UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN
INFORMACIJSKE TEHNOLOGIJE

Zaključna naloga
(Final project paper)

Simplicialni kompleksi, orientacija in klasifikacija ploskev

(Simplicial Complexes, Orientation and Classification of Compact Surfaces)

Ime in priimek: Ivan Bartulović
Študijski program: Matematika
Mentor: prof. dr. Bojan Kuzma

Koper, junij 2016

Ključna dokumentacijska informacija

Ime in PRIIMEK: Ivan BARTULOVIĆ

Naslov zaključne naloge: Simplicialni kompleksi, orientacija in klasifikacija ploskev

Kraj: Koper

Leto: 2016

Število listov: 44 Število slik: 10 Število referenc: 11

Mentor: prof. dr. Bojan Kuzma

Ključne besede: simpleks, triangulacija, realizacija, kompaktna ploskev

Math. Subj. Class. (2010): 57Q05, 55U10, 55U05, 57Q15.

Izvleček:
V glavnem delu naloge definiramo geometrijske simplicialne komplekse in obravnavamo topološke lastnosti simplicialnih politopov. Nato definiramo topološke in abstraktne simplicialne komplekse, ter dokažemo izrek o realizaciji, s pomočjo katerega naprej obravnavamo simplicialno orientacijo in orientacijo triangulabilnih ploskev. Kot zgled pokažemo da je valj neorientabilen in Möbiusov trak neorientabilen. V zadnjem poglavju dokažemo da je vsaka kompaktna ploskev homeomorfna kvocientnemu prostoru nekega poligona v ravnini ter formuliramo izrek o njihovi klasifikaciji.

Key words documentation

Name and SURNAME: Ivan BARTULOVIĆ

Title of final project paper: Simplicial Complexes, Orientation and Classification of Surfaces

Place: Koper

Year: 2016

Number of pages: 44 Number of figures: 10 Number of references: 11

Mentor: Prof. Bojan Kuzma, PhD

Keywords: simplex, triangulation, realization, compact surface

Math. Subj. Class. (2010): 57Q05, 55U10, 55U05, 57Q15.

Abstract: In the main part of the paper, we define geometric simplicial complexes and examine topological properties of simplicial polytopes. Then we define topological and abstract simplicial complexes and prove the realization theorem, which enables us to study simplicial orientation and orientation of triangulable surfaces. As an example, we prove that cylinder is orientable and that Möbius strip is not orientable. In the last chapter, we prove that every compact surface is homeomorphic to the quotient space of some plane polygon and state the classification theorem.

Acknowledgement

Thanks to professor Bojan Kuzma for his patience, help and devoted time.
I would also like to thank UP FAMNIT for giving many opportunities during all these years.
Finally, I thank to my family and friends for support.

Contents

List of Figures

1 Introduction

In the final project paper we describe some basic notions about simplicial complexes and present some of their applications. Specifically, the principal problem of topology is to determine which spaces are the same up to homeomorphism. The motivation to introduce simplicial complexes here arises from a common idea in many branches of mathematics: if possible, we break the space into simpler pieces and consider the way in which they are put together. Let us give an overview of the topics to be presented:

In Chapter 2 we recall some basic definitions and theorems which are going to be frequently used in the upcoming chapters.
In Chapter 3 we primarily define geometric simplexes and study topological properties of simplicial polytopes. Also, explained is a way to subdivide a given geometric simplicial complex. Furthermore, we define topological and abstract simplexes. The latter lead us to show the fact that every abstract simplicial complex has a geometric realization.
In Chapter 4 we define orientation of simplexes, introduce the notion of induced orientation and as an example prove that cylinder is orientable and that Möbius strip is nonorientable.
In Chapter 5 we prove an important step of classification of compact surfaces, the fact that each compact surface is a quotient space of some polygon in plane.

Unless otherwise stated, our work in Chapters 2 mostly follows from both [5] and [11], the main source of Chapter 3 is [5], while Chapters 4 and 5 are concentrated on [7] and [9], respectively. Figures 6, 7 and 8 are taken from [9], while figure 10 is from [6].

2 Preliminaries

Definition 2.1. Let $\mathbb{K}$ be a field. A vector space $(V, +, \cdot)$ over the field $\mathbb{K}$ is an additive abelian group, together with a multiplication of elements of V by elements of $\mathbb{K}$, i.e. an operation $\cdot : \mathbb{K} \times V \to V$ satisfying the following conditions:

1. If 1 is unity of $\mathbb{K}$, then $1v = v$ for all $v \in V$.
2. If $c \in \mathbb{K}$ and $v, w \in V$, then $c(v + w) = cv + cw$.
3. If $x, y \in \mathbb{K}$ and $v \in V$, then $(x + y)v = xv + yv$.
4. If $x, y \in \mathbb{K}$ and $v \in V$, then $(xy)v = x(yv)$.

Definition 2.2. Subset W of V is said to be subspace of V, if it is subgroup of $(V, +)$ and if $c \in \mathbb{K}$ and $v \in W$, then $cv \in W$.

Definition 2.3. Vector space V over $\mathbb{K}$ is said to be normed if there exist a map $\|\cdot\| : V \to \mathbb{R}$ with the following properties:

1. $\|x\| = 0 \iff x = 0$
2. $\|\lambda x\| = |\lambda|\|x\| \quad \forall x \in V, \forall \lambda \in \mathbb{K}$
3. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.

Definition 2.4. Subset M of $\mathbb{R}^n$ containing elements of form $(1 - \lambda)x + \lambda y$ where $x, y \in M$ and $\lambda \in \mathbb{R}$ is called an affine set.

Theorem 2.5. *If V is a finite dimensional normed vector space over $\mathbb{K}$, then any two norms on V are equivalent.*

Proof. See [2], page 69. □

Theorem 2.6. *Subspaces of $\mathbb{R}^n$ are precisely those affine sets which contain 0.*

Proof. If M is subspace of V, it contains 0 and since it is closed under addition and scalar multiplication, it is in particular an affine set.
Conversely, suppose M is an affine set containing 0. Pick $x, y \in M$ and $\lambda \in \mathbb{R}$ arbitrarily. Firstly, $\lambda x = (1 - \lambda)0 + \lambda x \in M$. Furthermore, $x + y = 2\left(\frac{1}{2}(x + y)\right) = 2\left(\frac{1}{2}x + \left(1 - \frac{1}{2}\right)y\right) \in M$. Hence, M is a subspace. □

Theorem 2.7. *For each nonempty affine set $M \subseteq \mathbb{R}^n$ there exist a unique subspace W of $\mathbb{R}^n$ such that $M = W + a$, for some $a \in M$.*

Proof. Choose an arbitrary $y \in M$. Then $\hat{L} = M + (-y) = M - y$ is an affine set: if $x-y, z-y \in M-y$ and $\lambda \in \mathbb{R}$, then $(1-\lambda)(x-y)+\lambda(z-y) = (1-\lambda)x+\lambda z-y \in M-y$. Also, it obviously contains 0. By theorem 2.6, $\hat{L}$ is subspace of R^n. Since y was arbitrary, we can create the desired subspace with $L = M - M$.
For the uniqueness part, take L_1 and L_2 to be subspaces with such property: $L_1 + a_1 = M = L_2 + a_2$ for some $a_1, a_2 \in M$. Then $L_2 = L_1 + a, a = a_1 - a_2 \in M$. Since $0 \in L_2$, then $-a \in L_1$ and consequently $a \in L_1$. Hence, $L_2 = L_1 + a \subseteq L_1$. By symmetry, we also get $L_1 \subseteq L_2$. The uniqueness follows. □

Definition 2.8. Let V and W be vector spaces over the same field $\mathbb{K}$. A map $A : V \to W$ is linear if for all $x, y \in V$ and each $\lambda \in \mathbb{K}$ we have

$$\begin{aligned} A(x+y) &= A(x) + A(y) \\ A(\lambda x) &= \lambda A(x) \end{aligned}$$

Definition 2.9. Let $M \subseteq \mathbb{R}^n$ be an affine set. A mapping $T : M \to \mathbb{R}^m$ is an affine map if $T((1-\lambda)x + \lambda y) = (1-\lambda)T(x) + \lambda T(y)$ holds for arbitrary $x, y \in M$.

Remark 2.10. An inductive argument shows that condition on mapping T in the definition 2.9 can be stated as

$$T\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i T(x_i), \text{ with } \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \in \mathbb{R}$$

Theorem 2.11. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is affine if and only if is of form $x \mapsto Ax+b$, where $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $b \in \mathbb{R}^m$.*

Proof. Mapping $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(x) = Ax + b$ is indeed an affine mapping:

$$\begin{aligned} T((1-\lambda)x + \lambda y) &= A((1-\lambda)x + \lambda y) + b \\ &= (1-\lambda)(Ax + b) + \lambda(Ay + b) \\ &= (1-\lambda)T(x) + \lambda T(y) \end{aligned}$$

Conversely, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map. Set $b = T(0)$ and say $A(x) = T(x) - b$. Then A maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and we note that it suffices to show that A is linear. Clearly, $A(0) = 0$. Also, A is affine:

$$\begin{aligned} A((1-\lambda)x + \lambda y) &= T((1-\lambda)x + \lambda y) - b \\ &= (1-\lambda)T(x) + \lambda T(y) - b \\ &= (1-\lambda)(T(x) - b) + b(1-\lambda) + \lambda T(y) - b \\ &= (1-\lambda)Ax + \lambda Ay \end{aligned}$$

Finally, $A(\lambda x) = A((1-\lambda)0+\lambda x) = (1-\lambda)A0+\lambda Ax = \lambda Ax$. Also, $A(x+y) = A\left(2\left(\frac{1}{2}(x+y)\right)\right) = 2A\left(\frac{1}{2}(x+y)\right) = 2A\left(\left(1-\frac{1}{2}\right)x+\frac{1}{2}y\right) = 2\left(\frac{1}{2}Ax+\frac{1}{2}Ay\right) = Ax+Ay$ □

Remark 2.12. Note that if affine map T is defined on the entire $\mathbb{R}^n$, then it is automatically continuous. Namely, by theorem 2.11, it is of form $x \mapsto Ax+b$. Furthermore, since $\mathbb{R}^n$ and $\mathbb{R}^m$ are normed finite dimensional vector spaces, each linear map between them is continuous (for a proof, see [2], page 70). So, $x \mapsto Ax+b$ is composition of continuous functions $x \mapsto Ax$ and $x \mapsto x+b$.

The remark 2.12 rises the following question: given an arbitrary affine map, does it have continuous extension? Also, does there exist an affine space containing certain points? If it does, is it unique? We will answer these questions soon in the following series of definitions and theorems:

Definition 2.13. A set of points H^k of $\mathbb{R}^n$ is called a k-dimensional hyperplane if there is a linearly independent set of vectors p_i for $1 \leq i \leq k, k < n$ and a vector p_0 such that each element $h \in H^k$ may be expressed as

$$h = p_0 + \sum_{i=1}^{k} \lambda_i p_i,\ \lambda_i \in \mathbb{R}$$

Example 2.14. If $p_0 = (0, 0, \ldots, 0)$, then it is easy to check that H^k is a k-dimensional vector subspace of $\mathbb{R}^n$. In general, H^k is a translation of some vector subspace.

Proposition 2.15. *Vectors $p_1-p_0, p_2-p_0, \ldots p_k-p_0 \in \mathbb{R}^n$ are linearly independent if and only if no $(k-1)$-dimensional hyperplane contains all the points $p_0, p_1, \ldots, p_k \in \mathbb{R}^n$.*

Proof. Suppose that $p_1-p_0, p_2-p_0, \ldots, p_k-p_0$ are linearly independent and that there is a $(k-1)$-dimensional hyperplane containing $p_0, p_1, \ldots, p_k$. This is equivalent with the existence of $(k-1)$-dimensional subspace V and $x \in \mathbb{R}^n$ such that $p_0-x, p_1-x, \ldots, p_k-x \in V$. Since V is a vector subspace, $p_i-p_0 = (p_i-x)-(p_0-x) \in V-V = V$; a contradiction since p_i-p_0 are linearly independent.
Conversely, suppose that there is no $(k-1)$-dimensional hyperplane containing all the points $p_0, p_1, \ldots, p_k \in \mathbb{R}^n$ and that $p_1-p_0, p_2-p_0, \ldots p_k-p_0$ are linearly dependent. Hence the subspace of $\mathbb{R}^n$ spanned by these vectors is of dimension at most $k-1$. Then, after eventually adding some vectors, we get $(k-1)$-dimensional subspace V containing all the vectors $p_1-p_0, p_2-p_0, \ldots, p_k-p_0$. But then p_0+V defines the $(k-1)$-dimensional hyperplane which obviously contains $p_0, p_1, \ldots, p_k$. □

Remark 2.16. Given $k+1$ points in $\mathbb{R}^n$. We say that they are geometrically independent, if they do not generate a $(k-1)$-dimensional hyperplane.

The proposition we have just proved motivates the following important definition:

Definition 2.17. A set $P = \{p_0, p_1, \ldots, p_k\}$ in $\mathbb{R}^n$ is affinely independent if vectors $p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0$ are linearly independent.

Let us show that the choice of p_0 in definition 2.17 is not important:

Lemma 2.18. *Suppose $P = \{p_0, p_1, \ldots, p_k\}$ is affinely independent set of vectors in $\mathbb{R}^n$. Then the vectors $p_i - p_j$ for any j fixed and $i \neq j$ are also linearly independent.*

Proof. First suppose that the following equation holds for some real scalars λ_i:

$$\sum_{\substack{i=0 \\ i\neq j}}^{k} \lambda_i (p_i - p_j) = 0$$

Then the left hand side can be written in the following way:

$$\sum_{\substack{i=0 \\ i\neq j}}^{k} \lambda_i (p_i - p_0 - (p_j - p_0)) = \sum_{\substack{i=0 \\ i\neq j}}^{k} \lambda_i (p_i - p_0) - \left(\sum_{\substack{i=0 \\ i\neq j}}^{k} \lambda_i \right) (p_j - p_0)$$

By the assumption, we have $\lambda_i = 0$ for all $i \neq j$. □

Corollary 2.19. *Any subset of affinely independent set $P = \{p_0, p_1, \ldots, p_k\}$ is itself affinely independent.*

Proof. Take an arbitrary subset S of affinely independent set P. By lemma 2.18, it is not important whether S contains p_0 or not. Evidently, as any subset of linearly independent set is again linearly independent, the statement follows. □

Lemma 2.20. *The set $P = \{p_0, p_1, \ldots, p_k\}$ in $\mathbb{R}^n$ is affinely independent if and only if the equations $\sum_{i=0}^{k} \lambda_i p_i = 0$ and $\sum_{i=0}^{k} \lambda_i = 0$ for $p_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ imply $\lambda_i = 0$ for all $0 \leq i \leq k$.*

Proof. From the given conditions, we notice that

$$0 = \sum_{i=0}^{k} \lambda_i p_i = \sum_{i=1}^{k} \lambda_i p_i + \lambda_0 p_0 = \sum_{i=1}^{k} \lambda_i p_i - \left(\sum_{i=1}^{k} \lambda_i \right) p_0 = \sum_{i=1}^{k} \lambda_i (p_i - p_0)$$

By definition, p_i are affinely independent if $p_i - p_0$ are linearly independent. This gives that $\lambda_i = 0$ for $1 \leq i \leq k$. But then from $\sum_{i=0}^{k} \lambda_i = 0$ we deduce that also $\lambda_0 = 0$. □

The following theorem gives an useful characterization of points in a hyperplane containing the given set of points:

Theorem 2.21. *If $P = \{p_0, p_1, \ldots, p_k\}$ for $k \leq n$ is an affinely independent set in $\mathbb{R}^n$, then there is unique k-dimensional hyperplane H^k containing P. Moreover, each element $h \in H^k$ may be uniquely expressed as $h = \sum_{i=0}^{k} \mu_i p_i$, where $\sum_{i=0}^{k} \mu_i = 1$ and $\mu_i \in \mathbb{R}$.*

Proof. Let H^k be the set of vectors, which all can be written in the form

$$h = p_0 + \sum_{i=1}^{k} \lambda_i (p_i - p_0), \; \lambda_i \in \mathbb{R}$$

By definition 2.13, H^k is a hyperplane. Furthermore, from the equations

$$p_j = p_0 + \sum_{i=0}^{k} \delta_{ij} (p_i - p_0)$$

it follows that this hyperplane contains the set P. Since the set $\left\{h - p_0 \mid h \in H^k\right\}$ is a k-dimensional subspace of $\mathbb{R}^n$ with basis $p_i - p_0, 1 \leq i \leq k$, this representation is unique. Now note that we can rearrange the equations into

$$h = p_0 + \sum_{i=1}^{k} \lambda_i(p_i - p_0) = \left(1 - \sum_{i=1}^{k} \lambda_i\right) p_0 + \sum_{i=1}^{k} \lambda_i p_i$$

So in fact, we can take $\mu_0 = 1 - \sum_{i=1}^{k} \lambda_i$ and $\mu_i = \lambda_i, 1 \leq i \leq k$ to see that both above written conditions are satisfied. Moreover, since λ_i's are unique, so are μ_i's.

For the uniqueness part, suppose that there exists another such k-dimensional hyperplane F^k containing P. Then there is a linearly independent set $B = \{b_1, \ldots, b_k\}$ and a vector b_0, such that f lies in F^k if and only if

$$f = b_0 + \sum_{i=1}^{k} \eta_i b_i, \eta_i \in \mathbb{R} \tag{2.1}$$

Since F^k contains P, there exist real numbers α_{ij} such that

$$p_j = b_0 + \sum_{i=1}^{k} \alpha_{ij} b_i, \; 0 \leq j \leq k$$

From this, we obtain the following system of equations for $b_i, 1 \leq i \leq k$:

$$p_j - p_0 = \sum_{i=1}^{k} (\alpha_{ij} - \alpha_{i0})\, b_i$$

Since B and $p_i - p_0, 1 \leq i \leq k$ are both assumed to be linearly independent sets, there exist unique solutions:

$$b_i = \sum_{i=1}^{k} \beta_{ij}(p_j - p_0), \; 1 \leq i \leq k$$

Plugging in $b_i, 0 \leq i \leq k$ into the equation 2.1, the conclusion that $H^k = F^k$ follows. $\square$

Now we can positively answer on the question on the existence of the continuous extension of an affine map:

Theorem 2.22. *Let $P = \{p_0, p_1, \ldots, p_k\}$ for $k \leq n$ be an affinely independent set in $\mathbb{R}^n$, and H^k the unique hyperplane containing P. Then any affine function $f : H^k \to \mathbb{R}^m$ has an affine continuous extension $\hat{f} : \mathbb{R}^n \to \mathbb{R}^m$.*

Proof. Since $P = \{p_0, p_1, \ldots, p_k\}$ is an affinely independent set, the vectors $p_i - p_0, 0 \leq i \leq k$ are linearly independent. Now add $\hat{w}_{k+1}, \hat{w}_{k+2}, \ldots, \hat{w}_n \in \mathbb{R}^n$ to get a basis $\{p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0, \hat{w}_{k+1}, \hat{w}_{k+2}, \ldots, \hat{w}_n\}$ of vector space $\mathbb{R}^n$.
Introduce $w_i = \hat{w}_i + p_0$ for $k+1 \leq i \leq n$ and define $f(w_i) = 0$. Now, we construct a linear function $A : \mathbb{R}^n \to \mathbb{R}^m$ first by giving the images of the basis vectors. $A(p_i - p_0) = f(p_i) - f(p_0), 0 \leq i \leq k$ and $A(\hat{w}_i) = A(w_i - p_0) = f(w_i) - f(p_0) = -f(p_0)$. Furthermore, pick an arbitrary $x \in \mathbb{R}^n$. Then it is written uniquely in form

$$x = \sum_{i=1}^{k} \lambda_i (p_i - p_0) + \sum_{i=k+1}^{n} \lambda_i (w_i - p_0),\ \lambda_i \in \mathbb{R}$$

Finally, the linear map determined by images of basis vector is

$$A(x) = \sum_{i=1}^{k} \lambda_i A(p_i - p_0) + \sum_{i=k+1}^{n} \lambda_i A(w_i - p_0)$$

At the last step, we define $\hat{f} : \mathbb{R}^n \to \mathbb{R}^m$ by $\hat{f}(x) = A(x - p_0) + f(p_0)$. As first, note that $\hat{f}$ is affine by the theorem 2.11. Hence, it is continuous by remark 2.12. We easily check that $\hat{f}(p_i) = A(p_i - p_0) + f(p_0) = f(p_i) - f(p_0) + f(p_0) = f(p_i)$ for $0 \leq i \leq k$ and $\hat{f}(w_i) = A(w_i - p_0) + f(p_0) = 0$. To show that $\hat{f} \equiv f$ on H^k we pick an arbitrary point $x \in H^k$. By the definition 2.13 one can write it uniquely in form $x = p_0 + \sum_{i=1}^{k} \lambda_i (p_i - p_0)$ for some $\lambda_i \in \mathbb{R}$. Then

$$\begin{aligned}
\hat{f}(x) &= A(p_0 + \sum_{i=1}^{k} \lambda_i (p_i - p_0) - p_0) + f(p_0) \\
&= \sum_{i=1}^{k} \lambda_i A(p_i - p_0) + f(p_0) \\
&= \left(1 - \sum_{i=1}^{k} \lambda_i\right) f(p_0) + \sum_{i=1}^{k} \lambda_i f(p_i) \\
&= f\left(\left(1 - \sum_{i=1}^{k} \lambda_i\right) p_0 + \sum_{i=1}^{k} \lambda_i p_i\right) = f(x)
\end{aligned}$$

where in the last line of the equation we use the fact that f is affine on H^k. This completes the proof. □

For the end of this section, let us recall some basic properties of convex sets:

Definition 2.23. A subset C of $\mathbb{R}^n$ is convex if $x, y \in C$ and $\lambda \in [0,1]$ imply $(1-\lambda)x + \lambda y \in C$.
Smallest convex subset containing $C \in \mathbb{R}^n$ is called convex hull of C, denoted by $co(C)$.

Proposition 2.24. *Let A and B be arbitrary subsets of $\mathbb{R}^n$ such that $A \subseteq B$. Then $co(A) \subseteq co(B)$.*

Proof. Since $co(A)$ is by the previous definition the smallest subset of $\mathbb{R}^n$ containing A, one concludes $co(A) \subseteq B$. But since also $B \subseteq co(B)$, we have $co(A) \subseteq co(B)$. □

In the following lemma, we prove an essential property about convex hulls of finite sets:

Lemma 2.25. *Convex hull of a finite set $P = \{p_0, p_1, \dots, p_k\}$ in $\mathbb{R}^n$ is a compact set.*

Proof. In $\mathbb{R}^n$ compact sets are precisely those which are bounded and closed. Note also, that by theorem 2.5 we have that all norms on a finite dimensional vector space are equivalent.
Firstly, let us show that P is bounded. Namely, $P \subseteq B(0, M)$, where $M = \max_{0 \leq i \leq k} \|p_i\|$. To prove this, choose an arbitrary $x \in co(P)$. Then it can be written as $x = \sum_{i=0}^{k} \lambda_i p_i$ with $\sum_{i=0}^{k} \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$. We estimate the distance between x and 0:

$$\|x - 0\| = \|\sum_{i=0}^{k} \lambda_i p_i\| \leq \sum_{i=0}^{k} |\lambda_i| \|p_i\| \leq M \sum_{i=0}^{k} \lambda_i = M$$

Hence, $co(P) \subseteq B(0, M)$.

To see that $co(P)$ is closed, note that it is enough to show that $\overline{co(P)} \subseteq co(P)$. Take an arbitrary point $x \in \overline{co(P)}$. Then x is the limit of a sequence $\{x_n\}_{n \in \mathbb{N}}$ lying in $co(P)$. So, $x_n = \sum_{i=0}^{k} \lambda_{in} p_i$ with $\sum_{i=0}^{k} \lambda_{in} = 1, 0 \leq \lambda_{in} \leq 1$.
By passing to a convergent subsequence, we can define $\mu_i = \lim_{j \to \infty} \lambda_{in_j}, 0 \leq i \leq k$. Since $[0,1]$ is a closed subset of complete space $\mathbb{R}$, we have $0 \leq \mu_i \leq 1$ for each μ_i. Furthermore, since $\sum_{i=0}^{k} \lambda_{in} = 1$ for each $n \in \mathbb{N}$, so it is in the limit: $\sum_{i=0}^{k} \mu_i = 1$.
Finally, as a normed vector space, $\mathbb{R}^n$ is also a linear topological space, which means that vector addition and scalar multiplication are continuous. Again, after passing to subsequences in each of the sequences λ_{in} for $0 \leq i \leq k$, we have

$$x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} \sum_{i=0}^{k} \lambda_{in} p_i = \sum_{i=0}^{k} \mu_i p_i$$

This shows that $x \in co(P)$ and consequently, $\overline{co(P)} \subseteq co(P)$. □

Lemma 2.26. *Given a continuous bijection $f : X \to Y$, where X is compact and Y is Hausdorff topological space. Then it is closed map. In particular, f is homeomorphism.*

Proof. $A \subset X$ be closed. Then A is compact as well, as a closed subset of compact set. By continuity, $f(A)$ is also compact. But then, since Y is Hausdorff space, $f(A)$ must be also closed. For the second part, just note that A closed in $X \Leftrightarrow f(A)$ closed in Y. Hence, f is homeomorphism. □

Proposition 2.27. *Let $U \subseteq \mathbb{R}^n, n \in \mathbb{N}$ be bounded convex open set and $w \in U$. Then each ray R from w intersects ∂U, the boundary of U in exactly one point. Moreover, there is a homeomorphism of $\overline{U}$ with the unit disk $D^n = \{y \in \mathbb{R}^n \mid \|y\| \leq 1\}$ which maps ∂U onto ∂D^n.*

Proof. (taken from [3])
Let R be a fixed ray from $w \in U$. Clearly, it is a set $\{w + tp \mid t \geq 0\}$ for some unit vector along R. Since U is bounded, R intersects ∂U in some point x, which can be written in form $x = w + t_1 p$ for some $t_1 > 0$.
Suppose that R intersects ∂U in $y \neq x$. Without loss of generality, we assume that $y = w + t_2 p$, where $t_2 > t_1$. Then $x = (1-t)w + ty$ for $t = \frac{t_1}{t_2} \in (0,1)$, i.e. x is on a line segment between w and y. As $y \in \partial U = \overline{U} \setminus U$, there is a sequence $\{y_n\}_{n \in \mathbb{N}} \in U$ converging to y. But, then the sequence $w_n = \frac{x - ty_n}{1-t} \in U$ converges to $w \in U$. This gives that $x = tw_n + (1-t)y_n$ for some $t \in (0,1)$ and $w_n, y_n \in U$, which implies that $x \in U$. This is a contradiction, since $x \in \partial U$ and $\partial U \cap U = \emptyset$.
For the second part of the statement, assume that w is the origin. Function $f : \mathbb{R}^n \setminus \{0\} \to \partial D^n$ defined by $f(x) = \frac{x}{\|x\|}$ is continuous. By the above arguments, f restricted to ∂U is bijection. Furthermore, since ∂U is compact, $f\restriction_{\partial U} : \partial U \to \partial D^n$ is homeomorphism by lemma 2.26. Let $g : \partial D^n \to \partial U$ be its inverse. To extend g bijectively on whole D^n, we map a line segment joining 0 and $u \in \partial D^n$ linearly onto a line segment joining 0 and $g(u) \in \partial U$. Formally, its extension $\hat{g} : D^n \to U$ is defined by

$$\hat{g}(x) = \begin{cases} x \cdot \|g\left(\frac{x}{\|x\|}\right)\| & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is obvious that $\hat{g}$ is continuous for $x \neq 0$. So it remains to examine continuity at $x = 0$: since g is continuous on compact set, there is a constant $M > 0$ such that $\|g(x)\| \leq M$ for each $x \in \partial D^n$. Let $\varepsilon > 0$ be an arbitrary real number and choose $0 < \delta < \frac{\varepsilon}{M}$. If $\|x\| < \delta$, then $\|\hat{g}(x) - \hat{g}(0)\| < M\delta < \varepsilon$ and hence, $\hat{g}$ is continuous at $x = 0$. □

3 Geometric, Topological and Abstract Simplicial Complexes

3.1 Geometric Simplicial Complexes

The theorem 2.21 naturally leads to something one might have already suspected: vectors in H^k, hyperplane containing given affinely independent set $P = \{p_0, p_1, \ldots, p_k\}$ are uniquely determined by vectors in $\mathbb{R}^k$ (as k-dimensional subspaces of $\mathbb{R}^n$ are isomorphic to $\mathbb{R}^k$). This means that for each $(\mu_1, \mu_2, \ldots, \mu_k)^T$ in $\mathbb{R}^k$, there is $h \in H^k$ such that $h = \sum_{i=0}^{k} \mu_i p_i$, where we define $\mu_0 = 1 - \sum_{i=1}^{k} \mu_i$. It is reasonable to formalize this idea by coordinates and mappings:

Definition 3.1. Given an affinely independent set $P = \{p_0, p_1, \ldots, p_k\}$ and the unique k-dimensional hyperplane H^k containing it. Then the real numbers $\mu_i, 0 \leq i \leq k$ are the barycentric coordinates of vector $h \in H^k$ with respect to P if

$$h = \sum_{i=0}^{k} \mu_i p_i \quad \text{and} \quad \sum_{i=0}^{k} \mu_i = 1.$$

Proposition 3.2. *Let $P = \{p_0, p_1, \ldots, p_k\}$ be an affinely independent set and let H^k be the unique k-dimensional hyperplane containing it. Then the transformation $b_P : \mathbb{R}^k \to H^k$ defined for $v = (v_1, v_2, \ldots, v_k)^T \in \mathbb{R}^k$ by*

$$b_P(v) = \left(1 - \sum_{i=1}^{k} v_i\right) p_0 + \sum_{i=1}^{k} v_i p_i$$

is affine, one-to-one, onto and its inverse is also affine.

Proof. Firstly, for $v, w \in \mathbb{R}^k$ and arbitrary real number λ an elementary calculation shows that $b_P((1-\lambda)v + \lambda w) = (1-\lambda)b_P(v) + \lambda b_P(w)$. By definition 2.9, b_P is affine map.
If $b_P(v) = b_P(w)$ for some v and w, where $v, w \in \mathbb{R}^k$. Then we can set $v = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$ and $w = (\mu_1, \mu_2, \ldots, \mu_k)^T$ for $\lambda_i, \mu_i \in \mathbb{R}$ and straightforward calculation shows

$$\sum_{i=1}^{k} (\lambda_i - \mu_i)(p_i - p_0) = 0$$

By affine independence of p_i, it is concluded that $\lambda_i = \mu_i$ for each $1 \leq i \leq k$. This means $v = w$ and consequently, b_P is one-to-one.
If we pick an arbitrary $h \in H^k$, then by theorem 2.21 it can be uniquely expressed as

$$h = \left(1 - \sum_{i=1}^{k} \mu_i\right) p_0 + \sum_{i=1}^{k} \mu_i p_i \tag{3.1}$$

for $\mu_i \in \mathbb{R}, 1 \leq i \leq k$. So the obvious choice $v = (\mu_1, \mu_2, \ldots, \mu_k)^T$ satisfies $b_P(v) = h$. Hence, b_P is also onto.
Since b_P is bijective, we can define inverse map $b_P^{-1} : H^k \to \mathbb{R}^k$. Again, we express an arbitrary $h \in H^k$ identically as in the equation and easily define

$$b_P^{-1}\left(\left(1 - \sum_{i=1}^{k} \mu_i\right) p_0 + \sum_{i=1}^{k} \mu_i p_i\right) = (\mu_1, \mu_2, \ldots, \mu_k)^T$$

It is a matter of simple calculation to show that b_P^{-1} is an affine map. □

By the theorem 2.21 and proposition 3.2, the next definition is unambiguous:

Definition 3.3. Let $P = \{p_0, p_1, \ldots, p_k\}$ be the set of $k + 1$ affinely independent points in $\mathbb{R}^n$. The k-dimensional geometric simplex, called geometric k-simplex in $\mathbb{R}^n$ generated by P, is the set of all points of the hyperplane H^k containing P having nonnegative barycentric coordinates with respect to P.
An open geometric k-simplex is defined similarly, except that all barycentric coordinates are required to be positive.
The geometric k-simplex generated by P is denoted by $\langle p_0 p_1 \ldots p_k\rangle$. In general, if vertices are not given explicitly, it will be denoted by s^k.

Remark 3.4. By proposition 3.2, barycentric coordinates with respect to P are unique.

Example 3.5. We examine some cases to see which object are simplexes of smaller dimensions: if $k = 0$ we get $\langle p_0\rangle = \left\{\sum_{i=0}^{0} \mu_i p_0 \mid \mu_0 \geq 0, \sum_{i=0}^{0} \mu_i = 1\right\} = \{p_0\}$. So, we get only the point p_0. For $k = 1$ we get $\langle p_0 p_1\rangle = \left\{\sum_{i=0}^{1} \mu_i p_0 \mid \mu_i \geq 0, \sum_{i=0}^{1} \mu_i = 1\right\} = \{(1 - \mu)p_0 + \mu p_1 \mid \mu \geq 0\}$, which is usual line segment. Now, for $k = 2$ we get $\langle p_0 p_1 p_2\rangle = \left\{\sum_{i=0}^{2} \mu_i p_0 \mid \mu_i \geq 0, \sum_{i=0}^{2} \mu_i = 1\right\}$. We show that this is a plane triangle. Take an arbitrary x from this set and assume that $x \neq p_0$. Consider that $x = \sum_{i=0}^{2} \mu_i p_i = \mu_0 a_0 + (1 - \mu_0)[\frac{\mu_1}{\lambda} p_1 + \frac{\mu_2}{\lambda} p_2]$ where $\lambda = 1 - \mu_0$ which is nonzero, since $x \neq p_0$. Notice that $p = \frac{\mu_1}{\lambda} p_1 + \frac{\mu_2}{\lambda} p_2$ is a point on line segment between p_1 and p_2 and hence, x is a point on a line segment between p_0 and p. Notice also that 2-simplex is a union of such segments, i.e. it is a plane triangle. It is similarly shown that for $k = 3$ we obtain a tetrahedron.

If one wants to see what are the geometric open simplex in each of these cases, one has to only exclude the possibility of having any barycentric coordinate equal to zero. Indeed, for $k = 0$ we get a point p_0 again. In case $k = 1$, we get all the points as before except p_0 and p_1 - so, we get an *open* line segment. For $k = 2$ we get *open* triangle, i.e. from the original triangle we obtained we exclude $\langle p_0\rangle, \langle p_1\rangle$ and $\langle p_2\rangle$, but also line segments $\langle p_0p_1\rangle$,$\langle p_1p_2\rangle$ and $\langle p_2p_0\rangle$. In case $k = 3$, obtained is the *open* tetrahedron in which are excluded $\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle$, line segments $\langle p_0p_1\rangle, \langle p_1p_2\rangle, \langle p_2p_3\rangle, \langle p_3p_0\rangle$ and also triangles $\langle p_0p_1p_2\rangle, \langle p_1p_2p_3\rangle, \langle p_2p_3p_0\rangle$ and $\langle p_0p_1p_3\rangle$.

Suppose $\langle p_0p_1 \cdots p_k\rangle$ is geometric k-simplex. Referring to corollary 2.19, it follows that any nonempty subset of set $\{p_0, p_1, \ldots, p_k\}$ is itself the set of vertices of a geometric simplex of certain dimension, subsimplex of the initial simplex.

Definition 3.6. Subsimplex of some simplex s^k is called a face of s^k.
In particular, $\langle p_0p_1 \cdots \hat{p_j} \ldots p_k\rangle$ will denote the face of $\langle p_0p_1 \cdots p_k\rangle$ obtained by deleting p_j from set of its vertices $\{p_0, p_1, \ldots, p_n\}$.

Example 3.7. Given geometric simplex $\langle p_0p_1 \ldots p_k\rangle$. From the definition 3.6 we conclude that each $\langle p_i\rangle, 0 \leq i \leq k$ is a face of it. Also, $\langle p_0p_1 \ldots p_k\rangle$ is a face of itself.

Remark 3.8. Let s^k be an arbitrary k-simplex. To generate a simplex of dimension $0 \leq i \leq k$, one has to choose $i+1$ points out of given $k+1$ points. Hence, the number of faces of s^k equals to number of ways to do this. By elementary calculation we get that this number equals to $\binom{k+1}{1} + \binom{k+1}{2} + \cdots + \binom{k+1}{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} - 1 = 2^{k+1} - 1$.

Lemma 3.9. *Let $x \in \langle p_0p_1 \cdots p_k\rangle$. Then there exist proper disjoint faces $\langle p_0p_1 \cdots p_r\rangle$ and $\langle p_{r+1}p_{r+2} \cdots p_k\rangle$ of it such that x lies on a segment connecting points of them, i.e. $x = \lambda_1 x' + \lambda_2 x''$ where $\lambda_1 + \lambda_2 = 1$ with $\lambda_1, \lambda_2 \in [0, 1]$ and $x' \in \langle p_0p_1 \cdots p_r\rangle, x'' \in \langle p_{r+1}p_{r+2} \cdots p_k\rangle$ for integer $0 < r \leq k$.*

Proof. First, write x in barycentric coordinates $x = \sum_{i=0}^{k} \mu_i p_i$, where $\sum_{i=0}^{k} \mu_i = 1$ and $0 \leq \mu_i \leq 1$. Choose an integer $0 < r \leq k$ and denote $\lambda_1 = \sum_{i=0}^{r} \mu_i$ and $\lambda_2 = \sum_{i=r+1}^{k} \mu_i$. If one of them is zero, say λ_2, then $\mu_i = 0$ for all $r+1 \leq i \leq k$ and $x = \sum_{i=0}^{k} \mu_i p_i = \sum_{i=0}^{r} \mu_i p_i$, so that we can put $x' = \sum_{i=0}^{r} \mu_i p_i$ and $x'' = 0$. Treatment of the case $\lambda_1 = 0$ is completely analogous.

In the case that both of them are nonzero real numbers, we can set

$$x' = \sum_{i=0}^{r} \frac{\mu_i}{\lambda_1} p_i \quad \text{and} \quad x'' = \sum_{i=r+1}^{k} = \frac{\mu_i}{\lambda_2} p_i$$

to conclude that $\frac{\mu_i}{\lambda_1}, 0 \leq i \leq r$ and $\frac{\mu_i}{\lambda_2}, r+1 \leq i \leq k$ are barycentric coordinates of affinely independent sets $\{p_0, p_1, \ldots, p_r\}$ and $\{p_{r+1}, p_{r+2}, \ldots, p_k\}$ respectively. And then, by separating sums and plugging x' and x'', we get $x = \lambda_1 x' + \lambda_2 x''$, with $\lambda_1 + \lambda_2 = 1$. □

Proposition 3.10. *The geometric k-simplex $\langle p_0 p_1 \cdots p_k \rangle$ generated by a set $P = \{p_0, p_1, \ldots, p_k\}$ of $k+1$ affinely independent points is exactly the convex hull of P, $co(P)$.*

Proof. Let S be an arbitrary convex set containing P. We claim that k-simplex generated by P is contained in S. We proceed by induction. For $k = 0$, it holds trivially. Suppose it is true up to $k-1$ and choose x in $\langle p_0 p_1 \cdots p_k \rangle$. By previous lemma applied on $\langle p_0 \rangle$ and $\langle p_1 p_2 \cdots p_k \rangle$, there exist a point $x'' \in \langle p_1 p_2 \cdots p_k \rangle$, such that x is on a line segment connecting p_0 with x''. By the induction hypothesis, $x'' \in S$. Therefore, x as a point on a segment connecting points from S is also in S, so $\langle p_0 p_1 \cdots p_k \rangle \subseteq S$. This means that $\langle p_0 p_1 \cdots p_k \rangle \subseteq co(P)$.

To see $co(P) \subseteq \langle p_0 p_1 \cdots p_k \rangle$, we first show that given simplex is convex set. Choose arbitrary $x, y \in \langle p_0 p_1 \cdots p_k \rangle$ and arbitrary $\lambda \in [0, 1]$. Then

$$(1-\lambda)x + \lambda y = (1-\lambda)\sum_{i=0}^{k} \mu_i p_i + \lambda \sum_{i=0}^{k} \eta_i p_i = \sum_{i=0}^{k}((1-\lambda)\mu_i + \lambda\eta_i)p_i$$

where $0 \leq \mu_i, \eta_i \leq 1$ with $\sum_{i=0}^{k} \mu_i = 1, \sum_{i=0}^{k} \eta_i = 1$. But then $(1-\lambda)x + \lambda y \in \langle p_0 p_1 \cdots p_k \rangle$, as $0 \leq (1-\lambda)\mu_i + \lambda\eta_i \leq 1$ and $\sum_{i=0}^{k}(1-\lambda)\mu_i + \lambda\eta_i = (1-\lambda)\sum_{i=0}^{k} \mu_i + \lambda \sum_{i=0}^{k} \eta_i = 1$. By definition 3.3 we have $P \subseteq \langle p_0 p_1 \cdots p_k \rangle$. Also, by proposition 2.24, $co(P) \subseteq co(\langle p_0 p_1 \cdots p_k \rangle)$ and consequently $co(P) \subseteq \langle p_0 p_1 \cdots p_k \rangle$. □

To investigate more general sets, we must be able to somehow *join* simplexes. And if we are able, they will be basic building blocks of polyhedra and spaces which are geometric realizations of them. To fruitfully work with it, mathematically more precise definitions of this notions are to be introduced:

Definition 3.11. Two geometric simplexes s^m and $s^n, m \leq n$, both lying in the same ambient space $\mathbb{R}^N$, are properly joined if $s^m \cap s^n = \emptyset$ or $s^m \cap s^n = s^k$ for some $k \leq m$, i.e. $s^m \cap s^n$ is a common face of s^m and s^n.

Since joining simplexes will give rise to some more complicated structure, this concept is formalized in the next definition:

Definition 3.12. A geometric simplicial complex K is finite collection of properly joined geometric simplexes lying inside $\mathbb{R}^N$ such that if s^n is any simplex of K, then each face of it also belongs to K.
Dimension of K is defined by $\dim K = \max_{s \in K} \dim s$, i.e. it is the maximum dimension among the dimensions of simplexes contained in it.

Remark 3.13. Geometric 0-,1- and 2-simplexes contained in an geometric simplical complex will be called vertices, edges and triangles respectively.

3.2 Topological Simplicial Complexes

Definition 3.14. Let X be a topological space. Topological n-simplex σ^n is a pair (A, h) consisting of a topological space $A \subseteq X$ and homeomorphism $h : s_A^n \to A$ where s_A^n denotes some geometric n-simplex. The space A is said to be the carrier of s_A^n.

Notion of face of a topological k-simplex is described naturally:

Definition 3.15. Let $\sigma^n = (A, h)$ be topological simplex with $h : s_A^n \to A$. Then the collection $\mathcal{F} \subseteq \mathcal{P}(A)$ of all its faces is defined by

$$\mathcal{F}_{\sigma^n} = \left\{ h(s_A^k) \mid s_A^k \text{ is a face of } s_A^n \right\}$$

Definition 3.16. Two topological simplexes (A, h) and (B, h') which lie in the same ambient space are properly joined if

1. $A \cap B$ is face of both A and B
2. if s_1 is the face of $s_A = h^{-1}(A)$ and s_2 is the face of $s_B = h'^{-1}(B)$ then there exist a linear bijection $l : s_1 \to s_2$ with the property $h\restriction_{s_1} = (h'\restriction_{s_2}) \circ l$.

Definition 3.17. A topological complex K is at most countable collection of properly joined simplexes such that each face of a simplex in K is itself simplex in K.

Remark 3.18. For certain class of spaces called compact surfaces, a topological complex will consist only of finitely many properly joined simplexes. This theory will be exposed in chapter 5.

3.3 Topological Properties of Geometric Complexes

Definition 3.19. Let $K \subseteq \mathbb{R}^n$ be a geometric simplicial complex. A simplicial polytope, denoted by $|K|$, is the union of all simplexes in K, i.e. $|K| = \cup_{s \in K} s$.
Geometric simplicial complex K is said to be a triangulation of topological space X if there exist a homeomorphism $h : |K| \to X$.

Theorem 3.20. *Let $P = \{p_0, p_1, \ldots, p_k\}$ be the set of $k+1$ affinely independent points in $\mathbb{R}^n$. Then an open geometric k-simplex is a relatively open subset in the hyperplane H^k containing P, with respect to subspace topology.*

Proof. Define $H = \left\{(\lambda_0, \lambda_1, \dots, \lambda_k) \mid \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \in \mathbb{R}\right\} \subset \mathbb{R}^{k+1}$ and function $f : H \cap \mathbb{R}^{k+1}_{\geq 0} \to co(P)$ with

$$f((\lambda_0, \lambda_1, \dots, \lambda_k)) = \sum_{i=0}^{k} \lambda_i p_i$$

where $\mathbb{R}^{k+1}_{\geq 0} = \{(r_0, r_1, \dots, r_k) \mid r_i \in \mathbb{R}, r_i \geq 0\}$. This function is well defined, i.e. maps onto $co(P)$ by proposition 3.10. Moreover, by proposition 3.2, it is an affine bijection and with affine inverse f^{-1}. By theorem 2.22 f is restriction of continuous function to $H \cap \mathbb{R}^{k+1}_{\geq 0}$ and hence, it is itself continuous. The same conclusion holds for its inverse. Thus, function f is homeomorphism of spaces $H \cap \mathbb{R}^{k+1}_{\geq 0}$ and $co(P)$.
Now, note that $f(\mathrm{Int}(H \cap \mathbb{R}^{k+1}_{\geq 0})) = \mathrm{Int}(co(P))$. But the left hand side of this equation equals $f(\mathrm{Int}(H \cap \mathbb{R}^{k+1}_{\geq 0})) = f(\mathrm{Int}(H) \cap \mathrm{Int}(\mathbb{R}^{k+1}_{\geq 0})) = f(H \cap \mathbb{R}^{k+1}_{>0})$. This leads to $f(H \cap \mathbb{R}^{k+1}_{>0}) = \mathrm{Int}(co(P))$, which implies that open geometric simplex generated by P coincides with the relative interior in H^k of simplex generated by P . □

Proposition 3.21. *Each point of $|K|$ lies in the relative interior of exactly one simplex.*

Proof. Suppose that s and t are simplexes of complex K whose relative interiors overlap. Since their intersection is evidently nonempty, they must intersect in a common face. Without loss of generality we can write $s = \langle a_0 a_1 \dots a_l a_{l+1} \dots a_k \rangle$ and $t = \langle a_0 a_1 \dots a_l b_{l+1} \dots b_n \rangle$ for $0 \leq l < k \leq n$. If $x \in \mathrm{Int}(s) \cap \mathrm{Int}(t)$, then

$$x = \sum_{i=0}^{l} \mu_i a_i + \sum_{i=l+1}^{k} \mu_i a_i = \sum_{i=0}^{l} \lambda_i a_i + \sum_{i=l+1}^{n} \lambda_i b_i$$

where $\sum_{i=0}^{k} \mu_i = 1$ and $\sum_{i=0}^{n} \lambda_i = 1$ with condition $0 < \lambda_i, \mu_i < 1$, by theorem 3.20. But then since $\{a_0, a_1, \dots a_l, a_{l+1}, \dots, a_k, b_{l+1}, \dots, b_n\}$ is affinely independent, $\lambda_i = \mu_i$ for $0 \leq i \leq l$ and $\lambda_i = 0, \mu_j = 0$ for $l+1 \leq i \leq k$ and $l+1 \leq j \leq n$ respectively, which contradicts the requirement that all λ_i and μ_i are positive. So, the only face of simplex that contains all the points of the relative interior is the whole simplex itself. Hence $s = t$. □

Example 3.22. Given is the geometric simplicial complex K consisting of two 2-simplexes s_1^2 and s_2^2 together with their faces. Suppose also that they intersect 0-simplex $\langle p \rangle$, their common face. Then $p \in |K|$ is in relative interior of $\langle p \rangle$.

Theorem 3.23. *Simplicial polytope of n-dimensional geometric simplicial complex $K \subseteq \mathbb{R}^N$ is compact space.*

Proof. Since $|K| = \cup_{s \in K} s$ is a union of finitely many simplexes in the same ambient space $\mathbb{R}^N$, it is enough to show compactness of only one simplex, as union of finitely many compact sets is again a compact set. But this fact follows by lemma 2.25, as each geometrical simplex is by proposition 3.10 a convex hull of finitely many points. □

Before we start discussion about connectedness, recall once again that we sometimes use vertex, edge, triangle instead of 0-,1- and 2- simplexes respectively.

Definition 3.24. Two geometric simplexes s_1 and s_2 are edge connected in geometric simplicial complex K if there exists a chain of edges s_i^1 for $1 \leq i \leq k$ such that:

1) $s_1 \cap s_1^1$ is a vertex of s_1
2) $s_2 \cap s_k^1$ is a vertex of s_2
3) $s_i^1 \cap s_{i+1}^1$ is a vertex of both of simplexes s_i and s_{i+1} for all $1 \leq i \leq k-1$.

Chain of such edges between the vertices $\langle v \rangle$ of s_1 and $\langle w \rangle$ of s_2 is called an edge path.

Remark 3.25. Edge connectedness in an equivalence relation on K. The equivalence classes with respect to relation of edge connectedness between simplexes of complex K are called combinatorial components. The complex is edge connected if there exist one combinatorial component.

Lemma 3.26. *A geometric simplicial complex K is edge connected if and only if each simplex $s_1 \in K$ has at least one common proper face with some simplex $s_2 \in K$.*

Proof. Assume that K is edge connected and that there exist simplex s_1 having no common proper faces with other simplexes $s_2, s_3, \ldots, s_m \in K$. Since K is simplicial complex, simplexes are properly joined and hence by definition 3.12 we have $s_1 \cap s_i = \emptyset$ for $2 \leq i \leq m$. But then there is no edge path between any vertex of s_1 and any vertex of s_i, which contradicts our assumption.
The converse is rather trivial, since if each simplex has at least one common proper face with some other simplex in K, then one can produce edge path between two arbitrary vertices in K. $\square$

Theorem 3.27. *Let K be an arbitrary geometric simplicial complex such that $|K| \subseteq \mathbb{R}^N$ is connected. Then it is path connected.*

Proof. First note that each geometric simplex in K has at least one common proper face with some other simplex of K. Otherwise we could separate $|K|$ into disjoint union of nonempty closed proper subsets, which is a contradiction with the assumption that $|K|$ is connected. By lemma 3.26, we conclude that K is edge connected.
Furthermore, pick an arbitrary $u \in |K|$ lying in some simplex s_1 of K and define a nonempty subset $\Omega_1 \subseteq K$ as follows

$$\Omega_1 = \{s \mid \text{there is a path between } u \text{ and vertex of } |s| \text{ in } |K|\}$$

and say $\Omega_2 = K \setminus \Omega_1$. Note that both Ω_1 and Ω_2 are simplicial complexes.

If $|\Omega_1| \cap |\Omega_2| = \emptyset$, then they are at positive distance, i.e. $d(|\Omega_1|, |\Omega_2|) > 0$. But then $|K| = |\Omega_1| \cup |\Omega_2|$, with $|\Omega_1|$ and $|\Omega_2|$ being nonempty disjoint closed proper subsets of $|K|$, which is a contradiction with assumption that $|K|$ is connected.
Now assume $|\Omega_1| \cap |\Omega_2| \neq \emptyset$, i.e. that there exists $x \in |\Omega_1| \cap |\Omega_2|$. Then x is in the interior of a unique simplex s_2 by proposition 3.21. If $v \in |\Omega_2|$, then by convexity we have path to the vertex of simplex containing v in its interior. Then, we have an edge path to vertex of simplex s_2 and hence, by convexity, we connect it to x. But $x \in |\Omega_1|$, so there is a path from x to u. Consequently $|\Omega_2| \subseteq |\Omega_1|$, which implies $|K| = |\Omega_1|$, and hence, $|K|$ is path connected. □

For the end of this section, we discuss maps between simplicial complexes and those between their simplicial polytopes. The general idea is to preserve the simplicial structure, i.e. to map simplexes onto simplexes and this will motivate us also in work with abstract simplicial complexes. However, since polytopes of geometric complexes are topological spaces, we can employ continuity:

Lemma 3.28. *Let K and L be geometric simplicial complexes and $f : K \to L$ a function satisfying the condition that if $\{p_0, p_1, \dots, p_k\}$ generates a geometric simplex of K, then also $\{f(p_0), f(p_1), \dots, f(p_k)\}$ generates a simplex of L. Then there is an induced continuous map $f^* : |K| \to |L|$, called simplicial map, defined by setting*

$$f^*(x) = \sum_{i=0}^{k} x_i f(p_i)$$

where x_i are barycentric coordinates of $x \in |K|$ with respect to p_i in geometric simplex $\langle p_0 p_1 \dots p_k \rangle$.

Proof. Suppose that $|K|$ is a union of m simplexes and that K is n-dimensional. Let $s_1 = \langle p_0 p_1 \dots p_k \rangle$ be an arbitrary geometric simplex in complex K. Then first construct an affine function defined only on simplicial polytope of this particular simplex: $f_1 : |\langle p_0 p_1 \dots p_k \rangle| \to |L|$ by $f_1(x) = \sum_{i=0}^{k} x_i f(p_i)$, x_i being barycentric coordinates of x with respect to $p_i, 0 \leq i \leq k$. As f_1 is by the theorem 2.22 a restriction of an affine function defined on R^N, it is continuous.
Now we do the same procedure and obtain the remaining functions $f_2, f_3, \dots, f_m$, continuous on simplicial polytopes of $s_2, s_3, \dots, s_m$ respectively. Consider properly joined simplexes s_i and s_j for $i \neq j$ intersecting in the proper face of each of them s_{ij}. Then restriction of f_i to s_{ij} depends only on vertices spanning s_{ij}, since barycentric coordinates with respect to the points not spanning s_{ij} are equal to zero. The same holds for f_j, which means that these functions agree on s_{ij}.
Finally, define the desired function $f^* : |K| \to |L|$ as $f^* = f_i$ on $s_i, 1 \leq i \leq m$. Since $|K|$ is the union of simplicial polytopes of s_i, which are all closed and since all f_i agree on the common intersections of any number of simplexes, f^* is continuous. □

Finally, we can define when two simplicial polytopes are *the same*:

Definition 3.29. Simplicial polytopes of two geometric simplicial complexes K and L are isomorphic if there exist a one-to-one simplicial map $f^* : |K| \to |L|$ such that the inverse mapping is also a simplicial map.

Remark 3.30. With the notation as in lemma 3.28, consider that $f(p_i), 0 \leq i \leq k$ are not necessarily distinct. For example, if $f(p_i) = f(p_j), 0 \leq i < j \leq k$ then the barycentric coordinate of $f(x)$ with respect to the vertex $f(p_i)$ is $x_i + x_j$.
Loosely speaking, it may happen that we map 2-simplex onto 1- simplex. This map will preserve simplicial structure, but indeed decrease the dimension.

3.3.1 Barycentric Subdivision

Before diving into further theory, let us briefly show a standard technique to make triangulation K finer in sense that the number of simplexes is increased. The simplicial polytope will remain as original, but with decreased diameters of simplexes in it. We describe the procedure in consecutive definitions:

Definition 3.31. Given an arbitrary geometric simplex $s^k = \langle p_0 p_1 \dots p_k \rangle$. A point of it with all barycentric coordinates equal,

$$\dot{s}^k = \frac{1}{k+1}\left(p_0 + p_1 + \cdots + p_k\right)$$

is called a barycenter of s^k.

Let s_1 and s_2 be geometric simplexes and write $s_1 \preceq s_2$ if s_1 is a (proper) face of s_2. Let K be a geometric simplicial complex consisting only of one simplex $s^k = \langle p_0 p_1 \dots p_k \rangle$. We describe its first barycentric subdivision, denoted by K^1, in the following way:
Consider the set $L = \left\{\dot{s^*} \mid s^* \preceq s^k\right\}$ of barycenters of all the faces of s^k. By the remark 3.8, it is concluded that $\mathrm{card}(L) = 2^{k+1} - 1$. Note that barycenter of any 0-simplex $\langle p_i \rangle$ is the point p_i itself. We say that a collection of barycenters $\dot{s}_0^*, \dot{s}_1^*, \dots, \dot{s}_l^* \in L$ where $0 \leq l \leq k$ form the new geometric l-simplex $\langle \dot{s}_0 \dot{s}_1 \dots \dot{s}_l \rangle$ if $s_0^* \prec s_1^* \prec \cdots \prec s_l^*$. So, the first barycentric subdivision of the simplicial complex $K = \left\{s^* \mid s^* \preceq s^k\right\}$ is a collection

$$K^1 = \left\{\langle \dot{s_0^*} \dot{s_1^*} \dots \dot{s_l^*} \rangle \mid s_i^* \preceq s^k, 0 \leq i \leq l \text{ and } s_0^* \prec s_1^* \prec \cdots \prec s_l^*, 0 \leq l \leq k\right\}$$

Lemma 3.32. *Given an arbitrary geometric simplicial complex K and let K^1 be its first barycentric subdivision, then each simplex of K^1 is contained in a simplex of K.*

Proof. Choose any simplex of the first barycentric subdivision $\hat{s} = \langle \dot{s}_0 \dot{s}_1 \dot{s}_2 \dots \dot{s}_{k-1} \dot{s} \rangle \in K^1$ where $s_0 \prec s_1 \prec s_2 \prec \cdots \prec s_{k-1} \prec s$. Since each point that generates $\hat{s}$ is a barycenter of some face of s, $\hat{s}$ is contained in s. Therefore, each simplex of K^1 is contained in some simplex of K. □

Theorem 3.33. *Given an arbitrary geometric simplicial complex K. The previously described procedure executed on each simplex of K gives a new geometric simplicial complex denoted by K^1, called the first barycentric subdivision of K. Moreover, $|K| = |K^1|$*

Proof. (taken from [1])
Choose an arbitrary geometric simplex $\hat{s} = \langle \dot{s}_0 \dot{s}_1 \dot{s}_2 \dots \dot{s}_{k-1} \dot{s} \rangle \in K^1$ where $s_0 \prec s_1 \prec s_2 \prec \cdots \prec s_{k-1} \prec s$. Note that each proper face of $\hat{s}$ is again a simplex of K^1, since each of them is created by deleting some barycenters of s and taking the convex hull of the remaining points.
Note that in order to show that K^1 is complex, it remains to see that its simplexes are properly joined. This fact, together with $|K| = |K^1|$ is proved by induction on the m, number of simplexes of K. If $m = 1$, then K consists of only one vertex, which is a barycenter of itself and the both results are trivially seen to be true. Suppose these are true also for all complexes which have less than m simplexes and form K, complex made of exactly m simplexes. Take out simplex of maximum dimension of it and name the new complex with L, i.e. $L = K \setminus s$, where s is a simplex of K for which $\dim s = \dim K = k$. By the inductive hypothesis, L^1 is simplicial complex and $|L| = |L^1|$, so it suffices to check whether the remaining simplexes, those in $K^1 \setminus L^1$, are properly joined.
So, let $u = \langle \dot{s}_0 \dot{s}_1 \dot{s}_2 \dots \dot{s}_{k-1} \dot{s} \rangle \in K^1 \setminus L^1$, where $s_0 \prec s_1 \prec s_2 \prec \cdots \prec s_{k-1} \prec s$. Note that $v = \langle \dot{s}_0 \dot{s}_1 \dot{s}_2 \dots \dot{s}_{k-1} \rangle$ forms face v of u with $v \in L^1$ and $|v| = |u| \cap |L^1|$. From this, we conclude that if u meets simplex in L^1, it does it in a face of v, or in other words, in one of its own faces. Now we check the intersections of the simplexes in $K^1 \setminus L^1$: let u and v be defined as in the preceding case and let $u' \in K^1 \setminus L^1$ and $v' \in L^1$ be another such pair. Also, suppose that $u \cap u' \neq \emptyset$, because otherwise, we can immediately say that they are properly joined. If $v \cap v' \neq \emptyset$, they intersect in a common face, since both of them are simplexes of L^1, which is by the hypothesis a simplicial complex. Else if $v \cap v' = \emptyset$, then $u \cap u' = \dot{s}$, again a common face of each. This finishes the induction argument.

It remains to show the second statement: since any simplex of K^1 is by lemma 3.32 contained a simplex of K, we have $|K^1| \subseteq |K|$, so it suffices to consider the reversed inclusion, under a hypothesis that $|L| = |L^1|$. Choose an arbitrary $x \in |K|$ and u, the unique simplex of K containing x in its relative interior (see proposition 3.21). If $x = \dot{u}$

is a barycenter of u, then $x \in |K^1|$ by definition. Else, join $\dot{u}$ to x by straight line and prolong it to intersect proper face v of u in the point $y \in |L| = |L^1|$, by the hypothesis. Hence, there is a $t \in [0,1]$ such that $x = (1-t)\dot{u} + ty$ and so $y \in \hat{v}$, for some simplex $\hat{v}$ in L^1. Then vertices of $\hat{v}$ and the point $\dot{u}$ make simplex of $|K^1|$ containing x by convexity. Therefore $x \in |K^1|$ and thus $|K| \subseteq |K^1|$. □

Remark 3.34. One can also further subdivide K^1 to obtain second barycentric subdivision of K, denoted by K^2, then from this to get K^3 and so on... Note that if we iterate the conclusion of the previous theorem, we obtain $|K| = |K^m|$ for any $m \in \mathbb{N}$.

3.4 Abstract Simplicial Complexes

In this section, we describe complexes abstractly. Given is an arbitrary simplicial complex K in $\mathbb{R}^n$. Two things characterize it: set of its vertices in $\mathbb{R}^n$ and subsets of $\mathbb{R}^n$ containing vertices which generate simplexes. But sometimes it is useful to begin with a nonempty finite set and a collection of its subsets and *realize* this pair as a geometric complex in some Euclidean space. Before starting it, let us define things that will make our use of language easier.

Definition 3.35. An abstract simplicial complex is a pair $\{V, S\}$, where V is a nonempty finite set of elements called vertices and $S \subseteq \mathcal{P}(V)$ collection of nonempty subsets of V, called abstract simplexes such that:

1. $v \in V \Rightarrow \sigma = \{v\} \in S$
2. $\sigma \in S$ and $\emptyset \neq \tau \subseteq \sigma \Rightarrow \tau \in S$.

A set $\tau \in S$ from the second condition is called a face of abstract simplex σ.
Dimension of abstract simplex $\sigma \in S$ is defined to be $\dim \sigma = \text{card}(\sigma) - 1$. For abstract simplicial complex $\{V, S\}$ it is defined as $\dim \{V, S\} = \max_{\sigma \in S} \dim \sigma$.
The abstract k-simplex generated by some set $\sigma \in S$ having $k+1$ elements and with vertices given explicitly will be often denoted by $\sigma^k = \langle v_0 v_1 \dots v_k \rangle$.

Remark 3.36. It is easily checked that every geometric simplicial complex K satisfies the above definition. Namely, one can take the points which generate 0-simplexes to correspond to elements of set V and simplexes of K to correspond to elements of collection S. More precisely, for every v in V we have $\{v\} \in S$, as $\{v\}$ defines 0-simplex. Moreover, any subset of $s \in S$ is again in S, since any face of each geometric simplex is itself simplex of geometric simplicial complex K.

As in is the case of geometric simplicial complexes, it is useful to present maps between different abstract simplicial complexes which preserve simplicial structure.

Definition 3.37. Given two abstract simplicial complexes $\{V, S\}$ and $\{V', S'\}$. A function $f : V \to V'$ such that if $\langle v_0 v_1 \cdots v_k \rangle$ is an abstract simplex of $\{V, S\}$, then $\langle f(v_0) f(v_1) \cdots f(v_k) \rangle$ is an abstract simplex of $\{V', S'\}$ is called the abstract simplicial map. Two abstract simplicial complexes $\{V, S\}$ and $\{V', S'\}$ are isomorphic if f is bijection.

3.4.1 Realization Theorem

What we now aim to show is sort of converse of remark 3.36 - the fact that is possible to realize abstract simplicial complex in some Euclidean space. However, to be precise, it still remains to precisely define what realization is.

Definition 3.38. A geometric simplicial complex K is realization of an abstract complex $\{V, S\}$ if there is a bijection f from V onto the set generating 0-simplexes of K, so that each element $\sigma \in S$ is mapped onto simplex of K generated by images of the function f restricted to set σ.

Theorem 3.39. *Every abstract simplicial complex $\{V, S\}$ has a realization, geometric simplicial complex K in some Euclidean space. Pair $\{V, S\}$ is called a vertex scheme of K.*

Proof. Let $\{V, S\}$ be abstract simplicial complex with set of vertices $V = \{v_0, v_1, \ldots, v_n\}$. Consider standard simplex: given $p_i \in \mathbb{R}^n, 0 \leq i \leq n$ such that p_0 is point of origin and $p_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where only nonzero coordinate is on i-th place. It is obvious that $p_1, p_2, \ldots, p_n$ are linearly independent. Since $p_i - p_0 = p_i$, we have also that $p_1 - p_0, p_2 - p_0, \ldots, p_n - p_0$ are linearly independent and hence $p_0, p_1, \ldots, p_n$ are affinely independent. By corollary 2.19, their any subcollection is also affinely independent and hence it makes geometric simplex, a face of n-simplex in $\mathbb{R}^n$.
We construct the realization K of $\{V, S\}$ by considering one-to-one correspondence $f : V \to \{p_0, p_1, \ldots, p_n\}$ defined by $f(v_i) = p_i$ for $0 \leq i \leq n$:

$$K = \{co(f(\sigma)) \mid \sigma \in S\}$$

and saying that $co(f(\sigma))$ is realization of abstract simplex $\sigma \in S$.
As first, let us show that K, defined as above is indeed a simplicial complex. Since there are only finitely many sets in S, K is also finite. Let $\sigma = \{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\} \in S$ be an arbitrary abstract simplex in $\{V, S\}$. By the above observations, the set $f(\sigma) = \{f(v_{i_0}), f(v_{i_1}), \ldots, f(v_{i_k})\} = \{p_{i_0}, p_{i_1}, \ldots, p_{i_k}\}$ generates k-simplex $co(f(\sigma))$ in $\mathbb{R}^n$. Furthermore, if τ is any nonempty subset of σ, then by definition 3.35 $\tau \in S$ and hence, $co(f(\tau)) \in K$. This means that realization of any face of abstract simplex σ is a face of geometric simplex $co(f(\sigma))$ in K. It remains to show that simplexes in K are properly joined. Consider arbitrary $\rho, \xi \in S$.

- Assume $\rho \cap \xi = \emptyset$. Then also $f(\rho) \cap f(\xi) = \emptyset$, as f is injective. Hence, $f(\rho) = \{p_{i_0}, p_{i_1}, \ldots, p_{i_k}\}$ and $f(\xi) = \{p_{j_0}, p_{j_1}, \ldots, p_{j_l}\}$. By assumption $p_{i_r} \neq p_{j_s}$ for $0 \leq r \leq k$ and $0 \leq s \leq l$. Then we claim that also $co(f(\rho)) \cap co(f(\xi)) = \emptyset$. If this would not be a case, then there would exist an $x \in \mathbb{R}^n$ with $x \in co(f(\rho)) \cap co(f(\xi))$ and we could express it as

$$x = \sum_{r=0}^{k} \mu_{i_r} p_{i_r} = \sum_{s=0}^{l} \mu_{j_s} p_{j_s}$$

where $\sum_{r=0}^{k} \mu_{i_r} = 1, \sum_{s=0}^{l} \mu_{j_s} = 1$ and $0 \leq \mu_{i_r} \leq 1, 0 \leq \mu_{j_s} \leq 1$. These equations imply

$$\sum_{r=0}^{k} \mu_{i_r} p_{i_r} + \sum_{s=0}^{l} (-\mu_{j_s}) p_{j_s} = 0$$

But since $f(\rho) \cup f(\xi)$ is subset of linearly independent set, it is itself linearly independent and hence $\mu_{i_r} = 0, \mu_{j_s} = 0$ for all $0 \leq r \leq k$ and $0 \leq s \leq l$, which contradicts the assumptions $\sum_{r=0}^{k} \mu_{i_r} = 1$ and $\sum_{s=0}^{l} \mu_{j_s} = 1$.

- Assume $\rho \cap \xi = \eta$. Then since η is subset of sets from collection S, then $\eta \in S$. Now we have to prove that $co(f(\eta)) = co(f(\rho)) \cap co(f(\xi))$. Since $\eta \subseteq \rho$ and $\eta \subseteq \xi$, we have $f(\eta) \subseteq f(\rho)$ and $f(\eta) \subseteq f(\xi)$ and consequently by proposition 2.24, $co(f(\eta)) \subseteq co(f(\rho))$ and $co(f(\eta)) \subseteq co(f(\xi))$, which further implies $co(f(\eta)) \subseteq co(f(\rho)) \cap co(f(\xi))$.

  Furthermore, suppose that $x \in co(f(\rho)) \cap co(f(\xi))$. Say that

$$\begin{aligned} f(\rho) &= \left\{p^*_{i_0}, p^*_{i_1}, \ldots, p^*_{i_m}, p_{i_{m+1}}, p_{i_{m+2}}, \ldots, p_{i_k}\right\} \\ f(\xi) &= \left\{p^*_{i_0}, p^*_{i_1}, \ldots, p^*_{i_m}, p_{j_{m+1}}, p_{j_{m+2}}, \ldots, p_{j_l}\right\} \end{aligned}$$

  and $f(\eta) = \left\{p^*_{i_0}, p^*_{i_1}, \ldots, p^*_{i_m}\right\}$ for $0 \leq m \leq \min\{k, l\}$. Then x can be written in the following forms:

$$x = \sum_{r=0}^{m} \mu_{i_r} p^*_{i_r} + \sum_{s=m+1}^{k} \mu_{i_s} p_{i_s} = \sum_{r=0}^{m} \lambda_{i_r} p^*_{i_r} + \sum_{s'=m+1}^{l} \lambda_{j_{s'}} p_{j_{s'}}$$

  provided $\sum_{r=0}^{m} \mu_{i_r} + \sum_{s=m+1}^{k} \mu_{i_s} = 1$ and $\sum_{r=0}^{m} \lambda_{i_r} + \sum_{s'=m+1}^{l} \lambda_{j_{s'}} = 1$ with $0 \leq \mu_{i_r}, \mu_{i_s}, \lambda_{i_r}, \lambda_{j'_s} \leq 1$. But these imply

$$\sum_{r=0}^{m} (\mu_{i_r} - \lambda_{i_r}) p^*_{i_r} + \sum_{s=m+1}^{k} \mu_{i_s} p_{i_s} + \sum_{s'=m+1}^{l} \left(-\lambda_{j_{s'}}\right) p_{j_{s'}} = 0$$

  Again, by the linear independence of $f(\rho) \cup f(\xi)$ we have $\mu_{i_r} = \lambda_{i_r}$ for $0 \leq r \leq m$ and $\mu_{i_s} = 0, \lambda_{j_{s'}} = 0$ for $m+1 \leq s \leq k$ and $m+1 \leq s' \leq l$, respectively. This implies that $x \in co(f(\eta))$, so $co(f(\rho)) \cap co(f(\xi)) \subseteq co(f(\eta))$, which yields the desired result.

So, in either case, simplexes in K above defined are properly joined. Hence, by the definition 3.12, K is a geometrical simplicial complex with vertex scheme $\{V, S\}$. $\square$

Is realization of abstract simplicial complex unique? The answer is in some sense positive and written in the following theorem:

Theorem 3.40. *Given an abstract simplicial complex $\{V, S\}$. Then the simplicial polytopes of any two realizations of it are isomorphic.*

Proof. Let K and L be realizations of vertex scheme $\{V, S\}$. Then there exist bijections $f : V \to \{p_0, p_1, \dots, p_n\}$ and $g : V \to \{q_0, q_1, \dots, q_n\}$, where $\{p_0, p_1, \dots, p_n\}$ and $\{q_0, q_1, \dots, q_n\}$ are both affinely independent sets. Define $h : K \to L$ mapping $h(p_i) = q_i, 0 \leq i \leq n$ and define h first on some geometric k-simplex $\langle p_{i_0} p_{i_1} \dots p_{i_k} \rangle$ by

$$h(x) = \sum_{r=0}^{k} \mu_{i_r} q_{i_r}$$

where $x = \sum_{r=0}^{k} \mu_{i_r} p_{i_r}$ with $\sum_{r=0}^{k} \mu_{i_r} = 1$ and $0 \leq \mu_{i_r} \leq 1$. Note that $q_{i_r}, 0 \leq r \leq k$ also generate k-simplex, as realizations of the same abstract simplex. By the lemma 3.28 there is continuous extension $h^* : |K| \to |L|$. So, in order to show that they are isomorphic, it suffices to check that h^* is one-to-one and that its inverse is continuous. Pick two arbitrary $x, y \in |K|$ lying in the same simplex $\langle p_{i_0} p_{i_1} \dots p_{i_k} \rangle$. Then $x = \sum_{r=0}^{k} \mu_{i_r} p_{i_r}$ with $\sum_{r=0}^{k} \mu_{i_r} = 1, 0 \leq \mu_{i_r} \leq 1$ and $y = \sum_{r=0}^{k} \lambda_{i_r} p_{i_r}$ with $\sum_{r=0}^{k} \lambda_{i_r} = 1, 0 \leq \lambda_{i_r} \leq 1$. If $h(x) = h(y)$ then

$$\sum_{r=0}^{k} \mu_{i_r} q_{i_r} = \sum_{r=0}^{k} \lambda_{i_r} q_{i_r}$$

which by linear independence means that $\mu_{i_r} = \lambda_{i_r}, 0 \leq r \leq k$ and hence $x = y$. Define inverse of h, $k : L \to K$ on simplex $\langle q_{i_0} q_{i_1} \dots q_{i_k} \rangle$ by

$$k(x) = \sum_{r=0}^{k} \mu_{i_r} p_{i_r}$$

where $x = \sum_{r=0}^{k} \mu_{i_r} q_{i_r}$ with $\sum_{r=0}^{k} \mu_{i_r} = 1$ and $0 \leq \mu_i \leq 1$. Let us check that this is indeed inverse of h:

$$h(k(x)) = h\left(k\left(\sum_{r=0}^{k} \mu_{i_r} q_{i_r}\right)\right) = h\left(\sum_{r=0}^{k} \mu_{i_r} p_{i_r}\right) = \sum_{r=0}^{k} \mu_{i_r} q_{i_r} = x$$

The case $k(h(x))$ is analogous. Therefore, $k = h^{-1}$ and $(h^{-1})^* : |L| \to |K|$ is continuous by lemma 3.28. $\square$

4 Simplicial Orientation

Definition 4.1. Let $\sigma^k = \langle v_0 v_1 \cdots v_k \rangle, v_i \in V$ be an abstract k-simplex. A $(k+1)$-tuple $(v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(k)})$, where π is a permutation of $\{0, 1, \ldots, k\}$ is called an ordering of vertices of σ^k.

Remark 4.2. There are $(k+1)!$ orderings of vertices of σ^k.

Definition 4.3. Let $\sigma^k = \langle v_0 v_1 \cdots v_k \rangle, v_i \in V$ be an abstract k-simplex and let two orderings of its vertices be equivalent if they differ by an even permutation. Orientation of σ^k is a choice of fixed ordering, i.e. equivalence class on the set of all $(k+1)$-tuples.

Example 4.4. Observe that since abstract 0-simplex $\sigma^0 = \langle v_0 \rangle$ is generated by only one vertex, classes of even and odd permutations on the set of orderings of its only vertex coincide and orientation is unambigously determined in only one way.
For 1-simplex $\sigma^1 = \langle v_0 v_1 \rangle$ the set of all orderings is $\{(v_0, v_1), (v_1, v_0)\}$, where each ordering is in its own equivalence class.
For 2-simplex, we have a set consisting of six orderings, again partitioned into two equivalence classes: $[(v_0, v_1, v_2)] = \{(v_0, v_1, v_2), (v_1, v_2, v_0), (v_2, v_0, v_1)\}$ and $[(v_0, v_2, v_1)] = \{(v_0, v_2, v_1), (v_2, v_1, v_0), (v_1, v_0, v_2)\}$.

To orient geometric simplex we follow the remark 3.36 which says that every geometric simplicial complex K satisfies the definition of abstract simplicial complex. Recall that we take the points generating 0-simplexes to correspond to elements of set V and simplexes of K to correspond to elements of collection S. On the other hand, we can employ the realization theorem 3.39 on the example 4.4 and get orientation of vertex, edge and triangle. This motivates the following definition:

Definition 4.5. Let s^k be a geometric k-simplex. Then its orientation is determined by the orientation of corresponding abstract simplex σ^k.

Definition 4.6. An oriented (geometric or abstract) k-simplex is simplex together with a choice of orientation. If $(v_0, v_1, \ldots, v_k)$ is ordering that is chosen, we denote the oriented simplex with $[v_0 v_1 \ldots v_k]$.
Equivalence class of even permutations of fixed ordering is positively oriented simplex, denoted by $+\sigma^k$ and equivalence class of odd permutations of this ordering is negatively oriented abstract simplex, denoted by $-\sigma^k$.

Remark 4.7. We sometimes loosely speak that we give the *positive* orientation to a geometric simplex, as we traverse through all the edges.

In fact, we can formally define *the minus operator* on arbitrarily chosen equivalence class of orderings of vertices:

Definition 4.8. Let C be the set of equivalence classes on a set of orderings of vertices of abstract simplex $\sigma^k = \langle v_0 v_1 \dots v_k \rangle$. Then $- : C \to C$ is defined by $-[v_0, v_1, \dots, v_k] = C \setminus [v_0, v_1, \dots, v_k]$, i.e. it transforms the chosen orientation into the opposite one.

Example 4.9. We consider abstract 2-simplex σ^2. Then $[v_0v_1v_2] = [v_1v_2v_0] = [v_2v_0v_1] = +\sigma^2$. On the other hand, $[v_1v_0v_2] = [v_0v_2v_1] = [v_2v_1v_0] = -\sigma^2$.
Now choose a geometric 2-simplex $s^2 = \langle p_0p_1p_2 \rangle$. Similarly as for abstract simplexes, $[p_0p_1p_2] = [p_1p_2p_0] = [p_2p_0p_1] = +s^2$. On the other hand, $[p_0p_2p_1] = [p_2p_1p_0] = [p_1p_0p_2] = -s^2$. So, getting an orientation of $s^2 = \langle p_0p_1p_2 \rangle$ is matter of choosing positive direction, while passing three edges of it. Roughly speaking, they correspond to opposite directions of traversing the boundary of s^2.

The second part of example 4.9 motivates us to decide what is a natural orientation of geometric simplex embedded in some Euclidean space:

Definition 4.10. An oriented geometric k-simplex $[p_0p_1 \dots p_k]$ is oriented naturally if and only if $\det(p_1 - p_0, p_2 - p_0, \dots, p_k - p_0) > 0$.

Proposition 4.11. *Notion of natural orientation is well defined.*

Proof. We have to prove that the above determinant does not change under even permutation of the ordering of the vertices of s^k (or abstract simplex corresponding to it). Denote matrix $A = (p_1 - p_0, p_2 - p_0, \dots, p_k - p_0)$. Since s^k is geometric simplex, vectors $p_1 - p_0, p_2 - p_0, \dots, p_k - p_0$ are linearly independent, so $\det A \neq 0$.
Consider interchanging two vertices in ordering $(v_0, v_1, \dots, v_k)$. If we interchange two vertices different than v_0, it will change the sign of $\det A$, as it acts like switching of two rows. Otherwise, if we interchange v_0 and v_i for $i > 0$, then we get $\hat{A} = (p_1 - p_i, p_2 - p_i, \dots, p_0 - p_i, \dots, p_k - p_i)$ and proceed as follows: multiply the i-th row with -1, which changes the sign of $\det \hat{A}$ and add it to each other row, which does not change its sign, to get original matrix A. Thus, a transposition of two arbitrary vertices in ordering $(v_0, v_1, \dots, v_k)$ change the sign of $\det A$ and so, an arbitrary permutation of ordering $(v_0, v_1, \dots, v_k)$ changes the sign of $\det A$ if and only if it is odd, i.e. it is invariant under even permutation of chosen ordering. $\square$

The rest of this section is dedicated to orientation of (abstract or geometric) simplicial complexes. To produce a good definition of orientation of given k-dimensional simplicial complex, we first determine the orientation of $(k-1)$-dimensional faces of k-simplexes:

Definition 4.12. Given an arbitrary oriented (abstract or geometric) k-simplex $\sigma^k = [v_0v_1 \dots v_k]$, the induced orientation of its $(k-1)$-dimensional face is defined as $\tau_i^{k-1} = (-1)^i [v_0v_1 \dots \hat{v}_i \dots v_k]$, where vertex $v_i, 0 \leq i \leq k$ is deleted from the vertex set.

Proposition 4.13. *The notion of induced orientation is well defined.*

Proof. We need to show that induced orientation is not changed under even permutation of ordering of vertices in σ^k from the previous definition. Since each permutation s can be written as a product of transpositions, it suffices to check that induced orientation changes when two adjacent vertices are switched. In case that neither of those two is vertex v_i, then it is clear that orientation changes. Otherwise, if we change v_i with v_{i-1} or v_{i+1}, then $\hat{\tau}_i^{k-1} = (-1)^{i-1}[v_0v_1 \dots v_{i-1}v_{i+1} \dots v_k]$ or $\hat{\tau}_i^{k-1} = (-1)^{i+1}[v_0v_1 \dots v_{i-1}v_{i+1} \dots v_k]$ respectively, which clearly changes the orientation. Therefore, induced orientation of $(k-1)$-dimensional face of σ^k changes only under odd permutation of ordering of vertices in σ^k. □

Definition 4.14. Let K be an arbitrary (abstract or geometric) k-dimensional simplicial complex with the property that every $(k-1)$-simplex is a face of no more than two k-simplexes. Given oriented simplexes σ_1^k and σ_2^k having simplex τ^{k-1} in common, we say that their orientations are consistent if they induce opposite orientations on τ^{k-1}.

An orientation on K is a choice of orientation on each of k-simplexes in such a way that any two simplexes that intersect in $(k-1)$-dimensional face are consistently oriented. If K admits orientation, it is said to be orientable. Its simplicial polytope $|K|$ is said to be orientable if K is orientable.

Lemma 4.15. *Let σ^k be an oriented abstract k-simplex with $k \geq 3$. Then every two $(k-1)$-faces, with orientation induced from σ^k, are oriented consistently, i.e. they induce opposite orientation on their common $(k-2)$-dimensional faces.*

Proof. Let τ_i^k, τ_j^k for $0 \leq i < j \leq k$ be two arbitrary $(k-1)$-dimensional faces of σ^k with their induced orientations. So, we can write $\tau_i^k = [v_0v_1 \dots \hat{v}_i \dots v_k]$ and $\tau_j^k = [v_0v_1 \dots \hat{v}_j \dots v_k]$. Denote their common face $(k-2)$-dimensional face by $\tau_i^k \cap \tau_j^k = \langle v_0v_1 \dots v_{i-1}v_{i+1} \dots v_{j-1}v_{j+1} \dots v_k \rangle$. If $\tau_i^k \cap \tau_j^k$ induces an orientation by τ_i^k, then $\tau_i^k \cap \tau_j^k = (-1)^{i+j-1} [v_0v_1 \dots v_{i-1}v_{i+1} \dots v_{j-1}v_{j+1} \dots v_k]$, since after deleting i, j is in the $(j-1)$-th place. On the other hand, if it induces an orientation by τ_j^k, then $\tau_j^k \cap \tau_j^k = (-1)^{i+j} [v_0v_1 \dots v_{i-1}v_{i+1} \dots v_{j-1}v_{j+1} \dots v_k]$, since the place of vertex i after deleting vertex j is unchanged. These orientations of $\tau_j^k \cap \tau_j^k$ are opposite, since $i + j - 1$ and $i + j$ are of different parity. Hence, they induce opposite orientations on their common $(k-2)$-dimensional faces. □

Remark 4.16. Given oriented k-simplex σ^k, with $k \geq 3$, we cannot consistently orient its faces of dimension less or equal to $(k-2)$. Namely, by the lemma 4.15 we have that $(k-2)$-dimensional faces induce opposite orientation by $(k-1)$-dimensional face of σ^k.

Example 4.17. In the figure 1 we illustrate the consistent and inconsistent orientations on a simple example of properly joined 2-simplexes:

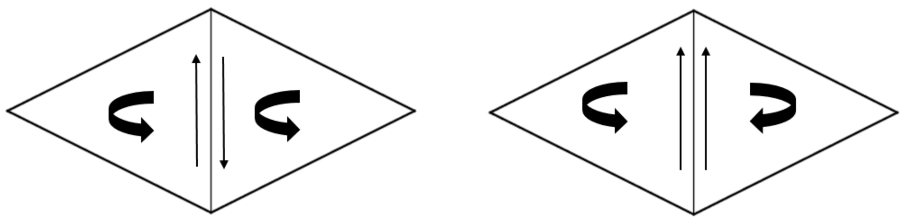

Figure 1: Consistent and inconsistent orientations of 2-simplexes

4.1 Cylinder versus Möbius strip

In this section we aim to use the theory about abstract simplicial complexes and orientation we developed until now. The general idea is to construct an abstract simplicial complex such that simplicial polytope of geometric realization K is homeomorphic to the object we want to study. Triangulations were taken from [3].

4.1.1 Cylinder

Cylinder is standardly obtained by the following procedure: Take for instance rectangle $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, 0 \leq y \leq 1\}$. We employ an equivalence relation on R and say that $(x,y) \sim (z,w)$ if and only if these two points are identified. Cylinder is obtained in partition into two types of sets:

1. Sets containing points of form $(0,y)$ and $(3,y)$ for $0 \leq y \leq 1$;

2. Singletons consisting only of point (x,y) for some x and y satisfying $0 < x < 3$ and $0 \leq y \leq 1$.

To examine its orientability, we use the realization theorem established in subsection 3.4.1. Construction is illustrated in the figure 2: given is the rectangle, opposite sides of which were identified. Abstract simplex generated by points c and d is identified with simplex generated by a_1 and b_1 and so $c \sim a_1$ and $d \sim b_1$.

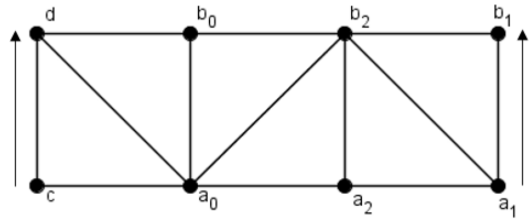

Figure 2: Abstract cylinder

The explicit abstract simplicial complex $\{V, S\}$ we are going to study is given as follows:

$$
\begin{aligned}
V &= \{a_0, a_1, a_2, b_0, b_1, b_2\} \\
S &= \{\{a_0\}, \{a_1\}, \{a_2\}, \{b_0\}, \{b_1\}, \{b_2\}, \{a_0, a_1\}, \{a_1, a_2\}, \{a_0, a_2\}, \{b_0, b_1\}, \\
&\quad \{b_1, b_2\}, \{b_0, b_2\}, \{a_0, b_1, b_0\}, \{a_0, a_1, b_1\}, \{a_1, b_1, b_2\}, \{a_1, a_2, b_2\}, \{a_0, a_1, b_1\}, \\
&\quad \{a_0, a_2, b_2\}, \{a_0, b_0, b_2\}\}
\end{aligned}
$$

Note that $\{V, S\}$ is indeed an abstract simplicial complex. V is finite and it is checked that any nonempty subset of $A \in S$ is again an element of S. Hence the theorem 3.39 applies. Realization K is 2-dimensional, compact and is of single combinatorial component - it is edge connected as geometric complex. Consider that also each 1-simplex of it is a face of no more than two 2-simplexes.

Regarding the orientation, observe the figure 3: if we have chosen oriented simplex $[cda_0]$, then we have to choose $[b_0a_0d]$, in order simplexes $\langle cda_0\rangle$ and $\langle b_0a_0d\rangle$ to induce opposite orientation on their common edge $\langle a_0d\rangle$. We continue in this fashion and get oriented 2-simplexes $[a_0b_0b_2]$, $[a_0b_2a_2]$, $[a_2b_2a_1]$, $[a_1b_2b_1]$ and it is easily checked that induced orientation on edges $\langle a_0b_0\rangle$, $\langle a_0b_2\rangle$, $\langle b_2a_2\rangle$, $\langle b_2a_1\rangle$ are opposite. Also, simplexes $[cda_0]$ and $[a_1b_2b_1]$ induce opposite orientation on $\langle a_1b_1\rangle = \langle cd\rangle$, the edge along which they are glued.

Note that no matter what simplex we choose at first and no matter which orientation prescribe on it, every other simplex can be oriented in consistent manner. Thus, K is orientable. If we embed K in some big enough ambient space, then we see that it is homeomorphic to cylinder, by the very construction of cylinder described in the beginning. This shows that cylinder is indeed orientable.

4.1.2 Möbius strip

Möbius strip is obtained in a similar way like cylinder, as a quotient space of rectangle. Start again with $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, 0 \leq y \leq 1\}$ and introduce equivalence relation on it, such that equivalence classes are sets of two types:

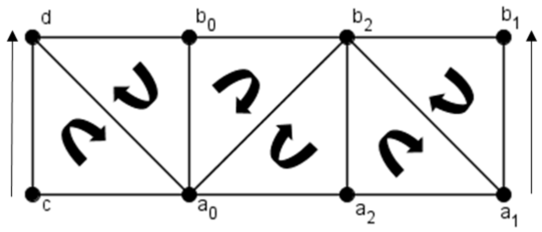

Figure 3: Cylinder is orientable

1. sets consisting of points of the form $(0, y)$ and $(3, 1 - y)$,
2. singletons containing only (x, y) for some x and y satisfying $0 < x < 3$ and $0 \leq y \leq 1$.

In other words, what is done is that we *flip* one of the vertical edges and attach to the second one. Other points remain the same. Question about its orientation is again translated into the language of abstract simplicial complexes.

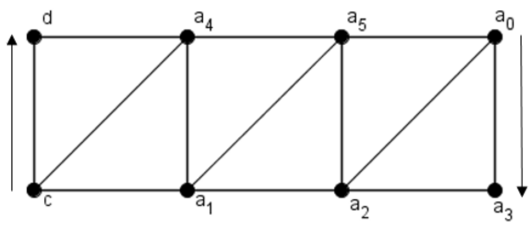

Figure 4: Abstract Möbius strip

The figure 4 shows one way of triangulating rectangle and consequently, after identifying vertical edges in opposite direction, so that $c \sim a_0$ and $d \sim a_3$, its quotient space, Möbius strip. Pair $\{V, S\}$ is composed in the following way:

$$\begin{aligned} V &= \{a_0, a_1, a_2, a_3, a_4, a_5\} \\ S &= \{\{a_0\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_0, a_1\}, \{a_0, a_2\}, \{a_0, a_3\}, \{a_0, a_4\}, \\ &\quad \{a_0, a_5\}, \{a_1, a_2\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_2, a_3\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_4, a_5\}, \\ &\quad \{a_0, a_1, a_4\}, \{a_0, a_2, a_3\}, \{a_0, a_2, a_5\}, \{a_0, a_3, a_4\}, \{a_1, a_2, a_5\}, \{a_1, a_4, a_5\}\} \end{aligned}$$

Again, since V is finite and since subset of $A \in S$ is again an element of S, $\{V, S\}$ is indeed an abstract simplicial complex and theorem 3.39 applies. Realization L is 2-dimensional, compact and connected as a complex. Notice that also each 1-simplex is a face of no more than two 2 simplexes.

Orientation of its simplicial polytope, surface homeomorphic to Möbius strip by its very construction, can be studied on abstract complex $\{V, S\}$, after *unfolding* it - passing to the rectangle, opposite sides of which were identified.

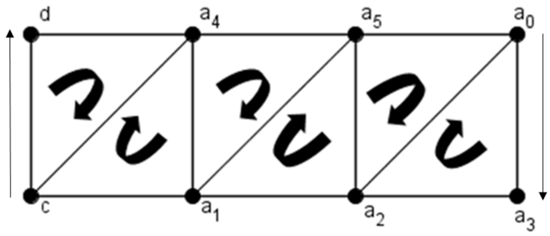

Figure 5: Möbius strip is nonorientable

In the figure 5, there is an unsuccessful attempt of giving consistent orientation on L. Suppose that Möbius strip is orientable. Then, no matter which 2-simplex we choose at first and no matter which orientation we prescribe on it, every other 2-simplex should be oriented in a consistent manner. For example, choose oriented simplex $[cda_4]$. Then we have to prescribe $[ca_4a_1]$, in order simplexes $\langle cda_4\rangle$ and $\langle ca_4a_1\rangle$ to induce opposite orientation on their common edge $\langle ca_4\rangle$. If we continue in this fashion, we obtain oriented 2-simplexes $[a_1a_4a_5]$, $[a_1a_5a_2]$, $[a_2a_5a_0]$, $[a_2a_0a_3]$ and it is easily verified that induced orientation on edges $\langle a_1a_4\rangle$, $\langle a_1a_5\rangle$, $\langle a_2a_5\rangle$, $\langle a_2a_0\rangle$ are opposite. Now note that simplexes $[cda_4]$ and $[a_2a_0a_3]$ induce the same orientation on $\langle a_0a_3\rangle = \langle cd\rangle$, the edge along which they are glued.

Thus, L is nonorientable. If we embed L in some big enough ambient space, then we see that it is homeomorphic to Möbius strip, by its construction described in the very beginning. Hence, Möbius strip is nonorientable surface.

5 Classification Theorem

Now we are ready to classify surfaces. We begin in general setting, but follow geometric intuition and theory developed, which will guide us to restrict to something nicer and more manageable at the moment. Recall that vertex, edge and triangle stand for 0-,1- and 2- simplexes. Let us first extend some definitions given in chapter 3:

Definition 5.1. An infinite geometric simplicial complex $K \subseteq \mathbb{R}^N$ is a countable collection of properly joined geometric simplexes such that there is a simplex of maximum dimension, each point of $\mathbb{R}^N$ intersects at most finite number of simplexes, and K is locally finite, i.e. each vertex is a face of finitely many simplexes.
Dimension of K is defined by $\dim K = \max_{s \in K} \dim s$. A simplicial polytope, denoted by $|K|$, is the union of all simplexes in K, i.e. $|K| = \cup_{s \in K} s$. A topological space is triangulable if there is a homeomorphism $h : X \to |K|$.

To reveal what spaces are triangulable, we first define spaces which are locally Euclidean:

Definition 5.2. An n-dimensional manifold, $n \in \mathbb{N}$ is a Hausdorff topological space such that each point x of it has an open neighbourhood homeomorphic to n-dimensional Euclidean unit ball $B^n = \{y \in \mathbb{R}^n \mid \|y\| < 1\}$.

Are all manifolds triangulable? Unfortunately, the answer is negative already in dimension four. Example is given in 1980 by Michael H. Freedman in [4]. Moreover, a research paper [8] in 2016 by C. Manolescu showed the existence of nontriangulable n-manifolds for $n \geq 5$. But what we can surely visualize are 2-manifolds. We do not require to much if we impose that they are also connected.

Definition 5.3. Connected 2-dimensional manifold is called a surface.

Theorem 5.4. *Every compact surface is homeomorphic to a polytope of some* 2 *- dimensional geometric simplicial complex.*

Proof. The first proof is given by Tibor Radó in 1925. For an outline of the proof, one can consult [10], Chapter 8. □

Remark 5.5. Theorem 5.4 reveals that each compact surface is triangulable. Moreover, the triangulation is of dimension 2.

Given a finite simplicial complex of dimension two, by theorem 3.23 we have that its simplicial polytope is compact. If we could prove the converse of it, we will benefit, since in this case we obtain surfaces which have triangulations out of finitely many triangles, which will be easier to control, as we have already examined their properties.

Theorem 5.6. *Surface is compact if and only if its any triangulation is composed of finitely many triangles.*

Proof. (taken from [6])
Suppose that compact surface S has triangulation of which simplicial complex K is infinite. By locally finitness of K, there are only finitely many triangles meeting at each vertex. Hence, if there are infinitely many triangles, there are infinitely many vertices as well. Now label these vertices by v_i and put them in a sequence $\{v_i\}_{i=1}^{\infty}$. By compactness, this sequence has a limit point in $v \in S$.
If v is in relative interior of some triangle, then it has a neighbourhood completely contained in triangle, so neighbourhood containing no other vertices. If v is on some edge, it has neighbourhood taken from two triangles and not containing any vertices. Else, if v is a vertex, it has a neighbourhood taken out of finitely many triangles and containing no other vertices. In each case, we have a contradiction with the definition of limit point. As we have said, the other implication follows from theorem 3.23. □

Now we give a more systematic approach for compact surfaces. The following definition summarizes useful concepts developed until now:

Definition 5.7. A triangulation T of compact surface $S \subseteq \mathbb{R}^N$ is a finite family $\{(T_i, \varphi_i) \mid 1 \leq i \leq n\}$ with $n \in \mathbb{N}$, such that $S = \cup_{i=1}^{n} T_i$, where each $T_i \subseteq \mathbb{R}^N$ is a closed subset and each φ_i a homeomorphism $\varphi_i : T_i' \to T_i$ for $1 \leq i \leq n$, T_i''s being triangles in $\mathbb{R}^2$. Moreover, T_i are also called triangles and subsets of T_i that are images of vertices and edges of T_i' under φ_i are called vertices and edges as well. Additionally, it is required that any two triangles T_i and T_j are either disjoint, have a single vertex in common, or have an entire edge in common.

Proposition 5.8. *Let S be a compact surface with triangulation T consisting of $n \in \mathbb{N}$ triangles. Then we can label triangles $T_1, T_2, \ldots, T_n$ in such a way that each T_i has an edge e_i common with at least one of the triangles $T_1, T_2, \ldots, T_{i-1}, 2 \leq i \leq n$.*

Proof. Label any of the given triangles with T_1. For T_2 choose any triangle having edge in common with T_1, for T_3 choose any triangle having edge in common with T_1 or T_2. If we could not continue the process at some stage, then we would have sets $\{T_1, T_2, \ldots, T_k\}$ and $\{T_{k+1}, T_{k+2}, \ldots, T_n\}$ such that no triangle of first set has an edge with any of the triangles in the second. But then $\cup_{i=1}^{k} T_i$ and $\cup_{i=k+1}^{n} T_i$ are both nonempty and closed sets (as finite union of closed sets). By construction, they are also disjoint and hence partition S. This contradicts the fact that S is connected. □

To show an important property of triangulation of compact surfaces, we recall the famous Jordan's curve theorem:

Definition 5.9. Simple closed curve is a continuous path $J : [0,1] \to \mathbb{R}^2$ with $J(0) = J(1)$ and property that $J\restriction_{[0,1)}$ is injective.

Theorem 5.10. *The image of a simple closed curve $J : [0,1] \to \mathbb{R}^2$ separates the plane, i.e. $\mathbb{R}^2 \setminus J([0,1]) = A \cup B, A \cap B = \emptyset$, where both A and B are connected, A bounded and B unbounded, with the property $\partial A = J([0,1]) = \partial B$.*

Proof. See [10], Chapter 4. □

Proposition 5.11. *Each edge in a triangulation of some compact surface S is face of at most two triangles.*

Proof. Suppose that there is a triangulation $|K|$ of S such that there exist an edge e, which is a face of k triangles, where $k \geq 3$. Take a point $x \in \mathrm{Int}(e)$. Then e separates $B^3_x \cap S$ for each sufficiently small ball B^3_x containing x in k connected components. On the other hand, since S is a 2-manifold, there exist a homeomorphism $F : B^3_x \to U$ with $U \subseteq \mathbb{R}^3$ such that $F(B^3_x \cap S) = (\mathbb{R}^2 \times \{0\}) \cap U$ and denote $C = (\mathbb{R}^2 \times \{0\}) \cap U$. In particular, choose ball $B^2_{F(x)} \subset \mathbb{R}^2 \times \{0\}$ around $F(x)$ sufficiently small that $\overline{B^2_{F(x)}} \subset C$. If necessary, make B^3_x smaller to have $B^3_x \cap S \subseteq F^{-1}(B^2_{F(x)})$. Note that after these adjustments e still separates $B^3_x \cap S$ into k connected components, while we claim that $F(B^3_x \cap S) \subseteq B^2_{F(x)} \subset C$ is separated by $F(e \cap B^3_x)$ into exactly 2 connected components: Notice that set $e \cap B^3_x$ is a curve. So, $F(e \cap B^3_x)$ is also a curve parametrized by $\alpha : [0,1] \to F(B^3_x \cap S)$. Since e intersects ∂B^3_x in exactly two points $\hat{y}$ and $\hat{z}$, also $\alpha([0,1])$ intersects $\partial F(B^3_x \cap S)$ in two points $y = \alpha(0)$ and $z = \alpha(1)$ and $\partial F(B^3_x \cap S) \setminus \{y, z\}$ is a union of exactly two curves parametrized by α_1 and α_2 with $\alpha_1(0) = \alpha_2(0) = z$ and $\alpha_1(1) = \alpha_2(1) = y$. Now introduce two new curves

$$\beta_i(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \alpha_i(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $i = 1, 2$, i.e. we add a lower and upper arc of $\partial F(B^3_x \cap S)$ to a curve α, in order to obtain two simple closed curves. Application of theorem 5.10 on simple closed curves β_1 and β_2 gives that $F(e \cap B^3_x)$ separates $F(B^3_x \cap S)$ into two components. This is a contradiction, as number of connected components is a topological invariant. □

The finiteness and nice combinatorial properties of triangulations of compact surfaces motivate us to *glue* triangles in $\mathbb{R}^2$ to obtain usual polygons, as simple models of them. Before facing a remarkable result about this connection, let us see some technical facts:

Definition 5.12. Subset D of $\mathbb{R}^2$ is called a topological disc, if it is homeomorphic to the unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 1\}$. Arc α is a subset of ∂D homeomorphic to interval $[0, 1]$.

Lemma 5.13. *A homeomorphism between two topological disks sends boundary to boundary.*

Proof. Suppose we have $h : D_1 \to D_2$ and that $h(x) \in \text{Int}(D_2)$ for $x \in \partial D_1$. Then h induces a homeomorphism $\hat{h} : D_1 \setminus \{x\} \to D_2 \setminus \{h(x)\}$. Now, note that $D_1 \setminus \{x\}$ is simply connected, but $\hat{h}(D_1 \setminus \{x\})$ is not. This is a contradiction, since simply connectedness is a topological invariant. □

Lemma 5.14. *Let D_1 and D_2 be topological disks and assume that $f : \partial D_1 \to \partial D_2$ be an arbitrary homeomorphism between their boundaries. Then there is a homeomorphism $\hat{f} : D_1 \to D_2$ with $\hat{f} = f$ on ∂D_1.*

Proof. (taken from [1]):
Assume $D \subseteq \mathbb{R}^2$ is a unit disk. Since D_1 and D_2 are topological disks, there exist homeomorphisms $h_1 : D_1 \to D$ and $h_2 : D_2 \to D$. Observe a homeomorphism $g = h_2 \circ f \circ h_1^{-1} : \partial D \to \partial D$. If we are able to extend g to homeomorphism $\hat{g} : D \to D$, then $\hat{f} = h_2^{-1} \circ \hat{g} \circ h_1 : D_1 \to D_2$ would extend f. Thus, it suffices to extend $g : \partial D \to \partial D$. Define

$$\hat{g}(x) = \begin{cases} \|x\| \cdot g\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Hence, $\hat{g}$ maps circles of radius $r > 0$ onto themselves, $\hat{g} : D \to D$ and $\hat{g} = g$ on ∂D. Since g is one-to-one, so is $\hat{g}$. For $x \neq 0$, it is obvious that $\hat{g}$ is continuous, so it remains to examine continuity at $x = 0$. Since g is continuous on compact set, there is a constant $M > 0$ such that $\|g(x)\| \leq M$ for each $x \in \partial D$. Let $\varepsilon > 0$ be an arbitrary real number and choose $0 < \delta < \frac{\varepsilon}{M}$. If $\|x\| < \delta$, then $\|\hat{g}(x) - \hat{g}(0)\| < M\delta < \varepsilon$ and hence, $\hat{g}$ is continuous at $x = 0$. □

Let us recall some properties about quotient topology:

Definition 5.15. A map $f : X \to Y$ is a quotient map if it is onto and if $U \subseteq Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open.

In particular, note that any onto function $f : X \to Y$ gives rise to a partition of X. Namely, say that $a \sim b$ if and only if $f(a) = f(b)$. This is an equivalence relation on X

and hence partitions it into the set of equivalence classes $X|_\sim = \{[x] \mid x \in X\}$, where $[x] = \{f^{-1}(y) \mid y \in Y\}$.

Theorem 5.16. *Let $q : X \to X|_\sim$ be a quotient map. If $g : X \to Z$ is a continuous map such that $a \sim b$ if and only if $g(a) = g(b)$ for all $a, b \in X$, then there exists a unique continuous map $f : X|_\sim \to Z$ such that $g = f \circ q$, i.e. the diagram*

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "g"'] & X|_\sim \arrow[d, "\exists! f"] \\ & Z \end{tikzcd}$$

commutes.

Proof. See [1], page 67. □

Lemma 5.17. *Quotient space of two topological disks glued along a pair of arcs in their boundaries is homeomorphic to a topological disk.*

Proof. (taken from [1])
Let D be a unit disk and D_1 and D_2 be arbitrary topological disks. Denote the arcs along which we glue by $\gamma_1 \subset D_1$ and $\gamma_2 \subset D_2$. Choose an arbitrary homeomorphism $h : \gamma_1 \to \gamma_2$ and determine the identification by relation $p \sim q$ if $h(p) = q$.
First define $f : \gamma_1 \cup \gamma_2 \to \{0\} \times [-1, 1]$. Say that f on γ_2 is an arbitrary homeomorphism onto $\{0\} \times [-1, 1]$ and for $p \in \gamma_1$, define $f(p) = f(h(p))$. Consider that $p \in \gamma_1$ and $q \in \gamma_2$ then $f(p) = f(q)$ if and only if $p \sim q$. Namely, if $f(p) = f(q)$, then $f(h(p)) = f(q)$. Since f is injective, we have $h(p) = q$ and hence $p \sim q$. Conversely, if $p \sim q$, then $h(p) = q$ for homeomorphism $h : \gamma_1 \to \gamma_2$ and $f(q) = f(h(p)) = f(p)$.
Now, we consider $\alpha = \partial D_1 \setminus \gamma_1$ and $\beta = \partial D_2 \setminus \gamma_2$, i.e. the complements of γ_i in ∂D_i having no endpoints. So, we can map α to $\{(x, y) \in \partial D \mid x > 0\}$ and β to $\{(x, y) \in \partial D \mid x < 0\}$. We obtained homeomorphisms $f_1 : \gamma_1 \cup \alpha \to \{(x, y) \in \partial D \mid x \geq 0\}$ and $f_2 : \gamma_2 \cup \beta \to \{(x, y) \in \partial D \mid x \leq 0\}$. By lemma 5.14 we are able to extend these maps with $\hat{f}_1 : D_1 \to \{(x, y) \in D \mid x \geq 0\}$ and $\hat{f}_2 : D_2 \to \{(x, y) \in D \mid x \leq 0\}$.
Finally, define the map $q : D_1 \cup D_2 \to (D_1 \cup D_2)|_\sim$ by

$$q(x) = \begin{cases} \hat{f}_1(x) & x \in D_1 \\ \hat{f}_2(x) & x \in D_2 \end{cases}$$

Obviously, q is onto. It is continuous, as $\hat{f}_1$ and $\hat{f}_2$ are continuous on closed sets D_1 and D_2, respectively. Hence, q is a quotient map. Consider now a continuous map $g : D_1 \cup D_2 \to D$ is a continuous function with $p \sim q$ if and only if $g(p) = g(q)$. By theorem 5.16, there exists a unique continuous function $f : (D_1 \cup D_2)|_\sim \to D$ such that $g = f \circ q$. Also, since $D_1 \cup D_2$ is compact, then also $(D_1 \cup D_2)|_\sim$ is compact. Notice also, that D is a Hausdorff space, so by lemma 2.26, we have that f is closed map and consequently a homeomorphism. Thus, the statement has been proven. □

Theorem 5.18. *Every compact surface S can be represented as a polygon with even number of vertices whose edges are identified in pairs.*

Proof. Since S is a compact surface, there exists its triangulation $\{(T_i, \varphi_i) \mid 1 \leq i \leq n\}$, T_i's and φ_i's being denoted exactly as in definition 5.7. Firstly, by proposition 5.8, we can use such reordering of the indices, that each T_i has an edge e_i with one of the triangles $T_1, T_2, \ldots, T_{i-1}$ for $2 \leq i \leq n$.
Furthermore, we may assume that triangles $T_i' \subset \mathbb{R}^2$ are disjoint. Namely, consider simplex s generated by points $p_0 = (0,0), p_1 = (1,0), p_2 = (0,1)$, i.e. and say $T_i' = \langle q_0^i q_1^i q_2^i \rangle$ with q_0^i, q_1^i, q_2^i being affinely independent points for each $1 \leq i \leq n$. Then functions $f_i : s \to T_i'$ defined by $f(\lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2) = \lambda_0 q_0^i + \lambda_1 q_1^i + \lambda_2 q_2^i$ with $\sum_{j=0}^{2} \lambda_j = 1$ and $0 \leq \lambda_j \leq 1$ are by proposition 3.2 affine bijections. Moreover, f_i and their inverses f_i^{-1} are continuous, as restrictions of continuous function on affine sets and hence f_i's are all homeomorphisms. Now denote $\hat{T}_i' = \langle t_0^i t_1^i t_2^i \rangle$, where $t_0^i = p_0 + (2(i-1), 0), t_1^i = p_1 + (2(i-1), 0)$ and $t_2^i = p_2 + (2(i-1), 0)$ for $1 \leq i \leq n$. Obviously, $\hat{T}_i' \cap \hat{T}_j' = \emptyset$ for $i \neq j$ and we replace the given triangulation (T_i, φ_i) with triangulation $(T_i, \hat{\varphi}_i)$ where $\hat{\varphi}_i : \hat{T}_i' \to T_i$ are homeomorphisms defined with $\hat{\varphi}_i = \varphi_i \circ f_i \circ h_i^{-1}$ and $h_i : s \to \hat{T}_i'$ are translation maps defined with $h_i(\lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2) = \lambda_0 t_0^i + \lambda_1 t_1^i + \lambda_2 t_2^i$ with $\sum_{j=0}^{2} \lambda_j = 1$ and $0 \leq \lambda_j \leq 1$.
So, by the above argument we assume that T_i' are disjoint and denote $T' = \cup_{i=1}^{n} T_i'$. T' is compact as a union of compact sets. Also, define $\varphi : T' \to S$ with $\varphi \restriction_{T_i} = \varphi_i$. Obviously, it is onto. It is also continuous, as each φ_i is continuous on $T_i', 1 \leq i \leq n$. By lemma 2.26, φ maps closed sets in T' to closed sets in S. This fact implies that S has quotient topology induced by φ.
The polygon we wish to obtain will be constructed inductively as a quotient space of T'. First, by proposition 2.27 we have that T_1' is a disk. Furthermore, there exists a triangle, denoted by T_2', which has an edge e_2' that is identified to an edge e_1' of T_1', since otherwise we would get a separation of a surface S. We will identify two points $x \in e_1'$ and $y \in e_2'$ if and only if $\varphi_1(x) = \varphi_2(y)$. However, T_2' is also a disk and since we glue disks along their arcs, the resulting quotient space $(T_1' \cup T_2')|_\sim$ is by lemma 5.17 homeomorphic to a disk. Now, by connectedness of surface S, we again find a triangle T_3' which has an edge e_3' to be glued on to an edge of either T_1' or T_2' other than e_1' and e_2', since otherwise we get a contradiction with the proposition 5.11. Continue this procedure until the end, when we get $D = (\cup_{i=1}^{n} T_i')|_\sim$. Note that D is homeomorphic

to a disk, as a space created by consecutively gluing finitely many disks along their arcs.

The map $\varphi : T' \to S$ induces an onto map $\psi : D \to S$ and S. Since D is compact as a quotient of compact space, and S is Hausdorff, ψ is closed. So, S has quotient topology determined by ψ, i.e. it is homeomorphic to D. Finally, we get the statement of the theorem by noticing that disk is by proposition 2.27 homeomorphic to closure of any bounded convex set. □

Now we review basic building blocks to form all compact surfaces:

Definition 5.19. Let S^1 be a unit circle in $\mathbb{R}^2$. Then, 2-dimensional sphere, denoted by S^2 is a space obtained by identifying points (x, y) and $(x, -y)$ for each $(x, y) \in S^1$.

Define $T = [0, 1] \times [0, 1]$. 2-dimensional torus is a space obtained first by identifying ordered pairs $(x, 0)$ and $(x, 1)$ for each $x \in [0, 1]$, and then also $(0, y)$ and $(1, y)$ for each $y \in [0, 1]$.

The real projective plane is a space obtained by identifying points (x, y) and $(-x, -y)$ for each $(x, y) \in S^1$.

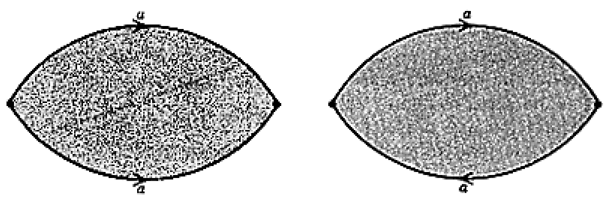

Figure 6: Sphere and projective plane schematically

Remark 5.20. From the definition for sphere and projective plane, we see that $(-1, 0)$ and $(0, 1)$ are fixed by identification. So we can illustrate it by 2-gons, as in figure 6. Since both S^1 and K are compact and connected, their quotient spaces are so as well. So sphere, projective plane and torus are compact surfaces. We emphasize those three compact surfaces, because they will be our basic building blocks in classification theorem.

Last sentence of remark 5.20 arises a question: How can one connect two of those surfaces?

Definition 5.21. Given disjoint compact surfaces S_1 and S_2. Choose disks $D_1 \subset S_1$ and $D_2 \subset S_2$ and denote $S_i' = S_i \setminus \mathrm{Int}(D_i), i = 1, 2$. Choose also a homeomorphism $h : \partial D_1 \to \partial D_2$. Connected sum of S_1 and S_2, denoted by $S_1 \# S_2$ is the quotient space obtained by identifying the points x and $h(x)$ for each $x \in \partial D_1$.

Remark 5.22. Intuitively speaking, we cut out the disk from each of the surfaces and then glue them together along the boundaries.

1. Let us see how to make the connected sum of two tori: Represent them first as a squares with opposite sides identified as in definition, then for convenience we cut the circular holes as in shaded regions and proceed as in Figure 7.

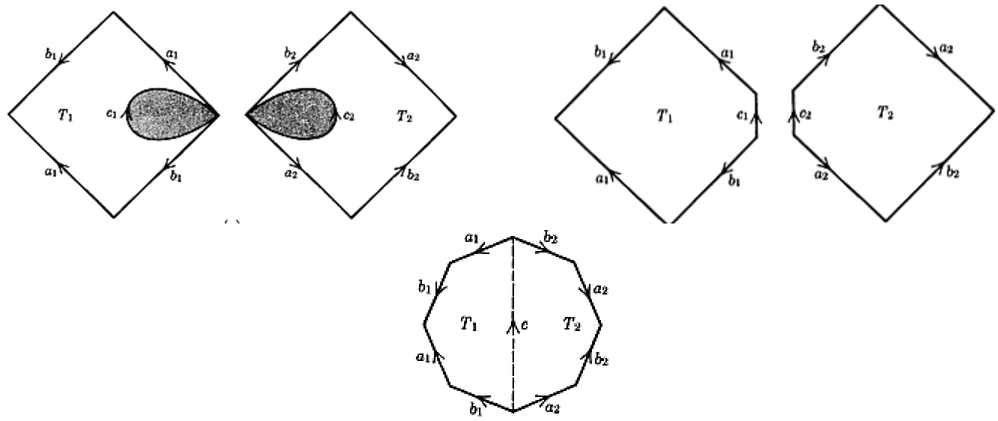

Figure 7: Connected sum of two tori

If we continue inductively, it can be seen that sum of n tori can be represented by $4n$-gon and with arrows set with the same pattern as above.

2. Suppose we want to make connected sum of two projective planes. We analogously cut out the circular holes, make necessary identifications to get a 4-gon as shown in figure 8.

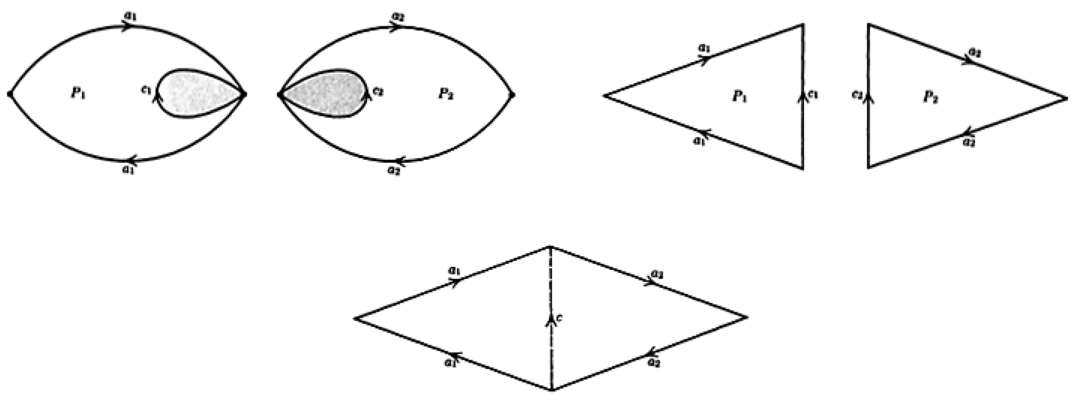

Figure 8: Glueing two projective planes

Again, if we continue inductively, it can be seen that sum of n projective spaces can be represented by $2n$-gons.

3. If we again look at 8 and switch the arrows in the way as spheres should be identified, we easily deduce that sum of spheres is again a sphere. Inductively, no matter how many spheres we sum, we again get only one.

Except for the geometric convenience of diagrams, they suggest us to write down the necessary identifications algebraically. Label edges of n-polygon with $a_i, 1 \leq i \leq n$. Observe a_1 in the diagram and traverse in chosen direction. If the next edge arrow agrees with the previous one, write down a_2^{+1}, or in simpler form a_2. Otherwise, write a_2^{-1}. Summary of this notation for the basic surfaces from above is:

- The sum of n-tori: $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$
- The sum of n-projective planes: $a_1a_1a_2a_2\dots a_na_n$
- The sphere: aa^{-1}

Now we introduce an application of this notation and concepts we introduced:

Definition 5.23. Given $K = [0,1] \times [0,1]$ be a compact subset of $\mathbb{R}^2$. The Klein bottle is a space obtained by identifying ordered pairs $(x, 0)$ and $(x, 1)$ for each $x \in [0, 1]$, and then also $(0, y)$ and $(1, y)$ for each $y \in [0, 1]$.

Remark 5.24. Since K is compact, its quotient space, Klein bottle, is a compact surface. Moreover, it is nonorientable, as it contains the Möbius strip.

Lemma 5.25. *Sum of two projective planes is equal to Klein bottle.*

Sketch of the proof. The figure 9 suggests a way to get a desired result in terms of the presented notation:

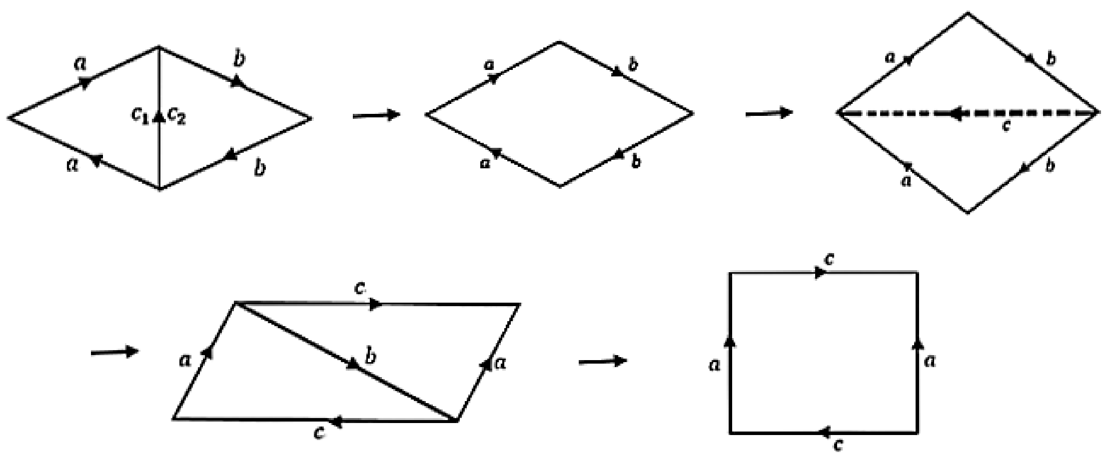

Figure 9: Sum of two projective planes

□

The following lemma is useful in the proof of the classification theorem:

Lemma 5.26. *Connected sum of torus and projective plane is equal to sum of three projective planes.*

Sketch of the proof. (taken from [6])
The figure 10 suggests a way to get the desired result in terms of the presented notation:

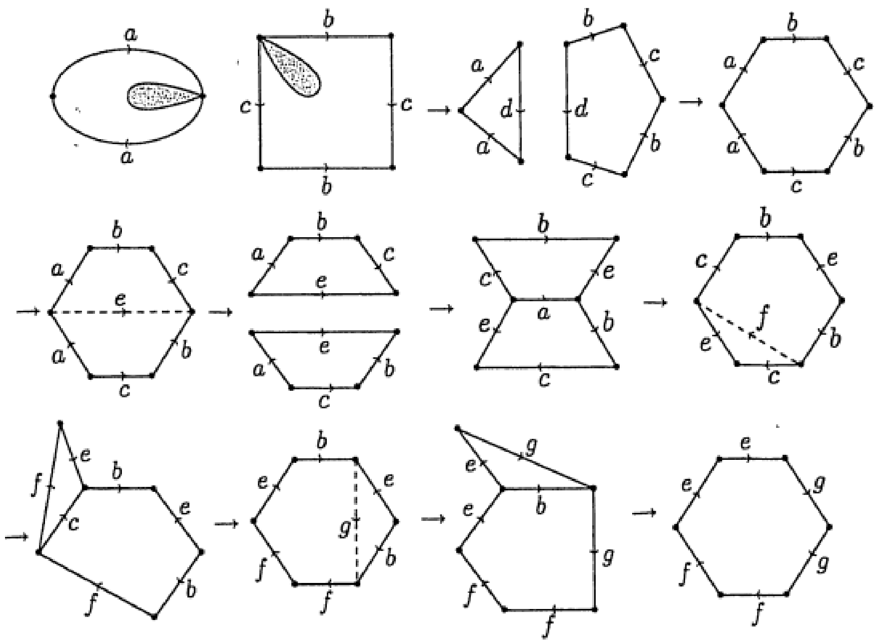

Figure 10: Sum of torus and projective plane

□

Finally, we can state the classification theorem:

Theorem 5.27. *Each compact surface is either homeomorphic to a sphere, or connected sum of tori, or connected sum of projective planes. Moreover, these are pairwise nonhomeomorphic.*

Proof. Proof of the theorem can be found in [9], Chapter 1. □

6 Conclusion

In this project paper we presented basic notions and results about simplicial complexes. Given an affinely independent set P in $\mathbb{R}^n$, there is a unique hyperplane H^k containing it. Geometric simplex generated by P is a set of points having nonnegative barycentric coordinate with respect to P. We also defined a notion of *subsimplex* called face. Two simplexes are properly joined if they are disjoint or meet in a common face. A geometric simplicial complex is a finite collection of properly joined simplexes.
Simplicial polytope $|K|$ of geometric simplicial complex is compact space. If $|K|$ is connected, then it is path connected. Introduced are maps between geometric simplicial complexes K and L, which have property that image of set of vertices spanning a simplex in K is a set of vertices spanning some simplex in L. It is shown that such map has a continuous extension between their simplicial polytopes. Barycentric subdivision of K is a simplicial complex K^1 having more simplexes than K, but with the same polytope, i.e. $|K| = |K^1|$.
Motivated to approach simplicial orientation abstractly, we defined abstract simplicial complexes, given by pair $\{V, S\}$, where V is a set of vertices in complex and S is a collection of all simplexes in it. The realization theorem gives us a way to transform an abstract complex into a geometric one.
Choose an arbitrary (abstract or geometric) simplex and consider the set of orderings of its vertices. Say that two orderings are equivalent if they differ by an even permutation. Orientation is a choice of equivalence class with respect to this relation. Simplicial complex K is orientable if simplexes of maximum dimension are consistently oriented, i.e. if they induce opposite orientations on their common faces.
Each compact connected 2-dimensional manifolds, also called compact surface is homeomorphic to a polytope of some 2-dimensional simplicial complex such that each 1-simplex is a face of at most two 2-simplexes. Finally, they admit a polygonal presentation, i.e. each compact surface is homeomorphic to a quotient space of some polygon having even number of sides. A reader interested particularly in this topic is referred to [9].

For further reading see [5] and [7]. For a geometric insight and motivation [1] and [6] were especially useful.

7 Povzetek naloge v slovenskem jeziku

Glavno vprašanje topologije je določiti, ali sta dva topološka prostora homeomorfna. Na to vprašanje je v splošnem težko odgovoriti, vendar če se omejimo na dovolj lepe prostore, lahko dobimo kakšen rezultat. Cilj naloge je bil predstaviti simplicialne komplekse, eno od koristnih orodij algebraične topologije. Pri tem smo predpostavljali, da bralec pozna osnove linearne algebre in splošne topologije.

V uvodnem poglavju definiramo hiperravnine, afine množice in afine preslikave. Pokažemo, da za vsako afino neodvisno množico $P = \{p_0, p_1, \dots, p_k\} \subset \mathbb{R}^n$ obstaja natanko ena hiperravnina H^k ki jo vsebuje, tako da vsako točko $x \in H^k$ enolično zapišemo kot $x = \sum_{i=0}^{k} \mu_i p_i$, kjer $\sum_{i=0}^{k} \mu_i = 1, \mu_i \in \mathbb{R}$. Pri tem številom μ_i rečemo baricentrične koordinate točke x glede na množico P. Dokažemo tudi, da je vsaka afina preslikava na hiperravnini zožitev afine preslikave, definirane na celotnem prostoru $\mathbb{R}^n$ in posledično je avtomatično zvezna. Na koncu poglavja si ogledamo osnovne pojme in rezulate o konveksnih množicah.

Naj bo $P = \{p_0, p_1, \dots, p_k\} \subset \mathbb{R}^n$ afino neodvisna množica. Geometrijski simpleks je množica točk, kjer so vse baricentrične koordinate glede na P pozitivne. Izkaže se, da je geometrijski simpleks konveksna ogrinjača množice P. Definiramo tudi njegova lica, simplekse generirane z nepraznimi podmnožicami množice P. Simpleksa, vložena v isti ambientni prostor $\mathbb{R}^N$ sta pravilno združena, če sta bodisi disjunktna, bodisi imata skupno lice. Simplicialni kompleks K je končna družina pravilno združenih simpleksov, ki so vsi vloženi v isti ambientni prostor $\mathbb{R}^N$, njegov simplicialni politop $|K|$ pa je unija vseh simpleksov, ki ležijo v K. Pokazali bomo, da je simplicialni politop $|K|$ vedno kompakten in če je povezan, potem je tudi s potmi povezan. Topološki prostor je triangulabilan, če je homeomorfen simplicialnem politopu nekega simplicialnega kompleksa. Ogledali smo si tudi preslikave med različnimi kompleksi K in L, ki slikajo množice oglišč, ki generirajo simplekse v K, v množice oglišč, ki generirajo simplekse v L. Vsaka taka preslikava ima zvezno razširitev $f^* : |K| \to |L|$, ki se imenuje simplicialna preslikava. Definirali smo tudi baricentrično subdivizijo K^1 podanega kompleksa K. Ta sestoji iz več simpleksov kot prvotni simplicialni kompleks, medtem pa je $|K| = |K^1|$. Nato še definiramo topološke in abstraktne simplicialne komplekse. Naj bo $\{V, S\}$ dan

abstraktni simplicialen kompleks, kjer je V množica oglišč, S pa množica abstraktnih simpleksov. Potem ga vedno lahko realiziramo v geometrijski simplicialni kompleks. Ideja je v tem, da si ogledamo bijekcijo f med V in poljubno afino neodvisno množico v $\mathbb{R}^n$, kjer je $n = \text{card}(V)$ in definiramo $K = \{co(f(\sigma)) \mid \sigma \in S\}$, njegovo realizacijo.
Da lahko definiramo orientacijo abstraktnega k-simpleksa, si bomo ogledali množico vseh urejanj vozlišč, ki ga generirajo in na njej vpeljali ekvivalenčno relacijo, tako da sta dva urejanja enaka, če se razlikujeta za sodo permutacijo. Orientacija je izbira ekvivalenčnega razreda glede na to relacijo. Definirali bomo tudi inducirano orientacijo njegovega $(k-1)$-dimenzionalnega lica. Simplicialni kompleks je orientabilen, če poljubna dva simpleksa maksimalne dimenzije inducirata nasprotno orientacijo na skupnemu licu. Kot zgled bomo pokazali, da je valj orientabilen in da Möbiusov trak ni orientabilen.
V zadnjem poglavju preučujemo kompaktne povezane 2-mnogoterosti, t.i. kompaktne ploskve. Najprej opazimo, da lahko definiramo simplicialne komplekse kot števne družine simpleksov, pri čemer zahtevamo da so lokalno končni, t.j. da ima vsako oglišče lice kvečjemu končno simpleksov. Privzamemo, da so kompaktne ploskve triangulabilne in pokažemo, da je vsaka taka triangulacija sestavljena iz končno trikotnikov. V nalogi je tudi pokazano, da je vsaka kompaktna ploskev homeomorfna kvocientnemu prostoru nekega poligona v ravnini, kjer paroma zlepimo njegove stranice. Na koncu formuliramo izrek o njihovi klasifikaciji: vsaka kompaktna ploskev je homeomorfna sferi, povezani vsoti torusov ali povezani vsoti projektivnih ravnin.

8 Bibliography

[1] M.A. Armstrong, *Basic Topology*, Springer-Verlag, New York, 1983. *(Cited on pages 19, 34, 35, and 41.)*

[2] J.B. Conway, *A course in Functional Analysis*, Springer-Verlag Inc., New York, 1994. *(Cited on pages 2 and 4.)*

[3] S. Deo, *Algebraic Topology, A Primer*, Hindustan Book Agency, 2006. *(Cited on pages 9 and 27.)*

[4] M.H. Freedman, The topology of four-dimensional manifolds. *Journal of Differential geometry, Vol. 17,* 3 (1982) 357–453. *(Cited on page 31.)*

[5] J.G. Hocking and G.S. Young, *Topology.* Addison-Wesley Publishing company, inc., 1961. *(Cited on pages 1 and 41.)*

[6] L.C. Kinsey, *Topology of Surfaces*, Springer-Verlag, New York, 1993. *(Cited on pages 1, 32, 40, and 41.)*

[7] J.M. Lee, *Introduction to Topological Manifolds*, Springer-Verlag, New York, 2000. *(Cited on pages 1 and 41.)*

[8] C. Manolescu, Pin (2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture. *Journal of the American Mathematical Society Vol. 29,* 2 (2016) 147–176. *(Cited on page 31.)*

[9] W.S. Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, 1977. *(Cited on pages 1, 40, and 41.)*

[10] E.E. Moise, *Geometric Topology in dimensions* 2 *and* 3, Springer-Verlag, New York, 1977. *(Cited on pages 31 and 33.)*

[11] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1970. *(Cited on page 1.)*