UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

Zaključna naloga
(Final project paper)

# Dinamični sistemi z uporabo v matematični biologiji

(Dynamical Systems with an Application in Mathematical Biology)

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**Koper, junij 2016**

# Ključna dokumentacijska informacija

Ime in PRIIMEK: Marko RAJKOVIĆ

Naslov zaključne naloge: Dinamični sistemi z uporabo v matematični biologiji

Kraj: Koper

Leto: 2016

Število listov: 63 Število slik: 9 Število referenc: 12

Mentor: izr. prof. dr. Jasna Prezelj

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Ključne besede: navadne diferencialne enačbe, obstoj in enoličnost, dinamični sistemi, asimptotsko obnašanje, Poincare–Bendixson teorija, tekmovalno izkljuevanje

Math. Subj. Class. (2010): 34A12, 37C10, 92D25

**Izvleček:**
V zaključni nalogi preučujemo lastnosti rešitev navadnih diferencialnih enačb: obstoj, enoličnost, odvisnost od začetnih pogojev ter asimptotsko obnašanje. Za ta namen uvedemo pojem dinamičnega sistema avtonomne enačbe kjer gledamo časovno evolucijo rešitve, ki se začne v določeni točki. Preučujemo stabilnost stacionarnih točk s pomočjo lokalne linearizacije in metode Ljapunova. Za ravninske dinamične sisteme predstavimo teorijo Poincaré–Bendixsona ki nam omogoča klasifikacijo asimptotskega obnašanja rešitev. Teoretične rezultate uporabimo za preučevanje sobivanja dveh uporabnikov na enem biotičnem viru.

# Key words documentation

Name and SURNAME: Marko RAJKOVIĆ

Title of final project paper: Dynamical Systems with an Application in Mathematical Biology

Place: Koper

Year: 2016

Number of pages: 63 Number of figures: 9 Number of references: 12

Mentor: Assoc. Prof. Jasna Prezelj, PhD

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Keywords: ordinary differential equations, existence and uniqueness, dynamical systems, asymptotical behaviour, Poincare–Bendixson theory, competitive exclusion

Math. Subj. Class. (2010): 34A12, 37C10, 92D25

**Abstract:** In the final project paper we study properties of solutions of ordinary differential equations: existence, uniqueness, dependence on the initial conditions and asymptotical behaviour. For these purposes, we introduce the notion of a dynamical system of autonomous equation where we consider the time evolution of a solution starting at a specific point. We study stability of fixed points by local linearization and via the Liapunov method. For planar dynamical systems we present the Poincaré–Bendixson theory, which classifies asymptotical behaviour of the solutions. We apply theoretical results to study coexistence of two consumers on a single biotic resource.

# Acknowledgement

I would like to thank to my mentor, professor Jasna Prezelj, and co-mentor, professor Barbara Boldin, for their help and useful advice during writing this paper. I also own great gratitude to all of my professors and stuff of UP FAMNIT for their support during my studies.
The biggest thanks goes to my family and friends for being my biggest support throughout my entire education.

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| | |
|---|---|
| *i.e.* | that is |
| *e.g.* | for example |
| *IVP* | initial value problem |
| *iff* | if and only if |

# 1 Introduction

Differential equations represent one of the most important tools in (applied) mathematics. They are interesting from both theoretical and applied point of view. An ordinary differential equation is defined as a functional relation of the form

$$F(t, x, x^{(1)}, \dots, x^{(k)}) = 0 \tag{1.1}$$

for the unknown function

$$x = (x_1, \dots, x_n) \in C^k(I, \mathbb{R}^n)$$

and its derivatives

$$x^{(j)}(t) = \left( \frac{d^j x_1(t)}{dt^j}, \dots, \frac{d^j x_n(t)}{dt^j} \right).$$

The highest derivative appearing in $F$ is called the **order** of the differential equation. If $F$ can be written in the form $F(x, x^{(1)}, \dots, x^{(k)}) = 0$, without explicit appearance of the **independent variable** $t$, we say that the differential equation is **autonomous**. The function $\phi(t)$ is said to be a solution of the equation (1.1) on an interval $J \subset I$ if $\phi \in C^k(J)$ and $F(t, \phi(t), \dots, \phi^{(k)}(t)) = 0,\ \ t \in J$.
Throughout this paper we work with differential equations in the explicit form, i.e. we suppose that the equation (1.1) can be solved in the highest derivative,

$$x^{(k)} = f(t, x, \dots, x^{(k-1)}). \tag{1.2}$$

Differential equations are typically studied by finding their solutions explicitly or by numerical approximation. However, one does not have to know the solution of the equation in order to know some of its properties: interval of existence and uniqueness, dependence on the initial condition and parameters, asymptotic behaviour. In this paper we mostly focus on these properties.
In Chapter 2 of this paper we introduce the notion of the initial value problem and study existence, uniqueness and dependence of solutions on the initial conditions and parameters.
In Chapter 3 we consider a special example of differential equations, linear differential equations. We give explicit solutions and study their asymptotic behaviour.
In Chapter 4 we introduce another notion of special interest in this paper, a dynamical system. With the help of the theory developed in Chapter 2 we study a special example

of a dynamical system, the flow of first order autonomous equations. We introduce the notion of a fixed point (steady state, equilibrium) as a point where $\dot{x} = 0$.
Motivated by the study of asymptotic behaviour of solutions of linear systems and with the help of the theory developed in Chapter 4, we study in Chapter 5 stability of fixed points and asymptotic behaviour of solutions of first order autonomous equations.
In Chapter 6 we study dynamical systems in two dimensions by classifying their possible asymptotic behaviour.
Finally, in Chapter 7 we give an application of the theory developed throughout the paper on an example from mathematical biology. We are interested in predator–prey problems, more precisely in competitive exclusion principle.

# 2 Initial value problems

The aim of this chapter is to prove basic existence and uniqueness results for ordinary differential equations. Of special interest is the initial value problem (IVP)

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{2.1}$$

We suppose that $f \in C(U, \mathbb{R}^n)$, $U$ an open set in $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$.
In addition to having a unique solution, IVP should be continuously dependent on initial conditions, i.e. small changes in the data should result in small changes of the solution.
If the IVP satisfies all the above conditions we say that it is **well-posed.**
For studying these properties we first formulate and prove fixed point theorems.

## 2.1 Fixed point theorems

For deriving the Banach fixed point theorem we first need some basic notions from real and functional analysis.

**Definition 2.1.** Let $(X, \|\cdot\|)$ be a normed vector space. A sequence $\{x_n\}$ **converges** to vector $x \in X$ if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. We denote this by $x_n \to x$.
A mapping $F : (X, \|\cdot\|) \to (Y, \|\cdot\|)$ is called **continuous** if $x_n \to x$ implies $F(x_n) \to F(x)$.
A sequence $\{x_n\}$ is called a **Cauchy sequence** if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) : (\forall m, n > N)\|x_m - x_n\| < \epsilon.$$

**Definition 2.2.** A normed space $(X, \|\cdot\|)$ is called **complete** if every Cauchy sequence in $X$ has a limit in $X$. A complete normed space is called a **Banach space**. If a Banach space $(X, \|\cdot\|)$ is also an algebra and $\|xy\| \leq \|x\|\|y\|$ for any $x, y \in X$ we call it a **Banach algebra**.

For our purposes we mostly use the following example of Banach spaces. Let $I$ be a closed interval and let $C(I)$ be the set of all real continuous functions on this interval. They form a vector space with operations defined pointwise. To get a normed space, we define a norm as: $\|x\| = \sup\limits_{t\in I} |x(t)| = \max\limits_{t\in I} |x(t)|$.

A sequence of functions $\{x_n\}$ converges to $x$ if and only if

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = 0.$$

Let $\{x_n\}$ be a Cauchy sequence in $C(I)$. For every $t \in I$ the sequence $\{x_n(t)\}$ is a Cauchy sequence in $\mathbb{R}$ and by completeness of $\mathbb{R}$ a convergent sequence having a limit $x(t)$. We have

$$|x_n(t) - x_m(t)| < \epsilon, \quad \forall n, m > N_\epsilon;\ t \in I$$

or in the limit

$$|x_n(t) - x(t)| \leq \epsilon, \quad \forall n > N_\epsilon;\ t \in I$$

which is the definition of uniform convergence. By the well known result from real analysis, the uniform limit of continuous functions is again continuous. Hence, every Cauchy sequence has a limit in $C(I)$ which gives us a Banach space.
To formulate the Banach's fixed point theorem, we need a special type of a mapping.

**Definition 2.3.** For a normed space $(X, \|\cdot\|)$ mapping $K : X \to X$ is said to be **Lipschitz continuous** if there exists a constant $L > 0$ such that

$$\|K(x) - K(y)\| \leq L\|x - y\|,\ \forall x, y \in X.$$

If $L < 1$ then we say that $K$ is a **contraction**.
We define the iteration of $K$ as: $K^n(x) = K(K^{n-1}(x)),\ K^0(x) = x$.

**Theorem 2.4.** *(Banach fixed point theorem or The Contraction principle) Let $C$ be a nonempty closed subset of a Banach space $X$ and let $K : C \to C$ be a contraction. Then $K$ has a unique fixed point $\bar{x}$. Moreover, for every $x \in C$ we have the estimate*

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta}\|K(x) - x\|.$$

*Proof.* We first prove uniqueness: let $x_0$ and $x_1$ be two different fixed points. Then by definition of a contraction we have: $\|x_0 - x_1\| = \|K(x_0) - K(x_1)\| \leq \theta\|x_0 - x_1\| < \|x_0 - x_1\|$, which is an obvious contradiction. Hence there can not be two different fixed points.
Take an arbitrary $x = x_0 \in X$ and define a sequence $x_n = K^n(x_0)$. We have

$$\|x_{n+1} - x_n\| \leq \theta\|x_n - x_{n-1}\| \leq \cdots \leq \theta^n\|x_1 - x_0\|$$

and by triangle inequality we estimate

$$\|x_n - x_m\| \leq \frac{\theta^m}{1-\theta}\|x_1 - x_0\| \tag{2.2}$$

which shows that $\{x_n\}$ is a Cauchy sequence in a Banach space. Therefore, the sequence has a limit $\bar{x}$. Furthermore:

$$0 = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|K(x_n) - x_n\| = \|K(\bar{x}) - \bar{x}\|$$

showing that $\bar{x}$ is a fixed point. By sending $n \to \infty$ in (2.2) we get the second claim of theorem. □

From the previous proof one can infer that $\theta^n$ can be replaced by any summable sequence $\{\theta_n\}$, giving the next theorem.

**Theorem 2.5.** *(Weissinger) Let $C$ be a nonempty closed subset of a Banach space $X$. Suppose that $K : C \to C$ satisfies: $\|K^n(x) - K^n(y)\| \leq \theta_n \|x - y\|$, $\forall x, y \in C, n \in \mathbb{N}$ with $\sum_{n=1}^{\infty} \theta_n < \infty$. Then $K$ has a unique fixed point $\bar{x}$ such that*

$$\|K^n(x) - \bar{x}\| \leq \left( \sum_{j=n}^{\infty} \theta_j \right) \|K(x) - x\|, \ \forall x \in C.$$

*Proof.* Suppose that we have two fixed points $x$ and $y$. Then we have $\|x - y\| = \|K^n(x) - K^n(y)\|$ or in the limit: $\|x - y\| \leq \lim_{n\to\infty} \theta_n \|x - y\| = 0$ since $\lim_{n\to\infty} \theta_n = 0$. To prove the existence of a solution, fix an arbitrary $x_0 \in C$ and generate a sequence $x_n = K^n(x_0)$. Using the above condition we can estimate

$$\|x_{n+1} - x_n\| = \|K^n(K(x_0)) - K^n(x_0)\| \leq \theta_n \|K(x_0) - x_0\|.$$

Furthermore, by triangle inequality we have

$$\|x_n - x_m\| \leq \left( \sum_{i=m}^{n-1} \theta_i \right) \|K(x_0) - x_0\| \tag{2.3}$$

and since sequence $\{\theta_n\}$ is summable we have that $\{x_n\}$ is a Cauchy sequence in a closed subset of Banach space and hence it is convergent with a limit $\bar{x} \in C$. Then we can write

$$0 = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|K(x_n) - x_n\| = \|K(\bar{x}) - \bar{x}\|$$

showing that $\bar{x}$ is a fixed point. By sending $n \to \infty$ in (2.3) we get the second claim of the theorem. □

## 2.2 The basic existence and uniqueness results

In this section we use results of the previous section to show existence and uniqueness results for the IVP in (2.1). First we extend the definition of Lipschitz continuous functions.

**Definition 2.6.** Let $U$ be an open set in $\mathbb{R}^{n+1}$. A function $f \in C(U, \mathbb{R}^n)$, is called locally Lipschitz continuous in the second argument, uniformly with respect to the first argument if for every compact set $V \subset U$ we have

$$L(V) = \sup_{(t,x)\neq(t,y)\in V} \frac{|f(t,x) - f(t,y)|}{|x-y|} < \infty$$

*Remark* 2.7. Since all the norms in $\mathbb{R}^n$ are equivalent, the Lipschitz condition does not depend on the choice of the norm. For convenience, we denote in the rest of this chapter the norm $|x| = |(x_1, \dots, x_n)| = \max_{i=1,\dots,n} |x_i|$ for $x \in \mathbb{R}^n$.

By integration, the IVP (2.1) is equivalent to $x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds$ since $f$ is continuous. The initial value $x_0$ can serve as an approximation of the solution for $t$ close to $t_0$. The next approximation is derived by putting $x_0(t)$ into the integral equation. In general we define

$$x_m(t) = K^m(x_0)(t); \quad K(x)(t) = x_0 + \int_{t_0}^t f(s, x(s))ds.$$

For convenience we take $t_0 = 0$ and $t \geq 0$. We want to have $K$ to be a contraction in a suitable Banach space $X$ and its closed subset $C$. We put $X = C([0,T], \mathbb{R}^n)$ with the usual norm, and $V = [0,T] \times \overline{B_\delta(x_0)}$ for suitable $T$. Since $V$ is compact we can define $M = \max_{(t,x)\in V} |f(t,x)|$. Taking $T_0 = \min\{T, \frac{\delta}{M}\}$ and $C = \overline{B_\delta(x_0)} = \{x \in X, \|x - x_0\| \leq \delta\}$ we get the desired spaces since

$$|K(x)(t) - x_0| \leq \int_0^t |f(s, x(s))|ds \leq tM \leq \delta, \quad \forall x \in C, \ t \leq T_0.$$

Hence, we have $K : C \to C$ and by taking $L$ as defined in (2.6) we get

$$|K(x)(t) - K(y)(t)| \leq \int_0^t |f(s, x(s)) - f(s, y(s))|ds \leq Lt \sup_{0\leq s\leq t} |x(s) - y(s)|, \ \forall x, y \in C.$$

In other words, $\|K(x) - K(y)\| \leq LT_0\|x - y\|$, $\forall x, y \in C$. If we take $T_1 < \min\{T_0, L^{-1}\}$ we have that $K$ is a contraction and we can apply the contraction principle.

**Theorem 2.8.** *Suppose $f \in C(U, \mathbb{R}^n)$ where $U$ is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$. If $f$ is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument then there exists a unique local solution $\bar{x}(t) \in C^1(I)$ of the IVP where $I$ is some interval containing $t_0$.*
*More precisely, if $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and $M$ denotes the maximum of $|f|$ on $V$ then the solution exists at least for $t \in [t_0, t_0 + T_0]$ and remains in $\overline{B_\delta(x_0)}$, where $T_0 < \min\{T, \delta/M, 1/L\}$ where $L$ is Lipschitz constant on $V$.*

*Remark* 2.9. Every $f \in C^1(U, \mathbb{R}^n)$ is locally Lipschitz since $\|f(x) - f(y)\| \leq L\|x - y\|$ where $L = \max_{\tau\in\Omega} \|J_f(\tau)\|$, $J_f$ is the Jacobian matrix of $f$ and $\Omega \subset U$ a convex set.

The previously described procedure of finding the solution of IVP is called **Picard iteration**. Although it is very useful in proving uniqueness and existence results, it is not useful in practice since it is often difficult to calculate integrals.

**Example 2.10.** Applying Picard iteration for IVP: $\dot{x} = ax;\ x(t_0) = x_0$ we get
$x_0(t) = x_0$
$x_1(t) = x_0 + \int_{t_0}^{t} ax_0(s)ds = x_0 + ax_0(t - t_0)$
$x_2(t) = x_0 + \int_{t_0}^{t} ax_1(s)ds = x_0 + ax_0(t - t_0) + a^2 x_0 \dfrac{(t - t_0)^2}{2}$.
By induction we conclude:

$$x_m(t) = \sum_{j=0}^{m} \frac{(t - t_0)^j}{j!} a^j x_0$$

which is exactly the $m'$th Taylor's polynomial of $x_0 e^{a(t-t_0)}$ and hence $x(t) = x_0 e^{a(t-t_0)}$.

Using the previous theorem we can prove

**Lemma 2.11.** *Suppose $f \in C^k(U, \mathbb{R}^n), k \geq 1$, where $U$ is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$. If $\bar{x}$ is a local solution of IVP (2.1) then $\bar{x} \in C^{k+1}(I)$ where $I$ is the interval of existence.*

*Proof.* For $k = 1$ we have $\bar{x} \in C^1$ by Theorem 2.8. Moreover, using $\dot{\bar{x}}(t) = f(t, x(t)) \in C^1$ we infer $\bar{x}(t) \in C^2$. The rest follows by induction. □

There is a classical method for numerical computation of solutions of differential equations, called Euler's method, which is based on approximation of the solution by piecewise linear functions.
Let IVP be of the form (2.1). If $\phi(t)$ is a solution, then by Taylor's theorem we have

$$\phi(t_0 + h) = x_0 + \dot{\phi}(t_0)h + o(h) = x_0 + f(t_0, x_0)h + o(h)$$

To approximate the solution we omit the error term and iterate the procedure:

$$x_h(t_{m+1}) = x_h(t_m) + f(t_m, x_h(t_m))h, \quad t_m = t_0 + mh$$

and use linear interpolation in between.
Euler's method can be used as a motivation for construction of solutions for an IVP. We need the following theorem.

**Theorem 2.12.** *(Arzela–Ascoli) Suppose that a sequence of functions $\{x_m(t)\} \in C(I, \mathbb{R}^n)$, $m \in \mathbb{N}$ on a compact interval $I$ is (uniformly)* ***equicontinuous****, i.e. for every $\epsilon > 0$ there is $\delta > 0$ (independent of $m$) such that $|x_m(t) - x_m(s)| < \epsilon$ for $|t - s| < \delta, m \in \mathbb{N}$. If the sequence $\{x_m\}$ is bounded then it has a uniformly convergent subsequence.*

**Theorem 2.13.** *(Peano's theorem) Let* $\dot{x} = f(x)$, $x(t_0) = x_0$ *be an autonomous IVP, where* $f \in C(\overline{U})$ *for an open, bounded subset* $U \subset \mathbb{R}^n$. *Then there exists a solution defined for time* $t_0 - \frac{D}{M\sqrt{n}} \leq t \leq t_0 + \frac{D}{M\sqrt{n}}$, *where* $D = \mathrm{dist}(x_0, \partial U)$ *and* $M = \|f\| = \max\limits_{i=1,\dots,n} |f_i|$ *on* $\overline{U}$.

*Proof.* Since $f$ is continuous on a compact set, it is uniformly continuous and hence for a fixed $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for $|x - y| < \delta$. For such $\delta$ we cut $U$ onto $\delta$-blocks. We start in the initial point and make an approximation by Euler's method

$$x(t) = x_0 + f(x_0)(t - t_0).$$

Let $t_1$ be the time when $x(t)$ hits the boundary of the $\delta$-block. We make a new approximation using that intersection as the initial condition. We repeat this procedure until we reach the boundary of $U$. The obtained piecewise linear curve is called an **Euler polygon**. If we travel with maximal velocity in the direction of the nearest boundary point then

$$|x(t) - x_0| \leq (t - t_0)\|f\|_2 \leq (t - t_0)M\sqrt{n} \leq \mathrm{dist}(x_0, \partial U) = D$$

and hence we get $t - t_0 \leq \frac{D}{M\sqrt{n}}$. We apply an analogous procedure for negative time direction.

For $t \in [t_i, t_{i+1}]$ we get

$$\begin{aligned} x(t) &= x_i + f(x_i)(t - t_i) = x_i + \int_{t_i}^{t} f(x_i)ds \\ x_i &= x_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} f(x_k)dt. \end{aligned}$$

We repeat this procedure on all parts of polygonal line and calculate

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))ds + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} (f(x_k) - f(x(s))ds + \int_{t_i}^{t} (f(x_i) - f(x(s)))ds.$$

The definition of the polygon gives $|f(x(s)) - f(x_i)| < \epsilon$ and hence

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))ds + (t - t_0)\theta, \ |\theta| < \epsilon.$$

Now we choose a decreasing sequence $\{\epsilon_n\}$ with limit 0 and consider the corresponding Euler polygons $\{x_n\}$ which satisfy

$$x_n = x_0 + \int_{t_i}^{t} f(x_n(s))ds + (t - t_0)\theta_n, \ |\theta_n| < \epsilon_n.$$

Since we are on a compact set $\overline{U}$ all this polygons are bounded by the same constant and they are equicontinuous i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that for every

$n \in \mathbb{N}$ we have $|x_n(t) - x_n(t')| < \epsilon$ for $|t - t'| < \delta$. Hence we can apply Arzela–Ascoli theorem to obtain a convergent subsequence with limit $x$. Since $|\theta_n| \to 0$ and $f$ is uniformly continuous, in the limit we have

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))ds$$

and hence $x(t)$ satisfies the IVP as we wanted to show. □

*Remark* 2.14. Continuity of $f$ is enough to guaranty existence of a solution, but not necessarily uniqueness. As an example of that, take $\dot{x} = |x|^{\frac{1}{2}}, x(0) = 0$ on $[0,1]$, where we have two solutions $x \equiv 0$ and $x = \frac{t^2}{4}$.
We can further use Weisinger's theorem to get less strict conditions for existence and uniqueness of solutions.

**Theorem 2.15.** *Suppose $f \in C(U, \mathbb{R}^n)$ where $U$ is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$ and $f$ is locally Lipschitz continuous in the second argument, uniformly with respect to the first argument. Let $\delta, T > 0$ such that $[t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$. Set*

$$M = \int_{t_0}^{t} \sup_{x \in B_\delta(x_0)} |f(s,x)|ds; \quad L(t) = \sup_{x \neq y \in B_\delta(x_0)} \frac{|f(t,x) - f(t,y)|}{|x-y|}$$

*and define $T_0 = \sup\{0 < t \leq T | M(t_0 + t) \leq \delta\}$. Suppose that $L_1(T_0) = \int_{t_0}^{t_0+T_0} L(t)dt$ is finite.*
*Then the unique solution of IVP is given by*

$$\bar{x} = \lim_{m \to \infty} K^m(x_0) \in C^1([t_0, t_0 + T_0], \overline{B_\delta(x_0)}).$$

*Proof.* Without loss of generality, we can take $t_0 = 0$. Set $X = C([0, T_0], \mathbb{R}^n)$ and $C = \overline{B_\delta(x_0)}$. Our choice of $T_0$ implies that $K : C \to C$ since we have

$$|K(x)(t) - x_0| \leq \int_0^t |f(s, x(s))|ds \leq M(t) \leq \delta, \ t \in [0, T_0].$$

In order to apply Weissinger theorem we have to show that

$$|K^m(x)(t) - K^m(y)(t)| \leq \frac{L_1(t)^m}{m!} \sup_{0 \leq s \leq t} |x(s) - y(s)|.$$

We prove this by induction. For $m = 1$ we have:

$$\begin{aligned}
|K(x)(t) - K(y)(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\
&\leq \int_0^t L(s)|x(s) - y(s)|ds \\
&\leq \sup_{0 \leq s \leq t} |x(s) - y(s)| \int_0^t L(s)ds
\end{aligned}$$

and we use the induction hypothesis and similar technique for the induction step:

$$\begin{aligned}
|K^{m+1}(x)(t) - K^{m+1}(y)(t)| &\leq \int_0^t |f(s, K^m(x)(s)) - f(s, K^m(y)(s))| ds \\
&\leq \int_0^t L(s)|K^m(x)(s) - K^m(y)(s)| ds \\
&\leq \int_0^t L(s) \frac{L_1(s)^m}{m!} \sup_{r \leq s} |x(r) - y(r)| \\
&\leq \sup_{r \leq t} |x(r) - y(r)| \int_0^t L_1'(s) \frac{L_1(s)^m}{m!} ds \\
&= \frac{L_1(t)^{m+1}}{(m+1)!} \sup_{0 \leq r \leq t} |x(r) - y(r)|.
\end{aligned}$$

We can apply Weissinger theorem since $\sum_{m=0}^{\infty} \frac{L_1(t)^m}{m!} = e^{L_1(t)} < \infty$ by definition of $L_1$ to get the desired result. □

For globally defined functions $f$ which are globally Lipschitz we can say even more.

**Corollary 2.16.** *Suppose* $[t_0, T] \times \mathbb{R}^n \subset U$ *and* $\int_{t_0}^T L(t)dt < \infty$, *where*

$$L(t) = \sup_{x \neq y \in \mathbb{R}^n} \frac{|f(t,x) - f(t,y)|}{|x - y|}.$$

*Then* $\bar{x}$ *is defined for all* $t \in [t_0, T]$. *If* $U = \mathbb{R}^{n+1}$, *with* $\int_{-T}^T L(t)dt < \infty$ *for all* $T > 0$, *then* $\bar{x}$ *is defined for all* $t \in \mathbb{R}$.

*Proof.* We can apply the previous theorem and since $B_\delta(x_0) = \mathbb{R}^n$ we have $\delta = \infty$ and existence of solution is guaranteed for all $[t_0, T]$. If furthermore $U = \mathbb{R}^{n+1}$, we can put $C = X = C([t_0, T], \mathbb{R}^n), T_0 = T$ for all $T > 0$ to get the global solution. □

## 2.3 Extensibility of solutions

The previous section showed that solutions of IVP might not exists for all $t \in \mathbb{R}$ even though the equation is defined everywhere. This section deals with the maximal interval where the solution of an IVP can be defined.
An extension of a solution can be accomplished by "glueing" two or more solutions. We will also consider the maximal interval where solutions coincide.
Suppose that solutions of IVP exist locally and are unique for any $(t_0, x_0)$ and let $\phi_1$ and $\phi_2$ be two solutions defined on open intervals $I_1$ and $I_2$, respectively. Then $I = I_1 \cap I_2 = (t_0, t_1)$ where $(t_0, t_1)$ is the maximal interval where solutions coincide.

To "glue" $\phi_1$ and $\phi_2$ we define:

$$\phi(t) = \begin{cases} \phi_1(t) & t \in I_1 \\ \phi_2(t) & t \in I_2. \end{cases}$$

By repeating this procedure we get the next theorem.

**Theorem 2.17.** *Suppose that IVP* (2.1) *has a unique local solution. Then there exists a unique solution defined on maximal interval* $I_{(t_0,x_0)} = (T_-(t_0,x_0), T_+(t_0,x_0))$.

The solution defined in the previous theorem is called the **maximal solution**. A solution defined for all $t \in \mathbb{R}$ is called a **global solution**.
The next lemma gives us the sufficient and necessary condition for a given solution to have an extension.

**Lemma 2.18.** *Let* $\phi(t)$ *be the unique solution of IVP defined on an interval* $(t_-, t_+)$, $t_+ < \infty$. *Then there exists an extension* $(t_-, t_+ + \epsilon)$ *for some* $\epsilon > 0$ *if and only if there exists a sequence* $\{t_m\} \in (t_-, t_+)$ *such that* $\lim_{m\to\infty}(t_m, \phi(t_m)) = (t_+, y) \in U$. *An analogous statement holds for* $(t_- - \epsilon, t_+)$.

*Proof.* If an extension exists, by continuity of $\phi$ the identity holds for any sequence $t_m \uparrow t$. Conversely, suppose that such a sequence exists. Since $U$ is open there is some $\delta > 0$ such that $V = [t_+ - \delta, t_+] \times \overline{B_\delta(U)} \subset U$ and $M = \max_{(t,x)\in V} |f(t,x)| < \infty$. Moreover, after maybe passing to a subsequence, we can assume that $t_m \in (t_+ - \delta, t_+)$, $\phi(t_m) \in B_\delta(U)$ and $t_m < t_{m+1}$. We prove that in this case we have $\lim_{t\uparrow t_+} \phi(t) = y$. Assume the contrary. Then we can find a sequence $\tau_m \uparrow t_+$ such that $|\phi(\tau_m) - y| \geq \gamma > 0$. Without loss of generality we can choose $\gamma < \delta$ and $\tau_m \geq t_m$. Moreover, by mean value theorem we can require $|\phi(\tau_m) - y| = \gamma$ and $|\phi(t) - y| < \delta$ for $t \in [t_m, \tau_m]$. But then

$$\begin{aligned} 0 \;&<\; \gamma = |\phi(\tau_m) - y| \leq |\phi(\tau_m) - \phi(t_m)| + |\phi(t_m) - y| \\ &\leq \int_{t_m}^{\tau_m} |f(s,\phi(s))| ds + |\phi(t_m) - y| \leq M|\tau_m - t_m| + |\phi(t_m) - y|, \end{aligned}$$

which is a contradiction since the right hand side converges to 0 as $m \to \infty$.
To make extension we glue $\phi$ and $\bar{\phi}$ defined as solution of IVP $x(t_+) = y$ on $(t_+ - \epsilon, t_+ + \epsilon)$. The new function is continuous by definition and differentiable by Peano's theorem since for fixed $(t_+, y) \in U$ there is a positive minimal time $\epsilon$ such that there is solution defined on $(t_+ + \epsilon, y)$. Thus we have defined a unique solution on $(t_-, t_+ + \epsilon)$. □

As corollary of this lemma we have a criterion that does not require a complete knowledge of solution.

**Corollary 2.19.** *Let $\phi(t)$ be solution of IVP defined on interval $(t_-, t_+)$. Suppose there is a compact set $[t_-, t_+] \times C \subset U$ such that $\phi(t_m) \in C$ for some sequence $\{t_m\} \in [t_0, t_+)$ converging to $t_+$. Then there exists an extension $(t_-, t_+ + \epsilon)$ for some $\epsilon > 0$.
In particular, if such an $C$ exists for every $t_+ > t_0$ then the solutions exist for all $t > t_0$. Analogous statement holds for $(t_- - \epsilon, t_+)$.*

*Proof.* Take $t_m \uparrow t_+$. By compactness $\{\phi(t_m)\}$ has a convergent subsequence and the claim follows from previous lemma. □

As the final goal of this section we can prove that there exists a global solution if $f$ has at most linear growth with respect to $x$. Before that we state a very useful inequality.

**Lemma 2.20.** *(Generalized Gronwall's inequality) Suppose $\alpha, \beta, \psi$ are real valued functions on some interval $[0, T]$ where $\beta$ is non-negative, $\beta, \psi$ are continuous and $\alpha$ is integrable on given interval. If $\psi(t)$ satisfies*

$$\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)ds, \; t \in [0, T],$$

*then*

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds, \; t \in [0, T]$$

*Proof.* Abbreviate $\phi(t) = \exp\left(-\int_0^t \beta(s)ds\right)$. Then one computes:

$$\frac{d}{dt}\left(\phi(t)\int_0^t \beta(s)\psi(s)ds\right) = \beta(t)\phi(t)\left(\psi(t) - \int_0^t \beta(s)\psi(s)ds\right) \leq \alpha(t)\beta(t)\phi(t).$$

Integrating this inequality with respect to $t$, dividing the result by $\phi(t)$ and finally adding $\alpha(t)$ on both sides proves the claim. □

We mostly use the following corollary of this lemma.

**Corollary 2.21.** *If $\psi(t) \leq \alpha + \int_0^t (\beta\psi(s) + \gamma)ds, t \in [0, T]$ for given constants $\alpha, \gamma \in \mathbb{R}$, $\beta \geq 0$ then*

$$\psi(t) \leq \alpha \exp(\beta t) + \frac{\gamma}{\beta}(\exp(\beta t) - 1), t \in [0, T]$$

*If $\beta = 0$ then $\psi(t) \leq \alpha + \gamma t$.*

*Proof.* Use the Generalized Gronwall's inequality for $\alpha, \beta$ and $\tilde{\psi}(t) = \psi(t) + \frac{\gamma}{\beta}$. □

We are now ready to prove the announced theorem.

**Theorem 2.22.** *Suppose $U = \mathbb{R} \times \mathbb{R}^n$ and for every $T > 0$ there are constants $M(T), L(T)$ such that: $|f(t, x)| \leq M(T) + L(T)|x|, (t, x) \in [-T, T] \times \mathbb{R}^n$. Then all solution are defined for all $t \in \mathbb{R}$.*

*Proof.* Without loss of generality $t_0 = 0$. By the above estimate:

$$|\phi(t)| \leq |x_0| + \int_0^t (M + L|\phi(s)|)ds, \quad t \in [0, T] \cap I.$$

By applying Corollary 2.21 we get

$$|\phi(t)| \leq |x_0|e^{LT} + \frac{M}{L}(e^{LT} - 1), \quad t \in [0, T] \cap I.$$

Thus, $\phi(t)$ lies in a compact ball and by Corollary 2.19 we can make extension of the interval of solution. Since the above holds for every $T > 0$ we conclude that solutions exist for all $t \in \mathbb{R}$. □

*Remark* 2.23. The above result is true if $|f(t, x)| \leq M(t) + L(t)|x|, \forall x \in \mathbb{R}^n$ where $M(t), L(t)$ are locally integrable.

## 2.4 Dependence on the initial conditions

The aim of this section is to give a criterion for a solution of the IVP to be continuously dependent on the initial conditions, i.e. the case when small changes in the data will result in small changes of the solution. Some of the proofs were taken from [7].

**Theorem 2.24.** *Let the function $f$ be continuous and Lipschitz with constant $L$. Then the solution of IVP is continuously dependent on the initial conditions. Moreover, by the results of previous sections we have a unique solution of IVP and hence IVP is well-posed.*

*Proof.* Without loos of generality, we can suppose $t_0 = 0$. Let $y$ be solution of IVP with $y(0) = x_0 + h$. Integrating the IVP, using the triangle inequality and the Lipschitz condition we get

$$\begin{aligned} |x(t) - y(t)| &\leq |h| + \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\ &\leq |h| + L \int_0^t |x(s) - y(s)|ds. \end{aligned}$$

Using the Gronwall's inequality for constant functions $\alpha$ and $\beta$ we get

$$|x(t) - y(t)| \leq |h|e^{L|t|}, \; t \in [-T, T].$$

□

We are now interested in dependence of solutions of IVP

$$\dot{x} = f(x, \mu), \quad x(0) = y \tag{2.4}$$

on initial conditions and a parameter $\mu \in \mathbb{R}^m$. We denote this solution as $u(t, y, \mu)$ and claim that it is as smooth as the function $f$.

**Theorem 2.25.** *Let $E \subset \mathbb{R}^n$ be an open set, $f \in C^k(E)$ and $x_0 \in E$. Then there exist $a > 0$ and $\delta > 0$ such that for every $y \in B(x_0, \delta)$ the IVP (2.4) has a unique solution $u(t, y) \in C^k(G)$ where $G = [-a, a] \times B(x_0, \delta)$. For fixed $y$ we have $u(t) \in C^{k+1}([-a, a])$.*

*Proof.* If $u$ solves (2.4) it also solves the integral equation $u(t, y) = y + \int_0^t f(u(s, y(s)))ds$. We use this to estimate

$$\begin{aligned} |u(t, y+h) - u(t, y)| &\leq |h| + \int_0^t |f(u(s, y+h)) - f(u(s, y))ds| \\ &\leq |h| + K \int_0^t |u(s, y+h) - u(s, y)|ds \end{aligned}$$

and by using Gronwall's lemma

$$|u(t, y+h) - u(t, y)| \leq |h|e^{Kt}, \ t \in [-a, a]. \tag{2.5}$$

Furthermore, from the above integral equation we see that if the partial derivative $\frac{\partial u}{\partial y}$ exists it satisfies the equation

$$\frac{\partial u(t, y)}{\partial y} = I + \int_0^t Df(u(s, y)) \frac{\partial u(s, y)}{\partial y} ds.$$

In other words, it is a solution of matrix IVP

$$\dot{\Phi} = Df(u)\Phi, \ \Phi(0, y) = I.$$

This IVP has a unique solution since $Df(u)$ is locally bounded and its norm can serve as the Lipschitz coefficient. We denote this solution by $V(t, x)$. Since we want to show that it is partial derivative, we calculate

$$\begin{aligned} |u(t, y+h) - u(t, y) - Vh| &\leq \int_0^t |f(u(s, y+h)) - f(u(s, y)) - Df(u(s, y))Vh|ds \\ &\leq \int_0^t |Df(u(s, y))(u(s, y+h) - u(s, y)) - Df(u(s, y))Vh|ds \\ &\quad + \int_0^t |R(u(s, y+h), u(s, y))|ds \\ &\leq \int_0^t |Df(u(s, y))||u(s, y+h) - u(s, y) - Vh|ds + \\ &\quad + \int_0^t |R(u(s, y+h), u(s, y))|ds, \end{aligned}$$

where $R$ is remainder of the Taylor expansion of $f$ up to the linear term. Thus for every $\epsilon > 0$ we can find a $\delta > 0$ such that if $|h| < \delta$ we have

$$|R(u(s, y+h), u(s, y))| \leq \epsilon |u(s, y+h) - u(s, y)|, \ s \in [-a, a]. \tag{2.6}$$

Let $|Df(u(s,y))| < M_1$. By using (2.5) and (2.6) we get

$$|u(t,y+h)-u(t,y)-Vh| \leq M_1 \int_0^t |u(s,y+h)-u(s,y)-Vh|ds+\epsilon a|h|e^{Ka},\ t \in [-a,a]$$

Denote $g(t) = |u(t,y+h)-u(t,y)-Vh|$. Then the above estimate rewrites into

$$g(t) \leq M_1 \int_0^t g(s)ds + a\epsilon|h|e^{Ka}.$$

We again use Gronwall's lemma and obtain $g(t) \leq \epsilon|h|ae^{Ka}e^{M_1 a},\ t \in [-a,a]$
We divide by $|h|$ to obtain that for $|h| < \delta$ we have

$$\frac{|u(t,y+h)-u(t,y)-Vh|}{|h|} \leq \epsilon a e^{Ka}e^{M_1 a}.$$

By sending $|h|$ to zero we get that $\dfrac{\partial u(t,y)}{\partial y} = V(t,h)$ and since $V \in C^{k-1}(G)$ the claim holds. □

We can similarly show the following theorem.

**Theorem 2.26.** *Let $U$ be an open set of $\mathbb{R}^{n+m}$, $f \in C^k(U)$ and $(x_0,\mu_0) \in U$. Then there exist $a > 0$ and $\delta > 0$ such that for every $y \in B(x_0,\delta)$ and $\mu \in B(\mu_0,\delta)$, the IVP (2.4) has a unique solution $u(t,y,\mu) \in C^k(G)$ defined on $G = [-a,a] \times B(x_0,\delta)$. For fixed $\mu$ and $y$ we have $u \in C^{k+1}([-a,a])$.*

# 3 Linear systems of equations

In this chapter we analyze IVPs of the form

$$\dot{x} = A(t)x + b(t), \quad x(0) = x_0$$

where $A : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n, x : \mathbb{R} \to \mathbb{R}^n, b : \mathbb{R} \to \mathbb{R}^n$. Such an IVP is called **linear**. If $b(t) \equiv 0$ it is called **homogeneous**, otherwise it is **non-homogeneous**. A special type of homogeneous equations is the one where $A$ is an $n \times n$ matrix of real (complex) numbers:

$$\dot{x} = Ax. \tag{3.1}$$

Applying Picard iteration similar as in (2.10) we get: $x(t) = \displaystyle\sum_{j=0}^{\infty} \frac{t^j}{j!} A^j x_0$ and hence

$$x(t) = \exp(tA)x_0, \tag{3.2}$$

where we define $\exp(A) = \sum_{j=0}^{\infty} \dfrac{A^j}{j!}$. In this case $A$ can be considered as a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$. The space of linear operators, in this case $n$ dimensional matrices, denoted by $B(\mathbb{C}^n)$, is a Banach algebra with the norm: $\|A\| = \sup_{\|x\|=1} |Ax|$. From here we see that the space of solutions of (3.1) is an $n$ dimensional vector space.

**Lemma 3.1.** *The sum* $\displaystyle\sum_{j=0}^{\infty} \frac{A^j}{j!}$ *converges for every* $A \in B(\mathbb{C}^n)$ *and the matrix exponential is well defined.*

*Proof.* Since $B(\mathbb{C}^n)$ is a Banach space it is enough to prove normal convergence: $\displaystyle\sum_{j=0}^{\infty} \frac{\|A^j\|}{j!}$. Since $B(\mathbb{C}^n)$ is a Banach algebra, we have: $\|A^j\| \leq \|A\|^j$ and convergence follows from convergence of the sum: $\displaystyle\sum_{j=0}^{\infty} \frac{\|A\|^j}{j!} = \exp(\|A\|)$. □

Full understanding of the matrix exponential comes from the Jordan canonical form. Before stating this well-known fact from linear algebra we need a definition.

**Definition 3.2.** A vector $x$ with $\|x\| = 1$ is called a **generalized eigenvector of rank m** of the matrix A corresponding to eigenvalue $\lambda$ if $(A - \lambda I)^m x = 0$ and $(A - \lambda I)^{m-1} x \neq 0$.

**Lemma 3.3.** *(Jordan canonical form) Let $A$ be a complex $n \times n$ matrix. Then there exists a matrix $U$ such that $A$ is transformed into a block diagonal matrix by*

$$U^{-1}AU = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$$

*where each block is of the form:*

$$J = \begin{pmatrix} \alpha & 1 & & & \\ & \alpha & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \alpha \end{pmatrix}$$

*and where $\alpha$ is some eigenvalue of $A$ and columns of $U$ are the corresponding generalized eigenvectors of $A$.*

We can calculate:

$$\exp(U^{-1}AU) = \sum_{j=0}^{\infty} \frac{(U^{-1}AU)^j}{j!} = \sum_{j=0}^{\infty} \frac{U^{-1}A^jU}{j!} = U^{-1}\exp(A)U$$

and thus $\exp(A) = U\exp(U^{-1}AU)U^{-1}$
Hence, the exponential of $A$ will be represented using generalized eigenvectors of $A$ and the exponential of its Jordan form.

## 3.1 Linear autonomous first-order systems

In this section we discuss properties of solutions of the system (3.1). Instead of considering directly the matrix $A$ we consider its Jordan form. We start by observing that

$$\exp(tU^{-1}AU) = \begin{pmatrix} \exp(tJ_1) & & & \\ & \exp(tJ_2) & & \\ & & \ddots & \\ & & & \exp(tJ_m) \end{pmatrix}$$

where for a block of dimension $n$ we have

$$\exp(tJ) = e^{\alpha t} \begin{pmatrix} 1 & t & \cdots & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}$$

If the corresponding generalized eigenvectors of the Jordan block for the eigenavalue $\alpha$ are $u_1, \dots u_{r(\alpha)}$ then every solution is a linear combination of terms of the type:

$$e^{\alpha t} \sum_{i=1}^{r(\alpha)} \sum_{j=1}^{i} u_j \frac{t^{i-j}}{(i-j)!} \tag{3.3}$$

If a real matrix $A$ has complex eigenvalue $\alpha$ with the corresponding generalized eigenvector $u$ then also $\bar{\alpha}$ is the eigenvalue with the corresponding generalized eigenvector $\bar{u}$. To form basis of the space of solutions we use $\omega = \mathrm{Re}(u)$ and $v = \mathrm{Im}(u)$ and get that components of every solution are linear combinations of the terms $u\, t^k e^{at} \cos bt$ and $u\, t^k e^{at} \sin bt$, where $a$ and $b$ are respectively real and imaginary part of some eigenvalue. If $a < 0$ the term converges to zero since $\exp(at)$ decays faster than any polynomial. If $a = 0$, $\exp(at)$ will remain bounded, but if $k > 0$ the solution diverges. However, in the case when the number of eigenvectors of rank 1 corresponds to algebraic multiplicity, there are no polynomial terms. In summary:

**Theorem 3.4.** *A solution of the linear system (3.1) converges to 0 as $t \to \infty$ if and only if $x_0$ lies in a subspace spanned by the generalized eigenvectors of the matrix $A$ corresponding to eigenvalues with a negative real part.*
*A solution remains bounded as $t \to \infty$ if and only if $x_0$ lies in the subspace spanned by the generalized eigenvectors of the matrix $A$ corresponding to eigenvalues with a negative real part and the eigenspaces of generalized eigenvectors of rank one corresponding to eigenvalues with vanishing real part.*

*Remark* 3.5. To get behavior as $t \to -\infty$ we just switch *negative* and *positive*.

We introduce the terminology for the above mentioned behavior of solutions.

**Definition 3.6.** A linear system is called **stable** if all solutions remain bounded as $t \to \infty$ and **asymptotically stable** if all solutions converge to 0 as $t \to \infty$.
We are actually discussing stability of the origin as a fixed point of the system (3.1). We are generalizing this in Chapter 5.

With this terminology we have

**Corollary 3.7.** *The linear system (3.1) is stable if and only if all eigenvalues of $A$ have a non-positive real part and for those with vanishing real part the corresponding algebraic and geometric multiplicity are equal.*
*The linear system (3.1) is asymptotically stable if and only if all eigenvalues $\alpha_j$ of $A$ satisfy* $\mathrm{Re}(\alpha_j) < 0$.

Generalized eigenvectors are a basis for $\mathbb{R}^n$. We decompose $\mathbb{R}^n$ in the following way:

$$E^{\pm}(e^A) = \mathrm{Lin}(u_i, v_i | u_i + iv_i \in Ker(A - \lambda I)^n \neq 0, \pm Re(\lambda) < 0)$$

$$E^0(e^A) = \mathrm{Lin}(u_i, v_i | u_i + iv_i \in Ker(A - \lambda I)^n \neq 0, Re(\lambda) = 0)$$

The spaces $E^+$, $E^-$ and $E^0$ are respectively called the **stable**, the **unstable** and the **central space**. As we already said $\mathbb{R}^n = E^+(e^A) \oplus E^-(e^A) \oplus E^0(e^A)$. If the initial value is in $E^+(e^A)$ (respectively $E^-(e^A)$) only the terms of type (3.3) for $\mathrm{Re}(\alpha) < 0$ (respectively $\mathrm{Re}(\alpha) > 0$) determine solution and we have that $x(t) \to 0$ as $t \to \infty$ (respectively $-\infty$).

Linear autonomous equations of the order $n$ (linear equations with constant coefficients) can be rewritten into linear autonomous first-order systems. The corresponding system for equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1\dot{x} + c_0 x = 0 \tag{3.4}$$

is given as

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{n-1} \end{pmatrix}, \quad x = \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n-1)} \end{pmatrix}.$$

Properties of the solutions of this system depend on eigenvalues of the matrix $A$.

**Theorem 3.8.** *Let $\alpha_j$, $1 \leq j \leq m$, be the zeros of the polynomial*
$P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$, *associated with the equation* (3.4), *where $a_j$ are their corresponding multiplicities. Then the functions:*

$$x_{j,k}(t) = t^k \exp(\alpha_j t); \quad 0 \leq k \leq a_j, \ 1 \leq j \leq m$$

*are $n$ linearly independent solutions of the equation* (3.4).

*Proof.* Let us look at the solution of the corresponding first-order system. By construction, the first component of every solution of the system will solve our $n'$th order equation. By collecting functions from each Jordan block this first component must be a linear combination of the functions $x_{j,k}(t)$. So the solution space is spanned by these functions. Since this space is $n$ dimensional, all functions must be present. In particular, these functions must be linearly independent. □

## 3.2 General linear first-order systems

In this section we return to the analysis of the system

$$\dot{x}(t) = A(t)x(t) + b(t); \quad A \in C(I, \mathbb{R}^n \times \mathbb{R}^n),\ b \in C(I, \mathbb{R}^n).$$

Regarding the existence of solutions of this system we observe the following:

**Theorem 3.9.** *The above defined system has a unique solution satisfying the initial condition $x(t_0) = x_0$. This solution is defined for all $t \in I$.*

*Proof.* This is a direct consequence of Theorem 2.22 and Corollary 2.16 since we can take $L(T) = \max_{t\in[t_0,T]} \|A(t)\|$ for every $T \in I$ to satisfy the Lipschitz condition. □

We will first consider a general linear homogeneous first-order system. These are systems of the form

$$\dot{x}(t) = A(t)x(t), \quad A \in C(I, \mathbb{R}^n \times \mathbb{R}^n). \tag{3.5}$$

We start by observing that linear combinations of solutions are again solutions. Hence the set of all solutions forms a vector space. This is often referred to as the **superposition principle**. In particular, solution for the initial condition $x(t_0) = x_0$ is given by:

$$\phi(t, t_0, x_0) = \sum_{j=1}^{n} \phi(t, t_0, \delta_j)x_{0,j},$$

where $\delta_j$ is the $j'$th canonical basic vector, $x_{0,j}$ the $j'$th component of $x_0$ and $\phi(t, t_0, \delta_j)$ is the value $x(t)$ for $x(t_0) = \delta_j$. Using matrix notation we can write

$$\phi(t, t_0, x_0) = \Pi(t, t_0)x_0$$

where $\Pi(t, t_0) = (\phi(t, t_0, \delta_1), \phi(t, t_0, \delta_2), \ldots, \phi(t, t_0, \delta_n))$. The matrix $\Pi(t, t_0)$ is called a **principal matrix solution**.

**Theorem 3.10.** *Solutions of the system* (3.5) *form an $n$ dimensional vector space. Moreover, there exists a matrix-valued solution $\Pi(t, t_0)$ such that the solution satisfying the initial condition $x(t_0) = x_0$ is given by $\Pi(t, t_0)x_0$.*

Since all columns of $\Pi(t, t_0)$ solve (3.5) we conclude:

$$\dot{\Pi}(t, t_0) = A(t)\Pi(t, t_0);\ \Pi(t_0, t_0) = I$$

In general, $\dot{X}(t) = A(t)X(t)$ iff all columns of $X$ solve the equation. Taking $n$ solutions as columns of a matrix we get the **Wronski matrix**: $U(t) = (\phi_1(t), \ldots, \phi_n(t))$. The determinant of $U(t)$ is called the **Wronski determinant**

$$W(U(t)) = \det(\phi_1(t), \ldots, \phi_n(t)).$$

If $\det U(t) \neq 0$ then $U$ is called a **fundamental matrix solution**, i.e its columns are linearly independent solutions. Every two fundamental matrix solutions $U(t), V(t)$ are connected by $V(t)V(t_0) = U(t)U(t_0)$ since a solution is uniquely determined by the initial conditions.

The next theorem shows that it is enough to check $\det U(t) \neq 0$ for one $t \in I$, i.e. solutions are linearly independent everywhere if they are independent at some point.

**Theorem 3.11.** *(Liouville's formula)*

$$W(U(t)) = W(U(t_0)) \exp\left(\int_{t_0}^{t} \mathrm{Tr}(A(s))ds\right).$$

*Proof.* To prove this result we need the following lemma

**Lemma 3.12.** *(Jacobi's formula) For a differentiable map $A : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$ we have*

$$\frac{d\det(A)}{dt} = \mathrm{Tr}(\mathrm{Adj}(A)\frac{dA}{dt}).$$

*Proof.* We start by observing that $\sum_i \sum_j A_{ij}B_{ij} = \mathrm{Tr}(A^T B)$ for any two same dimensional square matrices.

By definition $\det(A) = \sum_{j=1}^{n} A_{ij}(\mathrm{Adj}A^T)_{ij}$ for a fixed row $i$. If we consider the determinant as a function of entries: $\det(A) = F(A_{11}, \ldots, A_{nn})$, it will be multilinear function of $n^2$ independent variables. By chain rule we get

$$d\det(A) = \sum_i \sum_j \frac{\partial F}{\partial A_{ij}} dA_{ij}. \tag{3.6}$$

Furthermore, by the definition of determinant

$$\frac{\partial F}{\partial A_{ij}} = \sum_{k=1}^{n} \frac{\partial A_{ik}(\mathrm{Adj}A)^T_{ik}}{\partial A_{ij}}$$

where we calculated the determinant by expansion with respect to $i'$th row. By the product rule

$$\frac{\partial F}{\partial A_{ij}} = \sum_k \frac{\partial A_{ik}}{\partial A_{ij}}(\mathrm{Adj}A)^T_{ik} + \sum_k A_{ik}\frac{\partial(\mathrm{Adj}A)^T_{ik}}{\partial A_{ij}}.$$

Since in calculating $(\mathrm{Adj}A)^T_{ij}$ we do not use $i$'th row and $j$'th column of $A$ we have: $\dfrac{\partial(\mathrm{Adj}(A)^T)_{ik}}{\partial A_{ij}} = 0$. Since variables are independent we have $\dfrac{\partial A_{ik}}{\partial A_{ij}} = \delta_{jk}$ where $\delta_{jk}$ is

Kronecker's delta. Hence, $\dfrac{\partial \det(A)}{\partial A_{ij}} = (\mathrm{Adj}A)^T_{ij}$ and by inserting in (3.6):

$$d\det(A) = \sum_i \sum_j (\mathrm{Adj}A)^T_{ij} dA_{ij} = \mathrm{Tr}(\mathrm{Adj}(A)dA)$$

where we used the identity from the beginning of the proof. □

For proving Liouville's formula we will use Jacobi's formula:

$$\begin{aligned} \frac{d\det(U)}{dt} &= \mathrm{Tr}(\mathrm{Adj}(U)\frac{dU}{dt}) \\ &= \mathrm{Tr}(\mathrm{Adj}(U)AU) \\ &= \mathrm{Tr}(U\,\mathrm{Adj}(U)A) \\ &= \det(U)\,\mathrm{Tr}(A). \end{aligned}$$

Solving this equation gives us the desired identity. □

We return to the general system

$$\dot{x}(t) = A(t)x(t) + b(t); \quad A \in C(I, \mathbb{R}^n \times \mathbb{R}^n),\ b \in C(I, \mathbb{R}^n) \tag{3.7}$$

Since the difference of two solutions of the non-homogeneous system solves the corresponding homogeneous system we conclude that solutions of non-homogeneous systems form $n$ dimensional affine space over the solutions of the corresponding homogeneous linear equation. It therefore suffices to find one particular solution. Put:

$$x(t) = \Pi(t, t_0)c(t); \quad c(t_0) = x(t_0) = x_0.$$

This method is known as **variation of constants**. By inserting this into our equation, we calculate $\dot{x}(t) = A(t)x(t) + \Pi(t, t_0)\dot{c}(t)$ giving: $\dot{c}(t) = \Pi(t, t_0)^{-1}b(t)$. Solving this equation and plugging into the initial yields:

**Theorem 3.13.** *The solution of system $\dot{x}(t) = A(t)x(t) + b(t)$ corresponding to the initial condition $x(t_0) = x_0$ is given by*

$$x(t) = \Pi(t, t_0)x_0 + \int_{t_0}^{t} \Pi(s, t)^{-1} b(s) ds,$$

*where $\Pi(t, t_0)$ is the principal matrix solution of the corresponding homogeneous system.*

The above theory can be used in solving linear equations of order $n$,

$$x^{(n)} + q_{n-1}(t)x^{(n-1)} + \cdots + q_1(t)\dot{x} + q_0(t)x = 0 \tag{3.8}$$

where $q_i(t)$ are continuous functions. A solution is uniquely determined by initial conditions: $x^{(i)}(t_0) = x_i$ and as in case of constant coefficients we rewrite into a linear system:

$$A(t) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -q_0(t) & -q_1(t) & \cdots & \cdots & -q_{n-1}(t) \end{pmatrix}$$

and the principal matrix solution is given by

$$\Pi(t,t_0) = \begin{pmatrix} \phi_1(t,t_0) & \cdots & \phi_n(t,t_0) \\ \dot{\phi}_1(t,t_0) & \cdots & \dot{\phi}_n(t,t_0) \\ \vdots & \vdots & \vdots \\ \phi_1^{(n-1)}(t,t_0) & \cdots & \phi_n^{(n-1)}(t,t_0) \end{pmatrix}$$

where $\phi_j(t,t_0)$ is solution corresponding to the initial condition

$$(x(t_0), \dot{x}(t_0)\ldots, x^{(n-1)}(t_0)) = \delta_j.$$

As a consequence of Theorem 3.13 we have

**Theorem 3.14.** *The solution of a nonhomogeneous equation*

$$x^{(n)} + q_{n-1}(t)x^{(n-1)} + \cdots + q_1(t)\dot{x} + q_0(t)x = b(t)$$

*satisfying initial condition* $x(t_0) = x_0, \dot{x}(t_0) = x_1, \ldots, x^{(n-1)}(t_0) = x_{n-1}$ *is given by*

$$x(t) = x_0\phi_1(t,t_0) + \cdots + x_{n-1}\phi_n(t,t_0) + \int_{t_0}^{t} \phi_n(t,s)b(s)ds,$$

*where* $\phi_j(t,t_0),\ 1 \le j \le n$ *are the solutions corresponding to the initial conditions* $(\phi_j(t_0,t_0), \ldots, \phi(t_0,t_0)^{(n-1)}) = \delta_j$

Having $n$ solutions $f_1, \ldots, f_n$ of (3.8) we define the Wronskian as

$$W(f_1,\ldots,f_n)(t) = \det \begin{pmatrix} f_1 & \cdots & f_n \\ \dot{f}_1 & \cdots & \dot{f}_n \\ \vdots & \cdots & \vdots \\ f_1^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{pmatrix}$$

and Louiville's formula is $W(\phi_1,\ldots,\phi_n)(t) = W(\phi_1,\ldots,\phi_n)(t_0)\exp(-\int_{t_0}^{t} -q_{n-1}(s)ds)$.

# 4 Dynamical systems

In this chapter we develop the tools for a different approach to study the behaviour of solutions of differential equations. We are again interested in the dependence of solutions on initial conditions. We also study the stability properties of the system. In order to achieve all this we introduce the notion of a dynamical system.

**Definition 4.1.** A **dynamical system** is a triple $(T, X, \Phi)$ where $T$ is a time set, $X$ is the state space and $\Phi = \{\phi^t\}_{t\in T}$ is the family of operators

$$\begin{aligned} \Phi \quad : \quad & T \times X \to X \\ & (t, x) \mapsto \phi^t(x) \end{aligned}$$

satisfying

$$\begin{aligned} \phi^t(\phi^s(x)) &= \phi^{t+s}(x) \\ \phi^0(x) &= x. \end{aligned}$$

If $T$ is a group we say that the system is **invertible**. Dynamical systems with $T = \mathbb{N}_0$ or $T = \mathbb{Z}$ are called **discrete dynamical systems**. The systems for $T = \mathbb{R}^+$ or $T = \mathbb{R}$ are called **continuous dynamical systems**.

**Example 4.2.** A typical example of a discrete dynamical system is an iterated map. Let $f$ be a map of a set $I$ into itself and consider

$$\phi^n(f) = f^n = f(f^{n-1}),\ T = \mathbb{N}_0$$

An example of a continuous dynamical system is a homogeneous linear differential equation $\dot{x} = Ax$. We have $\phi^t := e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ i.e.

$$\phi^t(x) = e^{At}x,\ t \in \mathbb{R}.$$

## 4.1 The flow of an autonomous equation

In this section we have a closer look at the solutions of an autonomous system

$$\dot{x} = f(x),\ x(0) = x_0, \tag{4.1}$$

where $f \in C^k(M, \mathbb{R}^n), k \geq 1$ and $M$ is an open subset of $\mathbb{R}^n$.
Such a system can be regarded as a vector field on $\mathbb{R}^n$. Solutions are curves in $M \subset \mathbb{R}^n$ which are tangent to this vector field at every point. Solutions are called *integral curves* or *trajectories.*
We say that $\phi$ is an integral curve at $x_0$ if it satisfies $\phi(0) = x_0$. By Theorem 2.17 there is a (unique) **maximal integral curve** $\phi_x$ at every point $x$ defined on a maximal interval $I_x = (T_-(x), T_+(x))$. We call $T_+$ and $T_-$ the positive and the negative **lifetime** of $x$, respectively. We say that $x$ is $\sigma$ **complete** ($\sigma \in \{-,+\}$) if $T_\sigma = \sigma\infty$. If it is both $\pm$ complete it is called complete.
By introducing the set

$$W = \bigcup_{x \in M} I_x \times \{x\} \subset \mathbb{R} \times M \tag{4.2}$$

we define the flow of our differential equation to be the map

$$\Phi : W \to M;\ (t, x) \mapsto \phi(t, x), \tag{4.3}$$

where $\phi(t, x)$ is the value at time $t$ on the maximal integral curve at $x$. We will also use the notation $\Phi_t(x) = \Phi(t, x)$.
If $\phi(.)$ is the maximal integral curve at $x$ then $\phi(. + s)$ is the maximal integral curve at $y = \phi(s)$ and $I_x = s + I_y = \{t | t = s + t_1, t_1 \in I_y\}$. Hence, we conclude that for $x \in M$ and $s \in I_x$ we have

$$\Phi(s + t, x) = \Phi(t, \Phi(s, x)),\ \forall t \in I_x - s. \tag{4.4}$$

In conclusion, we have the following theorem.

**Theorem 4.3.** *Suppose that we have an IVP of the form* (4.1). *For all $x \in M$ there exists an interval $I_x \subset \mathbb{R}$ containing 0 and a corresponding unique maximal integral curve $\Phi(t, x) \in C^k(I_x, M)$ at $x$. Moreover, the set $W$ defined in* (4.2) *is open and $\Phi \in C^k(W, M)$ satisfies $\Phi(0, x) = x$ and*

$$\Phi(s + t, x) = \Phi(t, \Phi(s, x)),\ x \in M,\ s, t + s \in I_x.$$

*Proof.* Regarding previous observations, we are left to show that $\Phi \in C^k(W, M)$. Fix a point $(x_0, t_0)$ and set $\gamma = \Phi_{x_0}([0, t_0])$. By Lemma 2.11, since $f \in C^k(M, \mathbb{R}^n)$, there is a neighborhood $(-\epsilon(x), \epsilon(x)) \times U(x)$ of $(0, x)$ for every point $x \in \gamma$ such that $\Phi \in C^k$ on this neighborhood. Since $\gamma$ is compact we can cover it with finitely many of the neighborhoods $U(x)$ and by choosing minimal $\epsilon$ corresponding to that cover we get that $\Phi$ is defined on $(-\epsilon, \epsilon) \times U_0$ where $U_0$ is open neighborhood of $\gamma$. Next we pick $m \in \mathbb{N}$ satisfying $\frac{t_0}{m} < \epsilon$ and define $K : U_0 \to M, K(x) = \Phi(\frac{t_0}{m}, x)$. Observe that $K \in C^k(U_0, M)$ since $\Phi$ is locally $C^k$. By iterating $K$ we get that $K^j \in C^k(U_j, M), 0 \leq$

$j \leq m, U_j = K^{-j}(U_0) \subset U_0$. None of the sets $U_j$ is empty since $x_0 = K^{-j}(\Phi(\frac{jt_0}{m}), x) \in \gamma \subset U_j$. In summary,

$$\Phi(t, x) = \Phi(t - t_0, \Phi(t_0, x)) = \Phi(t - t_0, K^m(x)) \in C^k((t_0 - \epsilon, t_0 + \epsilon) \times U_m, M)$$

Since $(t_0, x_0)$ was arbitrary we get that $\Phi \in C^k(W, M)$. □

Furthermore, by taking $s = -t$ and $t = -s$ in (4.4) we get that $\Phi_t(.)$ is a local diffeomorphism with inverse $\Phi_{-t}(.)$. Therefore, $\Phi$ is a local one-parameter group of diffeomorphisms.
If the vector field $f$ is complete then $\Phi(s, \cdot) : M \to M$ is an automorphism of $M$.
The definition of the flow of an autonomous equation allows us to define generalization of Liouville's formula from Theorem 3.11.

**Theorem 4.4.** *Let $\dot{x} = f(x)$ be a dynamical system on $\mathbb{R}^n$ with the corresponding flow $\Phi(t, x)$. Let $U$ be a bounded, open subset of $\mathbb{R}^n$. Denote by $U(t) = \Phi(t, U) = \{\Phi(t, x), x \in U, t \in I_x\}$ the flow on the set $U$ and its volume by $V(t) = \int_{U(t)} dx$. Then*

$$\dot{V}(t) = \int_{U(t)} \operatorname{div}(f(x))dx. \tag{4.5}$$

*Proof.* By the change of variable formula we have

$$V(t) = \int_{U(t)} dx = \int_U \det(D_x\Phi(t, x))dx \tag{4.6}$$

where $D_x\Phi(t, x)$ is the Jacobian matrix of function $\Phi(t, x)$. Since $\Phi(t, x)$ is a solution of $\dot{x} = f(x)$ we have $f(\Phi(t, x)) = \dot{\Phi}(x, t)$ and after differentiation of both sides with respect to $x$ we have that $\Pi_x(t) = D_x(\Phi(t, x))$ satisfies

$$\dot{\Pi}_x(t) = Df(\Phi(t, x))\Pi_x(t)$$

and hence $D_x\Phi_t(x)$ is a solution of the matrix linear equation $\dot{X} = Df(\Phi(t, x))X$. By Liouville's formula we have

$$\det(D_x\Phi(t, x)) = \det(D_x\Phi(0, x)) \exp\left(\int_0^t \operatorname{div}(f(\Phi(s, x)))\right) = \exp\left(\int_0^t \operatorname{div}(f(\Phi(s, x)))\right)$$

since $\det(D_x\Phi(0, x)) = I$ and $\operatorname{Tr}(Df(x)) = \operatorname{div}(f(x))$.
By putting this in (4.6) and differentiating we get

$$\dot{V}(t) = \int_U \operatorname{div}(f(\Phi(t, x))) \det(D_x\Phi(t, x))dx$$

Applying the change of variable formula one again yields the desired result. □

## 4.2 Orbits and invariant sets

In this section we introduce some necessary terminology for studying the flow of an autonomous system.

**Definition 4.5.** The **orbit** of $x$ is defined as $\gamma(x) = \Phi(I_x \times \{x\}) \subset M$. The forward/backward orbit of $x$ are defined as: $\gamma_{\pm}(x) = \Phi((0, T_{\pm}(x)) \times \{x\})$.
If $\gamma(x) = \{x\}$ then $x$ is called a **fixed point** (a **steady state** or an **equilibrium**). Otherwise, $x$ is called **regular**.

*Remark* 4.6. By Theorem 4.3 if $y \in \gamma(x)$ then $\gamma(x) = \gamma(y)$. Hence, two orbits either coincide or are disjoint.

Away from fixed points all vector fields locally look the same.

**Lemma 4.7.** *(Straightening out of vector fields) Let $f(x_0) \neq 0$. Then there is a local coordinate transform $y = \varphi(x)$ such that $\dot{x} = f(x)$ is transformed to $\dot{y} = (1, 0, \dots, 0)$.*

*Proof.* Without loss of generality we can suppose $x_0 = 0$ and $f(0) = \delta_1 = (1, 0, \dots, 0)$ since otherwise we can linearly change coordinates to obtain these conditions. The desired function $\varphi$ should satisfy

$$\varphi(\Phi(t, (0, x_2, \dots, x_n))) = \varphi(\phi(t)) = y(t) = (0, x_2, \dots, x_n) + t(1, 0, \dots)$$

where $\phi$ is integral curve at $(0, x_2, \dots, x_n)$. Hence $\varphi$ should be the inverse function of $\psi((x_1, \dots, x_n)) = \Phi(x_1, (0, x_2, \dots, x_n))$. Using the chain rule we obtain the Jacobian matrix of $\dfrac{\partial \psi}{\partial x}$ at $x = 0$ as $\left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial x_2} \dots, \frac{\partial \Phi}{\partial x_n}\right)|_{t=0,x=0} = I$ since $\frac{\partial \Phi}{\partial x}|_{x=0,t=0} = I_n$ and $\frac{\partial \Phi}{\partial t}|_{t=0,x=0} = f(0) = \delta_1$.
Jacobian determinant is non-zero and hence by inverse mapping theorem $\psi$ is a local diffeomorphism with $y = \psi^{-1}(x)$ Furthermore, we have: $\frac{\partial \psi}{\partial x}\delta_1 = \frac{\partial \psi}{\partial x_1} = f(\psi(x))$ and finally our vector field in new coordinates satisfies: $\dot{y} = (\frac{\partial \psi}{\partial x})^{-1}|_{y=\psi^{-1}(x)}\dot{x} = (\frac{\partial \psi}{\partial x})^{-1}|_{x=\psi(y)}f(x) = \delta_1$. □

**Definition 4.8.** We say that $x \in M$ is a **periodic point** of $\Phi$ if there is some $T > 0$ such that $\Phi(T, x) = x$. The minimal of all such $T$ is called the **period** of $x$, that is $T(x) = \inf\{T > 0 | \Phi(T, x) = x\}$. By continuity of $\Phi$ we have $\Phi(T(x), x) = x$ and by property of flow $\Phi(t + T(x), x) = \Phi(t, x)$. Hence, if one point on an orbit is periodic, than every point is periodic. Such orbits are called **periodic orbits**.

From the above definition it follows that $\gamma(x)$ is periodic iff $\gamma_-(x) \cap \gamma_+(x) \neq \emptyset$ and hence periodic orbits are also called **closed** orbits.
As a consequence of Corollary 2.19 we have the following lemma.

**Lemma 4.9.** *Let $x \in M$ and suppose that the forward (backward) orbit lies in some compact set $C$. Then $x$ is + (respectively -) complete. A periodic point is complete.*

Another notion of special interest is defined in sequel.

**Definition 4.10.** A set $U$ is called $\sigma \in \{+,-\}$ **invariant** if $\gamma_\sigma(x) \subset U, \forall x \in U$ and invariant if it is both $\pm$ invariant.

By the above lemma, if $C \subset M$ is a compact $\sigma$ invariant set then all points $x \in M$ are $\sigma$ complete.
The next lemma gives some useful properties of $\sigma$ invariant sets.

**Lemma 4.11.** *Arbitrary intersections and unions of $\sigma$ invariant sets are $\sigma$ invariant. The closure of $\sigma$ invariant set is $\sigma$ invariant. If $U, V$ are two invariant sets then $U/V$ is also invariant.*

*Proof.* The first proposition is obvious from definition. To prove the second take $x \in \overline{U}$ and a sequence $\{x_n\} \in U$ such that $x_n \to x$. Fix $t \in I_x$. Since $W$ is open, there exists a $N \in \mathbb{N}$ such that $t \in I_{x_n}$ for $n > N$ and $\Phi(t,x) = \lim_{n\to\infty} \Phi(t,x_n) \in \overline{U}$. To prove the third proposition, take $x \in U/V$. Then, if $y \in \gamma(x) \cap V \neq \emptyset$ by property of orbits and invariance of V, we have $\gamma(y) = \gamma(x) \subset V$ which is contradiction with $x \notin V$. Hence, $\gamma(x) \in U/V$. □

One of our main aims is to describe the long-time dynamics of solutions. For this we next introduce the set where an orbit eventually accumulates.

**Definition 4.12.** The $\omega_\pm-$**limit set** of $x$, denoted by $\omega_\pm(x)$ is the set of all points $y \in M$ for which there exists a sequence $t_n \to \pm\infty$ such that $\Phi(t_n,x) \to y$. Clearly, $\omega_\sigma$ is empty unless $x$ is $\sigma$ complete. If $y \in \gamma(x)$ then $\omega_\pm(x) = \omega_\pm(y)$ since $\Phi(t_n,y) = \Phi(t_n+t,x)$ for $y = \Phi(t,x)$.

*Remark* 4.13. The $\omega_-$ set is also called $\alpha$ limit set in some of the literature, whereas the $\omega_+$ set is called $\omega$ limit set.

**Lemma 4.14.** *The set $\omega_\pm(x)$ is a closed invariant set.*

*Proof.* Take $y$ in closure of $\omega_\pm(x)$, meaning that there is $y_n \to y$, $y_n \in \omega_\pm(x)$ with $|y-y_n| < 1/(2n)$. Since $y_n \in \omega_\pm(x)$ there is $t_n \to \pm\infty$ such that $|\Phi(t_n,x)-y_n| < 1/(2n)$. Then, by triangle inequality $|\Phi(t_n,x)-y| < 1/n$ and we have $y \in \omega_\pm(x)$. If $\Phi(t_n,x) \to y$ then $\Phi(t_n+t,x) = \Phi(t,\Phi(t_n,x)) \to \Phi(t,y)$, confirming that $\Phi(t,y) \in \omega_\pm(x), \forall y \in \omega_\pm(x), \forall t \in I_y$ and $\omega_\pm(x)$ is invariant. □

**Lemma 4.15.** *If $\gamma_\sigma(x)$ is contained in some compact set, then $\omega_\sigma(x)$ is non-empty, compact and connected.*

*Proof.* By Lemma 4.9, $x$ is $\sigma$ complete so there exists a sequence $\Phi(t_n, x)$ with $t_n \to \sigma\infty$. By compactness we have a convergent subsequence and hence $\omega_\sigma(x)$ is non-empty. It is compact as a closed subset of compact set. If it is disconnected, we can split it into two disjoint closed sets $\omega_{1,2}$. Let $\delta = \mathrm{dist}(\omega_1, \omega_2)$. Taking all points that are on a distance at most $\delta/2$ from these sets we obtain two disjoint neighborhoods $U_{1,2}$ of $\omega_{1,2}$ respectively. Now we choose a strictly monotone sequence $t_n \to \sigma\infty$ such that $\Phi(t_{2m+1}, x) \in U_1$ and $\Phi(t_{2m}, x) \in U_2$. By connectedness of $\Phi((t_{2m}, t_{2m+1}), x)$ we can find $t_{2m} < \tilde{t}_m < t_{2m+1}$ such that $\Phi(\tilde{t}_m, x) \in C \setminus (U_1 \cup U_2)$. Since $C \setminus (U_1 \cup U_2)$ is compact we can assume $\Phi(\tilde{t}_m, x) \to y \in C \setminus (U_1 \cup U_2)$. But $y$ must also be in $\omega_\sigma(x)$ which is a contradiction. □

A nonempty, compact, $\sigma$ invariant set $C$ is called minimal if it contains no proper $\sigma$-invariant subset possessing these three properties. Note that for such a minimal set we have $C = \omega_+(x) = \omega_-(x)$ for every $x \in C$.

## 4.3 Attracting sets

In this section we generalize the notions of the previous sections to develop the theory for studying the behaviour of all points starting in some set, which will be important in study of long-time behavior of the flow of a differential equation. For simplicity, we assume from now on that the flow is $\sigma$ complete.

**Definition 4.16.** Let $X \subset M$. We define $\gamma_\pm(X) = \bigcup\limits_{\pm t \geq 0} \Phi(t, X) = \bigcup\limits_{x \in X} \gamma_\pm(x)$ and $\omega_\pm(X) = \{y \in M | \exists t_n \to \infty, x_n \in X : \Phi(t_n, x_n) \to y\}$.

We observe that $\gamma_\sigma(X)$ is an invariant set, whose closure is a closed invariant set by Lemma 4.11 and that we have $\bigcup\limits_{x \in X} \omega_\sigma(x) \subset \omega_\sigma(X)$.
We can say more about the set $\omega_\sigma(X)$ to get an analogous statement of Lemmas 4.11 and 4.14.

**Lemma 4.17.** *The set $\omega_\sigma(X)$ is a closed invariant set given by*

$$\omega_\sigma(X) = \bigcap_{\sigma t \geq 0} \Phi(t, \overline{\gamma_\sigma(X)}) = \bigcap_{\sigma t \geq 0} \overline{\bigcup_{\sigma(s-t) \geq 0} \Phi(s, X)}.$$

*Proof.* We prove only $\sigma = +$ case. Since $\Phi(t, .)$ is a diffeomorphism we have

$$\Phi(t, \overline{\gamma_+(X)}) = \overline{\Phi(t, \gamma_+(X))} = \overline{\bigcup_{s \geq t} \Phi(s, X)}.$$

To prove that $\bigcap_{t \geq 0} \Phi(t, \overline{\gamma_+(X)}) \subset \omega_\sigma(X)$ we take $y \in \bigcap_{t \geq 0} \Phi(t, \overline{\gamma_+(X)})$. Then, by definition, for every $n \in \mathbb{N}$ we can find some $y_n = \Phi(n + s_n, x_n) \in \Phi(n, \gamma_+(X))$ such

that $y_n \to y$. Setting $t_n = n + s_n$ we have found a sequence $t_n \to \infty$ with $\Phi(t_n, x_n) \to y$ showing that $y \in \omega_+(X)$.
In the other direction, for $y \in \omega_+(X)$ we have a sequence $t_n \to \infty$ and $\{x_n\} \in X$ with $y_n = \Phi(t_n, x_n) \to y$. Then $y_n \in \Phi(t, \gamma_+(X))$ for $t_n > t$ and thus $y \in \overline{\Phi(t, \gamma_+(X))}$ for every $t > 0$.
We can prove invariance as in the previous section and $\omega_\sigma$ is then closed invariant as intersection of closed invariant sets. □

Similar arguments can be used to prove the analogue of Lemma 4.15:

**Lemma 4.18.** *For non-empty set $X$ for which $\overline{\gamma_\sigma(X)}$ is compact, $\omega_\sigma(X)$ is non-empty and compact. If $\overline{\gamma_\sigma(X)}$ is in addition connected then so is $\omega_\sigma(x)$.*

We have finally come to the definition of central importance in this section.

**Definition 4.19.** For a given invariant set $\Lambda \subset M$ the sets

$$W^{\pm}(\Lambda) = \{x \in M | \lim_{t \to \pm\infty} \operatorname{dist}(\Phi(t, x), \Lambda) = 0\}$$

are the **stable** respectively **unstable** sets of $\Lambda$. The invariant set $\Lambda$ is called **attracting** if $W^+(\Lambda)$ is neighborhood of $\Lambda$. In this case the set $W^+(\Lambda)$ is called the **domain of attraction** for $\Lambda$.

Moreover, for any positively invariant neighborhood $U$ we have $W^+(\Lambda) = \bigcup\limits_{t<0} \Phi(t, U)$. In particular, $W^+(\Lambda)$ is invariant and choosing $U$ open we see that it is also open.
We can use theory developed in this section to find attracting sets. We fist introduce a definition.

**Definition 4.20.** An open connected set $E$ with compact closure is called a **trapping region** for the flow if $\Phi(t, \overline{E}) \subset E$ for all $t > 0$.

**Lemma 4.21.** *Let $E$ be a trapping region. Then $\Lambda := \omega_+(E) = \bigcap\limits_{t \le 0} \Phi(t, E)$ is a non-empty, invariant, compact and connected attracting set.*

*Proof.* By definition we have $\Phi(t + \epsilon, \overline{E}) \subset \Phi(t, E) \subset \Phi(t, \overline{E})$ giving us

$$\bigcap_{t \ge 0} \Phi(t, E) = \bigcap_{t \ge 0} \Phi(t, \overline{E}) = \bigcap_{t \ge 0} \Phi(t, \overline{\gamma_+(E)}) = \omega_+(E).$$

Hence, $\omega_+(E)$ is a non-empty, invariant and compact and connected by Lemma 4.18. To see that it is attracting suppose there were $x \in E$ and a sequence $t_n \to \infty$ with $\operatorname{dist}(\Phi(t_n, x), \Lambda) \ge \epsilon > 0$. Then, since $\Phi(t_n, x)$ stays in compact set $\overline{E}$ we can suppose that it converges to some $y$ after passing to a subsequence. But then $y \in \omega_+(x) \subset \omega_+(E)$ which is a contradiction. □

To improve the definition of an attracting set, we must ensure that it can not be divided into smaller attracting sets. To achieve that we pose the following definition.

**Definition 4.22.** A closed invariant set $\Lambda$ is **topologically transitive** if for any two open sets $U, V \subset \Lambda$ there is some $t \in \mathbb{R}$ such that $\Phi(t, U) \cap V \neq \emptyset$.
An **attractor** is a topologically transitive attracting set.

In this way the attractor can not be split into smaller attracting sets. An example of an attractor is an attracting set containing a dense orbit.

# 5 Stability of fixed points

In Chapter 3 we introduced the notion of stability for linear systems. In this chapter we introduce the notion of stability of fixed points for autonomous systems with the help of the newly developed theory. This notion will help us to study the long-time behavior of dynamical systems.

**Definition 5.1.** A fixed point $x_0$ of the equation $\dot{x} = f(x)$ is called **stable** if for any given neighborhood $U(x_0)$ there exists some other neighborhood $V(x_0) \subset U(x_0)$ such that every solution starting in $V(x_0)$ stays in $U(x_0)$ for all $t \geq 0$. A fixed point which is not stable is called **unstable**.
A stable fixed point $x_0$ is called **asymptotically stable** if there is a neighborhood $U(x_0)$ such that $\lim\limits_{t\to\infty} |\phi(t,x) - x_0| = 0,\ \forall x \in U(x_0)$.
Finally, a fixed point $x_0$ is called **exponentially stable** if there are constants $\alpha, \delta, C > 0$ such that: $|\phi(t,x) - x_0| \leq Ce^{-\alpha t}|x - x_0|,\ \forall x : |x - x_0| < \delta$.

Obviously, exponential stability implies asymptotical stability.

**Example 5.2.** In the previous chapter we discussed stability of the origin for equation $\dot{x} = Ax$. The condition for stability was that all eigenvalues have non-positive real parts and for those with real part 0 algebraic and geometric multiplicities are equal. In this case $\|\exp(tA)\| \leq C, t \geq 0$ for some constant $C$, which corresponds to this new definition of stability.
For asymptotic stability we needed all eigenvalues to have negative real parts. In this case, for every $\alpha < \min\{-\mathrm{Re}(\alpha_j)\}$ there is a constant $C(\alpha)$ such that $\|\exp(tA)\| \leq C(\alpha)e^{-t\alpha}, t \geq 0$, showing that in this case we have even exponential stability.
As an example of this, we investigate the most simple autonomous system, the homogeneous system of two linear equations,

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

The characteristic polynomial of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two roots, $\lambda_{1,2}$. As we have shown in Chapter 3, if the matrix has two generalized eigenvectors of rank one, denoted by $v_{1,2}$, the general solution has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \tag{5.1}$$

Otherwise, if $v_1$ is of rank one and $v_2$ of rank two, the general solution is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[(c_1 + c_2 t)v_1 + c_2 v_2\right] e^{\lambda_1 t} \tag{5.2}$$

We consider the phase portrait of the system in several cases.

1. We first examine the case when both eigenvalues are real and non-zero.

   - $\lambda_1 < \lambda_2 < 0$

     In this case the origin is asimptotically stable by Corollary 3.7 and we say that the origin is a **stable node**. The phase portrait is depicted on Figure 1 on the left.

   - $0 < \lambda_2 < \lambda_1$

     In this case the origin is unstable and it is said to be an **unstable node**. The phase portrait is the same as on Figure 1 on the left, only the arrows point in different direction.

   - $\lambda_1 = \lambda_2 < 0$

     If we have two eigenvectors of rank one, from (5.1) we see that solutions are rays passing through the origin and approaching to it as $t \to \infty$.

     If there is only one eigenvector of rank one, from (5.2) we see that the solution approach to the origin along eigenvector of rank two.

     In both cases we have a stable node.

   - $0 < \lambda_1 = \lambda_2$

     We have similar observation as in the previous case, only that in this case solutions going away from the origin as $t \to \infty$ and we have an unstable node.

   - $\lambda_1 < 0 < \lambda_2$

     From the general solution (5.1) we see that for $c_2 = 0$ the exponential term decreases and the solution approaches to the origin. Otherwise, the exponential term by $c_2$ increases and to solution goes away from the origin. In this case we say that the origin is a **saddle node**. Phase portrait is depicted on Figure 1 on the right.

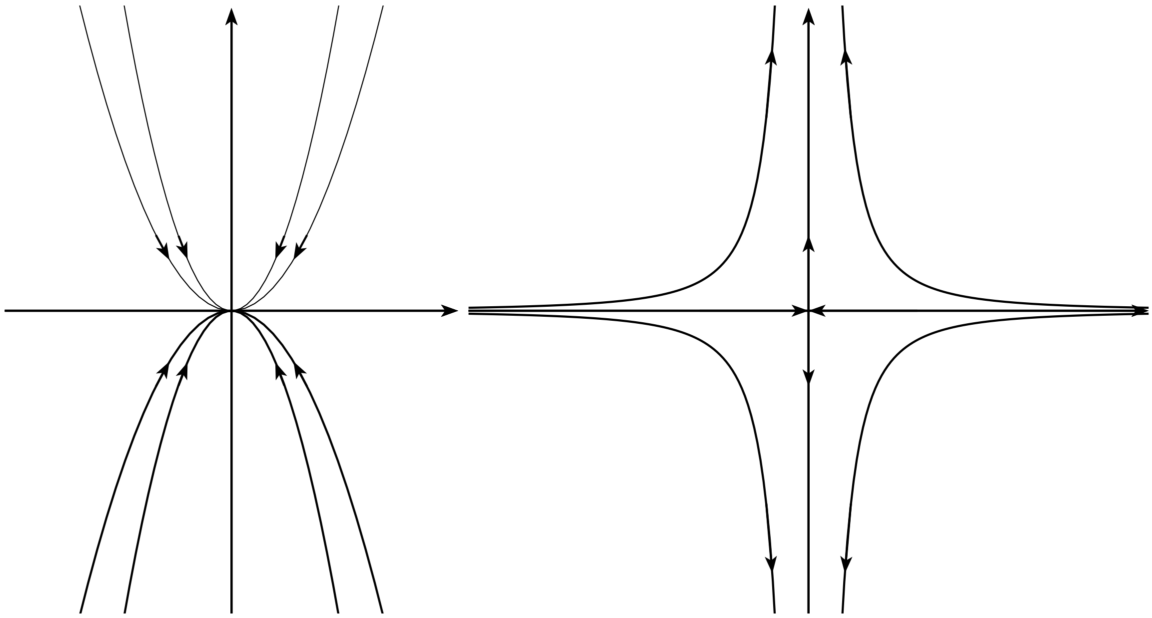

Figure 1: The origin as a stable node (left) and as a saddle node (right)

2. We proceed to the case of complex eigenvalues: $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta \neq 0$

   If $v = u + i\omega$ is the eigenvector corresponding to $\lambda_1$ we have that solution is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\alpha t}\{c_1(u\cos\beta t - \omega \sin\beta t) + c_2(u \sin\beta t + \omega\cos\beta t)\}$$

   For the corresponding values of $r_{1,2}$ and $\delta_{1,2}$ this can be rewritten into

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\alpha t}\begin{pmatrix} r_1\cos(\beta t - \delta_1) \\ r_2\cos(\beta t - \delta_2). \end{pmatrix} \tag{5.3}$$

   The vector on the left hand side is a periodic function of $t$ and it represent a closed curve around the origin.

   - In case of $\alpha < 0$ the exponential term decreases the distance of the curve from the origin. In this case the origin is asimptotically stable and it is said to be a **stable focus** or a **sink**. Phase portrait is depicted on Figure 2.
   - In case of $\alpha = 0$ we have closed curves as solutions and the origin is said to be a **center**.
   - In case of $\alpha > 0$ the exponential term increases the distance of the curve from the origin. In this case the origin is unstable and it is said to be an **unstable focus** or a **source**. Phase portrait is the same as on Figure 2 only the arrows point in different direction.

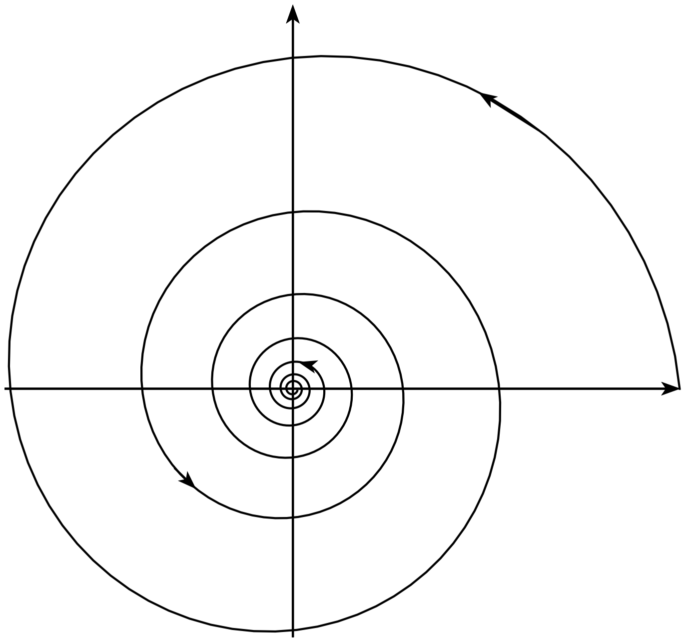

Figure 2: The origin as a stable focus

3. We are left with cases when the determinant of the matrix is equal to zero.

   - $\lambda_1 = 0,\ \lambda_2 > 0$

     In this case from 5.1 we see that solutions are rays starting on $v_1$ and having the direction $v_2$. For $t \to \infty$ the solution approaches to $c_1 v_1$ and every fixed point of the system is asimptotically stable.

   - $\lambda_1 = 0,\ \lambda_2 > 0$

     We have similar observations as before, only in this case solutions are going away from the starting point and every fixed point is unstable.

   - $\lambda_1 = \lambda_2 = 0$

     In this case the matrix $A$ is similar to the matrix $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and from solution of the system $\dot{x} = Bx$ we see that in this case we have lines parallel to the $x-$axis as solutions. Stationary point are all points on $x$-axis and they are unstable. In general example, solutions are parallel lines and fixed point are points on the lines passing through the origin.

## 5.1 Local behaviour near fixed points

Our aim in this section is to show that a lot of information on the stability of a flow near a fixed point can be inferred from the linearization of the system around the fixed point. On these linearized systems we will be able to apply the results from Section 3.1.

We start with an autonomous system

$$\dot{x} = f(x) \tag{5.4}$$

where $f \in C^k(M, \mathbb{R}^n)$ for $k \geq 1$ and $M$ is an open subset of $\mathbb{R}^n$.

By Taylor's theorem, every function $f \in C^1$ can be written in some neighborhood of some point $x_0$ as:

$$f(x) = f(x_0) + A(x - x_0) + o(|x - x_0|)$$

where $A$ is the Jacobian matrix of $f$ at $x_0$. If $x_0$ is a regular point of (5.4) we can straighten out the vector field near $x_0$ by Lemma 4.7 to get some information on the flow near this point. If $x_0$ is a fixed point we can write $f(x) = A(x - x_0) + o(|x - x_0|)$ and first observe the linear autonomous system

$$\dot{x} = Ax. \tag{5.5}$$

However, most of our results will hold only for systems for which none of the eigenvalues of matrix $A$ has a zero real part. Such systems are called **hyperbolic** and fixed points are **hyperbolic fixed points**.

The first result that connects the starting system with its linearization is:

**Lemma 5.3.** *Let $x_0$ be a fixed point of the autonomous system in* (5.4) *and $A$ its Jacobian matrix at $x_0$ which gives the linearized system in* (5.5). *Then we have one of the following cases:*

*(i) If* (5.5) *is asimptoticaly stable then $x_0$ is an asimptoticaly stable fixed point of* (5.4).

*(ii) If there is some eigenvalue of matrix $A$ with a positive real part then $x_0$ is an unstable fixed point.*

*(iii) If all eigenvalues of matrix $A$ have non-positive real parts and there is at least one with vanishing real part, we do not have information on stability of $x_0$ from linearization.*

Motivated by the linear case we can define for the fixed point $x_0$ of equation (5.4) the stable and the unstable set as the sets of all points converging to $x_0$ for $t \to \infty$ and $t \to -\infty$, respectively.

$$W^{\pm}(x_0) = \{x \in M | \lim_{t \to \pm\infty} |\Phi(t, x) - x_0| = 0\}.$$

Both sets are obviously invariant under the flow.
Furthermore, motivated by existence of stable and unstable manifolds for linear first order autonomous systems defined in Section 3.1 we will define their counterparts for general autonomous system.
Since our result is of local nature we fix a neighborhood $U(x_0)$ of $x_0$ and define:

$$M^{\pm\alpha}(x_0) = \{x|\gamma_\pm(x) \subset U(x_0) \wedge \sup_{\pm t \geq 0} e^{\pm\alpha t}|\Phi(t,x) - x_0| < \infty\}$$

to be the set of all points which converge to $x_0$ with some exponential rate $\alpha > 0$ as $t \to \pm\infty$. This is the counterpart of $E^{\pm\alpha}$: the space spanned by all eigenvectors of $A$ corresponding to eigenvalues with real part smaller/larger than $\alpha$.

**Definition 5.4.** We define the local stable and respectively unstable manifolds of a fixed point $x_0$ to be the set of all points which converge exponentially to $x_0$ as $t \to \infty$ and $t \to -\infty$, respectively:

$$M^\pm(x_0) = \bigcup_{\alpha > 0} M^{\pm\alpha}(x_0)$$

Both sets are $\pm$ invariant under the flow by construction.
The relation between above defined notions for system (5.4) and its linearization (5.5) is given in the following theorem.

**Theorem 5.5.** *Consider a system of the form* (5.4) *with $x_0$ as a fixed point for its flow and its linearization* (5.5). *With notation as introduced above, we have:*

*(i)* $M^\pm(x_0)$ *are smooth manifolds*

*(ii)* $E^\pm$ *is tangent to* $M^\pm(x_0)$ *at* $x_0$

*(iii) if* $x_0$ *is a hyperbolic fixed point then* $M^\pm(x_0) = W^\pm(x_0)$.

Finally, we state the theorem that shows the real nature of similarity of a local flow near a fixed point for hyperbolic systems and the flow of its linearization near origin:

**Theorem 5.6.** *(Hartman–Grobman) Suppose $f$ is a differentiable vector field with 0 as a hyperbolic fixed point. Denote by $\Phi(t,x)$ the corresponding flow and by $A$ the Jacobian matrix of $f$ at 0. Then there is a homeomorphism $\varphi(x) = x + h(x)$ with bounded function $h$ such that*

$$\varphi \circ e^{tA} = \Phi_t \circ \varphi$$

*in a sufficiently small neighborhood of 0.*

This shows that the orbits near a hyperbolic fixed point are locally just continuously deformed versions of their linear counterparts.
Proofs of Theorem 5.5 and 5.6 can be found in [10].

## 5.2 Liapunov method

In this section we present a very useful method to determine stability of a fixed point. It is based on the observation that for stable fixed points the distance to every orbit starting in some region is bounded and for asimptotically stable point it even converges to zero. This raises the idea of finding a function which has opposite direction from function $f$ on solution curves:

$$\mathrm{grad}(L)(\phi(t,x))f(\phi(t,x)) = \mathrm{grad}(L)(\phi(t,x))\dot{\phi}(t,x) = \frac{d}{dt}L(\phi(t,x)) \leq 0.$$

Hence we are looking for a function that will be non-increasing along solution curves. This condition will suffice even in case when we cannot calculate the gradient.

**Definition 5.7.** For a fixed point $x_0$ of (5.4) and its open neighborhood $U(x_0)$, a **Liapunov function** $L : U(x_0) \to \mathbb{R}$ is a continuous function such that: $L(x_0) = 0, L(x) > 0, x \neq x_0$ and $L(\phi(t_0)) \geq L(\phi(t_1)), t_0 < t_1, \phi(t_j) \in U(x_0)/x_0$ for any solution $\phi(t)$. It is called a **strict Liapunov function** if equality never occurs.

To make use of Liapunov functions we work with its sublevel sets. Let $S_\delta$ be the connected component of $\{x \in U(x_0) | L(x) \leq \delta\}$ containing $x_0$.

**Lemma 5.8.** *If $S_\delta$ is closed, then it is positively invariant.*

*Proof.* Suppose $\phi(t)$ leaves $S_\delta$ at $t_0$ with $x = \phi(t_0)$. Since $S_\delta$ is closed $x \in S_\delta \subset U(x_0)$ and since $U(x_0)$ is open there is a ball $B_r(x) \subset U(x_0)$ such that $\phi(t_0+\epsilon) \in B_r(x)/S_\delta$ for some $\epsilon > 0$. Then, since $S_\delta$ is full connected component we have $L(\phi(t_0+\epsilon)) > \delta = L(x)$ since otherwise we could extend $S_\delta$ by adding $\phi([t_0, t_0+\epsilon])$. This contradicts to non-increasing property. □

Moreover, $S_\delta$ is a neighborhood of $x_0$ which shrinks to a point when $\delta \to 0$.

**Lemma 5.9.** *For every $\delta > 0$ there is an $\epsilon > 0$ such that: $S_\epsilon \subset B_\delta(x_0)$ , $B_\epsilon(x_0) \subset S_\delta$.*

*Proof.* If the first claim were false, for every $n \in \mathbb{N}$ we would have $x_n \in S_{1/n}$ such that $|x_n - x_0| \geq \delta$. Since $S_{1/n}$ is connected we can take $|x_n - x_0| = \delta$ and by compactness of $\overline{B_\delta(x_0)}$ we have convergent subsequence $x_{n_m} \to y$. Then by continuity $L(y) = \lim_{m\to\infty} L(x_{n_m}) = 0$, implying $y = x_0$. This contradicts $|y - x_0| = \delta$.
If the second were false we could find a sequence $x_n$ such that $|x_n - x_0| \leq 1/n$ and $L(x_n) \geq \delta$. But then $\delta \leq \lim_{n\to\infty} L(x_n) = L(x_0) = 0$ which is an obvious contradiction. □

As a simple consequence of the above lemma we have:

**Theorem 5.10.** *(Liapunov) A fixed point $x_0$ for which there exists a Liapunov function $L$ is stable.*

*Proof.* By the above lemma, for any given neighborhood $V(x_0)$ we can find an $\epsilon > 0$ such that $S_\epsilon \subset V(x_0)$ is closed (if it shares common boundary with $V(x_0)$ we can decrease $\epsilon$) and hence positively invariant. □

Under some additional conditions we can claim asymptotic stability.

**Theorem 5.11.** *(Krasowski-LaSalle Principle) A fixed point $x_0$ of (5.4) for which there exists a Liapunov function $L$ which is not constant on any orbit lying entirely in $U(x_0)/\{x_0\}$ is asymptotically stable. Moreover, every such orbit converges to $x_0$.*

*Proof.* Take $x$ such that $\phi(t, x) \in U(x_0), t \geq 0$. Then the limit $\lim\limits_{t\to\infty} L(\phi(t, x)) = L_0(x)$ exists by monotonicity. For $y \in \omega_+(x)$ we can take $t_n \to \infty$ such that $\phi(t_n, x) \to y$. Then by continuity of $L$: $L_0(x) = \lim\limits_{n\to\infty} L(\phi(t_n, x)) = L(y)$ giving that $L(y)$ is constant for $y \in \omega_+(x)$, for arbitrary $x$. Since $L$ is not constant on any orbit and $x_0$ is stable, we have that $\omega_+(x) = x_0$ for any $x$ such that $\phi(t, x) \in U(x_0), t \geq 0$. □

We can use the same proof for generalization.

**Theorem 5.12.** *Let $L : U \to \mathbb{R}$ be continuous and bounded from bellow. If for some $x$ we have $\gamma_+(x) \subset U$ and $L(\phi(t_0, x)) \geq L(\phi(t_1, x)), t_0 < t_1$, then $L$ is constant on $\omega_+(x) \cap U$.*

Function $L$ satisfying above conditions is also called Liapunov function.
Most of Liapunov functions are differentiable. For these functions we can rewrite non-increasing condition the expression from start of section:

$$\frac{d}{dt}L(\phi(t, x)) = \mathrm{grad}(L)(\phi(t, x))\dot{\phi}(t, x) = \mathrm{grad}(L)(\phi(t, x))f(\phi(t, x)) \leq 0.$$

The expression $\mathrm{grad}L(x)f(x)$ is called the **Lie derivative** of $L$ along the vector field $f$. A function for which the Lie derivative vanishes is constant on every orbit and is hence called a **constant of motion**. We also note that in case of a strict Liapunov function its level sets are smooth manifolds by the implicit function theorem.

# 6 Planar Dynamical Systems

In this chapter we focus on dynamical systems in $\mathbb{R}^2$. We state certain results that explain why solutions in $\mathbb{R}^2$ behave quite regularly and allow us to classify the possible $\omega-$limit sets for planar systems.

## 6.1 The Poincaré map

In this section we present a useful method to study dynamical systems based on intersections of a periodic orbit with lower-dimensional subspace, called a **Poincaré section**. More precisely, one considers a periodic orbit with initial conditions within a section of the space, which leaves that section afterwards, and observes the point at which this orbit first returns to the section. We map first point to second with a map called **Poincaré map** or **First recurrence map**.
We start with the following definition.

**Definition 6.1.** A set $\Sigma \subset \mathbb{R}^n$ is called a **submanifold of codimension one** (i.e. its dimension is $n-1$), if it can be written as $\Sigma = \{x \in U | S(x) = 0\}$ where $U \subset \mathbb{R}^n$ is open, $S \in C^k(U)$ and $\frac{\partial S}{\partial x} \neq 0$ for all $x \in \Sigma$.
$\Sigma$ is said to be **transversal** to the vector field $f$ if $\frac{\partial S}{\partial x} f(x) \neq 0$ for all $x \in \Sigma$.
Since $\frac{\partial S}{\partial x}$ is orthogonal to the tangent plane of $\Sigma$ at every point, the direction of $f$ does not lie in this plane on any point and since $f$ is continuous it does not change direction with respect to $\Sigma$.

The next lemma proves existence of Poincaré map.

**Lemma 6.2.** *Suppose $x \in M$ and $T \in I_x$. Let $\Sigma$ be a submanifold of codimension one transversal to $f$ such that $\Phi(T, x) \in \Sigma$. Then there exists a neighborhood $U$ of $x$ and $\tau \in C^k(U)$ such that $\tau(x) = T$ and $\Phi(\tau(y), y) \in \Sigma$ for all $y \in U$.*

*Proof.* Consider the equation $S(\Phi(t, y)) = 0$ which holds for $(T, x)$. By transversality of $S$ to $f$ we have: $\frac{\partial}{\partial t} S(\Phi(t, y)) = \frac{\partial S}{\partial x}(\Phi(t, y)) f(\Phi(t, y)) \neq 0$, for $(t, y)$ in a neighborhood $I \times U$ of $(T, x)$. Then, by implicit function theorem, there is a mapping $\tau \in C^k(U)$ such that for all $y \in U$ we have $S(\Phi(\tau(y), y)) = 0$ i.e. $\Phi(\tau(y), y) \in \Sigma$. □

We finally define the Poincare map.

**Definition 6.3.** If $x$ is periodic and $T = T(x)$ is its period, then $P_\Sigma(y) = \Phi(\tau(y), y)$ is called the Poincare map. It maps $\Sigma$ into itself and every fixed point corresponds to a periodic orbit.

## 6.2 The Poincare–Bendixson Theorem

In this section we explain regular behavior of solutions in $\mathbb{R}^2$ and classify the possible $\omega-$limit sets for planar systems. We first present an important fact that differs $\mathbb{R}^2$ from $\mathbb{R}^n$, for $n \geq 3$.

**Theorem 6.4.** *(Jordan curve theorem) Every Jordan curve $J$ (i.e. homeomorphic image of the circle $S^1$) dissects $\mathbb{R}^2$ into two regions.*

Let $M \subset \mathbb{R}^2$ and $f \in C^1(M, \mathbb{R}^2)$ be given. By an **arc** $\Sigma \subset \mathbb{R}^2$ we mean a submanifold of dimension one given by a smooth map $t \mapsto s(t)$ with $\dot{s}(t) \neq 0$. Using this map $\Sigma$ can be ordered. For each regular $x \in M$, we can find a small arc $\Sigma$ containing $x$ transversal to $f$.
Given a regular point $x_0 \in \Sigma$ we can define the point of subsequent intersection of $\gamma_\sigma(x_0)$ with $\Sigma$ by $x_n = \Phi(t_n, x_0)$, $n \in \mathbb{N}_0$. This set can be finite or infinite but if it is infinite we have $t_n \to T_\sigma(x_0)$. Otherwise, if $t_n \to T \neq T_\sigma(x_0)$ we have that $y = \lim_{t \to T} \Phi(t, x)$ is finite and regular and hence we can straighten the vector field near $y$ to get that the difference between two consecutive points does not converge to 0, giving us a contradiction.

**Lemma 6.5.** *Let $x_0$ be a regular point and $\Sigma$ a transversal arc containing $x_0$. Denote by $x_n = \Phi(t_n, x)$, ordered according to $t_n$, the sequence of intersections of $\gamma_\sigma(x_0)$ with $\Sigma$. Then $\{x_n\}$ is monotone with respect to order of $\Sigma$. In other words, all points $x_n$ are from the same side of $x_0$ and $x_{n+1}$ is on the opposite side of $x_n$ from $x_{n-1}$.*

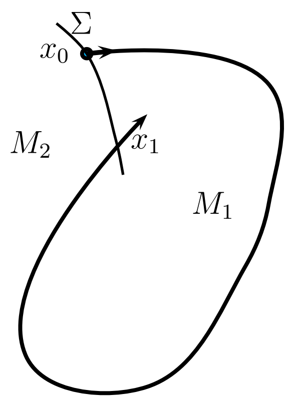


Figure 3: Proof of the Lemma 6.5

*Proof.* It is enough to consider $\sigma = +$ case. If $x_0 = x_1$ we are done. Otherwise, consider the curve $J$ from $x_0$ to $x_1$ along $\gamma_+(x_0)$ and part of $\tilde{\Sigma} \subset \Sigma$ from $x_1$ to $x_0$, as it is depicted in Figure 3. This is the image of a continuous bijection from $S^1$ and hence it is a Jordan curve giving $M/J = M_1 \cup M_2$, where $M_1, M_2$ are connected. Because of transversality, $f$ never changes sign on $\tilde{\Sigma}$. So, $\gamma_+(x_1)$ enters either $M_1$ or $M_2$ and stays there since it can not cross neither $\tilde{\Sigma}$ neither the orbit from $x_0$ to $x_1$, since $x_0$ is not periodic.So, either $\gamma_+(x_1) \subset M_1$ or $\gamma_+(x_1) \subset M_2$, i.e. either $x_0 < x_1$ and $\gamma_+(x_1)$ stays in the component with points $x > x_1$, or $x_0 > x_1$ and $\gamma_+(x_1)$ stays in the component with points $x < x_1$. Iterating this procedure proves the claim. □

Next, we can approximate every $y \in \Sigma \cap \omega_\sigma(x)$ by a sequence $\tilde{x}_n \in \Sigma \cap \gamma_\sigma(x)$. In fact, choose $t_n \to \sigma\infty$ such that $x_n = \Phi(t_n, x) \to y$. Then

$$\tilde{x}_n := \Phi(t_n + \tau(x_n), x) = \Phi(\tau(x_n), x_n) \to \Phi(\tau(y), y) = y$$

by continuity of $\Phi$ and $\tau$ and $\Phi(\tau(x_n), x_n) \in \Sigma$, where $\tau(x)$ is constructed as in Lemma 6.2 for $x = y, T = 0$.

**Corollary 6.6.** *Let $\Sigma$ be a transversal arc. Then $\omega_\sigma(x)$ intersects $\Sigma$ in at most one point.*

*Proof.* Suppose there exist two such points $y_1, y_2 \in \Sigma \cap \omega_\sigma(x)$. Then there are sequences $x_{1,n}, x_{2,n} \in \Sigma \cap \gamma_\sigma(x)$ converging to $y_1, y_2$, respectively. However, both of these sequences are monotone by Lemma 6.5 and hence we have a contradiction. □

**Corollary 6.7.** *Suppose $\omega_\sigma(x) \cap \gamma_\sigma(x) \neq \emptyset$. Then $x$ is periodic and $\omega_-(x) = \omega_+(x) = \gamma(x)$.*

*Proof.* By assumption there is an $y \in \omega_\sigma(x) \cap \gamma_\sigma(x)$. Then $\gamma(x) = \gamma(y) = \omega_\sigma(x)$ by property of orbits and invariance of $\omega_\sigma(x)$. If $y$ is fixed then $\gamma(x) = \{y\}$, thus $x = y$ and $x$ is fixed, i.e. periodic. If $y$ is regular, we pick a transversal arc containing $y$, together with a sequence $x_n \in \Sigma \cap \gamma_\sigma(x) \subset \Sigma \cap \omega_\sigma(x)$ converging to $y$. Then by the previous lemma $x_n = y$ constantly and $\gamma(x)$ is periodic. □

**Corollary 6.8.** *A minimal compact $\sigma-$invariant set $C$ is a periodic orbit.*

*Proof.* Take $x \in C$. Then $\omega_\sigma(x) = C$ by minimality and hence $\omega_\sigma(x) \cap \gamma_\sigma(x) \neq \emptyset$. Therefore $x$ is periodic by the previous corollary. □

Now we are ready to prove the first important fact about $\omega_\sigma$ sets.

**Lemma 6.9.** *(Poincare–Bendixson Theorem) If $\omega_\sigma(x) \neq \emptyset$ is compact and contains no fixed points, then it is a regular periodic orbit.*

*Proof.* Let $y \in \omega_\sigma(x)$ and take $z \in \omega_\sigma(y) \subset \omega_\sigma(x)$ which is not fixed by assumption. Then there exists a transversal arc $\sigma$ containing $z$ and $y_n \to z$ with $y_n \in \Sigma \cap \gamma_\sigma(y)$. By invariance of $\omega_\sigma(x)$ we have $y_n \in \Sigma \cap \gamma_\sigma(y) \subset \Sigma \cap \omega_\sigma(x) = \{z\}$ by Corollary 6.6 and hence $y_n = z$ and hence $\omega_\sigma(x)$ is a regular periodic orbit. □

We now give few lemmas to generalize the above proposition.

**Lemma 6.10.** *Suppose $\omega_\sigma(x)$ is connected and contains a regular periodic orbit $\gamma(y)$. Then $\omega_\sigma(x) = \gamma(y)$.*

*Proof.* If $\omega_\sigma(x)/\gamma(y) \neq \emptyset$ then by connectedness there is a point $\bar{y} \in \gamma(y)$ such that we can find a point $z \in \omega_\sigma(x)/\gamma(y)$ arbitrarily close to $\bar{y}$. Since $\gamma(y)$ is a regular orbit we can pick a transversal arc $\Sigma$ containing $\bar{y}$. By Lemma 6.2 we can find $\tau(z)$ such that $\Phi(\tau(z), z) \in \Sigma$. In that case we have $\Phi(\tau(z), z) \in \Sigma \cap \omega_\sigma(x) = \{\bar{y}\}$ by Corollary 6.6, thus $z \in \gamma(y)$, contradicting our assumption. □

**Lemma 6.11.** *Let $x \in M$ and suppose $\omega_\pm(x)$ are compact. Let $x_\pm \in \omega_\sigma(x)$ be two different fixed points. Then there exists at most one orbit $\gamma(y) \subset \omega_\sigma(x)$ with $\omega_\pm(y) = x_\pm$.*

*Proof.* Suppose there are two such orbits $\gamma(y_1), \gamma(y_2)$. Since $\lim_{t\to\pm\infty} \Phi(t, y_i) = x_\pm$ we can extend $\Phi(t, y_i)$ to the continuous function on $\mathbb{R} \cup \{\pm\infty\}$ by $\Phi(\pm\infty, y_i) = \pm x$. Hence the curve $J$ from $x_-$ to $x_+$ along $\gamma(y_1)$ and back from $x_+$ to $x_-$ along $\gamma(y_2)$, as in Figure 4, is a Jordan curve. Writing $M/J = M_1 \cup M_2$ we can suppose $x \in M_1$. Pick two transversal arcs $\Sigma_1, \Sigma_2$ containing $y_1, y_2$, respectively. Then $\gamma_\sigma(x)$ intersects $\Sigma_i$ in some points $z_i$. Now consider the Jordan curve from $y_1$ to $z_1$ to $z_2$ to $y_2$ to $x_+$ and back to $y_1$ (along $\Sigma_1$, $\gamma_\sigma(x)$, $\Sigma_2$, $\gamma(y_1)$, $\gamma(y_2)$). It dissects $M_1$ into two parts $N_1$ and $N_2$ such that $\gamma_\sigma(z_1)$ or $\gamma_\sigma(z_2)$ must remain in one of them, say $N_2$. But then $\gamma_\sigma(x)$ can not return close to points of $\gamma(y_i) \cap N_1$, contradicting our assumption. □

In other words, this lemma says that there cannot be two different orbits with same $\omega_\sigma$ limit sets represented by two different fixed points. However, it allows possibility of having two orbits $\gamma(y_1)$ and $\gamma(y_2)$ and two different fixed points $x_\pm$ with

$$\lim_{t\to\pm\infty} \Phi(t, y_1) = x_\pm, \quad \lim_{t\to\pm\infty} \Phi(t, y_2) = x_\mp.$$

On the other hand, for two different fixed points $x_\pm$ there can be a solution $\phi(t)$ satisfying $\lim_{t\to\pm\infty} = x_\pm$. Such solution is called a **heteroclinic orbit**. If $x_- = x_+$ then the solution is called a **homoclinic orbit**.
These preparations now yield the following theorem.

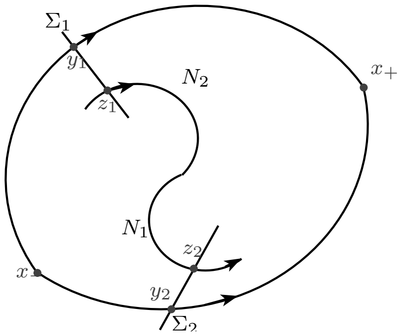


Figure 4: Proof of the Lemma 6.11

**Theorem 6.12.** *(generalized Poincare–Bendixson) Let $M$ be an open subset of $\mathbb{R}^2$ and $f \in C^1(M, \mathbb{R}^2)$. Fix $x \in M$, $\sigma = \pm$ and suppose $\omega_\sigma(x)$ is compact, connected and contains finitely many fixed points. Then one of the following cases holds:*

*(i)* $\omega_\sigma(x)$ *is a fixed point,*

*(ii)* $\omega_\sigma(x)$ *is a regular periodic orbit,*

*(iii)* $\omega_\sigma(x)$ *consists of finitely many fixed points $\{x_j\}$ and non-closed orbits $\gamma(y)$ such that $\omega_\pm(y) \in \{x_j\}$.*

*Proof.* If $\omega_\sigma(x)$ contains no fixed points it is a regular periodic orbit by Theorem 6.9. If it contains at least one fixed point, but no regular points, we have $\omega_\sigma(x) = \{x_1\}$ since fixed points are isolated and $\omega_\sigma(x)$ is fixed.
Now let us suppose that it contains both fixed and regular points. Let $y \in \omega_\sigma(x)$. We will prove that $\omega_\sigma(y)$ does not contain regular points. Let $z \in \omega_\sigma(y)$ be regular. Take a transversal arc $\Sigma$ containing $z$ and a sequence $y_n \to z$, $y_n \in \Sigma \cap \gamma(y)$. By Corollary 6.6 $\gamma(y) \subset \omega_\sigma(x)$ can intersect $\Sigma$ only in $z$. Hence $y_n = z$ and $\gamma(y)$ is regular periodic. Now Lemma 6.10 implies $\gamma(y) = \omega_\sigma(x)$ which is impossible since $\omega_\sigma(x)$ contains fixed points. □

Using the invariance of the domain bounded by a periodic orbit, one can show that the interior of every periodic orbit, contains at least one fixed point.
Of special interest are periodic orbits which attract other orbits. Such orbits are called $\omega_\sigma$ **limit cycles** (depending on whether they are positive or negative limit sets). The above presented theory allows us to tell something about such orbits in planar systems.

**Lemma 6.13.** *Let $\gamma(y)$ be an isolated regular periodic orbit (such that there are no other periodic orbits within a neighborhood). Then every orbit starting sufficiently close to $\gamma(y)$ will have either $\omega_+ = \gamma(y)$ or $\omega_- = \gamma(y)$.*

*Proof.* Choose a neighborhood $N$ of $\gamma(y)$ that does not contain other periodic orbits and a transversal arc $\Sigma$ containing $y$. Consider $x_0 \in \Sigma$ outside of $N_1$, the domain bounded by $\gamma(y)$. If this point is sufficiently close to $y$, then by continuity of function $\tau$ from lemma 6.2 its positive orbit will stay in $N$ and intersect $\Sigma$ in $x_1$. Without loss of generality we can take that $x_1$ is closer to $y$ than $x_0$ (otherwise we can reverse time). We use monotonicity proven in lemma 6.5 to see that the positive orbit of $x_1$ will stay in $M_1/N_1 \subset N$ and in limit the entire set $\omega_+(x)$ will stay in this area. Since this area does not contain other periodic orbits than $\gamma(y)$, nor fixed points, nor regular orbits, we must have $\omega_+(x) = \gamma(y)$. $\square$

Furthermore, from the above proof we notice that the set of points whose positive/negative orbits converges to some limit cycle $\gamma(y)$, excluding $y$, is and open set. This fact excludes the possibility of existence of a periodic orbit if we have a Liapunov function which is not constant on any orbit, since in that case all orbits near the fixed point inside of the periodic orbit would converge to that point, and not to the orbits.
We can give one more simple criterion for non-existence of periodic orbit.

**Lemma 6.14.** *(Dulac criterion) Let $R$ be a simply connected region in $\mathbb{R}^2$ and $\varphi(x,y) \in C^1(R)$ such that $\operatorname{div}(\varphi f) \neq 0$ almost everywhere on $R$. Then the system*

$$(\dot{x}, \dot{y}) = f(x,y) = (f_1(x,y), f_2(x,y)) \tag{6.1}$$

*has no periodic orbits nor homoclinic orbits lying entirely in $R$.*

*Proof.* Without loss of generality we can suppose that $\operatorname{div}(\varphi f) = \frac{\partial(\varphi f_1)}{\partial x} + \frac{\partial(\varphi f_2)}{\partial y} > 0$ almost everywhere. Let $C$ be periodic orbit and $D$ its interior. Then by Green's theorem:

$$0 < \int\int_D \frac{\partial(\varphi f_1)}{\partial x} + \frac{\partial(\varphi f_2)}{\partial y} dxdy = \oint_C (-\varphi f_2 dx + \varphi f_1 dy) = \oint_C \varphi(-\dot{y}\dot{x}dt + \dot{x}\dot{y}dt) = 0$$

since $C$ is a solution of (6.1). Thus, we have an obvious contradiction with the existence of a periodic orbit in $R$. $\square$

# 7 An example in Biology – Competitive exclusion

In this chapter we present an example of an application of the theory developed throughout previous chapters. Some of the best examples of using mathematical modelling can be found in biology where one tries to explain and predict behaviour of different biological subjects.

## 7.1 Motivation

The ecological principle of competitive exclusion asserts that two species cannot indefinitely occupy the same niche. The classical example is given by Volterra in [11] who considered a system of $n$ consumers with densities $x_i$ and a resource $R$:

$$\frac{dx_i}{dt} = x_i(\gamma_i R - \sigma_i) \tag{7.1}$$

$$R = R_{max} - F(x_1, \dots, x_n) \tag{7.2}$$

where $\gamma_i > 0$ relates increased resource abundance to increased growth of species $i$ and $\sigma_i > 0$ is the per capita death rate of individuals of species $i$. Here, $F$ is an unbounded increasing function of the population densities with $F(0, \dots, 0) = 0$. If we substitute (7.2) in (7.1) and write $\epsilon_i = \gamma_i R_{max} - \sigma_i$ we get Volterra's original equations:

$$\frac{dx_i}{dt} = x_i\left(\epsilon_i - \gamma_i F(x_1, \dots, x_n)\right), i = 1, \dots, n.$$

Volterra showed that, as $t \to \infty$, the species with largest $\gamma_i/\epsilon_i$ will approach a finite non-zero density and the remaining species will approach extinction, assuming that $\epsilon_i > 0$ and $x_i(0) \neq 0$ for the winning species.

Such a system is an example of a linear abiotic resource model. It is linear because the specific growth rates of the competitors are linear functions of resource densities and it describes an abiotic resource because the resource abundance changes according to an algebraic relationship. Furthermore, this model has several more simplifying assumptions:

(i) The organisms under consideration are "simple" in the sense that the dynamics of the system can be adequately described by the species densities $x_i$. Complications arising from age structure or physiological state are assumed unimportant.

(ii) The species interact only through the resource, so that their specific growth rates are functions of $R$ alone, not of the $x_i$.

(iv) The system under consideration is spatially homogeneous.

Ever since Volterra's original example there were attempts to state the principle of competitive exclusion as a theorem and generalize it to the case of $n$ consumers and $k < n$ resources. However, it was shown that after relaxing some of the restrains taken in basic models, such coexistence is possible, but not at fixed densities. In order to explain competitive exclusion properly, we must define this notion in a mathematical way. We start with similar notions as in Section 4.3 and make small adjustments.

**Definition 7.1.** A compact invariant set $K$ is called an **attractor** if there is an open set $U \supset K$ such that $\omega_+(U) = K$, where $\omega_+(U)$ is defined as in Section 4.3.

Now we consider a general $n$ species model

$$\frac{dx_i}{dt} = x_i g_i(x_1, \dots, x_n), \quad i = 1, \dots, n.$$

Here $x_i \geq 0$ is the population density of species $i$ and $g_i$ is a function representing the specific per capita growth rate of species $i$. These functions determine a vector field on $E^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_i \geq 0, \forall i\}$. We denote the space of smooth functions on this manifold $\mathcal{E}^n = C^\infty(E^n, \mathbb{R}^n)$. A function $g = (g_1, \dots, g_n) \in \mathcal{E}^n$ can be considered as $n$-species ecological community.
We are now ready to introduce mathematical definitions of the above mentioned ecological notions. These definitions are due to McGehee and Armstrong [5].

**Definition 7.2.** An ecological community $g \in \mathcal{E}^n$ is said to be **persistent** if $\phi_g$ has an attractor in $\mathrm{int}(E^n)$. We say that $g$ exhibits **exclusion** if it is not persistent. We say that a class of communities $\mathcal{C} \subset \mathcal{E}^n$ satisfies the **exclusion property** if for all $g \in \mathcal{C}$, $g$ exhibits exclusion.

By generalizing Volterra's model to include $k$ resources and by relaxing the assumption of linearity we get the class of abiotic resource models:

$$\frac{dx_i}{dt} = x_i u_i(R_1, \dots, R_k), \quad i = 1, \dots, n \tag{7.3}$$
$$R_j = R_{j,max} - F_j(x_1, \dots, x_n) = s_j(x_1, \dots, x_n), \quad j = 1, \dots, k \tag{7.4}$$

where we assume that $\dfrac{\partial u_i}{\partial R_j} \geq 0$ and $\dfrac{\partial s_j}{\partial x_i} \leq 0$ for all $i = 1, \dots, k$ and $j = 1, \dots, n$. With previously introduced terminology, we denote such a class of models by $\mathcal{F}_k^n \subset \mathcal{E}^n$:

$$\mathcal{F}_k^n = \{g = u \circ s : \ s \in C^\infty(E^n, \mathbb{R}^k), u \in C^\infty(\mathbb{R}^k, \mathbb{R}^n)\}.$$

The class of linear Volterra models is denoted by $\mathcal{LF}_k^n$. In case of $k \geq n$ we have $\mathcal{F}_k^n = \mathcal{E}^n$ and $\mathcal{F}_k^n$ does not show exclusion property. We are left to consider the case $k < n$. In this case, one can show that such a system can not have point attractors in $\mathrm{int}(E^n)$ and as a corollary we have that $\mathcal{F}_1^2$ shows exclusion property. However, this is one of the rare classes to be known to satisfy exclusion property. Even more, in case $3k \geq 2n$ one can show that $\mathcal{F}_k^n$ does not show exclusion property [5]. A corollary of Zicarelli's work [12] is that $\mathcal{F}_k^n$ does not satisfy the exclusion property for $k \geq 2$. Nitecki [6] constructed an example that shows that $\mathcal{F}_1^n$, $n \geq 3$ does not satisfy exclusion property. Thus, $\mathcal{F}_k^n$ satisfies exclusion property only for $n = 2, k = 1$. However, $\mathcal{LF}_k^n$ shows exclusion property for $k < n$.
Another important class of models concerns biotic resources, resources which regenerate according to their own differential equations, as would prey species. The defining equations for this class of models are

$$\frac{dx_i}{dt} = x_i u_i(R_1, \dots, R_k), \quad i = 1, \dots, n \tag{7.5}$$
$$\frac{dR_j}{dt} = R_j g_j(R_1, \dots, R_k, x_1, \dots, x_n), \quad j = 1, \dots, k \tag{7.6}$$

where we again have $\dfrac{\partial u_i}{\partial R_j} \geq 0$ and $\dfrac{\partial g_j}{\partial x_i} \leq 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, k$. The class of communities with $n$ consumers and $k$ biotic resources is denoted by $\mathcal{B}_k^n$, i.e we have

$$\mathcal{B}_k^n = \{s : E^n \times E^k \to \mathbb{R}^n \times \mathbb{R}^k; (y, x) \mapsto (u(x), g(y, x))\}$$

To use the results mentioned for the abiotic case we prove the following lemma due to McGehee and Armstrong [5].

**Lemma 7.3.** $\mathcal{B}_k^n \subset \mathcal{F}_{2k}^{n+k}$.

*Proof.* Let $g \in \mathcal{B}_k^n$ and write $g(y, x) = (u(x), s(y, x))$. Define

$$u^* : E^k \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k; (\mu, \nu) \mapsto (u(\mu), \nu)$$
$$r : E^n \times E^k \to E^k \times \mathbb{R}^k; (y, x) \mapsto (x, s(y, x))$$

Then $g = u^* \circ r \in \mathcal{F}_{2k}^{n+k}$. □

We use previous results to conclude that we can construct persistent communities when $2k \geq n$. Another corollary of Zicarelli's work is that $\mathcal{B}_k^n$ does not satisfy exclusion property for any $k, n \in \mathbb{N}$. However, $\mathcal{LB}_k^n$ satisfy exclusion property for $k < n$ as a consequence of the same fact for $\mathcal{LF}_{2k}^{n+k}$ [5].

## 7.2 Example of coexistence

In this section we give an example of coexistence of two competitors on a single biotic resource i.e. a community in $\mathcal{B}_1^2$.
For the biotic resource we presume logistic growth, which is the most often used model of bounded growth and corresponds to the situation where organisms are constrained with different limiting factors such as food and space. The density of the biotic resource is denoted by $N$.
Mutual relation between consumer and resource or predator and prey is described by **functional response**. By definition, it is the number of resource/prey consumed by one consumer/predator per unit of time, typically given as a function of resource density.
For one of the predators we presume a linear functional response which corresponds to the situation where the number of prey captured by a predator per unit of time is proportional to the prey density, say $C_2N$. Such a functional response is also reffered to as the Holling type I functional response [3].
For the second predator we use the Holling type II functional response which which has the form $F(N) = C_2N/(1 + hC_2N)$.
The Holling type II functional response can be derived in the following manner: suppose predators can be in one of two states, either searching for prey or handling the captured prey (e.g. eating, resting). Searchers capture the prey at a linear rate, $C_2N$. Upon capture, predators move into the class of handling predators. Handling predators spend $h$ units of time handling the prey and return to the searching state afterwards.
Suppose that a total of time $T$ is available for searching and handling. The number of prey caught by one predator in time of searching $T_s$ is $T_sC_2N$. Since $T_s = T - T_h$ (where $T_h$ is the time of handling of the captured prey) we have $T_s = T - hT_sC_2N$ and so $T_s = T/(1 + hC_2N)$. The functional response is therefore

$$F(N) = T_sC_2N/T = C_2N/(1 + hC_2N).$$

Let $P_1$ and $P_2$ denote, respectively, the density of predators with the Holling type I and II functional response and $N$ the density of prey. We describe the dynamics with the following system

$$\frac{dP_1}{dt} = P_1 (B_1C_1N - D_1) \tag{7.7}$$

$$\frac{dP_2}{dt} = P_2 \left(\frac{B_2C_2N}{1 + hC_2N} - D_2\right) \tag{7.8}$$

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - \frac{C_2NP_2}{1 + hC_2N} - C_1NP_1. \tag{7.9}$$

The above parameters have the following meaning:

$B_i$ conversion efficiency of food into offspring

$C_i$ searching consumers attack rate

$D_i$ per capita death rate of predator in case of extermination of resource

$h$ handling time for a resource item consumed by predator 2

$r$ intrinsic growth rate of the resource

$K$ carrying capacity of the resource.

We can simplify the system by rescaling to avoid dependence on units and thereby also decrease the number of parameters. We introduce $t' = rt$, $N' = N/K$, $P_1' = P_1/(B_1K)$, $P_2' = P_2/(B_2K)$ and after dropping the primes our system has the following form

$$\frac{dP_1}{dt} = P_1\left(a_1N - d_1\right) \tag{7.10}$$

$$\frac{dP_2}{dt} = P_2\left(\frac{a_2N}{1+bN} - d_2\right) \tag{7.11}$$

$$\frac{dN}{dt} = N(1-N) - \frac{a_2NP_2}{1+bN} - a_1NP_1, \tag{7.12}$$

were we have introduced new parameters $a_i = \dfrac{KB_iC_i}{r}$, $b = KC_2h$, $d_i = \dfrac{D_i}{r}$.
We first analyze the system where only one consumer is present. We start with the simpler case where only $P_1$ is present. Then we have the following system

$$\begin{aligned}\frac{dN}{dt} &= N(1-N) - a_1NP_1\\ \frac{dP_1}{dt} &= P_1(a_1N - d_1).\end{aligned}$$

We observe that we have $\dfrac{dN}{dt} \leq N(1-N)$ and hence we conclude $N(t) \leq \dfrac{\exp(t)C}{1+\exp(t)C}$ where $C = \dfrac{N(0)}{1-N(0)}$. This means that $N$ is bounded and if we start with $N(0) < 1$ we have $N(t) < 1$ for $t \geq 0$. We use this fact to show in a similar way that

$$\exp\left((a_1 - d_1)t\right)P_1(0) \geq P_1(t) \geq \exp(-d_1t)P_1(0).$$

We also observe that $(0, \exp\left(-d_1t\right)P_1(0))$ and $\left(\dfrac{\exp(t)C}{1+\exp(t)C}, 0\right)$ are solutions of the system and hence both of the axes are invariant.
Fixed points of this system are $(N, P_1) = \{(0,0), (1,0), \left(\dfrac{d_1}{a_1}, \dfrac{a_1-d_1}{a_1^2}\right)\}$ where the third point is biologically meaningful if and only if $a_1 \geq d_1$. This corresponds to the case where per capita number of offsprings of predator at carrying capacity of prey is bigger then per capita death rate of predators.

The Jacobian matrix of the system is given by $J = \begin{pmatrix} 1-2N-a_1P_1 & -a_1N \\ P_1a_1 & a_1N-d_1 \end{pmatrix}$. We calculate $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -d_1 \end{pmatrix}$ which is clearly unstable since $1 > 0$ and $d_1 \neq 0$.

For the point $(1,0)$ we have $J(1,0) = \begin{pmatrix} -1 & -a_1 \\ 0 & a_1-d_1 \end{pmatrix}$. Eigenvalues of this matrix satisfy $\lambda_1\lambda_2 = d_1 - a_1$ and $\lambda_1 + \lambda_2 = a_1 - d_1 - 1$. Therefore, the point $(1,0)$ is asimptotically stable if and only if $a_1 \leq d_1$, i.e. in case when the third fixed point is not biologically meaningful.

For the point $(N^*, P_1^*) = \left(\frac{d_1}{a_1}, \frac{a_1-d_1}{a_1^2}\right)$ we have $J(N^*, P_1^*) = \begin{pmatrix} -\frac{d_1}{a_1} & -d_1 \\ \frac{a_1-d_1}{a_1} & 0 \end{pmatrix}$. Its eigenvalues satisfy $\lambda_1 + \lambda_2 = -\frac{d_1}{a_1} < 0$ and $\lambda_1\lambda_2 = \frac{(a_1-d_1)d_1}{a_1} > 0$ and we conclude that $\lambda_{1,2} < 0$ showing that this point is asimptoticaly stable.

For the relation of the parameters $a_1 = d_1$ points $(1,0)$ and $(N^*, P_1^*)$ exchanged stability. This phenomenon is called a **transcritical bifurcation**.

The system where only the predator with Holling type II functional response and prey are present is far more complicated. This system is called the Rosenzweig–MacArthur system, after ecologists who where first to propose it in [8].

$$\begin{aligned} \frac{dN}{dt} &= N(1-N) - \frac{a_2NP_2}{1+bN} \\ \frac{dP_2}{dt} &= P_2\left(\frac{a_2N}{1+bN} - d_2\right). \end{aligned}$$

As in the above example we have $N(t) \leq \frac{\exp(t)C}{1+\exp(t)C}$ and

$$\exp\left(t\left(\frac{a_2}{b+1} - d_2\right)\right) P_2(0) \geq P_2(t) \geq \exp\left(-d_2t\right) P_2(0)$$

We also observe that $(0, \exp\left(-d_1t\right) P_1(0))$ and $\left(\frac{\exp(t)C}{1+\exp(t)C}, 0\right)$ are solutions of the system and hence both of the axis are invariant. Furthermore, for $N(0) > 0$ we have $N(t) > 0$ and similarly for $P_1$ and we conclude that the interior of the first quadrant, denoted by $Q$, is invariant. As for boundedness of solutions, we have the following lemma.

**Lemma 7.4.** *There exists a $S_0 > 0$ such that for all $S \geq S_0$ the triangle $T(S)$ with sides $N = 0$, $P_2 = 0$, $N + P_2 = S$ is invariant.*
*Since every initial point in $Q$ lies in a such triangle, every solution is bounded for $t \geq 0$.*

*Proof.* Since we have already shown the invariance of axis, we have to show that the solution can not leave the triangle through the hypotenuse $N + P_2 = S$. On a such line we have $\frac{dN}{dt} + \frac{dP_2}{dt} = N(1-N) - d_2P_2 = -N^2 + N(1+d_2) - d_2S$. It reaches a maximum, sign of which depends on disciriminant $D = (1+d_2)^2 - 4d_2S$. For $S$ large enough this maximum is negative and solution decreases along the hypotenuse and can not cross it. $\square$

We proceed to find the curves where the right hand sides of equations have zero value. Such curves are called **nullclines**. $P-$nullclines, i.e. the curves where $\frac{dP_2}{dt} = 0$ are $P_2 = 0$ and $N = \frac{d_2}{a_2 - bd_2}$ and $N-$nullclines, i.e. the curves where $\frac{dN}{dt} = 0$ are $N = 0$ and $P_2 = \frac{(1-N)(1+bN)}{a_2}$. We see that between lines $N = 0$ and $N = \frac{d_1}{a_1 - bd_1}$, P decreases and right from $N = \frac{d_1}{a_1 - bd_1}$ it increases. Similarly, between $P = 0$ and $P = \frac{(1-N)(1+bN)}{a}$, N increases and above $P = \frac{(1-N)(1+bN)}{a}$ it decreases. The areas of the first quadrant where the growth of both functions is monotone are denoted by $Q_i$, $i = 1, 2, 3, 4$. This situation is depicted in Figure 5.

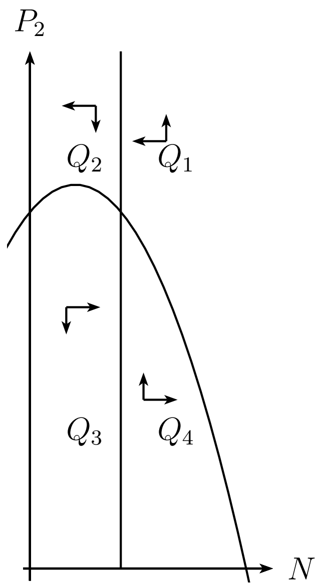


Figure 5: Nullclines of the Rosenzweig–MacArthur system

The fixed points of the system are found as intersections of $N-$nullclines and $P_2$-nullclines and are $(N, P_2) = \{(0,0), (1,0), \left(\frac{d_2}{a_2 - bd_2}, \frac{a_2 - bd_2 - d_2}{(a_2 - bd_2)^2}\right)\}$ where the third point makes biological sense only if $a_2 > d_2(1+b)$. This corresponds to the situation when per capita growth rate of the predator at carrying capacity of resource is positive.

Jacobian matrix of the system is $J = \begin{pmatrix} 1-2N-\dfrac{a_2P_2}{(1+bN)^2} & -\dfrac{a_2N}{1+bN} \\ \dfrac{a_2P_2}{(1+bN)^2} & \dfrac{a_2N}{1+bN}-d_2 \end{pmatrix}$.

We now get $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -d_2 \end{pmatrix}$ and $(0,0)$ is obviously unstable since $1 > 0$ and $d_2 \neq 0$. For the second point we have $J(1,0) = \begin{pmatrix} -1 & -\dfrac{a_2}{1+b} \\ 0 & \dfrac{a_2}{1+b}-d_2 \end{pmatrix}$ and we see that it is asimptotically stable when $\dfrac{a_2}{1+b} < d_2$ and unstable otherwise.

Now we consider the case $a_2 > d_2(1+b)$ i.e. the case when $(1,0)$ is unstable and the third point $(N^{**}, P_2^*) = \left(\dfrac{d_2}{a_2-bd_2}, \dfrac{a_2-bd_2-d_2}{(a_2-bd_2)^2}\right)$ is in the first quadrant. We have

$$J(N^{**}, P_2^*) = \begin{pmatrix} \dfrac{a_2d_2(b-1)-bd_2^2(b+1)}{a_2(a_2-bd_2)} & -d_2 \\ \dfrac{a_2-bd_2-d_2}{a} & 0 \end{pmatrix}.$$

Eigenvalues of this matrix satisfy $\lambda_1\lambda_2 = \dfrac{d_2(a_2-bd_2-d_2)}{a_2}$ and $\lambda_1+\lambda_2 = \dfrac{a_2d_2(b-1)-bd_2^2(b+1)}{a_2(a_2-bd_2)}$. In order to have $\lambda_1\lambda_2 > 0$ and $\lambda_1+\lambda_2 < 0$ we need to have $\dfrac{a_2}{b+1}\dfrac{b-1}{b} < d_2 < \dfrac{a_2}{b+1}$. In this case we have an asimptotically stable point. Observe that for the relation of parameters $\dfrac{a_2}{1+b} = d_2$ the two fixed points interchange stability and we have transcritical bifurcation.

Observe that the maximum of the non-trivial $N-$nullcline is reached for $N = \dfrac{b-1}{2b}$. If the point $(N^{**}, P_2^*)$ is asimptotically stable the above condition can be rewritten into $\dfrac{d_2}{a_2-bd_2} > \dfrac{b-1}{2b}$ showing that in this case the fixed point is on decreasing part of non-trivial $N-$nullcline.

We can show even more, that for the above relation of parameters every solution starting in the interior of $Q$ converges to the point $(N^{**}, P_2^*)$.

**Lemma 7.5.** *Let* $\dfrac{a_2}{b+1}\dfrac{b-1}{b} < d_2 < \dfrac{a_2}{b+1}$. *Then every solution starting in the interior of $Q$ converges to the fixed point* $(N^{**}, P_2^*) = \left(\dfrac{d_2}{a_2-bd_2}, \dfrac{a_2-bd_2-d_2}{(a_2-bd_2)^2}\right)$.

*Proof.* We will use the Lemma 6.14 (Dulac criterion) to show that no periodic orbit or homoclinic loop exist. Then, by the Poincare–Bendixson theorem, for each point $p$ in the open first quadrant $Q$, the $\omega-$limit set of $p$ must contain an equilibrium. If the $\omega_+$ set contains the sink $P_2^*$, then it can contain no other point and we are done. Otherwise, by the third case of Poincare–Bendixson theorem, the $\omega_+$ set of any orbit consists of two other fixed points and non-periodic orbits connecting them. The $\omega_+(p)$

set of points from $Q$ can be neither of two points since the stable manifolds of points are the $y$-axis and the $x$-axis, respectively. For the same reason, the heteroclinic orbit connecting these two points should belong to the $x$-axis and it cannot be the $\omega_+$ set of any $p \in Q$.
Following Hsu [4], we use the Dulac function of the form $g(N, P_2) = \dfrac{1+bN}{N} P_2^{\alpha-1}$ for the corresponding $\alpha$.
We calculate

$$\left(g\frac{dP_2}{dt}\right)_{P_2} + \left(g\frac{dN}{dt}\right)_N = \frac{P_2^{\alpha-1}}{N}\left[-2bN^2 + N(\alpha(a_2 - d_2 b) + b - 1) - \alpha d_2\right].$$

The term in the bracket has a maximum determined by the value of the discriminant $D = (\alpha(a_2 - d_2 b) + (b-1))^2 - 8\alpha d_2 b < (\alpha\frac{2a_2}{b+1} + b - 1)^2 - 8\alpha\frac{a_1}{b+1}(b-1) = (2\alpha\frac{a_1}{b+1} - (b-1))^2$.
Hence for $\alpha = \frac{b^2-1}{2a_2}$ we get that the maximum of the function is smaller then 0 and therefore $(g\frac{dP_2}{dt})_{P_1} + (g\frac{dN}{dt})_N < 0$ inside $Q$ and we can apply the Dulac criterion. □

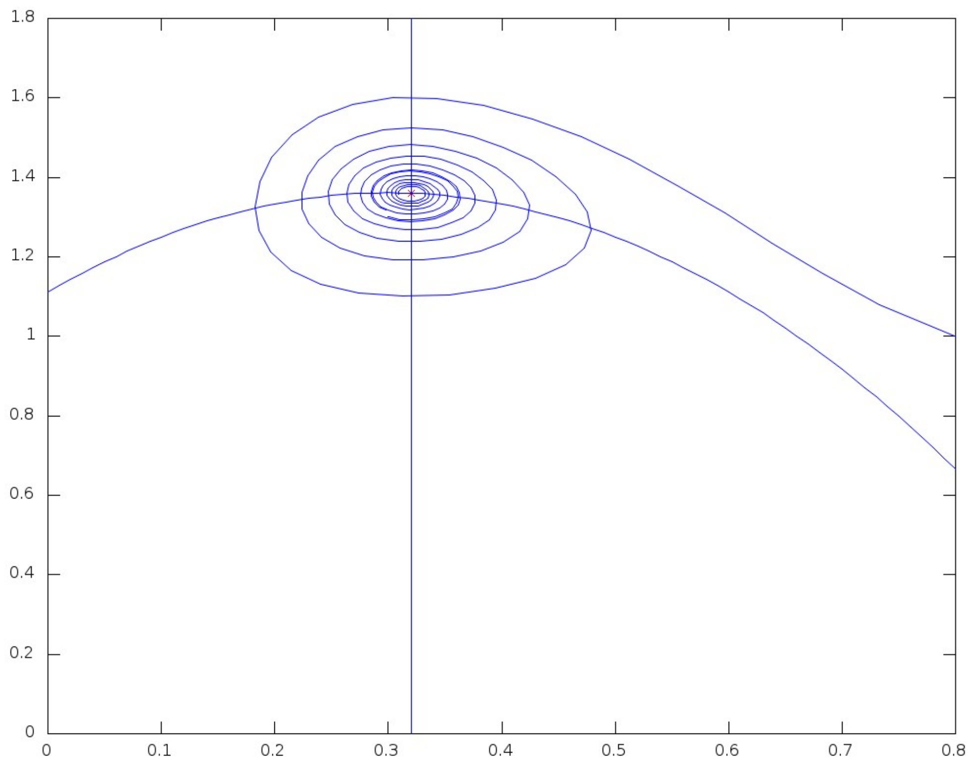


Figure 6: Two orbits of the Rosenzweig–MacArthur system (in coordinates $(N, P_2)$) for the values of parameters $a = 0.9$, $b = 2.5$, $d = 0.16$

We are left with the case $d_2 < \dfrac{a_2}{b+1}\dfrac{b-1}{b}$. This is the case when the positive fixed point is on the increasing part of the non-trivial $N-$nullcline. In this case none of the equilibria are stable. We can show that in this case there is a periodic orbit containing $(N^{**}, P_2^*)$.

**Lemma 7.6.** *Let $d_2 < \dfrac{a_2}{b+1}\dfrac{b-1}{b}$. Then there exists a periodic solution of Rosenzweig–MacArthur system. Furthermore, every solution starting in the interior of $Q$, except for the positive equilibrium, has the periodic orbit as its limit set.*

*Proof.* The main role in the proof is played by the unstable manifold of the point $(1, 0)$. We use the second claim of Theorem 5.5 to find the slope of its tangent at the point $(1, 0)$. If $d_2 < \dfrac{a_2}{b+1}\dfrac{b-1}{b}$ we find that this slope is less (more negative) then the slope of nullcline at the same point. This ensures that of the unstable manifold with $P_2 > 0$ lies above the nullcline.
By the above argument, the orbit of $M^-(1, 0)$ enters $Q_1$ as it leaves $(1, 0)$. Since by Lemma 7.4 all solutions are bounded, if $M^-(1, 0)$ stays in $Q_1$ then because of monotonicity of $N$ and $P_2$ the trajectory should converge to fixed point in $\overline{Q_1}$. However, $(N^{**}, P_2^*)$ is unstable and therefore, trajectory must go to $Q_2$. We apply similar arguments to show that the direction of the trajectory must be $Q_1 \to Q_2 \to Q_3 \to Q_4$.
We have traced the unstable manifold until its second intersection with the right part of nullcline. Observe the area $R$, closed by $\Gamma$, consisting of this part of unstable manifold an the part of nullcline connecting the intersection point and the point $(1, 0)$. It is compact, positively invariant, and contains only one unstable equilibrium $(N^{**}, P_2^*)$. By the Poincare–Bendixson theorem for each $p \in R$ distinct from $(N^{**}, P_2^*)$, the $\omega_+$ limit set of the trajectory through $p$ must be a periodic orbit in $R$.
Additionally, by the similar argument about crossing of the regions and negative invariance of the unstable manifold, we see that every non-fixed orbit starting outside of $R$ will cross the nullcline on the part belonging to $\Gamma$ and enter inside of $R$. Therefore, every non-fixed solution has the periodic orbit as its $\omega_+$ set. □

In this case linearly asimptotically stable point changed to unstable and in addition to an unstable equilibrium we have a limit cycle. In this case we say that for $d_2 = \dfrac{a_2}{b+1}\dfrac{b-1}{b}$ we have a **Hopf bifurcation**. Observe that for this relation of parameters the eigenvalues of the corresponding Jacobian matrix are purely imaginary.

We now proceed by considering coexistence of these two competitors. In order to survive, a small population $P_2$ must be able to invade equilibrium point of $N$ and $P_1$ i.e. we must have $\dfrac{dP_2}{dt} > 0$ near $(N^*, P_1^*)$. Because of continuity we must have $\dfrac{dP_2}{dt}(N^*, P_1^*) > 0$. Since $\dfrac{1}{P_2}\dfrac{dP_2}{dt}$ is strictly increasing and $\dfrac{dP_2}{dt}(N^{**}, P_1^*) = 0$ we must have $N^* > N^{**}$. However, with similar observations we see that if this relation holds

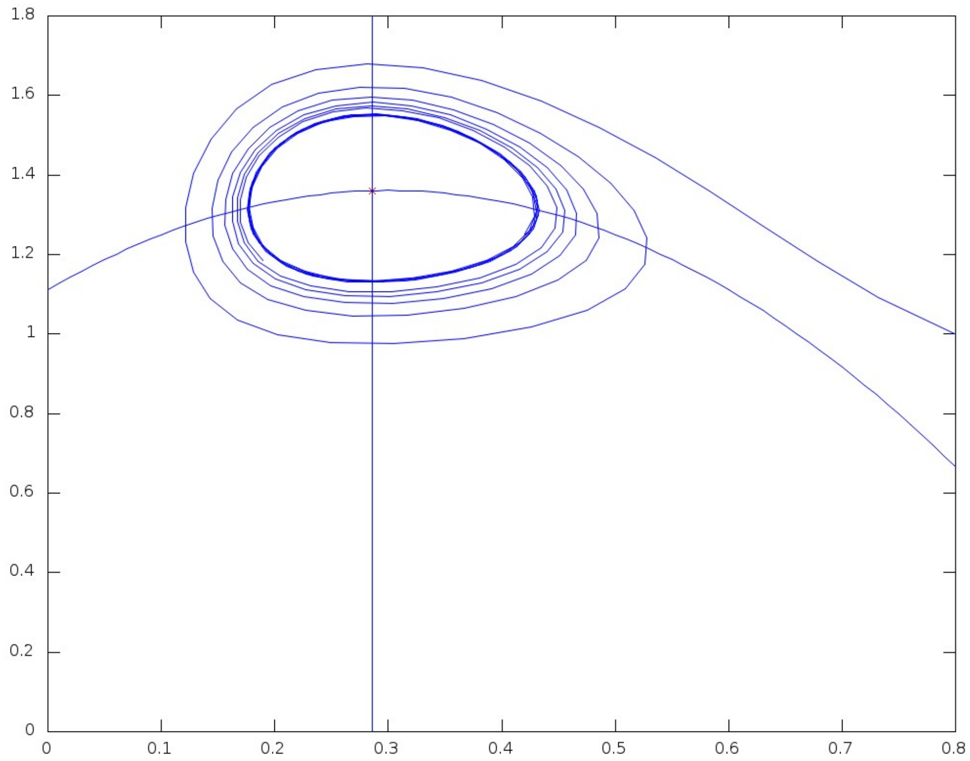


Figure 7: Two orbits of the Rosenzweig–MacArthur system (in coordinates $(N, P_2)$) for the values of parameters $a = 0.9$, $b = 2.5$, $d = 0.15$

$P_1$ can not invade equilibrium of $N$ and $P_2$ since $\dfrac{dP_1}{dt} < 0$ near $(N^{**}, P_2^*)$ . But we have shown that this point is not necessarily stable. The case where coexistence can happen is along the periodic orbit. In this case, in order for $P_1$ to invade $P_2$ and $N$ its average rate of increase along the orbit must be positive. That is, invasion is possible only if

$$\frac{1}{\tau}\int_0^\tau \frac{1}{P_1}\frac{dP_1}{dt}dt = \frac{1}{\tau}\int_0^\tau (a_1 N - d_1)\, dt = a_1\bar{N} - d_1 > 0$$

where $\bar{N} = \dfrac{1}{\tau}\int_0^\tau N(t)dt$. Since we have $a_1N^* - d_1 = 0$ it follows that $\bar{N} > N^*$. This is depicted in Figure 8.

Finally, the condition for coexistence is $\bar{N} > N^* > N^{**}$. To show that this condition is sufficient and that we really have coexistence, one has to consider global properties of the three-species system, as in McGehee and Armstrong [5]. The three dimensional portrait of the system is given in Figure 9.

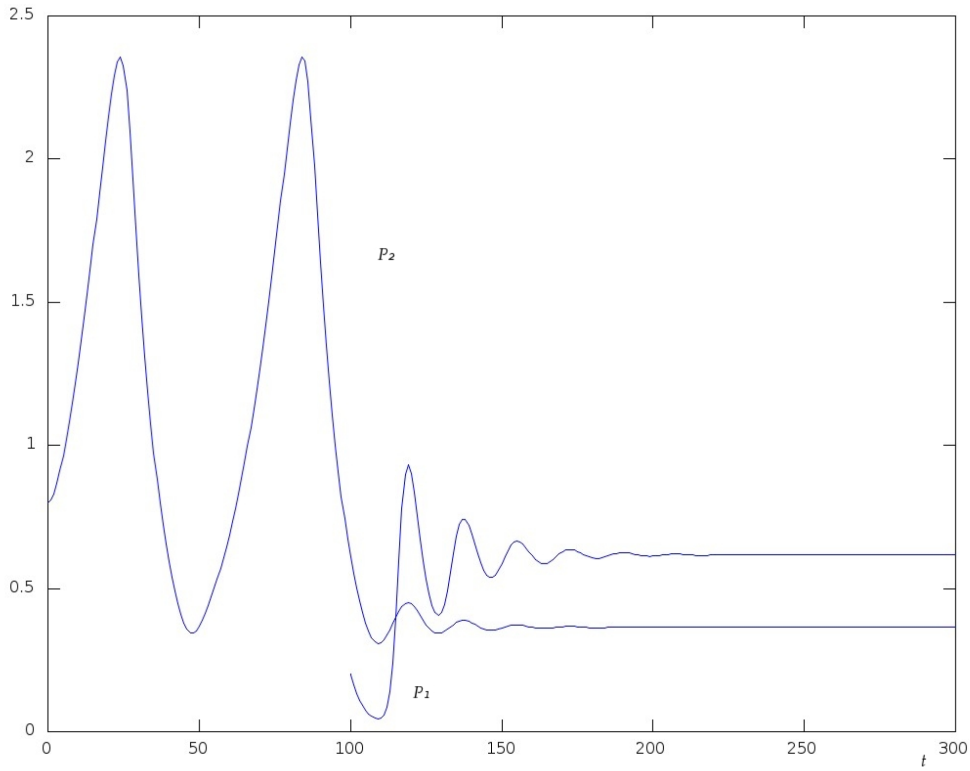


Figure 8: Impact of introduction of predator 1 onto periodic orbit of the prey and predator 2

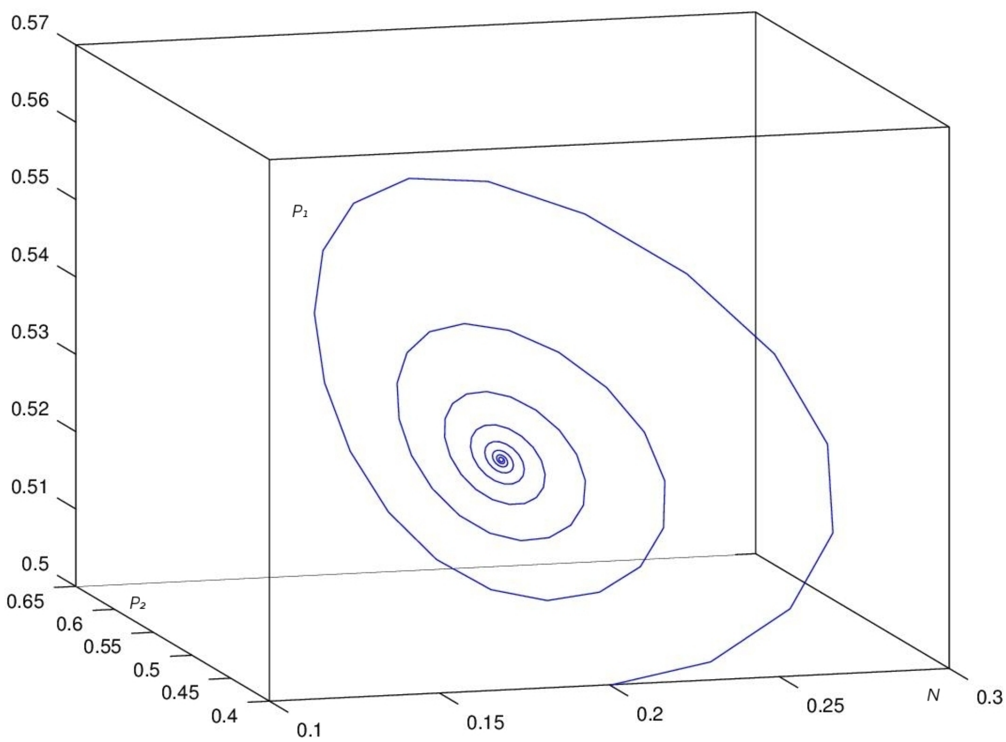


Figure 9: Three dimensional portrait of the system for the values of parameters $a_i = 1$, $d_1 = 0.1$, $d_2 = 0.2$, $b = 5$

# 8 Conclusion

In the final project we studied ordinary differential equations by considering existence and uniqueness of their solutions and their asymptotical behaviour. We were mostly following [10]. Theoretical results were applied on an example in mathematical biology. For the initial value problem of the form

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{8.1}$$

where $f \in C(U, \mathbb{R}^n)$, $U$ an open set in $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$ we used fixed point theorems to show local existence and uniqueness of solution if $f$ satisfies the Lipschitz condition. Peano's theorem guarantees local existence but not uniqueness if $f$ is a continuous function. If $f$ satisfies the Lipschitz condition we have shown that continuous dependence on the initial conditions and parameters which implies that such IVPs are well–posed.
Special attention was dedicated to linear systems of the form

$$\dot{x} = A(t)x + b(t), \quad x(0) = x_0, \tag{8.2}$$

where $A : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n, x : \mathbb{R} \to \mathbb{R}^n, b : \mathbb{R} \to \mathbb{R}^n$. We used results of the previous section to prove existence and uniqueness of solution for continuous functions $A(t)$ and $b(t)$. We gave a closed form of solutions for such systems and use these results to give solutions of higher order linear equations. For autonomous linear systems of the form

$$\dot{x} = Ax, \tag{8.3}$$

we have shown that asymptotical behaviour of solutions depends on eigenvalue decomposition of the matrix $A$.
Further study of solutions of differential equations was accomplished by introducing the notion of a dynamical system. For autonomous equations of the form

$$\dot{x} = f(x), \; x(0) = x_0, \tag{8.4}$$

where $f \in C^k(M, \mathbb{R}^n), k \geq 1$ and $M$ is an open subset of $\mathbb{R}^n$, we introduced the notion of the flow

$$\Phi(t, x) = \phi(t, x)$$

where $\phi(t, x)$ is the value at time $t$ on the maximal integral curve at $x$. We studied (8.4) by introducing notions from theory of dynamical systems: orbits, fixed points, invariant, limit and attracting sets.

We generalized the theory of asymptotic behavior of linear systems to general autonomous systems. We established the connection between a phase portrait of the starting system and its linearization and used this to study local behaviuor of hyperbolic systems near fixed points. The Liapunov method gave us another approach to the study of stability of a fixed point and its area of attraction.

Based on the Jordan curve theorem, we classified all possible limit sets for planar dynamical systems and gave some useful criteria for both existence and non-existence of periodic and homoclinic orbits. For dynamical systems in dimension bigger than 2, such theory does not exist and strange asymptotical behaviour of solutions can occur for readily simple systems.

Discrete dynamical systems are another example of dynamical systems where we are usually interested in behaviour of iterates of some function $f$. While we define and study similar notions as in the continuous case, the asymptotical behaviour can be strange even for continuous functions of one variable. Theory and examples of higher dimensional continuous systems and discrete dynamical systems can be found in [2, 9, 10].

Another important notion in theory of dynamical systems, only briefly mentioned in this paper, are bifurcations. The reader can find more on the theory of bifurcations in [9].

The theory developed throughout the paper was used in an example from the theory of competitive existence. We presented some results on possibility of coexistence of $n$ consumers on $k$ resources, both abiotic and biotic. We gave an example of coexistence of two consumers on a single biotic resource and studied the range of parameters under which the coexistence is possible. For predators we supposed Holling type I and II functional response, while the prey is following the logistic growth model. We proved that for coexistence predator 2 and the prey must coexists along periodic orbit and for certain values of parameters the predator 1 is able to invade that system.

A study of coexistence in $\mathcal{B}_1^2$ for other types of behaviour of species (functional responses, migrations) can be found in [1].

# 9 Povzetek naloge v slovenskem jeziku

V zaključni nalogi smo preučevali navadne diferencialne enačbe pri čemer smo študirali obstoj in enoličnost njihovih rešitev ter njihovo asimptotsko obnašanje. Teoretični rezultati so uporabljeni pri preučevanju primera v matematični biologiji.
Za začetno nalogo

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{9.1}$$

kjer je $f \in C(U, \mathbb{R}^n)$, $U$ odprta množica v $\mathbb{R}^{n+1}$ in $(t_0, x_0) \in U$, smo uporabili izreke o negibni točki za dokaz obstoja in lokalne enoličnosti rešitve za funkcije $f$, ki zadoščajo Lipschitzovemu pogoju. Za zvezne funkcije $f$ smo obstoj (ne pa enoličnosti) dobili iz Peanovega izreka. Za funkcije $f$, ki zadoščajo Lipschitzovemu pogoju smo pokazali zvezno in gladko odvisnost od začetnih pogojev in parametrov iz česar sledi, da je taka začetna naloga korektno zastavljena.
Posebej smo preučevali linearne sisteme oblike

$$\dot{x} = A(t)x + b(t), \quad x(0) = x_0, \tag{9.2}$$

kjer je $A : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n, x : \mathbb{R} \to \mathbb{R}^n, b : \mathbb{R} \to \mathbb{R}^n$. Za dokaz obstoja in enoličnosti rešitve za zvezne funkcije $A(t)$ in $b(t)$ smo uporabili rezultate prejšnjega poglavja. Dobili smo formulo za rešitve tega sistema in jo uporabili za izračun rešitev linearnih enačb višjega reda. Za avtonomne linearne sisteme oblike

$$\dot{x} = Ax. \tag{9.3}$$

smo preučevali asimptotsko obnašanje rešitev v odvisnosti od spektralne dekompozicije matrike $A$.
V nadaljevanju smo preučevali splošnejše sisteme diferencialnih enačb – dinamične sisteme. Omejili smo se na avtonomne sisteme. Cauchyjeva oz. začetna naloga za avtonomne sisteme je oblike

$$\dot{x} = f(x), \ x(0) = x_0, \tag{9.4}$$

kjer je $f \in C^k(M, \mathbb{R}^n), k \geq 1$ in $M$ odprta podmnožica $\mathbb{R}^n$. Uvedli smo pojem toka

$$\Phi(t, x) = \phi(t, x),$$

kjer je $\phi(t, x)$ vrednost v času $t$ na maksimalni integralski krivulji skozi $x$. Nalogo (9.4) smo preučevali z uvedbo pojmov iz teorije dinamičnih sistemov: orbit, stacionarnih točk, invariantnih, limitnih in privlačnih množic.

Teorijo asimptotskega obnašanja linearnih sistemov smo posplošili na splošne avotonomne sisteme. Pokazali smo zvezo med faznim portretom začetnega sistema in faznim portretom njegove linearizacije in to uporabili za preučevanje lokalnega obnašanja hiperboličnih sistemov v bližini stacionarnih točk. Metoda Ljapunova nam da še en pristop preučevanju stabilnosti stacionarnih točk in njihovih območj privlačnosti.

Na podlagi izreka o Jordanski krivulji za ravninske dinamične sisteme smo klasificirali vse možnosti za limitne množice in podali koristne kriterije za obstoj ali neobstoj periodičnih in homokliničnih orbit.

Teorijo, razvito skozi nalogo, smo uporabili na primeru iz teorije tekmovalnega izključevanja. Predstavili smo rezultate o možnostih sobivanja $n$ uporabnikov na $k$ virih, ki so lahko biotični in abiotični. Podali smo primer sobivanja dveh plenilcev na enem plenu in preučevali razpon parametrov, za katere je to mogoče. Za plenilce smo predpostavili funkcijski odziv tipov Holling I in Holling II, za plen pa logistični model rasti. Dokazali smo, da je za sobivanje potrebno, da plenilec 2 in plen sledita periodični orbiti ki jo plenilec 1 lahko "napade" za določene vrednosti parametrov.

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