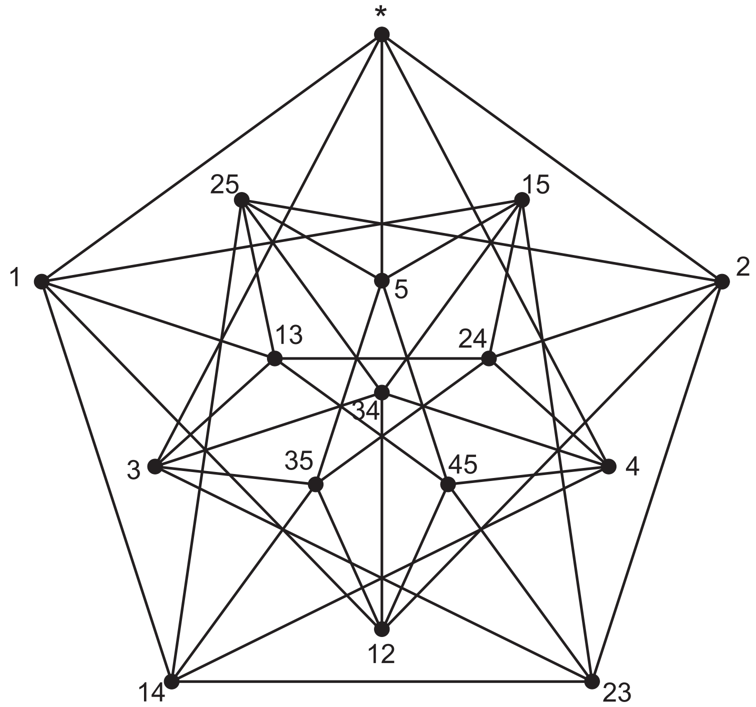

Figure 5: Clebsch graph

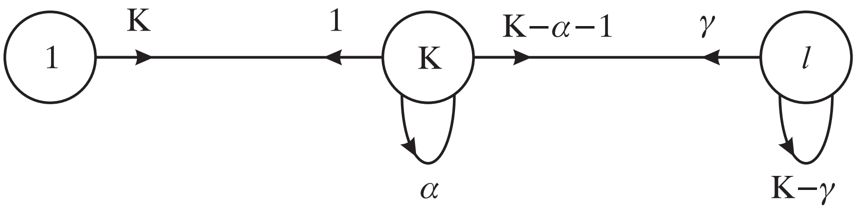

Figure 6: Diagram

We use next a diagram to find the parameters.

From the diagram we see that, if we select one vertex on the left, it is joined to k others. Each of these k vertices is joined to the initial vertex, and α vertices are adjacent to the initial one and to $k-\alpha-1$ others (since every vertex must have valency k). As for the remaining l vertices, each is adjacent to γ vertices in the central circle and to $k-\gamma$ vertices in its own circle. At the end we get the formula:

$$\gamma l = k(k-\alpha-1).$$

We show next that Γ_5 admits a rank 3 group of automorphism group. Let us denote this potential group by G. Then the two non-trivial orbits D_1 and D_2 of G acting on the Cartesian product $V(\Gamma) \times V(\Gamma)$ must consist precisely of the pairs of adjacent vertices, and the pairs of distinct non-adjacent vertices; the associated graph $\Gamma(D_1)$ is just Γ_5, and $\Gamma(D_2)$ is the complementary graph Γ_5^c. It is well-known that both graphs are strongly regular. And thus we will also prove that Γ_5 is indeed a strongly regular. The complementary graph Γ_5^c has parameters

$$\overline{k} = l = 10, \overline{\alpha} = l - k + \gamma - 1 = 6, \overline{\gamma} = l - k + \alpha + 1 = 6.$$

The complement graph Γ_5^c has parameters $(10, 6, 6)$, and it is known as the Halved 5-Cube Graph and it is strongly regular.

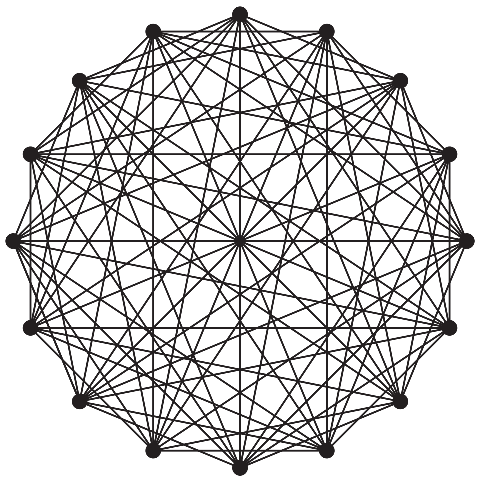

Figure 7: Halved 5-Cube graph

To define the potentional rank 3 group G let us consider the picture of Γ_5 shown in Fig. 4.3. For a permutation $\pi \in S_5$ we define the permutation $\bar{\pi}$ of the vertex set

$$V(\Gamma_5) = \{*\} \cup \{1, 2, 3, 4, 5\} \cup \{ij \mid i, j = 1, 2, 3, 4, 5 \text{ and } i \neq j\}$$

as follows

- $*^{\bar{\pi}} = *$.
- $i^{\bar{\pi}} = i^{\pi}$ where $i \in \{1, 2, 3, 4, 5\}$.
- $ij^{\bar{\pi}} = i^{\pi}j^{\pi}$ where $i, j \in \{1, 2, 3, 4, 5\}$ and $i \neq j$.

We prove that $\bar{\pi}$ is an automorphism of Γ_5. We use Definition 2.3.7. Let $\{u, v\} \in E(\Gamma_5)$. The following rules hold for Γ_5:

$$\{i, jk\} \in E(\Gamma_5) \iff i = j \text{ or } i = k, \text{ and}$$

$$\{ij, kl\} \in E(\Gamma_5) \iff \{i, j\} \cap \{k, l\} = \emptyset.$$

Therefore, we have three cases:

1. $u = *$ and $v = i$.

   In this case $u^{\bar{\pi}} = *$ and $v^{\bar{\pi}} \in \{1,2,3,4,5\}$. We get $\{u^{\bar{\pi}}, v^{\bar{\pi}}\} \in E(\Gamma_5)$.

2. $u = i$ and $v = ij$ where $i \neq j$.

   In this case $u^{\bar{\pi}} = i'$ and $v^{\bar{\pi}} = i'j'$ where $i' \neq j'$. We get $\{u^{\bar{\pi}}, v^{\bar{\pi}}\} \in E(\Gamma_5)$.

3. $u = ij$ and $v = kl$ where $\{i,j\} \cap \{k,l\} = \emptyset$.

   In this case $u^{\bar{\pi}} = i'j'$ and $v^{\bar{\pi}} = k'l'$ where $\{i',j'\} \cap \{k',l'\} = \emptyset$. We get $\{u^{\bar{\pi}}, v^{\bar{\pi}}\} \in E(\Gamma_5)$.

All these show that $\bar{\pi}$ is indeed an automorphism of Γ_5. Thus we can define the subgroup $G(*) \leq Aut(\Gamma_5)$ as

$$G(*) = \{\bar{\pi} \mid \pi \in S_5\}.$$

Now, we repeat the above argument for any vertex u of Γ_5 and in this way we define a subgroup $G(u) \leq Aut(\Gamma_5)$. As the last step, let G be the subgroup of $Aut(\Gamma_5)$ that is generated by all subgroups $G(u), u \in V(\Gamma_5)$.

We observe that G acts transitively on $V(\Gamma_5)$. Therefore, in order to see that G is indeed a rank 3 group we can use Theorem 2.2.9. This says that, G has rank 3 if and only if the stabilizer G_* has 3 orbits. It is true that $G(*) \leq G_*$. Also, it is obvious that the group $G(*)$ has the following orbits:

$$\{*\},\ \{1,2,3,4,5\} \text{ and } \{ij \mid i,j = 1,2,3,4,5 \text{ and } i \neq j\}.$$

On the other hand, every automorphism $\sigma \in G_*$ must fix $*$, and also σ cannot map a vertex i to a vertex jk because i is adjacent to $*$ but jk is not. This shows that G_* has the same orbits as $G(*)$, and so G is indeed a rank 3 group. $\square$

4.3 The strongly regular graph $(10, 0, 2)$

We define a t-design, but now, little bit differently from Definition 2.4.1.

Definition 4.3.1. A $t-(v,k,\lambda)$ **design** is a triple (X,B,I) with the following properties:

1. X is a v-set.

2. Given any 'block' $\beta \in B$, there are exactly k 'points' $x \in X$ such that $xI\beta$.

3. Given any t points $x_1, \ldots, x_t$, there are exactly λ blocks β such that $x_1 I\beta, \ldots, x_t I\beta$.

All parameters above are positive and $v > k \geq t$. It is obvious that we have trivial case when $k = t$. It is important that the members of B must be distinct, repeated blocks are not allowed.

In order to construct a strongly regular graph with parameters $(10, 0, 2)$ we need to define the so called Higman-Sims graph Σ. Recall definition of a projective plane and a hyperoval in Section 2.5. Let P be the set of points and Λ the set of lines of a projective plane of order 4. Let K be an 'extension class' of 56 hyperovals in (P, Λ); each member of K is a set of 6 points with property that no three are collinear, and any two members of K meet in 0 or 2 points. Finally, let 0 and ∞ be two symbols not in $P \cup \Lambda \cup K$.

Definition 4.3.2. The vertex set of the **Higman-Sims graph** Σ is

$$V = \{0, \infty\} \cup P \cup \Lambda \cup K,$$

and its edge set E is the union of the following seven subsets:

$$E_1 = \{0, \infty\}$$

$$E_2 = \{\{0, p\} \mid p \in P\}$$

$$E_3 = \{\{\infty, l\} \mid l \in \Lambda\}$$

$$E_4 = \{\{p, l\} \mid p \in P, l \in \Lambda \text{ and } p \in l\}$$

$$E_5 = \{\{p, H\} \mid p \in P, H \in K \text{ and } p \in H\}$$

$$E_6 = \{\{l, H\} \mid l \in \Lambda, H \in K \text{ and } l \cap H = \emptyset\}$$

$$E_7 = \{\{H, K\} \mid K \in K \text{ and } H \cap K = \emptyset\}.$$

The automorphism group $Aut(\Sigma)$, it is called the **Higman-Sims group**, has a remarkable property. It is a simple group of order $44352000 = 2^9 * 3^2 * 5^3 * 7 * 11$. It was found by Donald G. Higman and Charles C. Sims in 1968. This group is one of the so called 26 sporadic simple groups in the classification of finite simple groups.

We will use two lemmas about the Higman-Sims graph to solve our last problem.

Lemma 4.3.3. The Higman-Sims graph is a strongly regular graph with parameters $(22, 0, 6)$.

Lemma 4.3.4. Let $X_1 = P \cup \{\infty\}$ and $B_1 = K \cup \Lambda$, and define an incidence relations I_1 by the rule that

$$xI_1\beta \iff \{x, \beta\} \in E(\Sigma).$$

Then (X_1, B_1, I_1) is a $3 - (22, 6, 1)$ design.

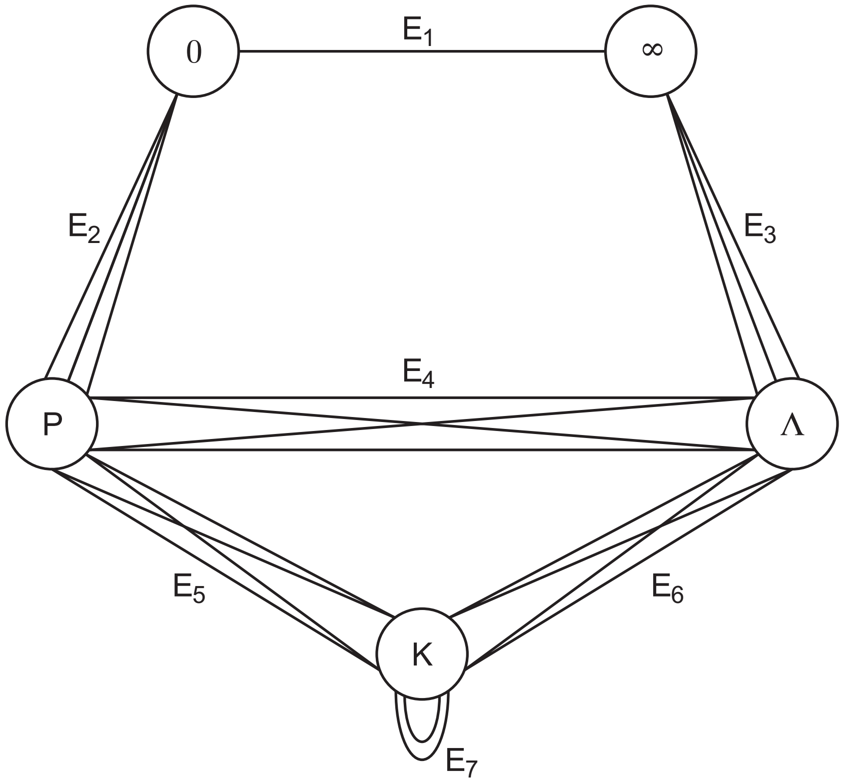

Figure 8: Higman-Sims graph

Problem 4.3.5. Define a graph Γ_{10} as follows. The vertices are the 56 hyperovals in $PG(2,4)$ which belong to a given 'extension class', and the edges join disjoint hyperovals. Prove that Γ_{10} is a strongly regular graph with parameters $(10,0,2)$.

Solution. We use the HINT that says: Γ_{10} is a subgraph of the Higman-Sims graph Σ. We also the following fact. This will be not proved here:

Fact: Every two hyperovals in K meet at 0 or 2 points.

We first find the degree of a vertex u of our graph Γ_{10}. Then $u = H$ where $H \in K$ is a hyperoval. Lemma 4.3.3 shows that u as a vertex of Σ has 22 neighbours. It follows from the definition of Σ that among these there are 6 neighbours in P and it is easy to compute that there are $21 - \binom{6}{2} = 6$ neighbours in Λ. Therefore, it has $22 - 6 - 6 = 10$ neighbours in K, which shows that u has degree 10. We proved that Γ_{10} is a 10-regular graph.

Now let u' be another vertex of Γ_{10}. Then $v = H'$ where $H' \in H$. We show that u and u' has no common neighbour in Γ_{10} when u and u' are adjacent, and exactly 2 common neighbours otherwise. By this we will prove that Γ_{10} is indeed a strongly regular graph with parameters $(10,0,2)$.

1. u and u' are adjacent.

It follows from Lemma 4.3.3 that as vertices of Σ, u and u' have no common neighbours. Since Γ_{10} is a subgraph of Σ we obtain that u and u' have no common neighbours in Γ_{10} as well.

2. u and u' are not adjacent.

   It follows from Lemma 4.3.3 that as vertices of Σ, u and u' have 6 neighbours. Among this there 2 neighbours in P.

   Next, we will show that they have 2 common neighbours in Λ. Let x_1 and x_2 be the common points of the hyperoval $H \cap H'$. Let y be an arbitrary point in H such that $y \neq x_1, x_2$. Let x_i' be the intersection of H' with the line through y and x_i for $i = 1, 2$. Let z_1, z_2 be the points $H' \setminus \{x_1, x_2, x_1', x_2'\}$. We show that the points y, z_1 and z_2 are collinear. It follows from Lemma 4.3.4 and the above FACT that there are 4 hyperoval in K that contains both y and z_1. It also follows from Lemma 4.3.4 that there is a unique hyperoval that contains y, z_1 and one point from $\{x_1, x_2, x_1', x_2'\}$. This shows that y, z_1 and z_2 are indeed collinear. But this means that through y there are two lines which has no intersection with H'. It can be obtained from this that there are exactly four lines in $l \in \Lambda$ which has the property that $|l \cap H| = 2$ and $l \cap H' = \emptyset$. Therefore, there $15 + 4$ lines in Λ which meets $H \cup H'$, and we conclude that there are indeed 2 common neighbours of u and u' in Λ.

   Now, it follows from the above observations and Lemma 4.3.3 that u and u' have $6 - 2 - 2 = 2$ common neighbours in Γ_{10}. □

5 Conclusion

In the final project we presented strongly regular graphs and feasibility conditions for them. Also we put our own solutions for four project problems. On these problems we try to explain conditions and try to get a feeling how the strongly regular graphs looks like. We obtain two types of strongly regular graphs with parameters $5, 0, 2$ and $10, 0, 2$.Also we mention here some of popular graphs, such as Petersen, Clebsch and Halved 5-Cube Graph. Furthermore we define edge and vertex set for Higman-Sims graph and construct it.

6 Povzetek naloge v slovenskem jeziku

Teorija grafov je zelo mlada in neraziskana znanost. Užitek ni le reševanje problemov, ampak odkrivanje neodkritega, odkrivanje skrivnosti grafov. Od tam se je žačela naša želja o grafih.

V uvodnem delu smo definirali nekaj zelo znanih in osnovnih definicij o grupah, o delovanju v grupah, kot tudi grafov in projektinih ravninah. Tudi smo omenili kaj je to t-design, kateri smo uporabljali v nalogah. Stvari katere se nahajajo na začetku so ene od osnovnih stvari, katere smo uporabljali skozi naše žaključno delo.

V diplomski nalogi smo definirali krepko-regularen graf z parametrima (k, α, γ). Krepko-regularni graf je regularni graf, kjer ima vsak sosednji par točk enako število skupnih sosednjih točk α, in vsak nesosednji par točk enako število skupnih sosednjih točk, γ. Mi smo se ukvarjali z krepko-regularnim grafima, kdaj je $\alpha = 0$.

V prvem delu našeg diplomskega dela smo rešili dva projektna problema 4.7.1 in 4.7.2 iz knjige [5]. Naloge niso preveč kompleksne. Za njihovo reševanje smo koristili določene tipe pogojev, ki se tičejo krepko-regularnih grafova. Pogoji so zelo koristni, ker jih lahko koristimo, da preverimo obstoj krepko regularnih grafov, kot tudi izračunavanje njegovih ostalih parametrov. Vsak od teh pogojev ima tudi svoj dokaz ki je prikazan v delu. Za dokaz teoreme 3.1.4 smo se spomnili tudi linearne algebre, o lastnih vrednostih in lastnih vektorjih. Po končanem prvem delu presli smo spet na krepko-regularne grafe.

V drugem delu nase teze, večjo pozornost posvetimo krepko regularnim grafom. To so grafovi z parametrima $(5, 0, 2)$ i $(10, 0, 2)$. Te parametre dobijemo iz enačbe pri problemu 3.2.2 iz knjige [5].

$$k = t^2 + 1, \text{ where } t \not\equiv 0 \,(mod\, 4).$$

mesto t smo pisali 2 oziroma 3. Tukaj sta se pojavila dva nova problema 4.7.3. i 4.7.4 iz iste knjige. Po reševanju drugega problema ki je malo težji od 4.7.3. pokažemo, da je bilo kateri krepko regularen-graf z parametrima $(5, 0, 2)$ ima rang 3 grupu automorfizmov (poglejte Problem 4.3.5)

U zadnjem delu smo rešili še en problem 4.7.5 iz iste knjige in konstruirali krepko-regularen graf z parametrima $10, 0, 2$ kot podgraf Higman-Simsovog grafa. Za konstrukcijo Higman-Simsovog grafa smo izračunali njegovo mnozico tock, tako kot njegovo mnozico povezav, katera se sastoji iz 7 podmnozic.

Zaključno delo vsebuje tudi nekaj slik, ki so služile lažjemu razumevanju.

7 Bibliography

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