UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN
INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA
(DOCTORAL THESIS)

O NEKATERIH PROBLEMIH, KI SO POVEZANI S TERWILLIGERJEVIMI ALGEBRAMI IN RAZDALJNO-URAVNOTEŽENIMI GRAFI

(ON CERTAIN PROBLEMS RELATED WITH TERWILLIGER ALGEBRAS AND DISTANCE-BALANCED GRAPHS)

BLAS FERNÁNDEZ

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BLAS FERNÁNDEZ

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My knowledge of math has been shaped by friends, family, teachers, professors, students, colleagues, and those with whom I have shared time in an office and at a university, both here and abroad. When it comes to doing math, inspiration and instruction spring from many sources, for math touches all of life. Let me therefore express my gratitude to those I have met along the way and who have supported me.

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Grief is like the ocean;
It comes in waves,
ebbing and flowing.
Sometimes the water is calm,
and sometimes it is overwhelming.
All we can do is learn to swim.
— Vicki Harrison

Abstract

On certain problems related with Terwilliger algebras and distance-blanced graphs

There has been a sizeable amount of research investigating (distance-regular) graphs that have a Terwilliger algebra T with, up to isomorphism, just a few irreducible T-modules of a certain endpoint, all of which are (non-)thin (with respect to a certain base vertex). These studies generally try to show that such algebraic conditions hold if and only if certain combinatorial conditions are satisfied. A natural follow-up to these results involving Terwilliger algebras of graphs which are not necessarily distance-regular is presented in the first part of this Ph.D. thesis.

Let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Firstly, we study the unique irreducible T-module with endpoint 0. We assume that this T-module is thin. We give a purely combinatorial characterization of this property. This characterization involves the number of walks of a certain shape between vertex x and vertices at some fixed distance from x. Secondly, we assume that x is not a leaf and that the unique irreducible T-module with endpoint 0 is thin. We find a combinatorial characterization of graphs, which also have, up to isomorphism, a unique irreducible T-module with endpoint 1, and this T-module is thin. The characterization of such graphs involves the number of some walks of a particular shape. Moreover, we give precise examples to construct many graphs which possess these properties from our general solution.

Throughout the second part of this Ph.D. thesis, we study certain problems related to the so-called *distance-balanced* graphs. A connected graph Γ is said to be *distance-balanced* if for any edge uv of Γ, the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. The family of distance-balanced graphs is very rich and its study is not only interesting from various purely graph-theoretic aspects, but also because the balancedness property of these graphs makes them very appealing in many research areas.

The notions of *nicely distance-balanced* graphs and *strongly distance-balanced* graphs appears quite naturally in the context of *distance-balanced* graphs as well. A connected graph Γ is called *nicely distance–balanced*, whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for any two adjacent vertices u, v of Γ there are exactly γ vertices of Γ which are closer to u than to v, and exactly γ vertices of Γ which are closer to v than to u. A graph Γ is said to be *strongly distance-balanced* if for any edge uv of Γ and any integer k, the number of vertices at distance k from u and at distance $k+1$ from v is equal to the number of vertices at distance $k+1$ from u and at distance k from v.

It is known that nicely distance-balanced graphs with diameter d and $\gamma = d$ are precisely complete graphs, complete multipartite graphs with parts of cardinality 2, and cycles of length $2d$ or $2d+1$. In this thesis, we classify regular nicely distance-balanced graphs with diameter d and $\gamma = d+1$. Moreover, we solve an open problem posed by Kutnar and Miklavič [57] by constructing several infinite families of nonbipartite nicely distance-balanced graphs which are not strongly distance-balanced. We disprove a conjecture regarding the characterization of strongly distance-balanced graphs posed by Balakrishnan et al. [3] by providing infinitely many counterexamples, and answer a question posed by Kutnar et al. in [55] regarding the existence of semisymmetric distance-balanced graphs which are not strongly distance-balanced by providing an infinite family of such examples. We also show that for a graph Γ with n vertices and m edges it can be checked in $O(mn)$ time if Γ is strongly distance-balanced and if Γ is nicely distance-balanced.

Mathematics Subject Classification: *05C12; 05C25; 05C75.*
Keywords: *distance-regularized vertex; pseudo-distance-regularized vertex; Terwilliger algebra; irreducible module; distance-balanced graph; nicely distance-balanced graph; strongly distance-balanced graph.*

Izvleček

O nekaterih problemih, ki so povezani s Terwilligerjevimi algebra in razdaljno-uravnoteženimi grafi

Mnogo raziskav Terwilligerjevih algeber je bilo do sedaj namenjeno raziskovanju (razdaljno-regularnih) grafov, katerih Terwilligerjeva algebra (glede na neko njihovo vozlišče) ima relativno malo nerazcepnih modulov z danim krajiščem, ter so vsi ti moduli (ne)tanki. V teh raziskavah raziskovalci ponavadi želijo pokazati, da je ta algebraičen pogoj izpolnjen če in samo če graf premore določene kombinatorične lastnosti. Naravno nadaljevanje teh raziskav so raziskave Terwilligerjevih algeber grafov, ki niso nujno razdaljno-regularni. Te raziskave so predstavljene v prvem delu te doktorske disertacije.

Naj bo Γ končen, enostaven in povezan graf. Izberimo si vozlišče x grafa Γ in naj bo $T = T(x)$ pripadajoča Terwilligerjeva algebra grafa Γ. Najprej bomo študirali enolično določen nerazcepen T-modul s krajiščem 0. Podali bomo povsem kombinatorično karakterizacijo lastnosti, da je ta T-modul tanek. V tej karakterizaciji nastopa število sprehodov (ki imajo določeno v naprej predpisano obliko) v grafu Γ med vozliščem x ter vozlišči na določeni fiksni razdalji od vozlišča x. V nadaljevanju bomo potem privzeli, da vozlišče x ni list grafa Γ, ter da je natančno določen nerazcepen T-modul s krajiščem 0 tanek. Podali bomo kombinatorično karakterizacijo lastnosti, da ima graf Γ do izomorfizma natančno en sam nerazcepen T-modul s krajiščem 1, ter je ta modul tanek. Tudi v tem primeru v karakterizaciji nastopa število sprehodov grafa Γ, ki so določene oblike. Podali bomo tudi konstrukcijo neskončne družine grafov, ki imajo opisano lastnost.

V drugem delu te doktorske disertacije bomo študirali nekatere probleme, ki so povezani s tako-imenovanimi *razdaljno-uravnoteženimi* grafi. Za povezan graf Γ rečemo, da je *razdaljno-uravnotežen*, če za vsako njegovo povezavo uv velja, da je število vozlišč grafa Γ, ki so bližja vozlišču u kot vozlišču v, enako številu vozlišč grafa Γ, ki so bližja vozlišču v kot vozlišču u. Družina razdaljno-uravnoteženih grafov je zelo bogata. Študij razdaljno-uravnoteženih grafov ni zanimiv samo iz čisto teoretičnega vidika, ampak tudi zato, ker so zaradi razdaljne-uravnoteženost ti grafi privlačni tudi na mnogih drugih raziskovalnih področjih.

Definiciji *lepo razdaljno-uravnoteženih* grafov in *krepko razdaljno-uravnoteženih* grafov se v kontekstu razdaljno-uravnoteženih grafov pojavita zelo naravno. Povezan graf Γ je *lepo razdaljno-uravnotežen*, če obstaja tako naravno število $\gamma = \gamma(\Gamma)$, da za vsako povezavo uv grafa Γ obstaja natanko γ vozlišč grafa Γ, ki so bližja vozlišču u kot vozlišču v, ter natanko γ vozlišč grafa Γ, ki so bližja vozlišču v kot vozlišču u. Graf Γ je *krepko razdaljno-uravnotežen*, če za vsako vozlišče uv grafa Γ in za vsako celo število k velja, da je število vozlišč grafa Γ, ki so na razdalji k od vozlišča u in na razdalji $k+1$ od vozlišča v, enako številu vozlišč grafa Γ, ki so na razdalji k od vozlišča v in na razdalji $k+1$ od vozlišča u.

Znano je, da za lepo razdaljno-uravnotežen graf Γ s premerom d velja $d \leq \gamma := \gamma(\Gamma)$, ter da so lepo razdaljno-uravnoteženi grafi z $d = \gamma$ natanko polni grafi, polni večdelni grafi $K_{t\times 2}$ $(t \geq 2)$, in pa cikli dolžine $2d$ oziroma $2d+1$. V tej doktorski disertaciji bomo klasificirali regularne lepo razdaljno-uravnotežene grafe, za katere velja $\gamma = d+1$. S konstrukcijo neskončnih družin nedvodelnih lepo razdaljno-uravnoteženih grafov, ki niso krepko razdaljno-uravnoteženi, bomo razrešili problem, ki sta ga v [57] postavila Kutnar in Miklavič. Ovrgli bomo domnevo o karakterizaciji krepko razdaljno-uravnoteženih grafov, ki so jo postavili Balakrishnan in ostali v [3]. Domnevo bomo ovrgli s konstrukcijo neskončno mnogo protiprimerov. Odgovorili bomo tudi na vprašanje Kutnar in ostalih v [55] glede obstoja semi-simetričnih razdaljno-uravnoteženih grafov, ki niso krepko razdaljno-uravnoteženi. Predstavili bomo namreč neskončno družino takih grafov. Pokazali bomo tudi, da če je Γ povezan graf z n vozlišči in m povezavami, potem lahko v času $O(mn)$ preverimo, ali je Γ krepko razdaljno-uravnotežen oziroma lepo razdaljno-uravnotežen.

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Ključne besede: *razdaljno-regularizirano vozlišče; pseudo-razdaljno-regularizirano vozlišče; Terwilligerjeva algebra; nerazcepen modul; razdaljno-uravnotežen graf; lepo razdaljno-uravnotežen graf; krepko razdaljno-uravnotežen graf.*

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Chapter 1

Introduction

Our research deals with certain combinatorial objects known as graphs. A graph $\Gamma = (X, \mathcal{R})$ is a mathematical structure consisting of a vertex set X and a set of edges $\mathcal{R}$ (or nonordered pairs of vertices). Normally, each vertex $x \in X$ is represented by a point and each edge $e = \{x, y\} \in \mathcal{R}$ by a line joining vertices x and y. Graph theory belongs to combinatorics, which is the part of mathematics that studies the structure and enumeration of discrete objects, in contrast to the continuous objects studied in mathematical analysis. In particular, graph theory is useful for studying any system with a certain relationship between pairs of elements, which give a binary relation. It is therefore not surprising that many problems and results can be formulated using these notions.

Throughout this Ph.D. dissertation, the interaction of these combinatorial objects together with certain algebraic methods is particularly strong and significant. Moreover, the main subject will have a special focus on the study of Terwilliger algebras of graphs which are not necessarily distance-regular as well as on some problems related to the so-called distance-balanced graphs.

The structure and content of this Ph.D. thesis is roughly divided into four parts: the introduction, parts A and B (where we show the description of the scientific background and the academic contributions), and the conclusion. The introduction consists of Chapter 1; where the basic concepts of the theory of Terwilliger algebras and distance-balanced graphs are discussed, and the goals and results of this thesis are explained. Part A, which includes Chapters 2-7, is called "On the Terwilliger algebra of a graph". Here, we present results echoing the surrounding literature on T-algebras of a distance-regular graph. Indeed, we compare these results and state our contributions in a more general setting. Throughout Chapters 2-7, our research is concentrated around irreducible T-modules with endpoint at most 1 of certain graphs, that are not necessarily distance-regular. The reader may bookmark Chapters 3 and 7 where we give our novel results on some combinatorial characterizations involving the number of certain walks in a graph, which are of a particular shape. Part B, which includes Chapters 8-10, is called "On distance-balanced graphs" and is devoted to the classification and the constructions of certain families of distance-balanced graphs which seem to be of interest from various purely graph-theoretic aspects. Finally, in the conclusion, which consists of Chapters 11 and 12, we briefly discuss our contributions to algebraic combinatorics and make some suggestions for further research.

During this Ph.D dissertation, we assume familiarity with the basic definitions coming from graph theory and algebraic combinatorics. We refer the reader to [6, 39, 40, 96] for additional background and notational conventions. We also point out several particular textbooks and research articles in which the reader can get acquainted with other aspects of the theory. Numbering of statements (notations, definitions, lemmas, propositions, theorems) is done in the thesis by sections. For instance, Theorem 7.2.5 denotes the fifth statement in Section 7.2 of Chapter 7. Moreover, all the original results are contained in research papers which are/will be published in specialized SCI journals; see [23, 24, 25, 26, 27, 28, 29] for more details.

Additionally, we point out that each chapter introduces the corresponding basic knowledge which is fundamental to understand the methods and results that are presented in this Ph.D. thesis. Although reading each chapter requires, of course, familiarity with basic concepts of abstract and linear algebra, and graph theory, we do not assume knowledge of any specific preliminary information, meaning that any experienced reader may read a chapter independently of the contents shown in the other ones. This method, in my opinion, allows a simple and clear approach to understand both classical and new results. Undoubtedly, specialists will notice the multiple presence of some definitions and results. Nevertheless, we hope that the style of presenting information will enable the reader to learn and understand our contributions and to acquire sufficient background to follow and be able to get familiar with contemporary investigations on algebraic combinatorics.

1.1 On the Terwilliger algebra of a graph

Let Γ be a graph and let G be a certain algebraic object, associated with Γ. In this case, one of the main motivations in our research is the following question: *What could we say about the combinatorial properties of* Γ*, if we know that* G *has certain algebraic properties?* And vice-versa: *What could we say about the algebraic properties of* G*, if we know that*

Γ *has certain combinatorial properties?* Perhaps the most well-known example of this interplay between combinatorics and algebra is obtained if G is the automorphism group of Γ. In this case there are many relations between combinatorial properties in Γ and algebraic properties of G. For example, if G acts transitively on the set of vertices of Γ, then Γ is regular, in the sense that every vertex of Γ has the same number of neighbours. Notice there are many more examples of this interplay available in the literature.

In this Ph.D. dissertation the algebraic object, associated with Γ, will not be its automorphism group, but rather a certain matrix algebra, called a *Terwilliger algebra of graph* Γ. The main motivation, however, remains the same: *What could we say about the combinatorial properties of* Γ*, if we know that the corresponding Terwilliger algebra has certain algebraic properties?* And vice-versa: *What could we say about the algebraic properties of the corresponding Terwilliger algebra of* Γ*, if we know that* Γ *has certain combinatorial properties?*

Terwilliger algebras of association schemes were defined by Terwilliger in [89, Definition 3.3], where they were called *subconstituent algebras*. These noncommutative algebras are generated by the Bose-Mesner algebra of the scheme, together with matrices containing local information about the structure with respect to a fixed vertex. Since then, numerous papers have appeared in which the Terwilliger algebra was successfully used for studying commutative association schemes and distance-regular graphs; see [43, 44, 60, 65, 68, 78, 79, 81, 84, 86] for the most recent research on the subject.

The algebra T was mainly used to study distance-regular graphs (see, for example, [6] for the definition of distance-regular graphs). This algebra has also been used to study the Q-polynomial distance-regular graphs [9, 11, 38, 47, 58, 72, 71] (see [6, page 135] for the definition of Q-polynomial distance-regular graphs), bipartite distance-regular graphs, almost-bipartite distance-regular graphs [13], group association schemes [4, 5], strongly regular graphs [95], Doob schemes [85] (see [6, page 27] for the definition of a Doob scheme), association schemes over the Galois rings of characteristic four [51], and has been even used in coding theory [37, 83].

Although the notion of a Terwilliger algebra could be easily generalized to an arbitrary finite, simple and connected graph, the state of the art regarding Terwilliger algebras of graphs, which are not distance-regular, is not so intensive. In [54, 61], the Terwilliger algebra of the incident graph of the so-called Johnson geometry was studied. In [94] the author studied the Terwilliger algebra of the incident graph of the Hamming graph. In [93] the relation between the Terwilliger algebra of a graph Γ and another matrix algebra associated with Γ, the so-called *quantum adjacency algebra of* Γ, was investigated. Moreover, in [59, 97] the authors studied the structure of certain T-algebras of finite trees. These results are the most recent research on the subject in this direction.

Throughout this section, let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ which is not a leaf and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. The algebra T is non-commutative and since it is closed under the conjugate-transpose map, any T-module is an orthogonal direct sum of irreducible T-modules. Therefore, in many instances this algebra can best be studied via its irreducible modules.

Assume now for a moment that Γ is distance-regular. It turns out that in this case the unique irreducible T-module with endpoint 0 is thin. Assume also that Γ is bipartite. It turns out that T has, up to isomorphism, a unique irreducible T-module with endpoint 1, and that this module is thin. It is for this reason that in this case irreducible T-modules with endpoint 2 were intensively studied; see for example [9, 11, 15, 16, 17, 18, 19, 20, 38, 62, 63, 66, 67, 69, 70, 81]. On the other hand, if Γ is nonbipartite, then the structure of irreducible T-modules of endpoint 1 is far more complicated than that of the bipartite case. For the relevant literature on this subject see, for example, [21, 47, 71, 72, 92].

Our research will be concentrated around irreducible T-modules with endpoint at most 1 of certain graphs, that are not necessarily distance-regular.

As already mentioned, there has been a sizeable amount of research investigating distance-regular graphs that have a Terwilliger algebra T with, up to isomorphism, just a few irreducible T-modules of a certain endpoint, all of which are (non-)thin (with respect to a certain base vertex); see, for example, [63, 64, 65, 66, 67, 68, 74, 81]. These studies generally try to show that such algebraic conditions hold if and only if certain combinatorial conditions are satisfied. A natural follow-up to these results involving Terwilliger algebras of non-distance-regular graphs is presented here.

It turns out that there exists a unique irreducible T-module with endpoint 0. It was already proved in [88] that this irreducible T-module is thin if Γ is distance-regular around x. The converse, however, is not true. Fiol and Garriga [33] later introduced the concept of *pseudo-distance-regularity* around vertex x, which is based on assigning weights to the vertices where the weights correspond to the entries of the (normalized) positive eigenvector. They showed that the unique irreducible T-module with endpoint 0 is thin if and only if Γ is pseudo-distance-regular around x (see also [30, Theorem 3.1]). To start our investigations, in Chapter 3 we give a purely combinatorial characterization of the property, that the unique irreducible T-module with endpoint 0 is thin. This characterization involves the number of walks of a certain shape between vertex x and vertices at some fixed distance from x.

Assume now that the unique irreducible T-module with endpoint 0 is thin, or equivalently, that x is pseudo-distance-regularized. The next goal is to find a combinatorial characterization of graphs, which also have a unique irreducible T-module with endpoint 1 (up to isomorphism), and this module is thin. If Γ is distance-regular, then this situation occurs if and only if Γ is bipartite or almost-bipartite [21, Theorem 1.3]. In Chapter 4 we show that if Γ is distance-biregular, then again Γ has (up to isomorphism) a unique irreducible T-module with endpoint 1, and this module is thin. The case when Γ is distance-regular around x but not necessarily distance-regularized (distance-regular or distance-biregular) is considered in Chapter 5 and in Chapter 6. Moreover, we generalize the above results to the case when Γ is not necessarily distance-regular around x in Chapter 7. The main result of this Ph.D. thesis is a combinatorial characterization of such graphs that involves the number of some walks in Γ of a particular shape. We remark that this result is a generalization of previous efforts in [13, 16, 21] to understand and classify graphs which are pseudo-distance-regular around a fixed vertex and also have a unique irreducible T-module (up to isomorphism) with endpoint 1, and this module is thin. Last but not least, we give precise examples to construct many graphs which possess these properties from our general solution.

1.2 On distance-balanced graphs

Let $\Gamma = (X, \mathcal{R})$ be a finite, simple, undirected, connected graph and let X and $\mathcal{R}$ denote the vertex set and the edge set of Γ, respectively. For $u, v \in X$, let $\partial(u, v) = \partial_\Gamma(u, v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of Γ we denote

$$W_{u,v} = \{x \in X \mid \partial(x, u) < \partial(x, v)\}.$$

We say that Γ is *distance–balanced* (DB for short) whenever for an arbitrary pair of adjacent vertices u and v of Γ we have that

$$|W_{u,v}| = |W_{v,u}|.$$

The investigation of distance-balanced graphs was initiated in 1999 by Handa [45], who considered distance-balanced partial cubes. The term itself was introduced by Jerebic, Klavžar and Rall in [52], who gave some basic properties and characterized Cartesian and lexicographic products of distance-balanced graphs. The family of distance-balanced graphs is very rich and its study is interesting from various purely graph-theoretic aspects where one focuses on particular properties of such graphs such as symmetry [55, 56, 98], connectivity [45, 75], or complexity aspects of algorithms related to such graphs [8]. However, the balancedness property of these graphs also makes them very appealing in areas such as mathematical chemistry and communication networks. For instance, the investigation of such graphs is highly related to the well-studied Wiener index and Szeged index (see [2, 52, 50, 87]) and they present very desirable models in various real-life situations related to (communication) networks [2]. Recently, the relations between distance-balanced graphs and the traveling salesman problem were studied in [12]. It turns out that these graphs can be characterized by properties that at first glance do not seem to have much in common with the original definition from [52]. For example, in [3] it was shown that the distance-balanced graphs coincide with the *self-median* graphs, that is, graphs for which the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex. Other such examples are *equal opportunity graphs* (see [2] for the definition). In [2] it is shown that distance-balanced graphs of even order are also equal to opportunity graphs. Finally, let us also mention that various generalizations of the distance-balanced property were defined and studied in the literature; see, for example, [1, 36, 49, 53, 76].

The notion of nicely distance-balanced graphs appears quite naturally in the context of DB graphs. We say that Γ is *nicely distance–balanced* (NDB for short) whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for an arbitrary pair of adjacent vertices u and v of Γ,

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds. Clearly, every NDB graph is also DB, but the opposite is not necessarily true. For example, if $n \geq 3$ is an odd positive integer, then the prism graph on $2n$ vertices is DB, but not NDB.

Assume now that Γ is NDB. Let us denote the diameter of Γ by d (the *diameter* of a graph is the maximum distance between two vertices). In [57], where these graphs were first defined, it was proved that $d \leq \gamma$ and NDB graphs with $d = \gamma$ were classified. It turns out that Γ is NDB with $d = \gamma$ if and only if Γ is either isomorphic to a complete graph on $n \geq 2$ vertices, a complete multipartite graph with parts of cardinality 2, or to a cycle on $2d$ or $2d+1$ vertices. In this Ph.D. thesis we study NDB graphs with $\gamma = d+1$. The situation in this case is much more complex than in the case $\gamma = d$. Therefore, we will concentrate our study on the class of regular NDB graphs with $\gamma = d+1$ in Chapter 9. The main result is shown in Theorem 9.7.1 where the classification of such graphs is given.

Another concept closely related to the concept of distance-balanced graphs is the one of strongly distance-balanced graphs. For an arbitrary pair of adjacent vertices u and v of a given graph Γ, and any two non-negative integers i,j, we let

$$D^i_j(u,v) = \{x \in X \mid \partial(u,x) = i \text{ and } \partial(v,x) = j\}.$$

A graph Γ is called *strongly distance-balanced* (SDB for short) if $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$ holds for every $i \geq 1$ and every pair of adjacent vertices u and v of Γ. It is easy to see that a strongly distance-balanced graph is also distance-balanced, but the converse is not true in general (see [55]). For more results on this and related concepts see, for example, [3, 8, 50, 57, 75].

Throughout Chapter 10 we focus our attention on some problems about distance-balanced graphs, especially on the construction of certain families of DB graphs which seem to be of interest in this area of research.

Our first construction is related to certain NDB graphs which are not SDB. Nicely distance-balanced graphs were studied in [57], where it is proved that in the class of bipartite graphs, the families of DB graphs and NDB graphs coincide, while there are examples of bipartite NDB graphs that are not SDB given by Handa [45]. Moreover, in [57], examples of nonbipartite SDB graphs that are not NDB were constructed. In Chapter 10 we solve [57, Problem 3.3] posed by Kutnar and Miklavič regarding the existence of nonbipartite NDB graphs which are not SDB by constructing several infinite families of such graphs.

Our second construction is related with a conjecture by Balakrishnan et al. about a characterization of SDB graphs. Let Γ be a graph, and let S be a subset of its vertex set. For a vertex v of Γ we define

$$\partial(v,S) = \sum_{x \in S} \partial(v,x).$$

Balakrishnan et al. [3] proved that a connected graph Γ is distance-balanced if and only if $\partial(v,X) = \partial(u,X)$ for all $u,v \in X$. Moreover, they conjectured that a graph Γ is strongly distance-balanced if and only if $\partial(u,W_{u,v}) = \partial(v,W_{v,u})$ holds for every pair of adjacent vertices u,v of Γ. It is clear that strongly distance-balanced graphs satisfy the above condition, but the question was if the converse also holds. In Chapter 10 we disprove [3, Conjecture 3.2] by providing infinitely many counterexamples.

Our third construction deals with the property of being (strongly) distance-balanced in the context of graphs enjoying certain special symmetry conditions. Kutnar et al. showed that vertex-transitive graphs are not only distance-balanced, they are also strongly distance-balanced (see [55]). Furthermore, since being vertex-transitive is not a necessary condition for a graph to be distance-balanced, it was therefore natural for the authors to explore the property of being distance-balanced within the class of *semisymmetric graphs*: a class of objects which are as close to vertex-transitive graphs as one can possibly get, that is, regular edge-transitive graphs which are not vertex-transitive. The smallest semisymmetric graph has 20 vertices and its discovery is due to Folkman [35], the initiator of this topic of research. A semisymmetric graph is necessarily bipartite, with the two sets of bipartition coinciding with the two orbits of the automorphism group. Consequently, semisymmetric graphs have no automorphisms which switch adjacent vertices, and therefore, may arguably be considered as good candidates for graphs which are not distance-balanced. Indeed, Kutnar et al. proved there are infinitely many semisymmetric graphs which are not distance-balanced, but there are also infinitely many semisymmetric graphs which are distance-balanced. In Chapter 10 we also answer [55, Question 4.6] posed by Kutnar et al. regarding the existence of semisymmetric DB graphs which are not SDB by providing infinite families of such graphs.

We conclude Chapter 10 by showing that for a graph Γ with n vertices and m edges it can be checked in $O(mn)$ time if Γ is strongly distance-balanced and if Γ is nicely distance-balanced.

Part A

On the Terwilliger algebra of a graph

Chapter 2

Overview

Terwilliger algebras of association schemes were defined by Terwilliger in [89], where they were called subconstituent algebras. These noncommutative algebras are generated by the Bose-Mesner algebra of the scheme, together with matrices containing local information about the combinatorial structure with respect to a fixed vertex. Numerous papers have appeared since then in which the Terwilliger algebra has been successfully used to study commutative association schemes and distance-regular graphs; see [10, 43, 44, 60, 63, 64, 65, 66, 67, 68, 73, 78, 79, 81, 84, 86] for the most recent research on the subject. However, the notion of a Terwilliger algebra can be easily generalized to an arbitrary finite, simple, and connected graph; see, for example, [26, 27, 30, 33, 59, 93, 94, 97], where Terwilliger algebras of non-distance-regular graphs were studied.

Let us first recall the definition of a Terwilliger algebra (see Section 3.1 for further details and formal definitions). Let Γ denote a finite, simple, connected graph with vertex set X. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Pick a vertex x of Γ and let $\epsilon(x)$ denote its eccentricity. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ and let E_i^* $(0 \leq i \leq \epsilon(x))$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose (y, y)-entry is equal to 1 if the distance between x and y is i, and 0 otherwise $(y \in X)$. We refer to matrices E_i^* $(0 \leq i \leq \epsilon(x))$ as *dual idempotents* of Γ with respect to x. The *Terwilliger algebra* $T = T(x)$ is a matrix subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix of Γ and the dual idempotents of Γ with respect to x. Algebra T acts on the space of all column vectors with coordinates indexed by X. Observe that T is closed under the conjugate-transpose map. Moreover, it follows that each T-module is a direct sum of irreducible T-modules. Therefore, in many instances the algebra T can best be studied via its irreducible modules. We now recall an important parameter which is assigned to every irreducible T-module. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i \mid 0 \leq i \leq \epsilon(x), E_i^* W \neq 0\}$. We say that W is *thin* if $\dim E_i^* W \leq 1$ for every $0 \leq i \leq \epsilon(x)$.

As previously stated, a substantial amount of research has been conducted on distance-regular graphs that have a Terwilliger algebra T with, up to isomorphism, just a few irreducible T-modules of a certain endpoint, all of which are (non)thin (with respect to a certain base vertex); see for example [63, 64, 65, 66, 67, 68, 74, 81]. These studies generally try to show that such algebraic conditions hold if and only if certain combinatorial conditions are satisfied. A natural follow-up to these results is presented here. Our research will be concentrated around irreducible T-modules with endpoint at most 1 of certain graphs, that are not necessarily distance-regular.

It turns out that there exists a unique irreducible T-module with endpoint 0. It was already proved in [88] that this irreducible T-module is thin if Γ is distance-regular around x. The converse, however, is not true. Fiol and Garriga [33] later introduced the concept of *pseudo-distance-regularity* around vertex x, which is based on assigning weights to the vertices where the weights correspond to the entries of the (normalized) positive eigenvector. They showed that the unique irreducible T-module with endpoint 0 is thin if and only if Γ is pseudo-distance-regular around x (see also [30, Theorem 3.1]). To start our investigations, in Chapter 3 we give a purely combinatorial characterization of the property, that the unique irreducible T-module with endpoint 0 is thin. This characterization involves the number of walks of a certain shape between vertex x and vertices at some fixed distance from x.

Assume now that the unique irreducible T-module with endpoint 0 is thin, or equivalently that x is pseudo-distance-regularized. The next goal is to find a combinatorial characterization of graphs, which also have a unique irreducible T-module with endpoint 1 (up to isomorphism), and this module is thin. If Γ is distance-regular, then this situation occurs if and only if Γ is bipartite or almost-bipartite [21, Theorem 1.3]. In Chapter 4 we show that if Γ is distance-biregular, then, again, Γ has (up to isomorphism) a unique irreducible T-module with endpoint 1, and this module is thin. The case when Γ is distance-regular around x but not necessarily distance-regularized (distance-regular or distance-biregular) will be considered in Chapter 5 and in Chapter 6. Moreover, we generalize the above results to the case when Γ is not necessarily distance-regular around x in Chapter 7. The main result of this Ph.D. thesis is a combinatorial characterization of such graphs that involves the number of some walks in Γ of a particular shape. Moreover, we give examples of graphs that possess the above-mentioned combinatorial properties. We remark that the main result of this Ph.D. thesis is a generalization of previous efforts in [13, 16, 21] to understand and classify graphs which are pseudo-distance-regular around a fixed vertex and also have a unique irreducible T-module (up to isomorphism) with endpoint 1, and this module is thin.

Chapter 3

On the trivial T-module of a graph

Let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. In this chapter we study the unique irreducible T-module with endpoint 0. We assume that this T-module is thin. The main result of the chapter is a combinatorial characterization of this property. This characterization involves the number of walks between vertex x and vertices at some fixed distance from x, which are of a certain shape.

The chapter is organized as follows. In Sections 3.1, 3.2 and 3.3 we recall basic definitions and results about Terwilliger algebras, non-negative irreducible matrices, and local (pseudo)-distance-regularity, respectively. In Section 3.4 we prove that the unique irreducible T-module with endpoint 0 is thin if and only if Γ is pseudo-distance-regular around the base vertex x. In Section 3.5 we present our main result, and we prove it in Section 3.6. We conclude the chapter with a couple of examples in Section 3.7.

The chapter is based on joint work with Štefko Miklavič. Our main results are currently published in The Electronic Journal of Combinatorics (2022); see [28] for more details.

3.1 Preliminaries

In this section we review some definitions and basic concepts. Here, we also provide proofs to some well-known results in the literature which will be frequently used throughout this Ph.D. dissertation. These proofs may serve as examples of ways to use the tools provided by Terwilliger in [88, 89]. We remark that there may exist more efficient ways to prove these results under certain particular assumptions such as considering association schemes or distance-regular graphs.

Throughout this chapter, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops and multiple edges, with vertex set X and edge set $\mathcal{R}$.

Let $x, y \in X$. The **distance** between x and y, denoted by $\partial(x,y)$, is the length of a shortest xy-path. The **eccentricity of** x, denoted by $\epsilon(x)$, is the maximum distance between x and any other vertex of Γ: $\epsilon(x) = \max\{\partial(x,z) \mid z \in X\}$. Let D denote the maximum eccentricity of any vertex in Γ. We call D the **diameter of** Γ. For an integer i we define $\Gamma_i(x)$ by

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}.$$

We will abbreviate $\Gamma(x) = \Gamma_1(x)$. Note that $\Gamma(x)$ is the set of neighbours of x. Observe that $\Gamma_i(x)$ is empty if and only if $i < 0$ or $i > \epsilon(x)$.

Let $\mathbb{C}$ denote the complex number field. A vector space $(V, +, \cdot)$ over $\mathbb{C}$ with a multiplication $\star : V \times V \to V$ is called a $\mathbb{C}$**-algebra** in case that $(V, +, \star)$ is a ring with identity where for every $\alpha \in \mathbb{C}$ and for all $u, v \in V$, the following hold:

$$(\alpha \cdot u) \star v = u \star (\alpha \cdot v) = \alpha \cdot (u \star v).$$

We now recall some definitions and basic results concerning a Terwilliger algebra of Γ. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let V denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication: if $B \in \mathrm{Mat}_X(\mathbb{C})$ and $v \in V$ then $Bv \in V$. We call V the **standard module**. We endow V with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^\top \overline{v}$ for $u, v \in V$, where $\top$ denotes transpose and $^-$ denotes complex conjugation. For $y \in X$, let $\widehat{y}$ denote the element of V with a 1 in the y-coordinate and 0 in all other coordinates. We observe that $\{\widehat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. That is, the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with entries given as follows:

$$(A)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = 1, \\ 0 & \text{if} \quad \partial(x,y) \neq 1, \end{cases} \qquad (x, y \in X).$$

The **adjacency algebra of** Γ, also called the **Bose-Mesner algebra of** Γ, is the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ.

We now recall the dual idempotents of Γ. To do this, fix a vertex $x \in X$ and let $d = \epsilon(x)$. We view x as a *base vertex*. For $0 \le i \le d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the **i-th dual idempotent of Γ with respect to x** [89, p. 378]. We also observe that (ei) $\sum_{i=0}^{d} E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le d)$; (eiii) $E_i^{*\top} = E_i^*$ $(0 \le i \le d)$; (eiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le d)$ where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. By these facts, matrices $E_0^*, E_1^*, \dots, E_d^*$ form a basis for the commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the **dual Bose-Mesner algebra of Γ with respect to x** [89, p. 378]. For convenience we define E_{-1}^* and E_{d+1}^* to be the zero matrix of $\mathrm{Mat}_X(\mathbb{C})$. Note that for $0 \le i \le d$ we have

$$E_i^* V = \mathrm{Span}\{\widehat{y} \mid y \in \Gamma_i(x)\},$$

and that

$$V = E_0^* V + E_1^* V + \cdots + E_d^* V \qquad (\textit{orthogonal direct sum}).$$

We call $E_i^* V$ the **i-th subconstituent of Γ with respect to x**.

We recall the definition of a Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the **Terwilliger algebra of Γ with respect to x**. Recall that M is generated by A. So, T is generated by A and the dual idempotents. We observe that T has finite dimension. In addition, since by construction T is generated by real-symmetric matrices, it follows that T is closed under the conjugate-transpose map.

We now recall the lowering, the flat and the raising matrix of T.

Definition 3.1.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$. Let $d = \epsilon(x)$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Define $L = L(x)$, $F = F(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by*

$$L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*, \qquad F = \sum_{i=0}^{d} E_i^* A E_i^*, \qquad R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*.$$

*We refer to L, F and R as the **lowering**, the **flat** and the **raising matrix with respect to** x, respectively. Note that $L, F, R \in T$. Moreover, $F = F^\top$, $R = L^\top$ and $A = L + F + R$.*

Observe that for $y,z \in X$ we have that the (z,y)-entry of L equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) - 1$, and 0 otherwise. The (z,y)-entry of F is equal to 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y)$, and 0 otherwise. Similarly, the (z,y)-entry of R equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) + 1$, and 0 otherwise. Consequently, for $v \in E_i^*V$ $(0 \le i \le d)$ we have

$$Lv \in E_{i-1}^*V, \qquad Fv \in E_i^*V, \qquad Rv \in E_{i+1}^*V. \tag{3.1}$$

For a vector subspace $W \subseteq V$, we denote by TW the subspace $\{Bw \mid B \in T, w \in W\}$. By a T**-module** we mean a subspace W of V, such that $TW \subseteq W$. Let W denote a T-module. Then W is said to be **irreducible** whenever W is nonzero and W contains no T-modules other than 0 and W.

Since the algebra T is closed under the conjugated-transpose map, it follows that the orthogonal complement of a T-module is also a T-module. In other words, if W is a T-module then the subspace $W^\perp = \{v \in V \mid \langle v,w\rangle = 0, \text{ for all } w \in W\}$ is a T-module. In fact, we notice that for every $v \in W^\perp$, $w \in W$ and every matrix $B \in T$,

$$\langle Bv, w\rangle = \left\langle v, \overline{B}^\top w \right\rangle = 0$$

since $\overline{B}^\top w \in W$ for every $B \in T$. This shows that $W^\perp$ is invariant under the action of any matrix in T and proves our claim.

Proposition 3.1.2. *Let $\Gamma = (X,\mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Let W_1 and W_2 denote T-modules such that $W_2 \subseteq W_1$. Then, $W_1 \cap W_2^\perp$ is a T-module and $W_1 = W_2 + (W_1 \cap W_2^\perp)$ (orthogonal direct sum).*

Proof. We observe that the intersection of T-modules is also a T-module. Therefore, since W_1 and W_2 are T-modules, it follows that $W_1 \cap W_2^\perp$ is a T-module. Now, let $P_{W_2} : V \to V$ be the orthogonal projection onto the subspace W_2. Recall that $P_{W_2}x = y$ if and only if $y \in W_2$ is the only vector such that $x - y \in W_2^\perp$. Then, for any $x \in W_1$ we have that $P_{W_2}x \in W_2$ and $x - P_{W_2}x \in W_2^\perp$. Consequently, since W_2 is a subspace of W_1 and $x = P_{W_2}x + (x - P_{W_2}x)$ it follows that $W_1 = W_2 + (W_1 \cap W_2^\perp)$ (orthogonal direct sum). ■

Since the algebra T is closed under the conjugate-transpose map, it turns out that any T-module is an orthogonal direct sum of irreducible T-modules.

Lemma 3.1.3. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Every nonzero T-module is an orthogonal direct sum of irreducibles T-modules.*

Proof. Let W denote a nonzero T-module. We proceed by induction on the dimension $\dim(W)$ of W. If $\dim(W) = 1$ then W is irreducible and by convention, it is assumed that W is itself a direct sum of irreducible modules. Assume now that $\dim(W) \geq 2$ and, by induction hypothesis, that every T-module with dimension strictly less than $\dim(W)$ is an orthogonal direct sum of irreducible T-modules. Now, if W is irreducible then the claim follows as it is assumed that W is itself a direct sum of irreducible modules. Otherwise, there exists a nonzero T-submodule W_1 with $\dim(W_1) < \dim(W)$. Then, by Proposition 3.1.2, there exists a nonzero T-module W_2 with $\dim(W_2) < \dim(W)$ such that $W = W_1 + W_2$ (orthogonal direct sum). Consequently, since by induction hypothesis, W_1 and W_2 are orthogonal direct sum of irreducible T-modules, the claim follows. ■

Proposition 3.1.4. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Every T-module W is orthogonal direct sum of the nonvanishing subspaces $E_i^* W$ $(0 \leq i \leq \epsilon(x))$.*

Proof. Pick $x \in X$ and let $d = \epsilon(x)$. Let W denote a T-module. Then, for $0 \leq i \leq d$, we have $E_i^* W \subseteq W$ and so $E_0^* W + E_1^* W + \cdots + E_d^* W \subseteq W$. Moreover, for every $w \in W$ we observe that

$$w = Iw = (E_0^* + E_1^* + \cdots + E_d^*)w = E_0^* w + E_1^* w + \cdots + E_d^* w,$$

where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. Therefore, $W \subseteq E_0^* W + E_1^* W + \cdots + E_d^* W$. Since $E_i^* E_i^* = E_i^*$ and $E_i^* E_j^* = 0$ for $0 \leq i, j \leq d$, $i \neq j$, the subspaces $E_i^* W$ are mutually orthogonal. This finishes the proof. ■

Let W be an irreducible T-module. By Proposition 3.1.4, we observe that W is an orthogonal direct sum of the nonvanishing subspaces $E_i^* W$ for $0 \leq i \leq \epsilon(x)$. Therefore, this fact motivates the next definitions which will be useful throughout this dissertation: by the **endpoint** of W we mean $r := r(W) = \min\{i \mid 0 \leq i \leq \epsilon(x),\ E_i^* W \neq 0\}$ and by the **diameter** of W, the scalar $d' := d'(W) = |\{i \mid 0 \leq i \leq \epsilon(x),\ E_i^* W \neq 0\}| - 1$. We also say that W is **thin** whenever the dimension of $E_i^* W$ is at most 1 for $0 \leq i \leq \epsilon(x)$.

Using the idea from [89, Lemma 3.9(ii)] we can easily determine which of the subspaces $E_i^* W$ $(0 \leq i \leq \epsilon(x))$ are nonzero.

Proposition 3.1.5. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Let W be an irreducible T-module with endpoint r and diameter d'. Then, $E_i^* W \neq 0$ if and only if $r \leq i \leq r + d'$ $(0 \leq i \leq \epsilon(x))$. Moreover,*

$$W = E_r^* W + E_{r+1}^* W + \cdots + E_{r+d'}^* W \qquad \textit{(orthogonal direct sum).}$$

Proof. Let W be an irreducible T-module with endpoint r and diameter d'. Pick an integer j $(0 \leq j \leq \epsilon(x))$. We first notice that, by the definition of the endpoint of W, we have $E_j^* W = 0$ for $0 \leq j < r-1$ and $E_r^* W \neq 0$. We next assume that the subspace $E_j^* W = 0$ and $j > r$. Now set $W[r, j-1] = E_r^* W + E_{r+1}^* W + \cdots + E_{j-1}^* W$. We observe that, by construction, $W[r, j-1]$ is a nonzero subspace of W which is invariant under the action of the dual idempotents E_i^* $(0 \leq i \leq \epsilon(x))$. Moreover, we claim that for any k $(0 \leq i \leq \epsilon(x))$ we have that

$$AE_k^* W \subseteq E_{k-1}^* W + E_k^* W + E_{k+1}^* W. \tag{3.2}$$

To prove our assertion (3.2), we recall that $E_0^* + E_1^* + \cdots + E_d^* = I$, where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$, and we observe that, for $0 \leq i, k \leq \epsilon(x)$, the matrix $E_i^* A E_k^*$ is zero if $|i - k| > 1$. Therefore, from the above comments, it follows that

$$\begin{aligned} AE_k^* W &= \sum_{i=0}^{\epsilon(x)} E_i^* A E_k^* W \\ &= E_{k-1}^* A E_k^* W + E_k^* A E_k^* W + E_{k+1}^* A E_k^* W \\ &\subseteq E_{k-1}^* W + E_k^* W + E_{k+1}^* W \end{aligned}$$

since W is a T-module. This proves our claim.

Equation 3.2 shows that the subspace $W[r, j-1]$ is A-invariant. Hence, $W[r, j-1]$ is a nonzero T-submodule of W which means that $W[r, j-1] = W$ as W is irreducible. We thus have that

$$d' + 1 = |\{i \mid 0 \leq i \leq \epsilon(x),\ E_i^* W \neq 0\}| \leq (j-1) - r + 1 = j - r,$$

which shows that $j > d' + r$. Therefore, either $0 \leq j < r$ or $j > d' + r$ if the subspace $E_j^* W$ is zero. Equivalently, if $r \leq j \leq d' + r$ then the subspace $E_j^* W$ is nonzero. This shows that the set $\{i \in \mathbb{Z} \mid r \leq i \leq r + d'\}$ of cardinality $d' + 1$ is a subset of $\{i \mid 0 \leq i \leq \epsilon(x),\ E_i^* W \neq 0\}$, which also has $d' + 1$ elements, by the definition of the diameter d' of W. Consequently, we have that

$$\{i \mid 0 \leq i \leq \epsilon(x),\ E_i^* W \neq 0\} = \{i \in \mathbb{Z} \mid r \leq i \leq r + d'\}.$$

This concludes the proof. ■

Let W and W' denote two irreducible T-modules. By a **T-isomorphism** from W to W' we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W and W' are said to be **T-isomorphic** (or simply **isomorphic**) whenever there exists a T-isomorphism $\sigma : W \to W'$.

We now present some well-known facts about (non-)isomorphic irreducible T-modules.

Proposition 3.1.6. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Any two non-orthogonal irreducible T-modules are T-isomorphic.*

Proof. Let U and W denote non-orthogonal irreducible T-modules. In particular, there exist $u \in U, w \in W$ such that $\langle u, w \rangle \neq 0$ and so, both U and W are nonzero T-modules. We observe that the standard module $V = U + U^{\perp}$ (orthogonal direct sum). Now let $P_U : W \to U$ be the orthogonal projection of W onto the subspace U. That is, for $x \in W$, $P_U x = y$ if and only if $y \in U$ is the only vector such that $x - y \in U^{\perp}$. We observe that P_U is well-defined since $W \subseteq V$ and so, for every vector $w \in W$ we have unique vectors $w_1 \in U$ and $w_2 \in U^{\perp}$ such that $w = w_1 + w_2$. Moreover, for every $w \in W$ we have that $w = P_U(w) + (w - P_U(w))$ with $P_U(w) \in U$ and $w - P_U(w) \in U^{\perp}$. Then, since U and $U^{\perp}$ are T-modules, for every matrix $B \in T$, we have that $Bw = BP_U(w) + B(w - P_U(w))$ with $BP_U(w) \in U$ and $B(w - P_U(w)) \in U^{\perp}$. This shows that $P_U(Bw) = BP_U(w)$ for every $w \in W$ and so $(BP_U - P_U B)W = 0$ for every $B \in T$. Then, since also P_U is a vector space homomorphism, it follows that P_U is a T-module homomorphism. Consequently, it holds that $\ker(P_U) = \{x \in W \mid P_U(x) = 0\}$ and $\operatorname{im}(P_U) = \{P_U(x) \mid x \in W\}$ are T-submodules of W and U, respectively. Now, recall that there exist $u \in U, w \in W$ such that $\langle u, w \rangle \neq 0$. This implies that

$$\langle u, P_U(w) \rangle = \langle u, P_U(w) \rangle + \langle u, w - P_U(w) \rangle = \langle u, P_U(w) + (w - P_U(w)) \rangle = \langle u, w \rangle \neq 0,$$

which shows that $P_U(x) \neq 0$ for some $x \in W$. Therefore, $\ker(P_U) \neq W$ and $\operatorname{im}(P_U) \neq 0$. Furthermore, since U and W are irreducible T-modules, $\ker(P_U) = 0$ and $\operatorname{im}(P_U) = U$. In other words, P_U is both a monomorphism and an epimorphism. This shows that P_U is a T-isomorphism and concludes the proof. ■

Corollary 3.1.7. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Any two non-isomorphic irreducible T-modules are orthogonal.*

Proof. Immediate from Proposition 3.1.6. ■

Proposition 3.1.8. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Any two isomorphic irreducible T-modules have the same endpoint and the same diameter.*

Proof. Let U and W denote two isomorphic irreducible T-modules. Then, there exists a T-isomorphism $\varphi : W \to U$. In particular we have $\ker(\varphi) = 0$ and $\mathrm{im}(\varphi) = U$.

For $0 \le i \le \epsilon(x)$, we next claim that $E_i^* U = 0$ if and only if $E_i^* W = 0$. To prove our assertion we notice that $E_i^* U = E_i^* \varphi(W) = \varphi(E_i^* W)$. Hence, if $E_i^* W$ is zero then $E_i^* U$ is zero. Conversely, if $E_i^* U = \varphi(E_i^* W)$ is zero then $E_i^* W \subseteq \ker(\varphi)$ which implies that $E_i^* W$ is zero.

It follows from the above comments that

$$\{i \mid 0 \le i \le \epsilon(x),\ E_i^* U \ne 0\} = \{i \mid 0 \le i \le \epsilon(x),\ E_i^* W \ne 0\}. \tag{3.3}$$

Therefore, from (3.3), it holds that U and W both have the same endpoint and the same diameter. ■

Recall that algebra T is closed under the conjugate-transpose map. So, in many instances, this algebra can best be studied via its irreducible modules. Since in particular the standard module decomposes as an orthogonal direct sum of irreducible T-modules, it is natural to consider certain algebraic properties on T-modules and try to investigate what these properties tell us about the combinatorial structure of a graph. Moreover, to study those graphs whose modules take 'simple' form could be of interest as well.

To start our investigations, we first consider modules whose algebraic properties and structure are as simple as possible.

Proposition 3.1.9. *Let $\mathcal{M}$ be a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ and let $v \in V$. Then, the subset $\mathcal{M}v := \{Bv \mid B \in \mathcal{M}\}$ is an $\mathcal{M}$-module.*

Proof. Clearly, $0 \in \mathcal{M}v$ as the zero matrix belongs to the subalgebra $\mathcal{M}$. So, $\mathcal{M}v$ is a nonempty subset of V. Let $\lambda \in \mathbb{C}$ and $w, z \in \mathcal{M}v$. So, there exist matrices $B_1, B_2 \in \mathcal{M}$

such that $w = B_1 v$ and $z = B_2 v$. Moreover, since $\mathcal{M}$ is a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ it holds that $B_1 + \lambda B_2 \in \mathcal{M}$. Consequently, $w + \lambda z = B_1 v + \lambda B_2 v = (B_1 + \lambda B_2)v \in \mathcal{M}v$. This shows that $\mathcal{M}v$ is a subspace of V. Furthermore, if $w \in \mathcal{M}v$ then $Cw = C(B_1 v) = CB_1 v$ for every $C \in \mathcal{M}$. So, $Cw \in \mathcal{M}v$ since $CB_1 \in \mathcal{M}$. We thus have that $\mathcal{M}v$ is invariant under the action of any matrix in $\mathcal{M}$. Hence, $\mathcal{M}v$ is an $\mathcal{M}$-module. ∎

Proposition 3.1.10. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Let W be an irreducible T-module with endpoint 0. Then, $T\widehat{x} := \{B\widehat{x} \mid B \in T\}$ is a subset of W.*

Proof. Since W is an irreducible T-module with endpoint 0, there exists a nonzero vector $w \in E_0^* W$. We know that $w = \sum_{y \in X} \alpha_y \, \widehat{y}$ for some scalars $\alpha_y \in \mathbb{C}$. We thus have that $w = E_0^* w = \alpha_x \, \widehat{x}$ with $\alpha_x \neq 0$. This yields that $\widehat{x} = \alpha_x^{-1} \, E_0^* w$ and so, $\widehat{x} \in E_0^* W \subseteq W$. Moreover, we notice that $B\widehat{x} \in W$ for every matrix $B \in T$ as W is a T-module. Therefore, the set $T\widehat{x} := \{B\widehat{x} \mid B \in T\}$ is a subset of W. ∎

Theorem 3.1.11. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. There exists a unique irreducible T-module with endpoint 0. Namely, the set $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$.*

Proof. We first observe that algebra T is a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$. Since $\widehat{x} \in V$ then, by Proposition 3.1.9 we have that $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$ is a T-module. Suppose now that W is an irreducible T-module with endpoint 0. Then, by Proposition 3.1.10, the set $T\widehat{x}$ is a subset of W. Since the identity matrix $I \in \mathrm{Mat}_X(\mathbb{C})$ belongs to T we have that $\widehat{x} = I\widehat{x}$ and so, the nonzero vector $\widehat{x} \in T\widehat{x}$. We thus have that $T\widehat{x}$ is a nonzero T-submodule of W. Consequently, $T\widehat{x} = W$ by the irreducibility of W. This finishes the proof. ∎

With reference to Theorem 3.1.11, the unique irreducible T-module with endpoint 0, the set $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$, is called the **trivial T-module**.

3.2 Non-negative irreducible matrices

In this section we recall couple of basic definitions and results about non-negative and irreducible matrices. The reader is refered to the book by Horn and Johnson for a review of these topics and further information; see [48].

We say that a matrix is **non-negative (positive)**, if all of its entries are non-negative (positive) real numbers, respectively. Similarly, a vector is **strictly positive** if all its entries are positive real numbers. Moreover, the **spectral radius** of a square matrix M, denoted by $\rho(M)$, is the maximum absolute value of the eigenvalues of M. A matrix is said to be **reducible** if it can be placed into block upper-triangular form by simultaneous row/column permutations. That is, an n-by-n matrix M is **reducible** if there exists an n-by-n permutation matrix P such that

$$P^\top MP = \begin{pmatrix} B & C \\ 0_{n-r,r} & D \end{pmatrix} \qquad (1 \leq r \leq n-1),$$

where $0_{n-r,r}$ denotes the $(n-r)$-by-r zero matrix. In the preceding definition, we do not insist that any of the blocks B, C, and D have nonzero entries. We require only that a lower-left $(n-r)$-by-r block of zero entries can be created by some sequence of row and column interchanges. However, we do insist that both of the square matrices B and D have size at least one, so no 1-by-1 matrix is reducible. We also say that a matrix is **irreducible** if it is not reducible.

We next recall a couple of definitions coming from graph theory. A **directed graph** $\overrightarrow{\Gamma}$ consists of a finite set of vertices together with a subset of ordered pairs of vertices called **arcs** or simply, directed edges. A **directed path** in a directed graph $\overrightarrow{\Gamma}$ is a sequence of directed edges in $\overrightarrow{\Gamma}$. The length of a directed path is the number of directed edges in the directed path if this number is finite; otherwise, the directed path is said to have infinite length. A directed graph $\overrightarrow{\Gamma}$ is **strongly connected** if between each pair of distinct vertices u and v there exists a directed path of finite length that begins at u and ends at v.

The notion of an irreducible matrix can be summarized visually by certain paths in a graph associated with its adjacency matrix.

Given an n-by-n matrix M, we say that $\overrightarrow{\Gamma}(M)$ is the **directed graph of** M if $\overrightarrow{\Gamma}(M)$ is the directed graph on n vertices $v_1, v_2, \cdots, v_n$ such that there is a directed edge in $\overrightarrow{\Gamma}(M)$ from v_i to v_j if and only if the (i,j)-entry of M is nonzero. By [48, Theorem 6.2.24] we have that M is irreducible if and only if $\overrightarrow{\Gamma}(M)$ is strongly connected.

The following theorem, known in the literature as Perron-Frobenius theorem, shows how much of Perron's theorem (see [48, Theorem 8.2.8]) generalizes to nonnegative irreducible matrices. The name of Frobenius is associated with generalizations of Perron's results about positive matrices to nonnegative matrices.

Theorem 3.2.1 ([48, Theorem 8.4.4])**.** *Let M be an irreducible and non-negative square matrix. Then the following (i), (ii) hold.*

(i) $\rho(M) > 0$ *and* $\rho(M)$ *is an algebraically simple eigenvalue of* M *(i.e., the corresponding eigenspace is one-dimensional).*

(ii) *There exists a strictly positive vector* υ *such that* $M\upsilon = \rho(M)\upsilon$.

We refer to vector υ *as a* ***Perron-Frobenius vector of the matrix*** M.

Throughout this section, let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph, with vertex set X and edge set $\mathcal{R}$. Let V denote the standard module of Γ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. We observe that $\overrightarrow{\Gamma}(A)$ is the directed graph obtained from Γ by replacing each edge uv of Γ by an arc from u to v and an arc from v to u. Since Γ is connected, there exists a path connecting any two vertices in Γ. Therefore, for each pair of distinct vertices u and v in $\overrightarrow{\Gamma}(A)$, there exists a directed path of finite length that begins at u and ends at v and so, $\overrightarrow{\Gamma}(A)$ is strongly connected. By [48, Theorem 6.2.24] we have that A is irreducible.

We also observe that A is a non-negative matrix and so, Theorem 3.2.1 applies. Throughout this section, let $\rho(A)$ denote the spectral radius of A and let υ denote a Perron-Frobenius vector of A.

Fix now a vertex $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Recall that $T\widehat{x}$ is the unique irreducible T-module with endpoint 0. Next, we show that $T\widehat{x} = T\upsilon$, where $T\upsilon = \{B\upsilon \mid B \in T\}$. This result was already proved by Terwilliger in [88], but for convenience of the reader we include a proof here. We first need the following lemma.

Lemma 3.2.2. *Let* $\Gamma = (X, \mathcal{R})$ *denote a finite, simple and connected graph. Pick* $x \in X$ *and let* $T = T(x)$ *be the Terwilliger algebra of* Γ *with respect to* x. *Let* υ *denote a Perron-Frobenius vector of the adjacency matrix* A *of* Γ. *Then,* $T\upsilon$ *is an irreducible* T*-module. Moreover,*

$$T\upsilon = T\widehat{x}.$$

Proof. Let υ denote a Perron-Frobenius vector of A and, for $z \in X$, let υ_z denote the z-coordinate of υ. By Theorem 3.2.1, we observe that $A\upsilon = \theta\upsilon$ for some $\theta > 0$. Let $B = \frac{\upsilon\upsilon^\top}{||\upsilon||^2}$. Note that B is the matrix representing the orthogonal projection onto the eigenspace belonging to θ. By the Spectral Decomposition Theorem (see e.g. [40, Theorem 5.1]), there exists a polinomial p with complex coefficients such that $p(A) = p(\theta)B$ with $p(\theta) \neq 0$.

In particular, we have that B belongs to T. Moreover, for $x, y \in X$, the (x,y)-entry of B is equal to $\frac{\upsilon_x \upsilon_y}{||\upsilon||^2}$. Hence, it follows that $||\upsilon||^2 B\widehat{x} = \upsilon_x \upsilon$ or, alternatively,

$$\upsilon = \frac{||\upsilon||^2}{\upsilon_x} B\widehat{x}.$$

Then, $\upsilon \in T\widehat{x}$ as B belongs to T. This implies that $T\upsilon \subseteq T\widehat{x}$. Furthermore, by Proposition 3.1.9 it holds that $T\upsilon$ is a T-module, which is nonzero as $\upsilon \in T\upsilon$. Therefore, as $T\widehat{x}$ is irreducible, it holds that $T\upsilon = T\widehat{x}$. The result follows. ■

3.3 Local (pseudo-)distance-regularity

Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. In this section we recall the notions of (local) distance-regularity and (local) pseudo-distance-regularity of Γ. To do this, fix $x \in X$ and let d denote the eccentricity of x.

Assume for a moment that $y \in \Gamma_i(x)$ $(0 \le i \le d)$ and let z be a neighbour of y. Then, by the triangle inequality,

$$\partial(x,z) \in \{i-1, i, i+1\},$$

and so $z \in \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$. For $y \in \Gamma_i(x)$ we therefore define the following numbers:

$$a_i(x,y) = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i(x,y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i(x,y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|.$$

We say that $x \in X$ is **distance-regularized** (or that Γ is **distance-regular around** x) if the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ do not depend on the choice of $y \in \Gamma_i(x)$ $(0 \le i \le d)$. In this case, the numbers $a_i(x) = a_i(x,y), b_i(x) = b_i(x,y)$ and $c_i(x) = c_i(x,y)$ are called the **intersection numbers of** x.

The concept of pseudo-distance-regularity around a vertex of a graph was introduced in [34] by Fiol, Garriga and Yebra as a natural generalization of distance-regularity around a vertex. We now recall this definition.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Let $\rho(A)$ denote the spectral radius of A and let $\upsilon \in V$ denote a Perron-Frobenius vector of A. For $z \in X$ let υ_z denote the z-coordinate of υ. For $y \in \Gamma_i(x)$ $(0 \le i \le d)$ we define numbers $a_i^*(x,y)$, $b_i^*(x,y)$ and $c_i^*(x,y)$

as follows:

$$a_i^*(x,y) = \sum_{z\in\Gamma(y)\cap\Gamma_i(x)} \frac{\upsilon_z}{\upsilon_y}, \quad b_i^*(x,y) = \sum_{z\in\Gamma(y)\cap\Gamma_{i+1}(x)} \frac{\upsilon_z}{\upsilon_y}, \quad c_i^*(x,y) = \sum_{z\in\Gamma(y)\cap\Gamma_{i-1}(x)} \frac{\upsilon_z}{\upsilon_y}.$$

Observe that $a_i^*(x,y)+b_i^*(x,y)+c_i^*(x,y) = \rho(A)$.

We say that vertex $x \in X$ is **pseudo-distance-regularized** (or that Γ is **pseudo-distance-regular around** x) if the numbers $a_i^*(x,y)$, $b_i^*(x,y)$ and $c_i^*(x,y)$ do not depend on the choice of y. In this case, they are denoted by $a_i^*(x)$, $b_i^*(x)$ and $c_i^*(x)$ and they are called the **pseudo-intersection numbers of** Γ **with respect to** x. Moreover, the array

$$\begin{Bmatrix} 0 & c_1^*(x) & \cdots & c_{d-1}^*(x) & c_d^*(x) \\ 0 & a_1^*(x) & \cdots & a_{d-1}^*(x) & a_d^*(x) \\ b_0^*(x) & b_1^*(x) & \cdots & b_{d-1}^*(x) & 0 \end{Bmatrix}$$

is called the **pseudo-intersection array of** Γ **with respect to** x.

Assume now that Γ is distance-regular around x. By [34, Proposition 3.2], Γ is also pseudo-distance-regular around x. However, the converse of this result is not true. In particular, it was shown in [34] by Fiol, Garriga and Yebra that the Cartesian product $P_3\square\cdots\square P_3$ of r paths of length 3 has pseudo-distance-regularized vertices which are not distance-regularized. For the convenience of the reader we would also like to present another example.

Example 3.3.1. *Let* Γ *be the connected graph with vertex set* $X = \{1,2,3,4,5,6\}$ *and edge set* $\mathcal{R} = \{\{1,2\},\{1,3\},\{2,4\},\{2,5\},\{3,5\},\{3,6\}\}$*. See Figure 3.1. Let* A *denote the adjacency matrix of* Γ*. It is easy to see that* $\rho(A) = \sqrt{5}$ *and* $\upsilon = (2\ \sqrt{5}\ \sqrt{5}\ 1\ 2\ 1)^\top$ *is a Perron-Frobenius vector of* A*. Consider vertex* $1 \in X$ *and note that* $\epsilon(1) = 2$*. It is straightforward to check that* Γ *is pseudo-distance-regular around* 1 *with the following pseudo-intersection array:*

$$\begin{Bmatrix} 0 & \frac{2}{\sqrt{5}} & \sqrt{5} \\ 0 & 0 & 0 \\ \sqrt{5} & \frac{3}{\sqrt{5}} & 0 \end{Bmatrix}$$

However, Γ *is not distance-regular around* 1*. Namely, vertex* $4 \in \Gamma_2(1)$ *has only one neighbour in* $\Gamma(1)$*, while vertex* $5 \in \Gamma_2(1)$ *has two neighbours in* $\Gamma(1)$*.*

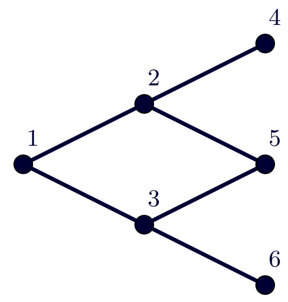

Figure 3.1: Graph Γ from Example 3.3.1.

3.4 Local pseudo-distance-regularity and the trivial module

As already mentioned, it was proved by Terwilliger in [88] that if Γ is distance-regular around x, then the trivial T-module is thin. Fiol and Garriga [33] later proved the following result (see also [30, Theorem 3.1]).

Theorem 3.4.1 ([30, Theorem 3.1])**.** *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Fix $x \in X$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. Then, the trivial T-module is thin if and only if Γ is pseudo-distance-regular around x.*

Throughtout this section we provide a proof of Theorem 3.4.1 with a slightly different approach.

3.4.1 Proof of Theorem 3.4.1: part 1

Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Fix a vertex $x \in X$. Assume that $\epsilon(x) = d$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. In this subsection we show that if Γ is pseudo-distance-regular around x, then the trivial T-module $T\hat{x}$ is thin. We start with the following comments and results which will be useful later for the proof of Theorem 3.4.1.

For an integer $0 \le i \le d$, a symmetric matrix $A_i^x \in \mathrm{Mat}_X(\mathbb{C})$ is said to be **x-local i-distance matrix**, if for any $y \in X$ the following holds:

$$(A_i^x)_{xy} = \begin{cases} \frac{\upsilon_y}{\upsilon_x} & \text{if} \quad \partial(x,y) = i, \\ 0 & \text{if} \quad \partial(x,y) \neq i. \end{cases}$$

We observe that the above definition gives no constraints on the entries which are not in the x-row or in the x-column of A_i^x. Thus, an example of an x-local 0-distance matrix is the identity matrix of $\mathrm{Mat}_X(\mathbb{C})$. Moreover, if Γ is regular, then the adjacency matrix A is an example of an x-local 1-distance matrix. An x-local i-distance matrix is called **proper**, if it is a polynomial of degree i in the adjacency matrix A of Γ. We remark that any proper x-local i-distance matrix is, by definition, an element of the Terwilliger algebra T.

The next theorem shows that, in the case of locally pseudo-distance-regularity, the proper distance matrices exist and satisfy a recurrence relation which is similar to that of the (standard) distance matrices of distance-regular graphs.

Theorem 3.4.2 ([34, Proposition 3.3])**.** *Let $\Gamma=(X,\mathcal{R})$ denote a finite, simple and connected graph. Fix $x\in X$ and let $\epsilon(x)=d$. Assume that Γ is pseudo-distance-regular around x. Let $a_i^*(x), b_i^*(x), c_i^*(x)$ $(0\le i\le d)$ denote the corresponding pseudo-intersection numbers of Γ with respect to x. For convenience set $b_{-1}^*(x)=c_{d+1}^*(x)=0$. Then, there exists a sequence $\{A_i^x\}_{i=0}^d$ of proper x-local i-distance matrices $A_0^x, A_1^x, \cdots A_d^x$. Moreover, the following holds for $0\le i\le d$:*

$$AA_i^x = b_{i-1}^*(x)A_{i-1}^x + a_i^*(x)A_i^x + c_{i+1}^*(x)A_{i+1}^x.$$

Pick now an integer $0\le j\le d$ and consider a proper x-local j-distance matrix $A_j^x\in\mathrm{Mat}_X(\mathbb{C})$. We observe that

$$A_j^x\hat{x} = \sum_{y\in\Gamma_j(x)} \frac{\upsilon_y}{\upsilon_x}\cdot\hat{y} = \frac{1}{\upsilon_x}E_j^*\upsilon.$$

Consequently, vector $A_j^x\hat{x}$ is non-zero and, for every $0\le i\le d$, the following also holds:

$$E_i^*A_j^x\hat{x} = \frac{1}{\upsilon_x}E_i^*E_j^*\upsilon = \delta_{i,j}A_j^x\hat{x}. \tag{3.4}$$

Proposition 3.4.3. *Let $\Gamma=(X,\mathcal{R})$ denote a finite, simple and connected graph. Assume that Γ is pseudo-distance-regular around x. Let $\{A_i^x\}_{i=0}^d$ be a sequence of proper x-local i-distance matrices. Then the vectors $A_i^x\hat{x}$ are pairwise orthogonal for $0\le i\le d$.*

Proof. Pick $0\le i,j\le d$. By (3.4) we have that

$$\langle A_i^x\hat{x}, A_j^x\hat{x}\rangle = \langle E_i^*A_i^x\hat{x}, A_j^x\hat{x}\rangle = \langle A_i^x\hat{x}, E_i^*A_j^x\hat{x}\rangle = \delta_{i,j}\|A_i^x\hat{x}\|^2.$$

Hence, the vectors $A_i^x\hat{x}$ are pairwise orthogonal for $0\le i\le d$. ■

We are now ready to prove the main result of this subsection.

Theorem 3.4.4. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Assume that Γ is pseudo-distance-regular around x. Let $\{A_i^x\}_{i=0}^d$ be a sequence of proper x-local i-distance matrices. Then, the following (i), (ii) hold.*

(i) The set $\{A_i^x \hat{x} \mid 0 \leq i \leq d\}$ is a basis for the trivial T-module $T\hat{x}$.

(ii) The trivial T-module $T\hat{x}$ is thin.

Proof. Consider the non-zero subspace $W \subseteq V$ generated by vectors $\{A_i^x \widehat{x} \mid 0 \leq i \leq d\}$. Recall that the matrices A_i^x $(0 \leq i \leq d)$ are elements of the algebra T, and so $W \subseteq T\widehat{x}$. By Theorem 3.4.2 the space W is invariant under the action of the adjacency matrix A. By (3.4), W is also invariant under the action of matrices E_i^* $(0 \leq i \leq d)$. It follows from the above comments that W is a T-module. Note that W is nonzero, and so $W = T\hat{x}$ by the irreducibility of $T\widehat{x}$. Recall that vectors $A_i^x \widehat{x}$ are nonzero and pairwise othogonal by Proposition 3.4.3 and so, they are linearly independent. Therefore, the set $\{A_i^x \widehat{x} \mid 0 \leq i \leq d\}$ is a basis for the trivial T-module $T\widehat{x}$. Moreover, for $0 \leq i \leq d$, we observe from (3.4) that the subspace $E_i^*(T\widehat{x})$ is spanned by the vector $A_i^x \widehat{x}$. The result follows. ■

Corollary 3.4.5. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Assume that Γ is distance-regular around x. Then, the trivial T-module $T\widehat{x}$ is thin.*

Proof. Recall that by [34, Proposition 3.2], Γ is also pseudo-distance-regular around x. The result now follows from Theorem 3.4.4. ■

However, we notice that the converse of Corollary 3.4.5 is not true in general. This is demonstrated in the following example.

Example 3.4.6. *Consider graph Γ from Example 3.3.1 (see also Figure 3.1). Fix vertex $1 \in X$ and note that $d = 2$. Consider the Terwilliger algebra of Γ with respect to vertex 1. Recall that we have shown in Example 3.3.1 that Γ is pseudo-distance-regular around 1. Then, by Theorem 3.4.4, the trivial T-module is thin. However, it also follows from Example 3.3.1 that Γ is not distance-regular around 1. This shows that the converse of Corollary 3.4.5 is not true.*

3.4.2 Proof of Theorem 3.4.1: part 2

Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Fix a vertex $x \in X$. Assume that $\epsilon(x) = d$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. In this subsection we show that if the trivial T-module $T\widehat{x}$ is thin, then Γ is pseudo-distance-regular around x.

Lemma 3.4.7. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Fix a vertex $x \in X$. Assume that $\epsilon(x) = d$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume the trivial T-module $T\hat{x}$ is thin. Then, the set $\{E_i^* \upsilon \mid 0 \le i \le d\}$ is a basis for $T\widehat{x}$ where υ denotes a Perron-Frobenius vector of the adjacency matrix of Γ.*

Proof. Since the identity matrix $I \in T$ we have that $\upsilon \in T\upsilon$. Then, by Lemma 3.2.2 we observe that $\upsilon \in T\widehat{x}$. Moreover, it follows that $E_i^* \upsilon \in T\widehat{x}$ for $0 \le i \le d$. Observe that

$$E_i^* \upsilon = \sum_{y \in \Gamma_i(x)} \upsilon_y \widehat{y},$$

and so $E_i^* \upsilon$ is nonzero for $0 \le i \le d$. Furthermore, for every $0 \le i, j \le d$, we observe that

$$\left\langle E_i^* \upsilon, E_j^* \upsilon \right\rangle = \left\langle \upsilon, E_i^* E_j^* \upsilon \right\rangle = \delta_{i,j} \cdot \left\| E_j^* \upsilon \right\|^2.$$

This shows that the vectors $E_i^* \upsilon$ $(0 \le i \le d)$ are pairwise orthogonal and consequently, they are linearly independent. Since the trivial T-module is thin and for every $0 \le i \le d$ the vector $E_i^* \upsilon \in E_i^*(T\widehat{x})$, it follows that $E_i^*(T\widehat{x})$ is spanned by $E_i^* \upsilon$ for $0 \le i \le d$. The claim follows. ■

Proposition 3.4.8. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Fix a vertex $x \in X$. Assume that $\epsilon(x) = d$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Pick $y, z \in X$. Then, the (y,z)-entry of $E_j^* A E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ equals 1 if $\partial(x,y) = j$, $\partial(y,z) = 1$ and $\partial(x,z) = i$, and 0 otherwise. In particular, if $|j - i| \ge 2$ then $E_j^* A E_i^* = 0$.*

Proof. Immediately from elementary properties of matrix multiplication. ■

Lemma 3.4.9. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Fix a vertex $x \in X$. Assume that $\epsilon(x) = d$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume the trivial T-module $T\widehat{x}$ is thin. Then, for every $0 \le i \le d$ there exist* *scalars* $\alpha_i(x), \beta_i(x), \gamma_i(x) \in \mathbb{C}$ *such that*

$$AE_i^* \upsilon = \beta_i(x) E_{i-1}^* \upsilon + \alpha_i(x) E_i^* \upsilon + \gamma_i(x) E_{i+1}^* \upsilon,$$

where υ *denotes a Perron-Frobenius vector of the adjacency matrix of* Γ*.*

Proof. Note that since $T\widehat{x}$ is a T-module, it is invariant under the action of A. It follows from Lemma 3.4.7 that for every $0 \leq i \leq d$, the vector $AE_i^* \upsilon$ is a linear combination of the vectors $E_j^* \upsilon$ $(0 \leq i \leq d)$. However, it follows from Proposition 3.4.8 that $AE_i^* \upsilon$ is a linear combination of just the vectors $E_{i-1}^* \upsilon$, $E_i^* \upsilon$ and $E_{i+1}^* \upsilon$. The claim follows. ∎

Lemma 3.4.10. *Let* $\Gamma = (X, \mathcal{R})$ *denote a finite, simple and connected graph. Fix a vertex* $x \in X$*. Assume that* $\epsilon(x) = d$ *and let* $T = T(x)$ *denote the Terwilliger algebra of* Γ *with respect to* x*. Let* $0 \leq i, j \leq d$ *and pick* $y \in \Gamma_j(x)$*. Let* υ *denote a Perron-Frobenius vector of the adjacency matrix of* Γ*. Then, the* y*-entry of the vector* $E_j^* A E_i^* \upsilon$ *equals*

$$\sum_{z \in \Gamma(y) \cap \Gamma_i(x)} \upsilon_z.$$

Proof. Elementary matrix multiplication. ∎

We are now ready to prove the main result of this subsection.

Theorem 3.4.11. *Let* $\Gamma = (X, \mathcal{R})$ *denote a finite, simple and connected graph. Fix a vertex* $x \in X$*. Assume that* $\epsilon(x) = d$ *and let* $T = T(x)$ *denote the Terwilliger algebra of* Γ *with respect to* x*. If the trivial* T*-module* $T\widehat{x}$ *is thin, then* Γ *is pseudo-distance-regular around* x*.*

Proof. Let υ denote a Perron-Frobenius vector of the adjacency matrix of Γ. Let $0 \leq i \leq d$. By Lemma 3.4.9, there exist scalars $\alpha_j(x), \beta_j(x), \gamma_j(x)$ for each $j \in \{i-1, i, i+1\}$ such that

$$\begin{aligned}
AE_{i-1}^* \upsilon &= \beta_{i-1}(x) E_{i-2}^* \upsilon + \alpha_{i-1}(x) E_{i-1}^* \upsilon + \gamma_{i-1}(x) E_i^* \upsilon, \\
AE_i^* \upsilon &= \beta_i(x) E_{i-1}^* \upsilon + \alpha_i(x) E_i^* \upsilon + \gamma_i(x) E_{i+1}^* \upsilon, \\
AE_{i+1}^* \upsilon &= \beta_{i+1}(x) E_i^* \upsilon + \alpha_{i+1}(x) E_{i+1}^* \upsilon + \gamma_{i+1}(x) E_{i+2}^* \upsilon.
\end{aligned}$$

Multiplying the above equalities with E_i^* and using property (eiv) in Section 3.1, we get

$$E_i^* A E_{i-1}^* \upsilon = \gamma_{i-1}(x) E_i^* \upsilon, \tag{3.5}$$
$$E_i^* A E_i^* \upsilon = \alpha_i(x) E_i^* \upsilon, \tag{3.6}$$
$$E_i^* A E_{i+1}^* \upsilon = \beta_{i+1}(x) E_i^* \upsilon. \tag{3.7}$$

Pick a vertex $y \in \Gamma_i(x)$. Computing the y-entry of (3.5), (3.6) and (3.7) using Lemma 3.4.10, we get

$$\sum_{z \in \Gamma(y) \cap \Gamma_{i-1}(x)} \upsilon_z = \gamma_{i-1}(x) \upsilon_y, \tag{3.8}$$
$$\sum_{z \in \Gamma(y) \cap \Gamma_i(x)} \upsilon_z = \alpha_i(x) \upsilon_y, \tag{3.9}$$
$$\sum_{z \in \Gamma(y) \cap \Gamma_{i+1}(x)} \upsilon_z = \beta_{i+1}(x) \upsilon_y. \tag{3.10}$$

Recall that the entries of the vector υ are positive. Therefore, it follows from (3.8), (3.9) and (3.10) that $c_i^*(x,y) = \gamma_{i-1}(x)$, $a_i^*(x,y) = \alpha_i(x)$ and $b_i^*(x,y) = \beta_{i+1}(x)$. This shows that the pseudo-intersection numbers of Γ with respect to x do not depend on the choice of $y \in \Gamma_i(x)$. Hence, Γ is pseudo-distance-regular around x. ■

3.5 The main result and some products in T

Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. In this section we state our main result. To do this we need the following definition.

Definition 3.5.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x, y, z \in X$ and let $P = [y = x_0, x_1, \ldots, x_j = z]$ denote a yz-walk. The **shape of** P **with respect to** x is a sequence of symbols $t_1 t_2 \ldots t_j$, where $t_i \in \{f, \ell, r\}$, and such that $t_i = r$ if $\partial(x, x_i) = \partial(x, x_{i-1}) + 1$, $t_i = f$ if $\partial(x, x_i) = \partial(x, x_{i-1})$ and $t_i = \ell$ if $\partial(x, x_i) = \partial(x, x_{i-1}) - 1$ $(1 \le i \le j)$. We use exponential notation for shapes containing several consecutive identical symbols. For instance, instead of $rrrrfff\ell\ell r$ we simply write $r^4 f^3 \ell^2 r$. For a positive integer i, let $r^i\ell(y)$, $r^i f(y)$ and $r^i(y)$ respectively denote the number of xy-walks of the shape $r^i\ell$, $r^i f$ and r^i with respect to x. We also define $r^0\ell(y) = r^0 f(y) = 0$ for every $y \in X$, and $r^0(y) = 1$ if $y = x$ and $r^0(y) = 0$ otherwise. See Figure 3.2 for an example.*

For the rest of the chapter we adopt the following notation.

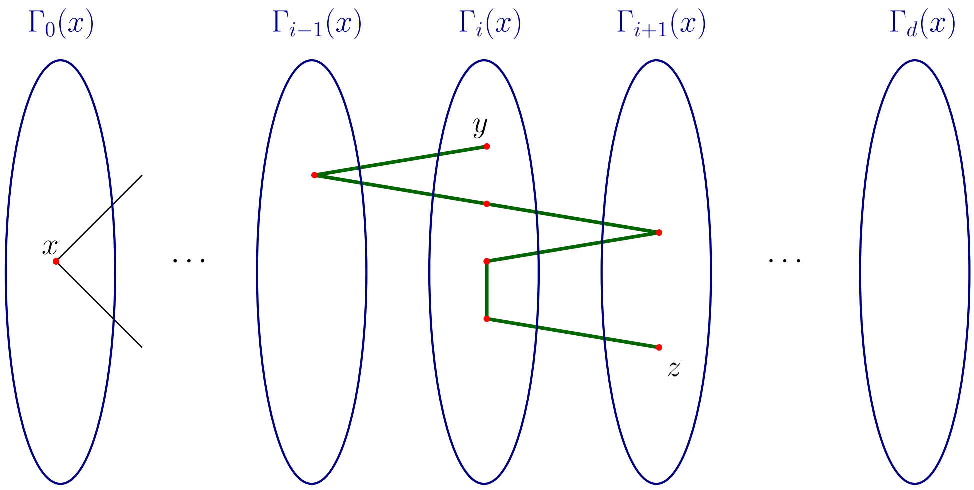

Figure 3.2: A yz-walk in a graph Γ of shape $\ell r^2 \ell f r$ with respect to x.

Notation 3.5.2. *Let Γ denote a finite, simple, connected graph with vertex set X. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix a vertex $x \in X$ and let d denote the eccentricity of x. Let $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \le i \le d)$ denote the dual idempotents of Γ with respect to x. Let V denote the standard module of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $T\widehat{x}$ denote the unique irreducible T-module with endpoint 0. Let $L = L(x)$, $F = F(x)$ and $R = R(x)$ denote the lowering, the flat and the raising matrix of T, respectively. For $y \in X$, let the numbers $r^i\ell(y)$, $r^i f(y)$ and $r^i(y)$ be as defined in Definition 3.5.1.*

We are now ready to state our main result which will be proved in Section 3.6.

Theorem 3.5.3. *With reference to Notation 3.5.2, the following (i)–(iii) are equivalent:*

(i) $T\widehat{x}$ *is thin.*

(ii) Γ *is pseudo-distance-regular around* x.

(iii) *For every integer i $(0 \le i \le d)$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ the following hold:*

$$r^{i+1}\ell(y) = \alpha_i\, r^i(y), \qquad r^i f(y) = \beta_i\, r^i(y).$$

Recall that the equivalency of (i) and (ii) of the above theorem was already proved in Section 3.4. Therefore, we will focus on the equivalency of (i) and (iii) in the rest of this chapter.

We first evaluate several products in the Terwilliger algebra T that we will need later for the proof of Theorem 3.5.3.

Lemma 3.5.4. *With reference to Notation 3.5.2, pick $y \in X$. Then, the following (i)–(iii) hold for an integer $i \geq 0$:*

(i) The y-entry of $R^i\widehat{x}$ is equal to the number $r^i(y)$.

(ii) The y-entry of $LR^i\widehat{x}$ is equal to the number $r^i\ell(y)$.

(iii) The y-entry of $FR^i\widehat{x}$ is equal to the number $r^if(y)$.

Proof. It immediately follows by using elementary matrix multiplication, comment below Definition 3.1.1, and (3.1). ∎

Proposition 3.5.5. *With reference to Notation 3.5.2, the vector $R^i\widehat{x}$ is nonzero for $0 \leq i \leq d$.*

Proof. Pick $0 \leq i \leq d$ and $y \in \Gamma_i(x)$ (note that $\Gamma_i(x)$ is nonempty). By Lemma 3.5.4(i), the y-entry of $R^i\widehat{x}$ is equal to the number $r^i(y)$. Note that by the definition of $r^i(y)$ and by the choice of y, we have that $r^i(y) > 0$. The result follows. ∎

3.6 Proof of the main theorem

With reference to Notation 3.5.2, in this section we prove Theorem 3.5.3. We also display a basis of $T\widehat{x}$ and the matrix representing the action of the adjacency matrix on this basis in the case when $T\widehat{x}$ is thin.

Lemma 3.6.1. *With reference to Notation 3.5.2, the following (i), (ii) are equivalent:*

(i) $T\widehat{x}$ is thin.

(ii) The set $\{R^i\widehat{x} : 0 \leq i \leq d\}$ is a basis of $T\widehat{x}$.

In particular, if the above equivalent conditions (i), (ii) hold, then $E^*_i(T\widehat{x})$ is spanned by $R^i\widehat{x}$ and $\dim(E^*_i(T\widehat{x})) = 1$ for $0 \leq i \leq d$.

Proof. As $R^i \in T$ for $0 \leq i \leq d$, we have that $R^i\widehat{x} \in T\widehat{x}$ for $0 \leq i \leq d$. Furthermore, by Proposition 3.5.5 and (3.1), the vectors $R^i\widehat{x}$ are nonzero, pairwise orthogonal and

$R^i\hat{x} \in E_i^*(T\hat{x})$ for $0 \le i \le d$. Assume first that $T\hat{x}$ is thin. Then $E_i^*(T\hat{x})$ is spanned by $R^i\hat{x}$ for $0 \le i \le d$. This proves that the set $\{R^i\hat{x} : 0 \le i \le d\}$ is a basis of $T\hat{x}$. Conversely, assume that $\{R^i\hat{x} : 0 \le i \le d\}$ is a basis of $T\hat{x}$. Then, the subspace $E_i^*(T\hat{x})$ is spanned by $R^i\hat{x}$, and so $\dim(E_i^*(T\hat{x})) = 1$ for $0 \le i \le d$. This implies that $T\hat{x}$ is thin. The result follows. ■

Let us give an examble about how to use Lemma 3.6.1 to prove that a trivial module is thin.

Example 3.6.2. *Consider graph Γ from Example 3.3.1 (see also Figure 3.1), and observe that Γ is bipartite. Fix vertex $1 \in X$ and note that $d = 2$. Let A denote the adjacency matrix of Γ. Consider the Terwilliger algebra of Γ with respect to vertex 1. Observe that Γ is bipartite and so, $F = 0$. Let W denote the vector subspace of V spanned by the vectors $R^i\hat{1}$ $(0 \le i \le 2)$. Since $\hat{1} \in E_0^*V$, it follows from* (3.1) *that $R^i\hat{1} \in E_i^*V$ for $0 \le i \le 2$. By construction and since $R^3\hat{1} = 0$, it is clear that W is closed under the action of R. Moreover, by* (eiv) *from Section 3.1, the subspace W is invariant under the action of the dual idempotents as well. From Definition 3.1.1, it is easy to see that $L\hat{1} = 0$, $LR\hat{1} = 2 \cdot \hat{1}$ and $LR^2\hat{1} = 3 \cdot R\hat{1}$. This implies that W is invariant under the action of L. Since $A = L + R$, it follows that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. This shows that W is a T-module. Note that $R^i\hat{1} \in T\hat{1}$ for $0 \le i \le 2$. We thus have $W \subseteq T\hat{1}$. Furthermore, by Proposition 3.5.5 and* (3.1)*, the vectors $R^i\hat{1}$ are nonzero, pairwise orthogonal and so, they are linearly independent. Consequently, it follows from the above comments that the set $\{R^i\hat{1} : 0 \le i \le 2\}$ is a basis of $T\hat{1}$. Therefore, by Lemma 3.6.1, the unique irreducible module with endpoint* 0 *is thin.*

Proof of Theorem 3.5.3

As already mentioned, the equivalency of Theorem 3.5.3(i) and Theorem 3.5.3(ii) follows from Theorem 3.4.1. We proceed by showing the equivalency of Theorem 3.5.3(i) and Theorem 3.5.3(iii).

(i) implies (iii)

Assume that $T\hat{x}$ is thin. Recall that by Lemma 3.6.1 the set $\{R^i\hat{x} : 0 \le i \le d\}$ is a basis of $T\hat{x}$, $E_i^*(T\hat{x})$ is spanned by $R^i\hat{x}$ and $\dim(E_i^*(T\hat{x})) = 1$ for $0 \le i \le d$. Consequently, by (3.1) and since $L, F \in T$, we have that

$$LR^{i+1}\hat{x} \in E_i^*(T\hat{x}), \qquad FR^i\hat{x} \in E_i^*(T\hat{x})$$

for every $0 \le i \le d$. It follows from the above comments that for every $0 \le i \le d$ there exist scalars α_i, β_i, such that

$$LR^{i+1}\widehat{x} = \alpha_i R^i \widehat{x}, \qquad FR^i \widehat{x} = \beta_i R^i \widehat{x}.$$

The result now follows from Lemma 3.5.4.

(iii) implies (i)

Let W denote the vector subspace of V spanned by the vectors $R^i\widehat{x}$ $(0 \le i \le d)$. Since $\widehat{x} \in E_0^*V$, it follows from (3.1) that $R^i\widehat{x} \in E_i^*V$ for $0 \le i \le d$. By construction and since $R^{d+1}\widehat{x} = 0$, it is clear that W is closed under the action of R. Moreover, by (eiv) from Section 3.1, the subspace W is invariant under the action of the dual idempotents as well. From Definition 3.1.1 and (3.1) it is easy to see that $L\widehat{x} = F\widehat{x} = 0$.

Recall that by the assumption, for every integer $0 \le i \le d$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ we have

$$r^{i+1}\ell(y) = \alpha_i \, r^i(y), \qquad r^i f(y) = \beta_i \, r^i(y).$$

It follows from Lemma 3.5.4 that $LR^{i+1}\widehat{x} = \alpha_i R^i\widehat{x}$ and $FR^i\widehat{x} = \beta_i R^i\widehat{x}$. Therefore, W is invariant under the action of L and F. Since $A = L + F + R$, it follows that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Hence, W is a T-module. Note that $R^i\widehat{x} \in T\widehat{x}$ for $0 \le i \le d$, and so $W \subseteq T\widehat{x}$. As W is nonzero and $T\widehat{x}$ is irreducible, we thus have $W = T\widehat{x}$. It is clear that W is thin, since by construction and (3.1), the subspace E_i^*W is spanned by $R^i\widehat{x}$. This finishes the proof. ■

Theorem 3.6.3. *With reference to Notation 3.5.2, assume that* Γ *satisfies the equivalent conditions of Theorem 3.5.3. Then the set*

$$\mathcal{B} = \left\{ R^i\widehat{x} \mid 0 \le i \le d \right\}$$

is a basis of $T\widehat{x}$*. Moreover, the matrix representing the action of* A *on* $T\widehat{x}$ *with respect to* *the (ordered) basis $\mathcal{B}$ is given by*

$$\begin{pmatrix} 0 & \alpha_0 & & & & \\ 1 & \beta_1 & \alpha_1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \alpha_{d-2} & \\ & & & 1 & \beta_{d-1} & \alpha_{d-1} \\ & & & & 1 & \beta_d \end{pmatrix}.$$

Proof. By Theorem 3.5.3(iii), for every integer $0 \le i \le d$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ we have

$$r^{i+1}\ell(y) = \alpha_i\, r^i(y), \qquad r^i f(y) = \beta_i\, r^i(y).$$

It follows from Lemma 3.5.4 that $LR^i\widehat{x} = \alpha_{i-1}R^{i-1}\widehat{x}$ and $FR^i\widehat{x} = \beta_i R^i\widehat{x}$. Recall that $A = L + F + R$, and so the result follows (note also that $\beta_0 = 0$). ■

3.7 Examples

With reference to Notation 3.5.2, in this section we present some examples. We first consider graphs which are distance-regular around certain vertex x.

3.7.1 Distance-regularized vertices

With reference to Notation 3.5.2, assume that Γ is distance-regular around x, with the corresponding intersection numbers $a_i(x), b_i(x), c_i(x)$ $(0 \le i \le d)$. Then it is easy to see that for every $y \in \Gamma_i(x)$ $(0 \le i \le d)$ we have

$$r^i(y) = \prod_{j=1}^{i} c_j(x), \qquad r^{i+1}\ell(y) = b_i(x)\prod_{j=1}^{i+1} c_j(x), \qquad r^i f(y) = a_i(x)\prod_{j=1}^{i} c_j(x).$$

Therefore, for every $y \in \Gamma_i(x)$ we have that $r^{i+1}\ell(y) = \alpha_i r^i(y)$ and $r^i f(y) = \beta_i r^i(y)$ with $\alpha_i = b_i(x)c_{i+1}(x)$ and $\beta_i = a_i(x)$. By Theorem 3.5.3, the trivial T-module $T\widehat{x}$ is thin.

3.7.2 Distance-regularized graphs

Recall graph $\Gamma = (X, \mathcal{R})$ from Notation 3.5.2. As we already mentioned, it was proved in [88] by Terwilliger that the unique irreducible T-module is thin if Γ is distance-regular around x; see also Subsection 3.7.1. However, as shown in Example 3.4.6, the converse is not true, i.e. if the trivial module $T\widehat{x}$ is thin then Γ is not neccesarily distance-regular around x. In spite of that, if for every vertex $x \in X$, the trivial module $T\widehat{x}$ is thin then, for every $x \in X$, we have that Γ is distance-regular around x. This condition was studied by Terwilliger in [88]. For the sake of completeness, we next present a proof of this result with a slightly different approach.

Theorem 3.7.1. *With reference to Notation 3.5.2, the following (i)–(iii) are equivalent:*

(i) *For every $x \in X$, the trivial module $T\widehat{x}$ is thin.*

(ii) *The vectors $s_i := s_i(x) = \sum_{y \in \Gamma_i(x)} \widehat{y}$ $(0 \le i \le d)$, form a basis of the trivial module.*

(iii) Γ *is distance-regularized.*

Moreover, if (i)–(iii) hold then Γ *is distance-regular or distance-biregular.*

Proof. We will show that (i) implies (ii), (ii) implies (iii) and (iii) implies (i).

<u>(i) implies (ii)</u>: Pick $x \in X$. If the trivial module $T\widehat{x}$ is thin, then, by Lemma 3.6.1, the subspace $E_1^*(T\widehat{x})$ is spanned by $R\widehat{x} = s_1$ and $\dim(E_1^*(T\widehat{x})) = 1$. Moreover, by Lemma 3.4.7 we also have that $E_1^*(T\widehat{x})$ is spanned by $E_1^* \upsilon$. Therefore, there exists $\alpha \in \mathbb{C}$ such that $E_1^* \upsilon = \alpha s_1$. This yields that

$$\sum_{y \in \Gamma_1(x)} (\upsilon_y - \alpha)\widehat{y} = 0,$$

and so, $\upsilon_y = \upsilon_z$ for every $y, z \in \Gamma_1(x)$. Since by assumption the trivial module $T\widehat{x}$ is thin for every vertex $x \in X$, it follows from the above comments that $\upsilon_y = \upsilon_z$ if $y, z \in X$ are connected by a path of even length. In particular, for every $y, z \in \Gamma_i(x)$ $(0 \le i \le d)$ we have $\upsilon_y = \upsilon_z$. Therefore, there exists a nonzero scalar α_i such that $E_i^* \upsilon = \alpha_i s_i$ $(0 \le i \le d)$. The claim now immediately follows from Lemma 3.4.7 as the set $\{E_i^* \upsilon \mid 0 \le i \le d\}$ is a basis for $T\widehat{x}$.

<u>(ii) implies (iii)</u>: Pick $x \in X$. Assume that the vectors $s_i := s_i(x)$ $(0 \le i \le d)$ form a basis of the trivial module. We observe that $E_j^* s_i = \delta_{i,j} s_i$ $(0 \le i, j \le d)$. Moreover, note that since $T\widehat{x}$ is a T-module, it is invariant under the action of A. It follows from Lemma 3.4.7

that for every $0 \le i \le d$, the vector $As_i = AE_i^* s_i$ is a linear combination of the vectors s_j $(0 \le j \le d)$. However, it follows from Proposition 3.4.8 that As_i is a linear combination of just s_{i-1}, s_i and s_{i+1}. Then, for every $0 \le i \le d$ there exist scalars $\alpha_i(x), \beta_i(x), \gamma_i(x) \in \mathbb{C}$ such that

$$As_i = \beta_i(x)s_{i-1} + \alpha_i(x)s_i + \gamma_i(x)s_{i+1},$$

where we assume that $s_i = 0$ whenever $i < 0$ or $i > d$. In particular,

$$\begin{aligned} AE_{i-1}^* s_{i-1} &= \beta_{i-1}(x)s_{i-2} + \alpha_{i-1}(x)s_{i-1} + \gamma_{i-1}(x)s_i, \\ AE_i^* s_i &= \beta_i(x)s_{i-1} + \alpha_i(x)s_i + \gamma_i(x)s_{i+1}, \\ AE_{i+1}^* s_{i+1} &= \beta_{i+1}(x)s_i + \alpha_{i+1}(x)s_{i+1} + \gamma_{i+1}(x)s_{i+2}. \end{aligned}$$

Multiplying the above equalities with E_i^* and using property (eiv) in Section 3.1, we get

$$E_i^* AE_{i-1}^* s_{i-1} = \gamma_{i-1}(x)s_i, \tag{3.11}$$
$$E_i^* AE_i^* s_i = \alpha_i(x)E_i^* s_i, \tag{3.12}$$
$$E_i^* AE_{i+1}^* s_{i+1} = \beta_{i+1}(x)s_{i+1}. \tag{3.13}$$

Pick a vertex $y \in \Gamma_i(x)$. Computing the y-entry of (3.11), (3.12) and (3.13), we get

$$\sum_{z \in \Gamma(y) \cap \Gamma_{i-1}(x)} 1 = \gamma_{i-1}(x), \tag{3.14}$$
$$\sum_{z \in \Gamma(y) \cap \Gamma_i(x)} 1 = \alpha_i(x), \tag{3.15}$$
$$\sum_{z \in \Gamma(y) \cap \Gamma_{i+1}(x)} 1 = \beta_{i+1}(x). \tag{3.16}$$

Therefore, it follows from (3.14), (3.15) and (3.16) that $c_i(x,y) = \gamma_{i-1}(x)$, $a_i(x,y) = \alpha_i(x)$ and $b_i(x,y) = \beta_{i+1}(x)$. This shows that the intersection numbers of Γ with respect to x do not depend on the choice of $y \in \Gamma_i(x)$. Hence, Γ is distance-regular around x. As $x \in X$ was arbitrary, it holds that Γ is distance-regularized.

(iii) implies (i): If Γ is distance-regularized then, Γ is distance-regular around every vertex. Therefore, it immediately follows from Subsection 3.7.1 that the trivial T-module $T\hat{x}$ is thin for every vertex $x \in X$.

The result now immediately follows from [41] as every distance-regularized graph is either distance-regular or distance-biregular. ■

3.7.3 Bipartite graphs

With reference to Notation 3.5.2, assume that Γ is bipartite. Observe that in this case $r^i f(y) = 0$ for every $0 \le i \le d$ and for every $y \in \Gamma_i(x)$. Therefore, we have the following result.

Corollary 3.7.2. *With reference to Notation 3.5.2, assume that Γ is bipartite. Then $T\widehat{x}$ is thin if and only if for $0 \le i \le d$ there exist scalars α_i, such that for every $y \in \Gamma_i(x)$ we have $r^{i+1}\ell(y) = \alpha_i r^i(y)$.*

Proof. Immediately from Theorem 3.5.3 and the above observation. ■

Example 3.7.3. *Consider graph Γ from Example 3.3.1 (see also Figure 3.1), and observe that Γ is bipartite. Fix vertex $1 \in X$ and note that $d = 2$. It is easy to see that for every $y \in \Gamma_i(1)$ $(0 \le i \le 2)$ we have $r^{i+1}\ell(y) = \alpha_i r^i(y)$, where $\alpha_0 = 2$, $\alpha_1 = 3$ and $\alpha_2 = 0$. As Γ is bipartite, it follows from Corollary 3.7.2 that the trivial module $T\widehat{1}$ is thin.*

3.7.4 Trees

With reference to Notation 3.5.2, assume that Γ is a tree. Observe that in this case (as Γ is also bipartite) we have $r^i(y) = 1$ and $r^i f(y) = 0$ for every $0 \le i \le d$ and for every $y \in \Gamma_i(x)$. Therefore, by Theorem 3.5.3, $T\widehat{x}$ is thin if and only if for $0 \le i \le d$ there exist scalars α_i, such that for every $y \in \Gamma_i(x)$ we have $r^{i+1}\ell(y) = \alpha_i$. Note however that $r^{i+1}\ell(y) = |\Gamma(y) \cap \Gamma_{i+1}(x)| = b_i(x,y)$. It follows that the trivial module $T\widehat{x}$ is thin if and only if the intersection numbers $b_i(x,y)$ do not depend on the choice of $y \in \Gamma_i(x)$. As $a_i(x,y) = 0$ and $c_i(x,y) = 1$ for every $y \in \Gamma_i(x)$, we have the following corollary of Theorem 3.5.3.

Corollary 3.7.4. *With reference to Notation 3.5.2, assume that Γ is a tree. Then $T\widehat{x}$ is thin if and only if Γ is distance-regular around x.*

3.7.5 Cartesian product $P_3 \square \cdots \square P_3$

Let us first recall the definition of Cartesian product of graphs. Let Γ_1 and Γ_2 be finite simple graphs with vertex set X_1 and X_2, respectively. Then the Cartesian product of Γ_1 and Γ_2, denoted by $\Gamma_1 \square \Gamma_2$, has vertex set $X_1 \times X_2$. Vertices (x_1, x_2) and (y_1, y_2) are adjacent in $\Gamma_1 \square \Gamma_2$ if and only if either $x_1 = y_1$ and x_2, y_2 are adjacent in Γ_2, or $x_2 = y_2$ and x_1, y_1 are adjacent in Γ_1.

With reference to Notation 3.5.2, in this subsection we consider graph $\Gamma = P_3 \square \cdots \square P_3$, the Cartesian product of n copies of the path P_3 on 3 vertices (cf. [34, p. 188]). Assume that the vertex set and the edge set of P_3 are $\{0,1,2\}$ and $\{\{0,1\},\{1,2\}\}$, respectively. Then the vertex set of Γ is

$$X = \{(y_1, y_2, \ldots, y_n) \mid y_i \in \{0,1,2\} \text{ for each } 1 \le i \le n\}.$$

Vertices $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ are adjacent in Γ if and only if y and z differ in exactly one coordinate (say coordinate i), and $|y_i - z_i| = 1$. Note that Γ is bipartite. We assume that vertex x from Notation 3.5.2 is vertex $x = (0,0,\ldots,0)$. Observe that $d = 2n$ and that for $0 \le i \le 2n$ we have

$$\Gamma_i(x) = \{(y_1, y_2, \ldots, y_n) \in X \mid y_1 + y_2 + \cdots + y_n = i\}.$$

For $1 \le i \le n$ let us denote by e_i the vertex of Γ, which has i-th coordinate equal to 1, and all other coordinates equal to 0. For vertices $y = (y_1, y_2, \ldots, y_n), z = (z_1, z_2, \ldots, z_n) \in X$ let $y+z$ denote the n-tuple $(y_1+z_1, y_2+z_2, \ldots, y_n+z_n)$. Note that $y+z$ is not necessarily contained in X. Furthermore, let us define $A(y) = \{j \mid 1 \le j \le n,\ y_j = 0\}$, $B(y) = \{j \mid 1 \le j \le n,\ y_j = 1\}$ and $C(y) = \{j \mid 1 \le j \le n,\ y_j = 2\}$. Note that

$$|A(y)| + |B(y)| + |C(y)| = n, \qquad |B(y)| + 2|C(y)| = \partial(x,y). \tag{3.17}$$

Assume now that $y = (y_1, y_2, \ldots, y_n) \in \Gamma_i(x)$. Then $r^i(y)$ is equal to the number of walks between x and y in the n-dimensional integer lattice, where for each step of the walk the only possible directions are along one of the "vectors" e_j $(0 \le j \le n)$. This shows that

$$\begin{aligned} r^i(y) &= \binom{i}{y_1}\binom{i-y_1}{y_2}\binom{i-y_1-y_2}{y_3}\cdots\binom{i-y_1-\cdots-y_{n-1}}{y_n} \\ &= \frac{i!(i-y_1)!(i-y_1-y_2)!\cdots(i-y_1-y_2-\cdots-y_{n-1})!}{y_1!(i-y_1)!y_2!(i-y_1-y_2)!\cdots y_{n-1}!(i-y_1-y_2-\cdots-y_{n-1})!y_n!} \\ &= \frac{i!}{y_1!y_2!\cdots y_{n-1}!y_n!} = \frac{i!}{2^{|C(y)|}}. \end{aligned}$$

Observe also that

$$\Gamma(y) \cap \Gamma_{i+1}(x) = \{y + e_j \mid j \in A(y)\} \cup \{y + e_j \mid j \in B(y)\}.$$

Moreover, for $j \in A(y)$ we have $|C(y+e_j)| = |C(y)|$, and for $j \in B(y)$ we have $|C(y+e_j)| = |C(y)|+1$. It follows that

$$\begin{aligned} r^{i+1}\ell(y) &= \sum_{j\in A(y)} r^{i+1}(y+e_j) + \sum_{j\in B(y)} r^{i+1}(y+e_j) \\ &= \frac{|A(y)|(i+1)!}{2^{|C(y)|}} + \frac{|B(y)|(i+1)!}{2^{|C(y)|+1}} = \frac{(i+1)!}{2^{|C(y)|}}\left(|A(y)| + \frac{|B(y)|}{2}\right). \end{aligned}$$

Finally, it follows from (3.17) that $|A(y)|+|B(y)|/2 = (2n-i)/2$, and so

$$r^{i+1}\ell(y) = \frac{(i+1)!(2n-i)}{2^{|C(y)|+1}}.$$

This shows that for every $y \in \Gamma_i(x)$ $(0 \le i \le 2n)$ we have $r^{i+1}\ell(y) = \alpha_i r^i(y)$, where $\alpha_i = (i+1)(2n-i)/2$ is independent on the choice of $y \in \Gamma_i(x)$. As Γ is bipartite, it follows from Corollary 3.7.2 that the trivial module $T\widehat{x}$ is thin.

3.7.6 A construction

In this subsection we show how to construct new graphs, that satisfy the equivalent conditions of Theorem 3.5.3 for a certain vertex. To do this, let Γ and Σ denote finite, simple graphs with vertex set X and Y, respectively. Assume that Γ is connected. Fix a vertex $x \in X$ and consider the Cartesian product $\Gamma \square \Sigma$. Let H denote a graph obtained by adding a new vertex w to the graph $\Gamma \square \Sigma$, and connecting this new vertex w with all vertices (x, y), where y is an arbitrary vertex of Σ. See for example Figure 3.3 below.

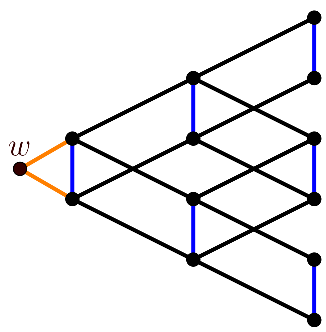

Figure 3.3: Graph H obtained from the Cartesian product $\Gamma \square P_2$ where Γ is the graph from Example 3.3.1 and P_2 denotes the path on 2 vertices.

Note that for an arbitrary vertex (x', y') of H different from w, the distance between w

and (x', y') in H is equal to the distance between x and x' in Γ plus one:

$$\partial_H(w, (x', y')) = \partial_\Gamma(x, x') + 1.$$

It follows that $d_H = d + 1$, where d_H is the eccentricity of w in H and d is the eccentricity of x in Γ. Moreover, for $1 \le i \le d_H$ we have

$$H_i(w) = \Gamma_{i-1}(x) \times Y = \{(u, y) \mid u \in \Gamma_{i-1}(x), y \in Y\}.$$

In what follows, we use subscripts to distinguish the number of walks of a particular shape in H and in Γ. For example, for $x' \in \Gamma_i(x)$, we denote the number of walks from x to x' of shape $r^{i+1}\ell$ with respect to x by $r^{i+1}\ell_\Gamma(x')$. For $(x', y') \in H_i(w)$, we denote the number of walks from w to (x', y') of shape $r^{i+1}\ell$ with respect to w by $r^{i+1}\ell_H((x', y'))$. It is easy to see that for $(x', y') \in H_i(w)$ $(1 \le i \le d_H)$ we have

$$\begin{aligned} r_H^i((x', y')) = r_\Gamma^{i-1}(x'), \qquad r^{i+1}\ell_H((x', y')) &= r^i\ell_\Gamma(x'), \\ r^i f_H((x', y')) &= r^{i-1} f_\Gamma(x') + |\Sigma(y')| r_\Gamma^{i-1}(x'), \end{aligned} \tag{3.18}$$

where $\Sigma(y')$ is the set of neighbours of y' in Σ. Assume now that for vertex x of Γ the equivalent conditions of Theorem 3.5.3 are satisfied, and that Σ is regular with valency k. It follows from (3.18) that for $1 \le i \le d_H$ and for every $(x', y') \in H_i(w)$ we have

$$r^{i+1}\ell_H((x', y')) = r^i\ell_\Gamma(x') = \alpha_{i-1} r_\Gamma^{i-1}(x') = \alpha_{i-1} r_H^i((x', y'))$$

and

$$r^i f_H((x', y')) = r^{i-1} f_\Gamma(x') + |\Sigma(y')| r_\Gamma^{i-1}(x') = (\beta_{i-1} + k) r_\Gamma^{i-1}(x').$$

As we also have $r\ell_H(w) = |Y| = |Y| r_H^0(w)$ and $f_H(w) = 0$, we see that vertex w of H satisfies the condition of Theorem 3.5.3(iii). Therefore, by Theorem 3.5.3, the trivial $T(w)$-module is thin.

Chapter 4

On the Terwilliger algebra of distance-biregular graphs

Let Γ denote a distance-biregular graph with vertex set X. Fix $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. In this chapter we consider irreducible T-modules with endpoint 1. We show that there are no such modules if and only if Γ is the complete bipartite graph $K_{1,n}$ $(n \geq 1)$ and x is a vertex of Γ with valency 1. If the valency of x is at least 2 then we show that, up to isomorphism, there is a unique irreducible T-module with endpoint 1, and this module is thin.

The chapter is organized as follows. In Sections 4.1 and 4.2 we recall basic definitions and results about distance-biregular graphs and Terwilliger algebras. In Section 4.2 we also prove that T has no irreducible modules with endpoint 1 if and only if x is of valency 1. In Section 4.3 we introduce the so-called intersection diagram of a distance-biregular graph. In Section 4.4 we evaluate certain products of matrices in algebra T. In Sections 4.5 and 4.6 we prove our main results.

The chapter is based on joint work with Štefko Miklavič. Our main results are currently published in Linear Algebra and its Applications (2020); see [26] for more details.

4.1 Preliminaries

In this section we review some definitions and basic concepts regarding distance-biregular graphs. Throughout this chapter, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops and multiple edges, with vertex set X and edge set $\mathcal{R}$.

Let $x, y \in X$. The **distance** between x and y, denoted by $\partial(x,y)$, is the length of a shortest xy-path. The **eccentricity of** x, denoted by $\epsilon(x)$, is the maximum distance between x and any other vertex of Γ: $\epsilon(x) = \max\{\partial(x,z) \mid z \in X\}$. Let D denote the maximum eccentricity of any vertex in Γ. We call D the **diameter of** Γ. For an integer i we define $\Gamma_i(x)$ by

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}.$$

We will abbreviate $\Gamma(x) = \Gamma_1(x)$. Note that $\Gamma(x)$ is the set of neighbours of x. Observe that $\Gamma_i(x)$ is empty if and only if $i < 0$ or $i > \epsilon(x)$. Assume for a moment that $y \in \Gamma_i(x)$ for some $0 \le i \le \epsilon(x)$ and let z be a neighbour of y. Then, by the triangle inequality,

$$\partial(x,z) \in \{i-1, i, i+1\},$$

and so $z \in \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$. For $y \in \Gamma_i(x)$ we therefore define the following numbers:

$$a_i(x,y) = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i(x,y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i(x,y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|.$$

We say that a vertex $x \in X$ is **distance-regularized** (or that Γ is **distance-regular around** x) if the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ do not depend on the choice of $y \in \Gamma_i(x)$ $(0 \le i \le \epsilon(x))$. In this case, the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ are simply denoted by $a_i(x), b_i(x)$ and $c_i(x)$ respectively, and are called the **intersection numbers of** x. Observe that if x is distance-regularized and $\epsilon(x) = d$, then $a_0(x) = c_0(x) = b_d(x) = 0$, $b_0(x) = |\Gamma(x)|$ and $c_1(x) = 1$. Note also that for every $1 \le i \le d$ we have that $b_{i-1}(x) > 0$ and $c_i(x) > 0$. For convenience we define $c_i(x) = b_i(x) = 0$ for $i < 0$ and $i > d$.

Graph Γ is said to be **distance-regularized** if each of its vertices is distance-regularized. A distance-regularized graph is called **distance-regular** if all of its vertices have the same intersection numbers, and is called **distance-biregular** otherwise. By [41] every distance-biregular graph is bipartite, and the vertices of the same bipartite class have the same intersection numbers. See [22, 31, 32, 77, 80] for further research on distance-biregular graphs.

Assume for the moment that Γ is distance-biregular and pick $x \in X$. As Γ is bipartite, we have $a_i(x) = 0$ for $0 \le i \le \epsilon(x)$ (otherwise there would exist a cycle of odd length in Γ). Furthermore, let Y and Y' be the bipartite parts of Γ. Note that all vertices from Y (Y', respectively) have the same eccentricity. We denote this common eccentricity by d (d', respectively). Observe that $|d - d'| \le 1$ and that $D = \max\{d, d'\}$. For $x \in Y$, $y \in Y'$ and an integer i we abbreviate $c_i := c_i(x)$, $b_i := b_i(x)$, $c'_i := c_i(y)$ and $b'_i := b_i(y)$.

4.2 The Terwilliger algebra

Recall graph $\Gamma = (X, \mathcal{R})$ from Section 4.1. In this section we recall some definitions and basic results concerning a Terwilliger algebra of Γ. We refer the reader to Section 3.1 for further details.

Let $\mathbb{C}$ denote the complex number field. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the **standard module**. We endow V with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^\top \overline{v}$ for $u, v \in V$, where $\top$ denotes transpose and $^-$ denotes complex conjugation. For $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y-coordinate and 0 in all other coordinates. We observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

We recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$, where D denotes the diameter of Γ, let $A_i \in \mathrm{Mat}_X(\mathbb{C})$ denote the matrix with (x, y)-entry defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = i, \\ 0 & \text{if} \quad \partial(x,y) \neq i, \end{cases} \qquad (x, y \in X).$$

The matrix $A := A_1$ is just the usual **adjacency matrix of** Γ. For notational convenience, set $A_i = 0$ for $i < 0$ and $i > D$. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{D} A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (aiv) $A_i^\top = A_i$ $(0 \le i \le D)$; where I (resp. J) denotes the identity matrix (resp. the all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. The **adjacency algebra of** Γ, also called the **Bose-Mesner algebra of** Γ, is the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ.

We now recall the dual idempotents of Γ. To do this, fix a vertex $x \in X$ and let $d = \epsilon(x)$. We view x as a *base vertex*. For $0 \le i \le d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ***i*-th dual idempotent of** Γ **with respect to** x [89, p. 378]. We also observe (ei) $\sum_{i=0}^{d} E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le d)$; (eiii) $E_i^{*\top} = E_i^*$ $(0 \le i \le d)$; (eiv)

$E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le d)$. By these facts, matrices $E_0^*, E_1^*, \dots, E_d^*$ form a basis for the commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the **dual Bose-Mesner algebra of Γ with respect to** x [89, p. 378]. For convenience we define E_{-1}^* and E_{d+1}^* to be the zero matrix of $\mathrm{Mat}_X(\mathbb{C})$. Note that for $0 \le i \le d$ we have

$$E_i^* V = \mathrm{Span}\{\hat{y} \mid y \in \Gamma_i(x)\}. \tag{4.1}$$

We call $E_i^* V$ the ***i*-th subconstituent of Γ with respect to** x. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_d^* V \qquad (\textit{orthogonal direct sum}).$$

Recall that the set $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. Therefore, for every $\upsilon \in V$ we have that $\upsilon = \sum_{y \in X} \upsilon_y \, \hat{y}$ for some $\upsilon_y \in \mathbb{C}$. In addition,

$$E_i^* \upsilon = \sum_{y \in X} \upsilon_y \, E_i^* \hat{y} = \sum_{y \in \Gamma_i(x)} \upsilon_y \, \hat{y}.$$

We recall the definition of a Terwilliger algebra of Γ. The Terwilliger algebra was first defined in [89, Definition 3.3], where it was called the **subconstituent algebra**. It was first defined for commutative association schemes, but the definition can be easily generalized to an arbitrary graph as follows. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the **Terwilliger algebra of Γ with respect to** x. Recall that M is generated by A. So, T is generated by A and the dual idempotents. We observe that T has finite dimension. In addition, since by construction T is generated by real-symmetric matrices, it follows that T is closed under the conjugate-transpose map. For a vector subspace $W \subseteq V$, we denote by TW the subspace $\{Bw \mid B \in T, w \in W\}$.

We now recall the lowering matrix and the raising matrix of the algebra T.

Definition 4.2.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Define $L = L(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by*

$$L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*, \qquad R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*.$$

*We refer to L and R as the **lowering** and the **raising matrix with respect to** x, respectively. Note that $R, L \in T$, $R = L^\top$ and $A = R + L$.*

Observe also that for $y,z \in X$ we have that the (z,y)-entry of L equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) - 1$, and 0 otherwise. Similarly, the (z,y)-entry of R equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) + 1$, and 0 otherwise.

By a **T-module** we mean a subspace of V which is B-invariant for every $B \in T$. Let W denote a T-module. Then W is said to be **irreducible** whenever W is nonzero and W contains no T-modules other than 0 and W. Since the algebra T is closed under the conjugate-transpose map, it turns out that any T-module is an orthogonal direct sum of irreducible T-modules. In particular, the standard module V is an orthogonal direct sum of irreducible T-modules.

Let W be an irreducible T-module. We observe that W is an orthogonal direct sum of the nonvanishing subspaces E_i^*W for $0 \le i \le d$. By the **endpoint** of W we mean $\min\{i \mid 0 \le i \le d,\ E_i^*W \neq 0\}$. We say that W is **thin** whenever the dimension of E_i^*W is at most 1 for $0 \le i \le d$.

Let W and W' denote two irreducible T-modules. By a **T-isomorphism** from W to W' we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W and W' are said to be **T-isomorphic** (or simply **isomorphic**) whenever there exists a T-isomorphism $\sigma : W \to W'$. We note that isomorphic irreducible T-modules have the same endpoint. It turns out that two non-isomorphic irreducible T-modules are orthogonal.

It is known that T has a unique irreducible T-module with endpoint 0, namely the subspace $T\hat{x} = \{B\hat{x} \mid B \in T\}$. We refer to $T\hat{x}$ as the **trivial T-module**. By Theorem 3.5.3 and Subsection 3.7.1, it turns out that if x is distance-regularized, the trivial T-module is thin. In this case vectors s_i $(0 \le i \le d)$, where

$$s_i = \sum_{y \in \Gamma_i(x)} \hat{y},$$

form a basis of the trivial T-module. In particular, if Γ is distance-biregular, we observe that the trivial T-module is thin.

In the rest of this chapter we will study irreducible T-modules with endpoint 1. Therefore, we will first characterize those distance-regularized vertices x of Γ, for which the corresponding Terwilliger algebra $T = T(x)$ has no irreducible T-modules with endpoint 1.

Proposition 4.2.2. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$ which is distance-regularized, and let $T = T(x)$ denote the corresponding Terwilliger* *algebra. Then, there are no irreducible T-modules with endpoint* 1 *if and only if* $|\Gamma(x)| = 1$.

Proof. Let V denote the standard module, and let W_0 denote the trivial T-module. Recall that W_0 is thin since x is distance-regularized.

Assume first that $|\Gamma(x)| = 1$ and let y denote the unique vertex in $\Gamma(x)$. Recall that by (4.1) we have that E_1^*V is spanned by $\hat{y}$. As vectors s_i $(0 \le i \le \epsilon(x))$ form a basis for W_0, we have that $E_1^*W_0$ is also spanned by $\hat{y}$. Suppose now that W_1 is an irreducible T-module with endpoint 1 and pick a nonzero vector $w \in E_1^*W_1$. As $E_1^*W_1 \subseteq E_1^*V$, we have that $w = \alpha\,\hat{y}$ for some scalar $\alpha \in \mathbb{C}$. However, W_0 and W_1 are not isomorphic (they have different endpoints), and are therefore orthogonal. This implies that $\alpha = 0$, a contradiction.

Assume next that T has no irreducible modules with endpoint 1. Recall that V is an orthogonal direct sum of irreducible T-modules. As none of these modules has endpoint 1 and as W_0 is the unique irreducible T-module with endpoint 0, we therefore have that $\dim(E_1^*V) = \dim(E_1^*W_0) = 1$. It follows from (4.1) that $|\Gamma(x)| = 1$. ■

Observe that the only distance-biregular graphs which have vertices with valency 1, are the complete bipartite graphs $K_{1,n}$ for $n \ge 1$.

4.3 The intersection diagrams

Throughout this section let $\Gamma = (X, \mathcal{R})$ denote a distance-biregular graph. We define certain partition of X, that we will find useful later.

Definition 4.3.1. *Let* $\Gamma = (X, \mathcal{R})$ *denote a distance-biregular graph with diameter* D*. Pick* $x, y \in X$*, such that* $y \in \Gamma(x)$*. For integers* i, j *we define sets* $D_j^i := D_j^i(x,y)$ *as follows:*

$$D_j^i = \Gamma_i(x) \cap \Gamma_j(y).$$

Observe that $D_j^i = \emptyset$ *if* $i < 0$ *or* $j < 0$*. Similarly,* $D_j^i = \emptyset$ *if* $i > \epsilon(x)$ *or* $j > \epsilon(y)$*. Furthermore, by the triangle inequality we have that* $D_j^i = \emptyset$ *if* $|i - j| \ge 2$*. Note also that as* Γ *is bipartite, the set* D_i^i *is empty for* $0 \le i \le D$*. The collection of all the subsets* D_{i-1}^i $(1 \le i \le \epsilon(x))$ *and* D_i^{i-1} $(1 \le i \le \epsilon(y))$ *is called the* ***intersection diagram of*** Γ ***with respect to the edge*** xy*. See Figure 4.1 for an example.*

For the rest of the chapter we adopt the following notation.

Notation 4.3.2. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-biregular graph, with vertex set X, edge set $\mathcal{R}$ and diameter D. Let $X = Y \cup Y'$ be a bipartition of Γ. Let d (d', respectively) denote the eccentricity of vertices from Y (Y', respectively). Let c_i, b_i be the intersection numbers of the vertices from Y. Similarly, let c'_i, b'_i be the intersection numbers of the vertices from Y'. Let $A_i \in \mathrm{Mat}_X(\mathbb{C})$ denote the i-th distance matrix of Γ. We abbreviate $A := A_1$. Fix $x \in X$ with $|\Gamma(x)| \geq 2$. Without loss of generality we assume that $x \in Y$. Let $E^*_i \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \leq i \leq d)$ denote the dual idempotents of Γ with respect to x. For convenience we set $E^*_{d+1} = 0$. Let V denote the standard module of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $L = L(x)$ and $R = R(x)$ denote the lowering and the raising matrix of T, respectively. Let J denote the all 1's matrix in $\mathrm{Mat}_X(\mathbb{C})$. For $y \in \Gamma(x)$ let the sets $D^i_j = D^i_j(x,y)$ be as defined in Definition 4.3.1.*

The proofs of the following lemmas are straightforward and therefore left to the reader.

Lemma 4.3.3. *With reference to Notation 4.3.2, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Then the following (i)–(iv) hold for $1 \leq i \leq D$.*

(i) *If $w \in D^i_{i-1}$ then $\Gamma(w) \subseteq D^{i-1}_{i-2} \cup D^{i-1}_i \cup D^{i+1}_i$.*

(ii) *If $w \in D^{i-1}_i$ then $\Gamma(w) \subseteq D^{i-2}_{i-1} \cup D^i_{i-1} \cup D^i_{i+1}$.*

(iii) $\Gamma_i(x) = D^i_{i-1} \cup D^i_{i+1}$ *and* $\Gamma_i(y) = D^{i-1}_i \cup D^{i+1}_i$.

(iv) *If $D^i_{i+1} \neq \emptyset$ ($D^{i+1}_i \neq \emptyset$, respectively) then $D^j_{j+1} \neq \emptyset$ ($D^{j+1}_j \neq \emptyset$, respectively) for every $0 \leq j \leq i$.*

Lemma 4.3.4. *With reference to Notation 4.3.2, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Assume that $z \in D^i_{i-1}$ $(1 \leq i \leq d)$. Then, the following (i)–(iii) hold.*

(i) $\left|\Gamma(z) \cap D^{i-1}_{i-2}\right| = c'_{i-1}$.

(ii) $\left|\Gamma(z) \cap D^{i+1}_i\right| = b_i$.

(iii) $\left|\Gamma(z) \cap D^{i-1}_i\right| = c_i - c'_{i-1} = b'_{i-1} - b_i$.

Lemma 4.3.5. *With reference to Notation 4.3.2, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Assume that $z \in D^{i-1}_i$ $(1 \leq i \leq d')$. Then, the following (i)–(iii) hold.*

(i) $\left|\Gamma(z) \cap D^{i-2}_{i-1}\right| = c_{i-1}$.

(ii) $\left|\Gamma(z) \cap D^i_{i+1}\right| = b'_i$.

(iii) $\left|\Gamma(z)\cap D^i_{i-1}\right| = c'_i - c_{i-1} = b_{i-1} - b'_i$.

With reference to Notation 4.3.2, recall that $d' \in \{d-1, d, d+1\}$. In Figure 4.1, a graphical representation of an intersection diagram for the case $d' = d+1$ is presented. A line between D^i_j and $D^{i'}_{j'}$ indicates the possibility of existence of edges between these two sets. The intersection diagrams for the other two cases (that is, for the cases $d' = d-1$ and $d' = d$) are similar and we will not present them here.

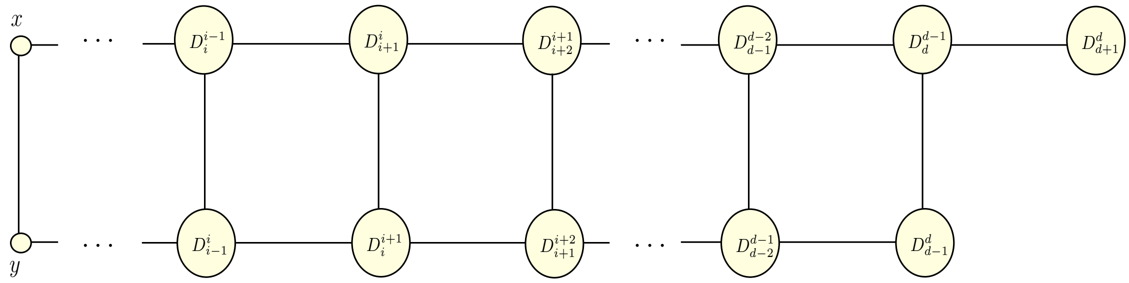

Figure 4.1: The intersection diagram of a distance-biregular graph Γ where $d' = d+1$.

4.4 Some products in the Terwilliger algebra

With respect to Notation 4.3.2, in this section, we evaluate several products in the Terwilliger algebra T that we will need later in this chapter.

Proposition 4.4.1. *With reference to Notation 4.3.2, pick $y,z \in X$. The (z,y)-entry of $E^*_i A_j E^*_k$ equals 1 if $\partial(x,z) = i$, $\partial(y,z) = j$ and $\partial(x,y) = k$, and 0 otherwise. In particular, the following (i), (ii) hold:*

(i) if one of i,j,k is greater than the sum of the other two, then $E^*_i A_j E^*_k = 0$;

(ii) if $i+j+k$ is odd, then $E^*_i A_j E^*_k = 0$.

Proof. It suffices to observe that $(E^*_i A_j E^*_k)_{zy} = (E^*_i)_{zz}(A_j)_{zy}(E^*_k)_{yy}$. Part (i) now follows from the definition of matrices E^*_i, A_j, E^*_k and the triangle inequality, and part (ii) holds since Γ is bipartite. ∎

Corollary 4.4.2. *With reference to Notation 4.3.2, pick $y,z \in X$. Then the (z,y)-entry of $E^*_i A_j E^*_1$ equals 1 if $y \in \Gamma(x)$ and $z \in D^i_j(x,y)$, and 0 otherwise. Moreover, if either $|i-j| > 1$ or $i = j$ then $E^*_i A_j E^*_1 = 0$.*

Proof. Immediately from Preposition 4.4.1. ■

Proposition 4.4.3. *With reference to Notation 4.3.2, pick* $y,z \in X$*. If* $y \in \Gamma(x)$ *then the* (z,y)*-entry of* $AE_i^* A_j E_1^*$ *equals* $\left|\Gamma(z) \cap D_j^i(x,y)\right|$.

Proof. By Corollary 4.4.2 and elementary matrix multiplication. ■

Recall that a sequence of vertices $[y_0, y_1, \cdots, y_t]$ of Γ is a **walk** if $y_{i-1}y_i$ is an edge of Γ for $1 \le i \le t$.

Proposition 4.4.4. *With reference to Notation 4.3.2, pick* $y,z \in X$ *and let* m *be a positive integer. Assume that* $y \in \Gamma_i(x)$*. Then the* (z,y)*-entry of the matrix* $R^m L$ *is equal to the number of walks* $[y = y_0, y_1, \cdots, y_{m+1} = z]$ *such that* $y_1 \in \Gamma_{i-1}(x)$ *and* $y_j \in \Gamma_{i+j-2}(x)$ *for every* $2 \le j \le m+1$.

Proof. Immediate from Definition 4.2.1 and elementary matrix multiplication. ■

Proposition 4.4.5. *With reference to Notation 4.3.2, pick* $y,z \in X$*. Then the following holds for* $1 \le i \le d$*:*

$$\left(E_i^* A^{i-1} E_1^*\right)_{zy} = \begin{cases} \prod_{k=1}^{i-1} c'_k & \text{if } y \in \Gamma(x) \text{ and } z \in D_{i-1}^i(x,y), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is straightforward to check that the (z,y)-entry of $E_i^* A^{i-1} E_1^*$ is zero if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$. Suppose now that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$. Then, it follows that $\left(E_i^* A^{i-1} E_1^*\right)_{zy} = \left(A^{i-1}\right)_{zy}$, which is further equal to the number of walks of length $i-1$ between y and z. Observe that by the triangle inequality and since Γ is bipartite we have that $\partial(y,z) \in \{i+1, i-1\}$. Therefore, if $\partial(y,z) = i+1$, we have that $\left(E_i^* A^{i-1} E_1^*\right)_{zy} = 0$. Moreover, if $\partial(y,z) = i-1$ then by Lemma 4.3.4(i) there are precisely $c'_{i-1} \cdots c'_1$ walks of length $i-1$ between y and z. The result follows. ■

Lemma 4.4.6. *With reference to Notation 4.3.2, the following holds for* $1 \le i \le d$*:*

$$E_i^* A_{i-1} E_1^* = \left(\prod_{k=1}^{i-1} \frac{1}{c'_k}\right) E_i^* A^{i-1} E_1^*.$$

In particular, $E_i^* A_{i-1} E_1^* \in T$.

Proof. Straightforward from Corollary 4.4.2 and Proposition 4.4.5. ■

Proposition 4.4.7. *With reference to Notation 4.3.2, pick $y,z \in X$. Then the following holds for $1 \le i \le d$:*

$$(AE_i^* A_{i-1} E_1^*)_{zy} = \begin{cases} b_{i-1} & \text{if } y \in \Gamma(x) \text{ and } z \in D_{i-2}^{i-1}(x,y), \\ b_{i-1} - b_i' & \text{if } y \in \Gamma(x) \text{ and } z \in D_i^{i-1}(x,y), \\ c_i' & \text{if } y \in \Gamma(x) \text{ and } z \in D_i^{i+1}(x,y), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Immediately from Preposition 4.4.3 and Lemmas 4.3.4 and 4.3.5. ■

Corollary 4.4.8. *With reference to Notation 4.3.2, the following (i)–(iii) hold for $1 \le i \le d$:*

(i) $AE_i^* A_{i-1} E_1^* = b_{i-1} E_{i-1}^* A_{i-2} E_1^* + \left(b_{i-1} - b_i'\right) E_{i-1}^* A_i E_1^* + c_i' E_{i+1}^* A_i E_1^*$.

(ii) $LE_i^* A_{i-1} E_1^* = b_{i-1} E_{i-1}^* A_{i-2} E_1^* + \left(b_{i-1} - b_i'\right) E_{i-1}^* A_i E_1^*$.

(iii) $RE_i^* A_{i-1} E_1^* = c_i' E_{i+1}^* A_i E_1^*$.

Proof. Straightforward from Proposition 4.4.7 and Definition 4.2.1. ■

Proposition 4.4.9. *With reference to Notation 4.3.2, pick $y,z \in X$. Then the following holds for $0 \le i \le d$:*

$$\left(E_i^* R^i L E_1^*\right)_{zy} = \begin{cases} \prod_{k=1}^{i} c_k & \text{if } y \in \Gamma(x) \text{ and } z \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is straightforward to check that the (z,y)-entry of $E_i^* R^i L E_1^*$ is zero if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$. Suppose now that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$. Then, it holds that $\left(E_i^* R^i L E_1^*\right)_{zy} = \left(R^i L\right)_{zy}$. By Proposition 7.2.3, the (z,y)-entry of $R^i L$ is equal to the number of paths of length i from z to x. Since x is distance-regularized we observe that there are precisely $c_i c_{i-1} \cdots c_1$ such paths. The claim follows. ■

Corollary 4.4.10. *With reference to Notation 4.3.2, the following holds for $0 \le i \le d$:*

$$E_i^* R^i L E_1^* = \left(\prod_{k=1}^{i} c_k\right) E_i^* J E_1^*.$$

In particular, $E_i^* J E_1^* \in T$.

Proof. Immediately from Proposition 4.4.9. ■

Lemma 4.4.11. *With reference to Notation 4.3.2, the following holds for* $0 \leq i \leq d$*:*

$$E_i^* A_{i+1} E_1^* = E_i^* J E_1^* - E_i^* A_{i-1} E_1^*.$$

In particular, $E_i^* A_{i+1} E_1^* \in T$*.*

Proof. By (aii) in Section 4.2 and Corollary 4.4.2 we have that

$$E_i^* J E_1^* = \sum_{k=0}^{d} E_i^* A_k E_1^* = E_i^* A_{i-1} E_1^* + E_i^* A_{i+1} E_1^*.$$

The second part of the claim follows from Lemma 4.4.6 and Corollary 4.4.10. ■

4.5 Irreducible T-modules with endpoint 1

With reference to Notation 4.3.2, let W denote an irreducible T-module with endpoint 1. In this section we show that W is thin and find a basis for W. We start with the following lemma.

Lemma 4.5.1. *With reference to Notation 4.3.2, pick* $w \in E_1^* V$*,* $w \neq 0$*, which is orthogonal to* s_1*. Then the following holds for* $0 \leq i \leq d$*:*

$$E_i^* A_{i+1} E_1^* w = -E_i^* A_{i-1} E_1^* w.$$

Proof. By Lemma 4.4.11 we have

$$E_i^* A_{i+1} E_1^* w = E_i^* J E_1^* w - E^* A_{i-1} E_1^* w.$$

However, as w and s_1 are orthogonal and $E_1^* w = w$, we have that $E_i^* J E_1^* w = 0$. The result follows. ■

Corollary 4.5.2. *With reference to Notation 4.3.2, pick* $w \in E_1^* V$*,* $w \neq 0$*, which is orthogonal to* s_1*. Then the following (i)–(iii) hold for* $1 \leq i \leq d$*:*

(i) $A E_i^* A_{i-1} E_1^* w = b_i' E_{i-1}^* A_{i-2} E_1^* w + c_i' E_{i+1}^* A_i E_1^* w$.

(ii) $L E_i^* A_{i-1} E_1^* w = b_i' E_{i-1}^* A_{i-2} E_1^* w$.

(iii) $R E_i^* A_{i-1} E_1^* w = c_i' E_{i+1}^* A_i E_1^* w$.

Proof. Straightforward from Corollary 4.4.8, Lemma 4.5.1 and Definition 4.2.1. ■

Proposition 4.5.3. *With reference to Notation 4.3.2, pick $w \in E_1^*V$, $w \neq 0$, which is orthogonal to s_1. Let W denote the vector subspace of V spanned by the vectors $\{E_i^* A_{i-1} E_1^* w \mid 1 \leq i \leq d\}$. Then W is a thin irreducible T-module with endpoint* 1.

Proof. Observe first that by (eiv) in Section 4.2, the subspace W is invariant under the action of the dual idempotents. By Corollary 4.5.2 it follows that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Therefore, W is a T-module. Let us now show that W is irreducible. Recall that W is an orthogonal direct sum of irreducible T-modules. Since E_0^*W is the zero subspace and $E_1^* A_0 E_1^* w = w \neq 0$, there exists an irreducible T-module W', such that the endpoint of W' is 1 and $W' \subseteq W$. Consequently, $E_1^*W' \subseteq E_1^*W$. However, the dimension of E_1^*W is 1, and so $E_1^*W' = E_1^*W$. But now we have

$$W = TE_1^*W = TE_1^*W' \subseteq W',$$

implying that $W = W'$. Hence, W is irreducible and its endpoint equals 1. It is also clear that W is thin. ■

Lemma 4.5.4. *With reference to Notation 4.3.2, pick $w \in E_1^*V$, $w \neq 0$, which is orthogonal to s_1. Then the following (i), (ii) hold:*

(i) $c'_i \left\| E_{i+1}^* A_i E_1^* w \right\|^2 = b'_{i+1} \left\| E_i^* A_{i-1} E_1^* w \right\|^2$ $(1 \leq i \leq d)$.

(ii) $\left\langle E_i^* A_{i-1} E_1^* w, E_j^* A_{j-1} E_1^* w \right\rangle = \delta_{ij} \prod_{k=1}^{i-1} \frac{b'_{k+1}}{c'_k} \|w\|^2$ $(1 \leq i, j \leq d)$.

Proof. (i) Pick $1 \leq i \leq d$. By Corollary 4.5.2 we have

$$\begin{aligned}
c'_i \left\| E_{i+1}^* A_i E_1^* w \right\|^2 &= \left\langle c'_i E_{i+1}^* A_i E_1^* w,\ E_{i+1}^* A_i E_1^* w \right\rangle \\
&= \left\langle R E_i^* A_{i-1} E_1^* w,\ E_{i+1}^* A_i E_1^* w \right\rangle \\
&= \left\langle E_i^* A_{i-1} E_1^* w,\ L E_{i+1}^* A_i E_1^* w \right\rangle \\
&= \left\langle E_i^* A_{i-1} E_1^* w,\ b'_{i+1} E_i^* A_{i-1} E_1^* w \right\rangle \\
&= b'_{i+1} \left\| E_i^* A_{i-1} E_1^* w \right\|^2.
\end{aligned}$$

(ii) If $i \neq j$, then the result follows by (eii), (eiii) and (eiv) from Section 4.2. Otherwise, the claim follows from (i) above by a straightforward induction argument. ■

Corollary 4.5.5. *With reference to Notation 4.3.2, pick $w \in E_1^*V$, $w \neq 0$, which is orthogonal to s_1. Then the following (i)–(iii) hold:*

(i) If $d' = d-1$ then $E^*_i A_{i-1} E^*_1 w \neq 0$ $(1 \leq i \leq d-2)$ and

$$E^*_{d-1} A_{d-2} E^*_1 w = E^*_d A_{d-1} E^*_1 w = 0.$$

(ii) If $d' = d$ then $E^*_i A_{i-1} E^*_1 w \neq 0$ for $1 \leq i \leq d-1$ and $E^*_d A_{d-1} E^*_1 w = 0$.

(iii) If $d' = d+1$ then $E^*_i A_{i-1} E^*_1 w \neq 0$ for $1 \leq i \leq d$.

Proof. We first recall that $b'_{d'} = b'_{d'+1} = 0$ and $b'_{i-1} \neq 0, c'_i \neq 0$ for $1 \leq i \leq d'$. The result now follows immediately from Lemma 4.5.4. ■

Theorem 4.5.6. *With reference to Notation 4.3.2, pick $w \in E^*_1 V$, $w \neq 0$, which is orthogonal to s_1. Let W denote the vector subspace of V spanned by the vectors $E^*_i A_{i-1} E^*_1 w$ $(1 \leq i \leq d)$. Then W is a thin irreducible T-module with endpoint 1 and the vectors $\{E^*_i A_{i-1} E^*_1 w \mid 1 \leq i \leq d'-1\}$ form an orthogonal basis of W. In particular, the dimension of W is $d'-1$.*

Proof. The first part of the claim follows from Proposition 4.5.3. We observe that vectors $E^*_i A_{i-1} E^*_1 w$ $(1 \leq i \leq d'-1)$ are linearly independent since they are nonzero and pairwise orthogonal by Lemma 4.5.4 and Corollary 4.5.5. The result follows from Corollary 4.5.5. ■

Theorem 4.5.7. *With reference to Notation 4.3.2, let W denote an irreducible T-module with endpoint 1. Then W is thin with dimension $d'-1$. Moreover, for $w \in E^*_1 W$, $w \neq 0$, the vectors $\{E^*_i A_{i-1} E^*_1 w \mid 1 \leq i \leq d'-1\}$ form an orthogonal basis of W.*

Proof. Let W' denote the vector subspace of V spanned by vectors $E^*_i A_{i-1} E^*_1 w$ $(1 \leq i \leq d'-1)$. Recall that the unique irreducible T-module with endpoint 0 and W are not isomorphic, and so w is orthogonal to s_1. By Theorem 4.5.6, W' is a T-module. Note that W' is nonzero and contained in W. As W is irreducible, we have that $W = W'$. The result now follows from Theorem 4.5.6 ■

4.6 The isomorphism class and the action of the adjacency matrix

With reference to Notation 4.3.2, in this section we first show that any two irreducible T-modules with endpoint 1 are isomorphic. We also display a matrix representing the action of matrix A on an irreducible T-module with endpoint 1 with respect to a basis given in the statement of Theorem 4.5.7.

Theorem 4.6.1. *With reference to Notation 4.3.2, there is, up to isomorphism, a unique irreducible T-module with endpoint* 1.

Proof. Let W and W' be irreducible T-modules with endpoint 1, and pick any nonzero vectors $w \in E_1^*W$ and $w' \in E_1^*W'$. By Theorem 4.5.7, it follows that the sets

$$\left\{E_i^* A_{i-1} E_1^* w \mid 1 \le i \le d' - 1\right\} \text{ and } \left\{E_i^* A_{i-1} E_1^* w' \mid 1 \le i \le d' - 1\right\}$$

are orthogonal bases of W and W', respectively. Hence, the linear map $\sigma : W \to W'$, defined by $\sigma\left(E_i^* A_{i-1} E_1^* w\right) = E_i^* A_{i-1} E_1^* w'$ is a vector space isomorphism. Furthermore, σ is a T-module isomorphism since by Corollary 4.5.2(i) and (eiv) from Section 4.2, it commutes with A and E_i^* $(0 \le i \le d)$. Thus W and W' are T-isomorphic. ■

Theorem 4.6.2. *With reference to Notation 4.3.2, let W denote an irreducible T-module with enpoint* 1*. Pick $w \in E_1^*W$, $w \neq 0$, and recall that*

$$\mathcal{B} = \left\{E_i^* A_{i-1} E_1^* w \mid 1 \le i \le d' - 1\right\}$$

is a basis of W. Then the matrix representing the action of A on W with respect to the (ordered) basis $\mathcal{B}$ is given by

$$\begin{pmatrix} 0 & b'_2 & & & & \\ c'_1 & 0 & b'_3 & & & \\ & c'_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & b'_{d'-2} & \\ & & & c'_{d'-3} & 0 & b'_{d'-1} \\ & & & & c'_{d'-2} & 0 \end{pmatrix}.$$

Proof. Immediately from Corollary 4.5.2(i). ■

Chapter 5

On bipartite graphs with exactly one irreducible T-module with endpoint 1, which is thin: the case when the base vertex is distance-regularized

Let Γ denote a finite, simple, connected and bipartite graph. Fix a vertex x of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume that x is a distance-regularized vertex, which is not a leaf. We consider the property that Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and that this T-module is thin. The main result of the chapter is a combinatorial characterization of this property.

The chapter is organized as follows. In Sections 5.1 and 5.2 we recall basic definitions and results about distance-regularity around a vertex, about Terwilliger algebras and about intersection diagrams. In Section 5.3 we then state our main result in Theorem 5.3.4. In Section 5.4, we prove that certain matrices of the Terwilliger algebra are linearly dependent, and we use this in Sections 5.5 and 5.6 to prove our main result. We present some examples in Section 5.7.

The chapter is based on joint work with Štefko Miklavič. Our main results are currently published in European Journal of Combinatorics (2021); see [27] for more details.

5.1 Preliminaries

In this section we review some definitions and basic concepts. Throughout this chapter, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{R}$.

Let $x, y \in X$. The **distance** between x and y, denoted by $\partial(x, y)$, is the length of a shortest xy-path. The **eccentricity of** x, denoted by $\epsilon(x)$, is the maximum distance between x and any other vertex of Γ: $\epsilon(x) = \max\{\partial(x, z) \mid z \in X\}$. Let D denote the maximum eccentricity of any vertex in Γ. We call D the **diameter of** Γ. For an integer i we define $\Gamma_i(x)$ by

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}.$$

We will abbreviate $\Gamma(x) = \Gamma_1(x)$. Note that $\Gamma(x)$ is the set of neighbours of x. Observe that $\Gamma_i(x)$ is empty if and only if $i < 0$ or $i > \epsilon(x)$. Assume for a moment that $y \in \Gamma_i(x)$ for some $0 \le i \le \epsilon(x)$ and let z be a neighbour of y. Then, by the triangle inequality,

$$\partial(x, z) \in \{i - 1, i, i + 1\},$$

and so $z \in \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$. For $y \in \Gamma_i(x)$ we therefore define the following numbers:

$$a_i(x, y) = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i(x, y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i(x, y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|.$$

We say that $x \in X$ is **distance-regularized** (or that Γ is **distance-regular around** x) if the numbers $a_i(x, y), b_i(x, y)$ and $c_i(x, y)$ do not depend on the choice of $y \in \Gamma_i(x)$ ($0 \le i \le \epsilon(x)$). In this case, the numbers $a_i(x, y), b_i(x, y)$ and $c_i(x, y)$ are simply denoted by $a_i(x), b_i(x)$ and $c_i(x)$ respectively, and are called the **intersection numbers of** x. Observe that if x is distance-regularized and $\epsilon(x) = d$, then $a_0(x) = c_0(x) = b_d(x) = 0$, $b_0(x) = |\Gamma(x)|$ and $c_1(x) = 1$. Note also that for every $1 \le i \le d$ we have that $b_{i-1}(x) > 0$ and $c_i(x) > 0$, and that $a_i(x) = 0$ if Γ is bipartite. For convenience we define $c_i(x) = b_i(x) = 0$ for $i < 0$ and $i > d$.

We now recall some definitions and basic results concerning a Terwilliger algebra of Γ. Let $\mathbb{C}$ denote the complex number field. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the **standard module**. We endow V with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v\rangle = u^\top \overline{v}$ for $u, v \in V$, where $\top$ denotes transpose and $^-$ denotes complex conjugation. For $y \in X$, let $\widehat{y}$ denote the element of V with a 1 in the y-coordinate and 0 in all other coordinates. We observe that $\{\widehat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. That is, the matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose entries are given as follows:

$$(A)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = 1, \\ 0 & \text{if} \quad \partial(x,y) \neq 1, \end{cases} \qquad (x, y \in X).$$

The **adjacency algebra of** Γ, also called the **Bose-Mesner algebra of** Γ, is the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ.

We now recall the dual idempotents of Γ. To do this fix a (not necessarily distance-regularized) vertex $x \in X$ and let $d = \epsilon(x)$. We view x as a *base vertex*. For $0 \leq i \leq d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if} \;\; \partial(x,y) = i, \\ 0 & \text{if} \;\; \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ***i*-th dual idempotent of** Γ **with respect to** x [89, p. 378]. We also observe (ei) $\sum_{i=0}^d E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \leq i \leq d)$; (eiii) $E_i^{*\top} = E_i^*$ $(0 \leq i \leq d)$; (eiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leq i, j \leq d)$. By these facts, matrices $E_0^*, E_1^*, \ldots, E_d^*$ form a basis for the commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the **dual Bose-Mesner algebra of** Γ **with respect to** x [89, p. 378]. Note that for $0 \leq i \leq d$ we have

$$E_i^* V = \mathrm{Span}\{\widehat{y} \mid y \in \Gamma_i(x)\}.$$

We call $E_i^* V$ the ***i*-th subconstituent of** Γ **with respect to** x. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_d^* V \qquad (\textit{orthogonal direct sum}).$$

For convenience we define E_{-1}^* and E_{d+1}^* to be the zero matrix of $\mathrm{Mat}_X(\mathbb{C})$.

We recall the definition of a Terwilliger algebra of Γ. The Terwilliger algebra was first defined in [89, Definition 3.3], where it was called the **subconstituent algebra**. It was first defined for commutative association schemes, but the definition can be easily generalized to an arbitrary graph as follows. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the **Terwilliger algebra of Γ with respect to x**. Recall that M is generated by A. So, T is generated by A and the dual idempotents. We observe that T has finite dimension. In addition, since by construction T is generated by real-symmetric matrices, it follows that T is closed under the conjugate-transpose map. For a vector subspace $W \subseteq V$, we denote by TW the subspace $\{Bw \mid B \in T, w \in W\}$.

We now recall the lowering matrix and the raising matrix of the algebra T.

Definition 5.1.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple, connected and bipartite graph. Pick $x \in X$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Define $L = L(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by*

$$L = \sum_{i=1}^{d} E^*_{i-1} A E^*_i, \qquad R = \sum_{i=0}^{d-1} E^*_{i+1} A E^*_i.$$

*We refer to L and R as the **lowering** and the **raising matrix with respect to** x, respectively. Note that $R, L \in T$, $R = L^\top$ and $A = R + L$.*

Observe also that for $y, z \in X$ we have that the (z,y)-entry of L equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) - 1$, and 0 otherwise. Similarly, the (z,y)-entry of R equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) + 1$, and 0 otherwise. Consequently, for $v \in E^*_i V$ $(0 \le i \le d)$ we have

$$Lv \in E^*_{i-1} V, \qquad Rv \in E^*_{i+1} V. \tag{5.1}$$

By a **T-module** we mean a subspace of V which is B-invariant for every $B \in T$. Let W denote a T-module. Then W is said to be **irreducible** whenever W is nonzero and W contains no T-modules other than 0 and W. Since the algebra T is closed under the conjugate-transpose map, it turns out that any T-module is an orthogonal direct sum of irreducible T-modules.

Let W be an irreducible T-module. We observe that W is an orthogonal direct sum of the nonvanishing subspaces $E^*_i W$ for $0 \le i \le d$. By the **endpoint** of W we mean $\min\{i \mid 0 \le i \le d,\ E^*_i W \ne 0\}$. We say that W is **thin** whenever the dimension of $E^*_i W$ is at most 1 for $0 \le i \le d$.

Let W and W' denote two irreducible T-modules. By a **T-isomorphism** from W to W' we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W and W' are said to be **T-isomorphic** (or simply **isomorphic**) whenever there exists a T-isomorphism $\sigma : W \to W'$. We note that isomorphic irreducible T-modules have the same endpoint. It turns out that two non-isomorphic irreducible T-modules are orthogonal.

It is known that T has a unique irreducible T-module with endpoint 0, namely the subspace $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$. We refer to $T\widehat{x}$ as the **trivial T-module**. It was proved in [88] by Terwilliger that the trivial T-module is thin if x is distance-regularized (see also Theorem 3.5.3 and Subsection 3.7.1). In this case vectors s_i $(0 \le i \le d)$, where

$$s_i = \sum_{y \in \Gamma_i(x)} \widehat{y},$$

form a basis of the trivial T-module.

In the rest of this chapter we will study irreducible T-modules with endpoint 1 in the case when Γ is distance-regular around x. By Proposition 4.2.2, there are no irreducible T-modules with endpoint 1 if and only if x is a leaf of Γ, that is, if and only if $|\Gamma(x)| = 1$ (see also [26, Proposition 3.2]). Therefore, we will assume for the rest of this chapter that $|\Gamma(x)| \ge 2$.

5.2 The intersection diagrams

Throughout this section let $\Gamma = (X, \mathcal{R})$ denote a bipartite graph. We define a certain partition of X that we will find useful later in this chapter.

Definition 5.2.1. *Let $\Gamma = (X, \mathcal{R})$ denote a bipartite graph with diameter D. Pick $x, y \in X$, such that $y \in \Gamma(x)$. For integers i, j we define sets $D^i_j := D^i_j(x, y)$ as follows:*

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(y).$$

*Observe that $D^i_j = \emptyset$ if $i < 0$ or $j < 0$. Similarly, $D^i_j = \emptyset$ if $i > \epsilon(x)$ or $j > \epsilon(y)$. Furthermore, by the triangle inequality we have that $D^i_j = \emptyset$ if $|i - j| \ge 2$. Note also that as Γ is bipartite, the set D^i_i is empty for $0 \le i \le D$. The collection of all the subsets D^i_{i-1} $(1 \le i \le \epsilon(x))$ and D^{i-1}_i $(1 \le i \le \epsilon(y))$ is called the **distance partition of Γ with respect to the edge $\{x, y\}$**. See Figure 5.1 for an example.*

The proof of the following lemma is immediate and left to the reader.

Lemma 5.2.2. *Let $\Gamma = (X, \mathcal{R})$ denote a bipartite graph with diameter D. Pick $x, y \in X$, such that $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Then the following (i)–(iv) hold for $1 \leq i \leq D$.*

(i) If $w \in D^i_{i-1}$ then $\Gamma(w) \subseteq D^{i-1}_{i-2} \cup D^{i-1}_i \cup D^{i+1}_i$.

(ii) If $w \in D^{i-1}_i$ then $\Gamma(w) \subseteq D^{i-2}_{i-1} \cup D^i_{i-1} \cup D^i_{i+1}$.

(iii) $\Gamma_i(x) = D^i_{i-1} \cup D^i_{i+1}$ and $\Gamma_i(y) = D^{i-1}_i \cup D^{i+1}_i$.

(iv) If $D^i_{i+1} \neq \emptyset$ ($D^{i+1}_i \neq \emptyset$, respectively) then $D^j_{j+1} \neq \emptyset$ ($D^{j+1}_j \neq \emptyset$, respectively) for every $0 \leq j \leq i$.

A graphical representation of a distance partition for the case when the eccentricity of a vertex $y \in \Gamma(x)$ is equal to $\epsilon(x)+1$ is presented in Figure 5.1. A line between D^i_j and $D^{i'}_{j'}$ indicates the possibility of existence of edges between these two sets. Such a graphical representation of a distance partition is called the **intersection diagram of Γ with respect to the edge** $\{x, y\}$.

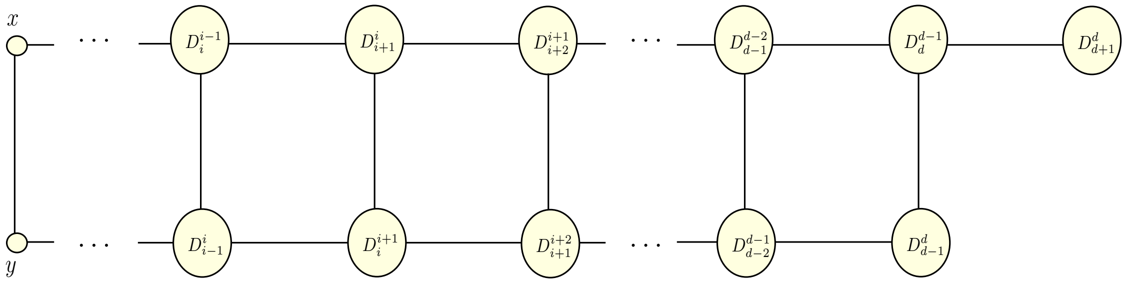

Figure 5.1: The intersection diagram of a bipartite graph Γ where $\epsilon(y) = \epsilon(x)+1 = d+1$.

The proof of the following lemma is straightforward and therefore left to the reader.

Lemma 5.2.3. *Let $\Gamma = (X, \mathcal{R})$ denote a bipartite graph with diameter D. Pick $x \in X$ and assume that x is distance-regularized. Pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Then, the following (i), (ii) hold.*

(i) Assume that $z \in D^i_{i-1}$ $(1 \leq i \leq D)$. Then $\left|\Gamma(z) \cap D^{i-1}_{i-2}\right| + \left|\Gamma(z) \cap D^{i-1}_i\right| = c_i(x)$ and $\left|\Gamma(z) \cap D^{i+1}_i\right| = b_i(x)$.

(ii) Assume that $z \in D^i_{i+1}$ $(0 \leq i \leq D)$. Then $\left|\Gamma(z) \cap D^{i+1}_i\right| + \left|\Gamma(z) \cap D^{i+1}_{i+2}\right| = b_i(x)$ and $\left|\Gamma(z) \cap D^{i-1}_i\right| = c_i(x)$.

Lemma 5.2.4. *Let $\Gamma = (X, \mathcal{R})$ denote a bipartite graph with diameter D. Pick $x \in X$ and assume that x is distance-regularized. Pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Assume that $D^i_{i+1} \neq \emptyset$, where $1 \leq i \leq D$. Then $D^i_{i-1} \neq \emptyset$. In particular, $D^i_{i-1} \neq \emptyset$ for $1 \leq i \leq \epsilon(x)$.*

Proof. If $i=1$ then the result holds as $D_0^1=\{y\}$. Assume therefore that $i \geq 2$ and that $D_{i+1}^i \neq \emptyset$. Suppose to the contrary that $D_{i-1}^i = \emptyset$ and let t be the greatest integer such that $D_{t-1}^t \neq \emptyset$. Observe that $b_t(x)=0$ by Lemma 5.2.3(i), which is impossible as $t < \epsilon(x)$. To prove the last part of the lemma observe that $\Gamma_i(x) = D_{i+1}^i \cup D_{i-1}^i$ (disjoint union) is nonempty for $0 \leq i \leq \epsilon(x)$. ■

5.3 The main result

Throughout this section let $\Gamma=(X,\mathcal{R})$ denote a bipartite graph. In this section we state our main result. To do this we need the following definition.

Definition 5.3.1. *Let $\Gamma=(X,\mathcal{R})$ denote a bipartite graph. Pick $x,y,z \in X$ and let $P=[y=x_0,x_1,\ldots,x_j=z]$ denote a yz-walk. The **shape of** P **with respect to** x is a sequence of symbols $t_1t_2\ldots t_j$, where $t_i \in \{\ell, r\}$, and such that $t_i=r$ if $\partial(x,x_i)=\partial(x,x_{i-1})+1$ and $t_i=\ell$ if $\partial(x,x_i)=\partial(x,x_{i-1})-1$, $(1 \leq i \leq j)$. We use exponential notation for shapes containing several consecutive identical symbols. For instance, instead of $rrrr\ell\ell r$ we simply write $r^4\ell^2r$. Analogously, $r^0\ell=\ell$ is also conventional. For a non-negative integer m, let $r^m\ell(y,z)$ and $r^m(y,z)$ respectively denote the number of yz-walks of the shape $r^m\ell$ and r^m with respect to x, where $r^0(y,z)=1$ if $y=z$ and $r^0(y,z)=0$ otherwise. See Figure 5.2 for an example.*

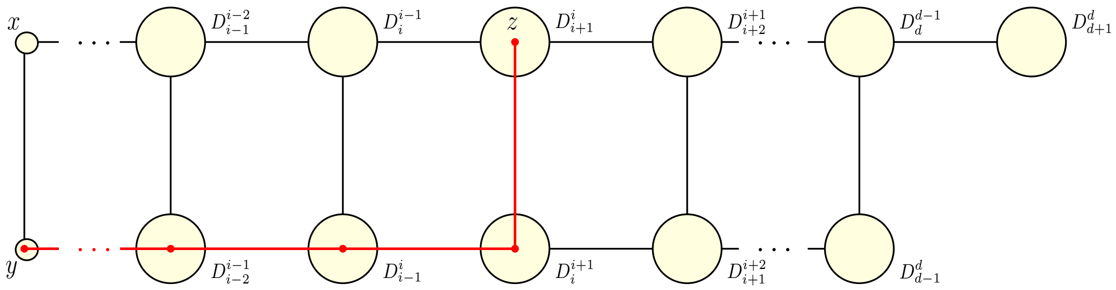

Figure 5.2: A yz-walk of the shape $r^i\ell$ for $y \in \Gamma(x)$ and $z \in D_{i+1}^i$ in a bipartite graph Γ where $\epsilon(y)=\epsilon(x)+1=d+1$.

The following observation is straightforward to prove (using elementary matrix multiplication and (5.1)).

Lemma 5.3.2. *Let $\Gamma=(X,\mathcal{R})$ denote a bipartite graph. Pick $x \in X$ and let $T=T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $L=L(x)$ and $R=R(x)$ denote the lowering and the raising matrix of T, respectively. Pick $y,z \in X$ and let m be a positive integer. Then the following (i)–(iii) hold:*

(i) *The* (z,y)*-entry of* R^m *is equal to the number* $r^m(y,z)$ *with respect to* x.

(ii) *The* (z,y)*-entry of* R^mL *is equal to the number* $\ell r^m(y,z)$ *with respect to* x.

(iii) *The* (z,y)*-entry of* LR^m *is equal to the number* $r^m\ell(y,z)$ *with respect to* x.

For the rest of the chapter we adopt the following notation.

Notation 5.3.3. *Let* $\Gamma = (X, \mathcal{R})$ *denote a finite, simple, connected, bipartite graph with vertex set* X*, edge set* $\mathcal{R}$ *and diameter* D*. Let* $A \in \mathrm{Mat}_X(\mathbb{C})$ *denote the adjacency matrix of* Γ*. Fix a distance-regularized vertex* $x \in X$ *with* $|\Gamma(x)| \geq 2$*. Let* d *denote the eccentricity of* x*, and let* $b_i(x), c_i(x)$ $(0 \leq i \leq d)$ *denote the intersection numbers of* x*. Let* $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \leq i \leq d)$ *denote the dual idempotents of* Γ *with respect to* x*. For convenience we set* $E_{-1}^* = E_{d+1}^* = 0$*. Let* V *denote the standard module of* Γ *and let* $T = T(x)$ *denote the Terwilliger algebra of* Γ *with respect to* x*. Let* $L = L(x)$ *and* $R = R(x)$ *denote the lowering and the raising matrix of* T*, respectively. Let* J *denote the all* 1*'s matrix in* $\mathrm{Mat}_X(\mathbb{C})$*. Recall that the unique irreducible* T*-module with endpoint* 0 *is thin. We denote this* T*-module by* V_0*. For* $y \in \Gamma(x)$ *and* $z \in X$ *let the sets* $D^i_j = D^i_j(x,y)$ *be as defined in Definition 5.2.1, and let the numbers* $r^m\ell(y,z)$ *and* $r^m(y,z)$ *be as defined in Definition 5.3.1.*

We are now ready to state our main result.

Theorem 5.3.4. *With reference to Notation 5.3.3, the following (i), (ii) are equivalent:*

(i) Γ *has, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin.*

(ii) *For every integer* $1 \leq i \leq d$ *there exist scalars* κ_i, μ_i*, such that for every* $y \in \Gamma(x)$ *the following (a), (b) hold:*

  (a) *For every* $z \in D^i_{i+1}(x,y)$ *we have that* $r^i\ell(y,z) = \mu_i$*. In particular,* $r^i\ell(y,z)$ *does not depend on the choice of* y, z*.*

  (b) *For every* $z \in D^i_{i-1}(x,y)$ *we have*

$$r^i\ell(y,z) = \kappa_i \, r^{i-1}(y,z) + \mu_i.$$

We finish this section with the following observation.

Proposition 5.3.5. *With reference to Notation 5.3.3, the following holds for* $0 \le i \le d$*:*

$$\left(E_i^* R^i L E_1^*\right)_{zy} = \begin{cases} \prod_{k=1}^{i} c_k(x) & \text{if } y \in \Gamma(x) \text{ and } z \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is straightforward to check that the (z,y)-entry of $E_i^* R^i L E_1^*$ is zero if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$. It is also straightforward to check that the result is true if $i = 0$. Suppose now that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$ with $i \ge 1$. Then $\left(E_i^* R^i L E_1^*\right)_{zy} = \left(R^i L\right)_{zy}$. By Lemma 5.3.2$(ii)$, the (z,y)-entry of $R^i L$ is equal to the number of paths of length i from z to x. Since x is distance-regularized we observe that there are precisely $c_i(x) c_{i-1}(x) \cdots c_1(x)$ such paths. The claim follows. ■

Corollary 5.3.6. *With reference to Notation 5.3.3, the following holds for* $0 \le i \le d$*:*

$$E_i^* R^i L E_1^* = \left(\prod_{k=1}^{i} c_k(x) \right) E_i^* J E_1^*.$$

In particular, $E_i^* J E_1^* \in T$.

Proof. Immediately from Proposition 5.3.5. ■

5.4 Linear dependency

With reference to Notation 5.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. In this section we show that certain matrices of T are linearly dependent.

Lemma 5.4.1. *With reference to Notation 5.3.3, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. Let* W *denote an irreducible* T*-module with endpoint* 1*. Pick matrices* $F_1, F_2, F_3 \in T$ *and an integer* i $(1 \le i \le d)$*. Then there exist scalars* λ_j $(1 \le j \le 3)$*, not all zero, such that*

$$\lambda_1 E_i^* F_1 E_1^* v + \lambda_2 E_i^* F_2 E_1^* v + \lambda_3 E_i^* F_3 E_1^* v = 0$$

for every $v \in E_1^* V_0 \cup E_1^* W$.

Proof. Pick nonzero vectors $v_0 \in E_1^* V_0$ and $v_1 \in E_1^* W$. Recall that $\dim(E_i^* V_0) = 1$. Let u_0 be an arbitrary nonzero vector of $E_i^* V_0$. We define vector u_1 as follows: if $E_i^* W = 0$, then let $u_1 = 0$; otherwise, let u_1 be an arbitrary nonzero vector of $E_i^* W$. As modules V_0 and W are thin, there exist scalars $r_{0,j}$, $r_{1,j}$ $(1 \le j \le 3)$ such that

$$E_i^* F_j E_1^* v_0 = r_{0,j}\, u_0 \qquad \text{and} \qquad E_i^* F_j E_1^* v_1 = r_{1,j}\, u_1. \tag{5.2}$$

Consider now the following homogeneous system of linear equations:

$$\begin{pmatrix} r_{0,1} & r_{0,2} & r_{0,3} \\ r_{1,1} & r_{1,2} & r_{1,3} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Observe that the above system has a nontrivial solution, and so there exist scalars λ_i $(1 \le i \le 3)$, not all zero, such that

$$\sum_{j=1}^{3} \lambda_j\, r_{0,j} = 0 \qquad \text{and} \qquad \sum_{j=1}^{3} \lambda_j\, r_{1,j} = 0. \tag{5.3}$$

Pick a vector $v \in E_1^* V_0 \cup E_1^* W$. Since T-modules V_0 and W are thin, there exists a scalar λ such that $v = \lambda v_k$ for some $k \in \{0,1\}$. Therefore, by (5.2) and (5.3) we have

$$\sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* v = \lambda \sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* v_k = \lambda \sum_{j=1}^{3} \lambda_j r_{k,j} u_k = 0.$$

■

Corollary 5.4.2. *With reference to Notation 5.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Let V_1 denote the subspace of V spanned by all irreducible T-modules with endpoint 1. Pick matrices $F_1, F_2, F_3 \in T$ and an integer i $(1 \le i \le d)$. Then there exist scalars λ_i $(1 \le i \le 3)$, not all zero, such that*

$$\lambda_1 E_i^* F_1 E_1^* v + \lambda_2 E_i^* F_2 E_1^* v + \lambda_3 E_i^* F_3 E_1^* v = 0$$

for every $v \in E_1^* V_0 \cup E_1^* V_1$.

Proof. Let $\{W^t \mid t \in \mathcal{I}\}$ be the set of all irreducible T-modules with endpoint 1, where $\mathcal{I}$ is an index set. Pick a T-module W^s, $s \in \mathcal{I}$. By Lemma 5.4.1, there exist scalars

λ_j $(1 \le j \le 3)$, not all zero, such that

$$\lambda_1 E_i^* F_1 E_1^* v + \lambda_2 E_i^* F_2 E_1^* v + \lambda_3 E_i^* F_3 E_1^* v = 0 \tag{5.4}$$

for every $v \in E_1^* V_0 \cup E_1^* W^s$. We claim that equality (5.4) holds for every $v \in E_1^* V_0 \cup E_1^* V_1$. Note that we could assume that $v \in E_1^* V_1$. Observe that v can be written as a sum

$$v = \sum_{t \in \mathcal{I}} v_t, \tag{5.5}$$

where $v_t \in E_1^* W^t$ for every $t \in \mathcal{I}$.

As any two irreducible T-modules with endpoint 1 are isomorphic, for every $t \in \mathcal{I}$ there exists a T-isomorphism $\sigma_t : W^s \to W^t$. Let $w_t \in W^s$ be such that $v_t = \sigma_t(w_t)$. It is easy to see that $w_t \in E_1^* W^s$ as $v_t \in E_1^* W^t$. By Lemma 5.4.1, there exist scalars λ_i $(1 \le i \le 3)$, not all zero, such that

$$\sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* w = 0$$

for every $w \in E_1^* W^s$. Therefore, for every $t \in \mathcal{I}$ we have

$$\sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* v_t = \sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* \sigma_t(w_t) = \sigma_t \left(\sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* w_t \right) = 0.$$

The claim now follows from (5.5). ■

Theorem 5.4.3. *With reference to Notation 5.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Pick matrices $F_1, F_2, F_3 \in T$ and an integer i $(1 \le i \le d)$. Then the matrices $E_i^* F_1 E_1^*$, $E_i^* F_2 E_1^*$ and $E_i^* F_3 E_1^*$ are linearly dependent.*

Proof. By Corollary 5.4.2 there exist scalars λ_i $(1 \le i \le 3)$, not all zero, such that

$$\sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* v = 0 \tag{5.6}$$

for every $v \in E_1^* V_0 \cup E_1^* V_1$, where V_1 denotes the sum of all irreducible T-modules with endpoint 1. Pick now an arbitrary vector $w \in V$ and observe that $E_1^* w = w_0 + w_1$ for some

$w_0 \in V_0$ and $w_1 \in V_1$. By (5.6) we have

$$\sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* w = \sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* w_0 + \sum_{j=1}^{3} \lambda_j E_i^* F_j E_1^* w_1 = 0.$$

As w was arbitrary, the result follows. ■

5.5 Algebraic condition implies combinatorial properties

With reference to Notation 5.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. In this section we prove that in this case combinatorial conditions (a), (b) described in part (ii) of Theorem 5.3.4 hold.

Lemma 5.5.1. *With reference to Notation 5.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Then for every i $(1 \leq i \leq d)$ there exist scalars κ_i, μ_i, such that*

$$E_i^* L R^i E_1^* = \kappa_i E_i^* R^{i-1} E_1^* + \mu_i E_i^* J E_i^*. \tag{5.7}$$

Proof. Pick i $(1 \leq i \leq d)$ and observe that by Definition 5.1.1 and Corollary 5.3.6, the matrices LR^i, R^{i-1} and $E_i^* J E_1^*$ are elements of algebra T. Therefore, by (eiv) from Section 5.1 and Theorem 5.4.3, there exist scalars $\lambda_j = \lambda_j^{(i)}$ $(1 \leq j \leq 3)$, not all zero, such that

$$\lambda_1 E_i^* L R^i E_1^* + \lambda_2 E_i^* R^{i-1} E_1^* + \lambda_3 E_i^* J E_1^* = 0.$$

Assume for the moment that $\lambda_1 \neq 0$. Then (5.7) holds with $\kappa_i = -\lambda_2/\lambda_1$ and $\mu_i = -\lambda_3/\lambda_1$. Now, assume that $\lambda_1 = 0$. We first claim that in this case we have that $D^i_{i+1}(x,y) = \emptyset$ for every $y \in \Gamma(x)$. Indeed, suppose to the contrary that there exists $y \in \Gamma(x)$ such that the set $D^i_{i+1}(x,y) \neq \emptyset$. In this case, observe that $D^i_{i-1}(x,y) \neq \emptyset$ by Lemma 5.2.4. Pick $z \in D^i_{i+1}(x,y)$ and note that it follows from Lemma 5.3.2(i) that the (z,y)-entry of $E_i^* R^{i-1} E_1^*$ is 0, while the (z,y)-entry of $E_i^* J E_1^*$ is 1. This implies that $\lambda_3 = 0$. Pick now $z \in D^i_{i-1}(x,y)$ and note that it follows from Lemma 5.3.2(i) that the (z,y)-entry of $E_i^* R^{i-1} E_1^*$ is nonzero. This implies $\lambda_2 = 0$, contradicting the fact that the scalars λ_j $(1 \leq j \leq 3)$ are not all zero. This proves our claim.

We next claim that $\lambda_2 \neq 0$. Suppose to the contrary that $\lambda_2 = 0$. Pick $y \in \Gamma(x)$ and consider the sets $D^k_j(x,y)$ for $0 \leq k,j \leq D$. As $D^i_{i+1}(x,y) = \emptyset$, we clearly have that $D^i_{i-1}(x,y) \neq \emptyset$. Pick $z \in D^i_{i-1}(x,y)$ and observe that the (z,y)-entry of $E^*_i J E^*_1$ is equal to 1, which forces $\lambda_3 = 0$, again contradicting the fact that the scalars λ_j $(1 \leq j \leq 3)$ are not all zero. It follows that $\lambda_2 \neq 0$. Therefore, we have that

$$E^*_i R^{i-1} E^*_1 = -\frac{\lambda_3}{\lambda_2} E^*_i J E^*_1, \tag{5.8}$$

and so for any $y \in \Gamma(x)$ and for any $z \in D^i_{i-1}(x,y)$, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ is equal to $-\lambda_3/\lambda_2$. In other words, for any $y \in \Gamma(x)$ and for any $z \in D^i_{i-1}(x,y)$ there are exactly $-\lambda_3/\lambda_2$ walks of the shape r^{i-1} from y to z.

Pick again any $y \in \Gamma(x)$. Observe that since x is distance-regularized and since the set $D^i_{i+1}(x,y) = \emptyset$, Lemma 5.2.3 implies that every $z \in D^i_{i-1}(x,y)$ has exactly $b_i(x)$ neighbours in $D^{i+1}_i(x,y)$, and that every $z \in D^{i+1}_i(x,y)$ has exactly $c_{i+1}(x)$ neighbours in $D^i_{i-1}(x,y)$. It follows from the above comments that for any $z \in D^i_{i-1}(x,y)$ there are exactly $-b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$ walks of the shape $r^i\ell$ from y to z. We now claim that (5.7) holds for any κ_i, μ_i such that $\lambda_3\kappa_i - \lambda_2\mu_i = b_i(x)c_{i+1}(x)\lambda_3$. For example, we may let either $\kappa_i = b_i(x)c_{i+1}(x)$ and $\mu_i = 0$, or $\kappa_i = 0$ and $\mu_i = -b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$. Indeed, pick any $y, z \in X$. If either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$, then the (z,y)-entry of both sides of (5.7) equals 0. If $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$, then $z \in D^i_{i-1}(x,y)$ as $D^i_{i+1}(x,y) = \emptyset$. The (z,y)-entry of the left-hand side of (5.7) equals the number of yz-walks of the shape $r^i\ell$, which equals $-b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$ by the above comments. However, it follows from Lemma 5.3.2 and (5.8) that also the (z,y)-entry of the right-hand side of (5.7) equals $-b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$, and the result follows. ■

Theorem 5.5.2. *With reference to Notation 5.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. For every integer $1 \leq i \leq d$ there exist scalars κ_i, μ_i, such that for every $y \in \Gamma(x)$ the following (a), (b) hold:*

(a) For every $z \in D^i_{i+1}(x,y)$ we have that $r^i\ell(y,z) = \mu_i$. In particular, $r^i\ell(y,z)$ does not depend on the choice of y,z.

(b) For every $z \in D^i_{i-1}(x,y)$ we have thaz

$$r^i\ell(y,z) = \kappa_i\, r^{i-1}(y,z) + \mu_i.$$

Proof. Pick an integer i $(1 \leq i \leq d)$ and recall that by Lemma 5.5.1 equation (5.7) holds.

Pick $y \in \Gamma(x)$.

(a) Pick $z \in D^i_{i+1}(x,y)$ and observe that by Lemma 5.3.2 the (z,y)-entry of the left-hand side of (5.7) equals $r^i\ell(y,z)$. On the other hand, again by Lemma 5.3.2, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ equals 0, while the (z,y)-entry of $E^*_i J E^*_1$ is obviously equal to 1. Therefore, the (z,y)-entry of the right-hand side of (5.7) equals μ_i, and so $r^i\ell(y,z)$ does not depend on the choice of y,z.

(b) Pick now $z \in D^i_{i-1}(x,y)$ and observe that by Lemma 5.3.2 the (z,y)-entry of the left-hand side of (5.7) equals $r^i\ell(y,z)$. On the other hand, again by Lemma 5.3.2, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ equals $r^{i-1}(y,z)$, while the (z,y)-entry of $E^*_i J E^*_1$ is obviously equal to 1. Therefore, the (z,y)-entry of the right-hand side of (5.7) equals $\kappa_i \, r^{i-1}(y,z) + \mu_i$.

The result follows. ■

5.6 Combinatorial properties imply algebraic condition

With reference to Notation 5.3.3, assume that Γ satisfies part (ii) of Theorem 5.3.4. In this section we prove that in this case Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. We also display a basis of this module and the matrix representing the action of the adjacency matrix on this basis.

Proposition 5.6.1. *With reference to Notation 5.3.3, assume that Γ satisfies part (ii) of Theorem 5.3.4. Then for every integer i $(1 \le i \le d)$, the following equality holds:*

$$E^*_i L R^i E^*_1 = \kappa_i E^*_i R^{i-1} E^*_1 + \mu_i E^*_i J E^*_1. \tag{5.9}$$

Proof. Pick an integer i $(1 \le i \le d)$ and vertices $y,z \in X$. We will show that the (z,y)-entries of both sides of (5.9) agree. Observe first that if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$, then the (z,y)-entry of both sides of (5.9) equals 0. Therefore, assume that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$. Abbreviate $D^k_j(x,y) = D^k_j$ for $0 \le k,j \le D$ and observe that $\Gamma_i(x) = D^i_{i-1} \cup D^i_{i+1}$.

Assume first that $z \in D^i_{i+1}$ and note that the (z,y)-entry of $E^*_i L R^i E^*_1$ is equal to the number $r^i\ell(y,z)$, while the (z,y)-entries of $E^*_i R^{i-1} E^*_1$ and $E^*_i J E^*_1$ are 0 and 1 respectively. As $r^i\ell(y,z) = \mu_i$ by the assumption, the (z,y)-entries of both sides of (5.9) agree.

Assume next that $z \in D^i_{i-1}$ and note that the (z,y)-entry of $E^*_i L R^i E^*_1$ ($E^*_i R^{i-1} E^*_1$, respectively) is equal to the number $r^i \ell(y,z)$ ($r^{i-1}(y,z)$, respectively). The (z,y)-entry of $E^*_i J E^*_1$ is of course equal to 1. By the assumption we have that $r^i \ell(y,z) = \kappa_i \, r^{i-1}(y,z) + \mu_i$, and so the (z,y)-entries of both sides of (5.9) agree. This finishes the proof. ■

Lemma 5.6.2. *With reference to Notation 5.3.3, assume that* Γ *satisfies part (ii) of Theorem 5.3.4. Pick* $w \in E^*_1 V$, $w \neq 0$, *which is orthogonal to* s_1. *Then* $Lw = 0$ *and* $LR^i w = \kappa_i R^{i-1} w$ *for every* $1 \le i \le d$.

Proof. As $w \in E^*_1 V$ we have that $E^*_1 w = w$ and so,

$$\langle \boldsymbol{j}, w \rangle = \langle \boldsymbol{j}, E^*_1 w \rangle = \langle E^*_1 \boldsymbol{j}, w \rangle = \langle s_1, w \rangle = 0,$$

where $\boldsymbol{j}$ denotes the all 1's vector in V. This implies $Jw = 0$. By elementary matrix multiplication it is easy to see $E^*_0 A E^*_1 = E^*_0 J E^*_1$. Therefore, by Definition 5.1.1 and the above comments we have $Lw = E^*_0 A E^*_1 w = E^*_0 J E^*_1 w = E^*_0 J w = 0$. In addition, by (5.1) and Proposition 5.6.1,

$$LR^i w = E^*_i L R^i E^*_1 w = \kappa_i E^*_i R^{i-1} E^*_1 w + \mu_i E^*_i J E^*_1 w = \kappa_i E^*_i R^{i-1} E^*_1 w = \kappa_i R^{i-1} w.$$

The result follows. ■

Lemma 5.6.3. *With reference to Notation 5.3.3, assume that* Γ *satisfies part (ii) of Theorem 5.3.4. Pick* $w \in E^*_1 V$, $w \neq 0$, *which is orthogonal to* s_1. *Then the following (i)–(iii) hold:*

(i) $\left\| R^i w \right\|^2 = \kappa_i \left\| R^{i-1} w \right\|^2$ $(1 \le i \le d)$.

(ii) $\left\langle R^i w, R^j w \right\rangle = \delta_{ij} \prod_{l=1}^{i} \kappa_l \left\| w \right\|^2$ $(0 \le i,j \le d)$.

(iii) There exists i $(1 \le i \le d)$ *such that* $\kappa_i = 0$.

Proof. (i) Pick $1 \le i \le d$. Then by Lemma 5.6.2 we have

$$\left\| R^i w \right\|^2 = \left\langle R^i w, \; R^i w \right\rangle = \left\langle LR^i w, \; R^{i-1} w \right\rangle = \kappa_i \left\| R^{i-1} w \right\|^2.$$

(ii) If $i \neq j$, then the result follows from (eii), (eiii) and (eiv) below the definition of the dual idempotents in Section 5.1 and from (5.1). If $i = j$ then the result follows from (i) above by a straightforward induction argument.

(*iii*) Immediate from (*ii*) above since by (5.1) we have $R^d w = 0$ and w is a nonzero vector. ■

Theorem 5.6.4. *With reference to Notation 5.3.3, assume that Γ satisfies part (ii) of Theorem 5.3.4. Pick $w \in E_1^* V$, $w \neq 0$, which is orthogonal to s_1. Let W denote the vector subspace of V spanned by the vectors $R^i w$ $(0 \le i \le d)$. Let s $(1 \le s \le d)$ be the least integer such that $\kappa_s = 0$. Then W is a thin irreducible T-module with endpoint 1 and the vectors $\{R^{i-1} w \mid 1 \le i \le s\}$ form an orthogonal basis of W. In particular, the dimension of W is s.*

Proof. Observe that by (5.1) and since $RE_d^* = 0$, the subspace W is invariant under the action of the dual idempotents. By construction and since $R^d w = 0$ by (5.1) it is also clear that W is closed under the action of R. Moreover, it follows from Lemma 5.6.2 that W is invariant under the action of L. Since $A = L + R$, it turns out that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Therefore, W is a T-module. It is clear that W is thin, since by construction, (5.1) and Lemma 5.6.2, the subspace $E_i^* W$ is generated by $R^{i-1} w$.

Now, let us show that W is irreducible. Note that $w \in W$ and so W is non-zero. Recall that W is an orthogonal direct sum of irreducible T-modules. Since $E_0^* W$ is the zero subspace and $E_1^* w = w \neq 0$, there exists an irreducible T-module W', such that the endpoint of W' is 1 and $W' \subseteq W$. Consequently, $E_1^* W' \subseteq E_1^* W$. However, the dimension of $E_1^* W$ is 1, and so $E_1^* W' = E_1^* W$. But now we have that

$$W = TE_1^* W = TE_1^* W' \subseteq W',$$

implying that $W = W'$. Hence, W is irreducible and its endpoint equals 1.

Finally, notice that $R^s w = 0$ by Lemma 5.6.3(*i*). Furthermore, it holds that vectors $\{R^{i-1} w \mid 1 \le i \le s\}$ are nonzero and pairwise orthogonal by Lemma 5.6.3(*ii*) and the definition of number s. The result follows. ■

Theorem 5.6.5. *With reference to Notation 5.3.3, assume that Γ satisfies part (ii) of Theorem 5.3.4. Let W denote an irreducible T-module with endpoint 1. Let s $(1 \le s \le d)$ be the least integer such that $\kappa_s = 0$. Pick $w \in E_1^* W$, $w \neq 0$. Then the vectors $\{R^{i-1} w \mid 1 \le i \le s\}$ form an orthogonal basis of W. In particular, W is thin with dimension s.*

Proof. Let W' denote the vector subspace of V spanned by the vectors $\{R^{i-1} w \mid 1 \le i \le d\}$. Recall that W and the unique irreducible T-module with endpoint 0 are not isomorphic, and so w is orthogonal to s_1. By Theorem 5.6.4, W' is a T-module. Note that W' is nonzero and contained in W. As W is irreducible, we have that $W = W'$. The result now follows from Theorem 5.6.4. ■

Theorem 5.6.6. *With reference to Notation 5.3.3, assume that* Γ *satisfies part (ii) of Theorem 5.3.4. Then there is, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin.*

Proof. Let W and W' be irreducible T-modules with endpoint 1, and pick any nonzero vectors $w \in E_1^*W$ and $w' \in E_1^*W'$. Let s $(1 \le s \le d)$ be the least integer such that $\kappa_s = 0$. By Theorem 5.6.5, the vectors

$$\left\{R^{i-1}w \mid 1 \le i \le s\right\} \text{ and } \left\{R^{i-1}w' \mid 1 \le i \le s\right\}$$

are orthogonal bases of W and W', respectively. Hence, the linear map $\sigma : W \to W'$, defined by $\sigma\left(R^{i-1}w\right) = R^{i-1}w'$ is a vector space isomorphism. It is clear that σ commutes with R. By Lemma 5.6.2 it follows that σ also commutes with L. Since $A = L + R$, it turns out that σ commutes with A as well. Furthermore, σ is a T-module isomorphism since by (eiv) from Section 5.1, it commutes also with E_i^* $(0 \le i \le d)$. Thus W and W' are T-isomorphic. ■

Theorem 5.6.7. *With reference to Notation 5.3.3, assume that* Γ *satisfies part (ii) of Theorem 5.3.4. Let* W *denote an irreducible* T*-module with endpoint* 1*. Pick* $w \in E_1^*W$*,* $w \neq 0$*, and recall that*

$$\mathcal{B} = \left\{R^{i-1}w \mid 1 \le i \le s\right\}$$

is a basis of W*, where* s *is the least integer such that* $\kappa_s = 0$ $(1 \le s \le d)$*. Then the matrix representing the action of* A *on* W *with respect to the (ordered) basis* $\mathcal{B}$ *is given by*

$$\begin{pmatrix} 0 & \kappa_1 & & & & \\ 1 & 0 & \kappa_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \kappa_{s-2} & \\ & & & 1 & 0 & \kappa_{s-1} \\ & & & & 1 & 0 \end{pmatrix}.$$

Proof. Recall that $A = R + L$. The result now follows from Lemma 5.6.2. ■

5.7 Examples

In this section we present several examples of bipartite graphs for which the equivalent conditions of Theorem 5.3.4 hold for a certain vertex x. We first focus on the case where the partition from Definition 5.2.1 is equitable for every $y \in \Gamma(x)$, and the parameters of this partition do not depend on the choice of $y \in \Gamma(x)$ (see for example [39, Subsection 9.3] for the definition of equitable partitions). More precisely, we have the following definition.

Definition 5.7.1. *With reference to Notation 5.3.3 we say that* Γ *is* 1-homogeneous with respect to x *(in the sense of Curtin and Nomura [21]), whenever for all integers* h,i,j,k $(0 \leq h,i,j,k \leq D)$ *there is a structure constant* $\gamma^{h,i}_{j,k}(x)$ *such that for all vertices* y *and* z *of* Γ *with* $\partial(x,z) = h$, $\partial(y,z) = i$, $\partial(y,x) = 1$, *the number*

$$|\{w \in X \mid \partial(x,w) = j, \partial(z,w) = 1, \partial(y,w) = k\}| = \gamma^{h,i}_{j,k}(x).$$

With reference to Notation 5.3.3, assume for a moment that Γ is a bipartite graph which is 1-homogeneous with respect to x and pick $y \in \Gamma(x)$. Pick also $1 \leq i \leq d$ and vertices $z_1 \in D^i_{i+1}(x,y)$ and $z_2 \in D^i_{i-1}(x,y)$. It is clear from Definition 5.7.1, that the number of yz_1-walks of the shape $r^i\ell$ with respect to x does not depend on the choice of y and z_1. Similarly, the number of yz_2-walks of the shape r^{i-1} ($r^i\ell$, respectively) with respect to x also does not depend on the choice of y and z_2. Thus, it is clear that in this case conditions (a), (b) described in part (ii) of Theorem 5.3.4 are satisfied, and so Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and this module is thin. We would like to point out that if Γ is a bipartite distance-regular graph or distance-biregular graph, then Γ is 1-homogeneous with respect to every vertex (see [6] for the definition of distance-regular and distance-biregular graphs).

Our next example shows that there exist graphs which admit vertex x, such that there is, up to isomorphism, a unique irreducible $T(x)$-module of endpoint 1, and this module is thin, but the corresponding partitions from Definition 5.2.1 are not equitable.

Let Γ denote the graph in Figure 5.3 and let $x = 1$. It is easy to check that Γ is bipartite and distance-regular around vertex 1. Let $T = T(1)$ be the Terwilliger algebra of Γ with respect to vertex 1.

Consider vertex $2 \in \Gamma(1)$. The intersection diagram for the distance partition with respect to the edge $\{1,2\}$ is presented in Figure 5.4.

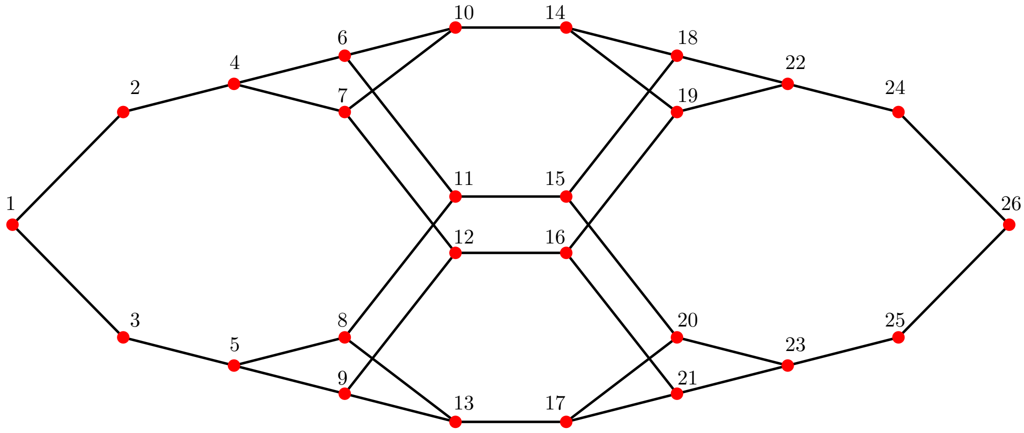

Figure 5.3: Graph Γ which has, up to isomorphism, exactly one irreducible $T(1)$-module with endpoint one, and this module is thin.

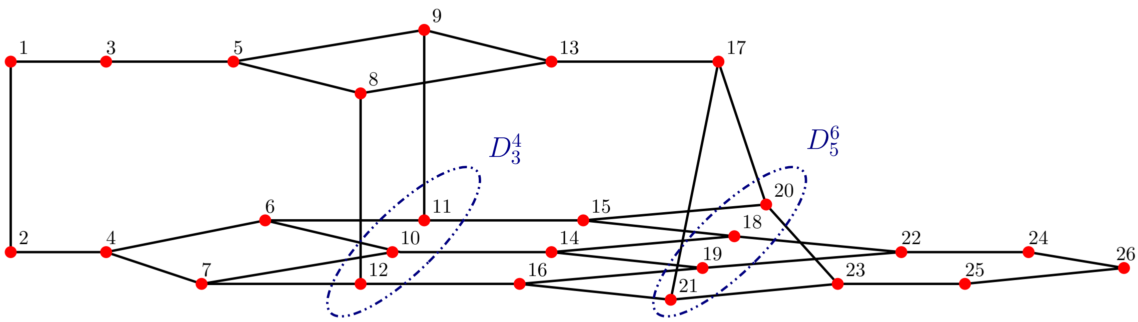

Figure 5.4: Distance partition of Γ with respect to the edge $\{1,2\}$.

Consider vertex $3 \in \Gamma(1)$. The intersection diagram for the distance partition with respect to the edge $\{1,3\}$ is similar and is presented in Figure 5.5.

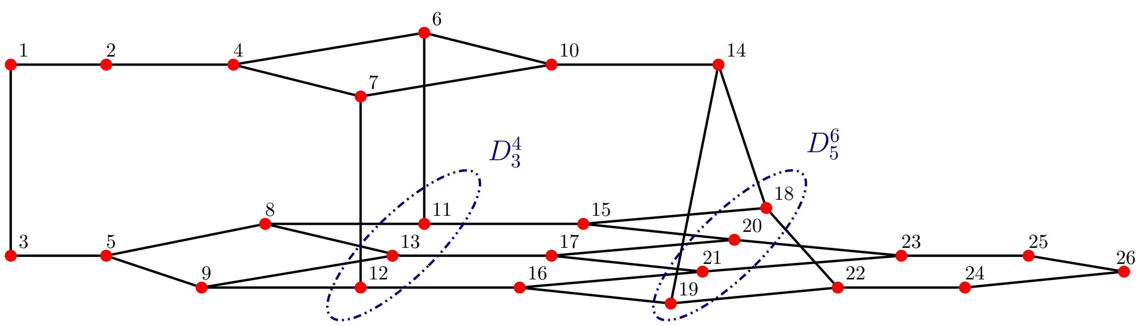

Figure 5.5: Distance partition of Γ with respect to the edge $\{1,3\}$.

It is now straightforward to check that properties (a), (b) described in part (ii) of Theorem 5.3.4 hold with the following values of κ_i, μ_i $(1 \leq i \leq 9)$ as presented in Table 5.1.

Consequently, by Theorem 5.3.4, it holds that Γ has, up to isomorphism, a unique

i	1	2	3	4	5	6	7	8	9
κ_i	1	2	2	1	2	2	1	0	0
μ_i	0	0	1	0	2	0	0	8	0

Table 5.1: Values of scalars κ_i and μ_i, $(1 \leq i \leq 9)$.

irreducible T-module with endpoint 1, and this module is thin. Moreover, this T-module has dimension $s = 8$. Note also that the partitions presented by the intersection diagrams in Figures 5.4 and 5.5 are not equitable, and so Γ is not 1-homogeneous with respect to vertex 1.

Chapter 6

Graphs with exactly one irreducible T-module with endpoint 1, which is thin: the distance-regularized case

Let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume that x is a distance-regularized vertex, which is not a leaf. We consider the property that Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and that this T-module is thin. The main result of the chapter is a combinatorial characterization of this property.

The chapter is organized as follows. In Sections 6.1 and 6.2 we recall basic definitions and results about distance-regularity around a vertex, about Terwilliger algebras and about intersection diagrams. In Section 6.3 we then state our main result in Theorem 6.3.4. We use the fact that certain matrices of the Terwilliger algebra are linearly dependent in Sections 6.4 and 6.5 to prove the main result. In Section 6.6, we have some comments about certain distance partitions of a graph which has, up to isomorphism, exactly one irreducible T-module with endpoint 1 (with respect to some base vertex), which is thin. We finish the chapter presenting some examples of such graphs.

The chapter is based on a solo article. The main results are currently published in Journal of Algebraic Combinatorics (2022); see [23] for more details.

6.1 Preliminaries

In this section we review some definitions and basic concepts. Throughout this chapter, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops and multiple edges, with vertex set X and edge set $\mathcal{R}$.

Let $x, y \in X$. The **distance** between x and y, denoted by $\partial(x,y)$, is the length of a shortest xy-path. The **eccentricity of** x, denoted by $\epsilon(x)$, is the maximum distance between x and any other vertex of Γ: $\epsilon(x) = \max\{\partial(x,z) \mid z \in X\}$. Let D denote the maximum eccentricity of any vertex in Γ. We call D the **diameter of** Γ. For an integer i we define $\Gamma_i(x)$ by

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}.$$

We will abbreviate $\Gamma(x) = \Gamma_1(x)$. Note that $\Gamma(x)$ is the set of neighbours of x. Observe that $\Gamma_i(x)$ is empty if and only if $i < 0$ or $i > \epsilon(x)$. Assume for a moment that $y \in \Gamma_i(x)$ for some $0 \le i \le \epsilon(x)$ and let z be a neighbour of y. Then, by the triangle inequality,

$$\partial(x,z) \in \{i-1, i, i+1\},$$

and so $z \in \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$. For $y \in \Gamma_i(x)$ we therefore define the following numbers:

$$a_i(x,y) = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i(x,y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i(x,y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|.$$

We say that $x \in X$ is **distance-regularized** (or that Γ is **distance-regular around** x) if the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ do not depend on the choice of $y \in \Gamma_i(x)$ $(0 \le i \le \epsilon(x))$. In this case, the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ are simply denoted by $a_i(x), b_i(x)$ and $c_i(x)$ respectively, and are called the **intersection numbers of** x. Observe that if x is distance-regularized and $\epsilon(x) = d$, then $a_0(x) = c_0(x) = b_d(x) = 0$, $b_0(x) = |\Gamma(x)|$ and $c_1(x) = 1$. Note also that for every $1 \le i \le d$ we have that $b_{i-1}(x) > 0$ and $c_i(x) > 0$, and that $a_i(x) = 0$ if Γ is bipartite. For convenience we define $c_i(x) = b_i(x) = 0$ for $i < 0$ and $i > d$.

We now recall some definitions and basic results concerning a Terwilliger algebra of Γ. Let $\mathbb{C}$ denote the complex number field. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the **standard module**. We endow V with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v\rangle = u^{\top}\overline{v}$ for $u, v \in V$, where $\top$ denotes transpose and $^{-}$ denotes complex conjugation. For $y \in X$, let $\widehat{y}$ denote the element of V with a 1 in the y-coordinate and 0 in all other coordinates. We observe that $\{\widehat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. That is, the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with entries given as follows:

$$(A)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = 1, \\ 0 & \text{if} \quad \partial(x,y) \neq 1, \end{cases} \qquad (x, y \in X).$$

The **adjacency algebra of** Γ, also called the **Bose-Mesner algebra of** Γ, is the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ.

We now recall the dual idempotents of Γ. To do this fix a (not necessarily distance-regularized) vertex $x \in X$ and let $d = \epsilon(x)$. We view x as a *base vertex*. For $0 \leq i \leq d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y,y)-entry as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the ***i*-th dual idempotent of Γ with respect to x** [89, p. 378]. We also observe (ei) $\sum_{i=0}^{d} E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \leq i \leq d)$; (eiii) $E_i^{*\top} = E_i^*$ $(0 \leq i \leq d)$; (eiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leq i, j \leq d)$ where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. By these facts, matrices $E_0^*, E_1^*, \ldots, E_d^*$ form a basis for the commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the **dual Bose-Mesner algebra of Γ with respect to x** [89, p. 378]. Note that for $0 \leq i \leq d$ we have that

$$E_i^* V = \mathrm{Span}\{\widehat{y} \mid y \in \Gamma_i(x)\}.$$

We call $E_i^* V$ the ***i*-th subconstituent of Γ with respect to x**. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_d^* V \qquad (\textit{orthogonal direct sum}).$$

For convenience we define E_{-1}^* and E_{d+1}^* to be the zero matrix of $\mathrm{Mat}_X(\mathbb{C})$.

We recall the definition of a Terwilliger algebra of Γ. The Terwilliger algebra was first defined in [89, Definition 3.3], where it was called the **subconstituent algebra**. It was first defined for commutative association schemes, but the definition could be easily generalized to an arbitrary graph as follows. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the **Terwilliger algebra of Γ with respect to x**. Recall M is generated by A. So, T is generated by A and the dual idempotents. We observe T has finite dimension. In addition, since by construction T is generated by real-symmetric matrices, it follows that T is closed under the conjugate-transpose map. For a vector subspace $W \subseteq V$, we denote by TW the subspace $\{Bw \mid B \in T, w \in W\}$.

We now recall the lowering matrix, the flat matrix and the raising matrix of the algebra T.

Definition 6.1.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$. Let d denote the eccentricity of x and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Define $L = L(x)$, $F = F(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by*

$$L = \sum_{i=1}^{d} E^*_{i-1} A E^*_i, \qquad F = \sum_{i=0}^{d} E^*_i A E^*_i, \qquad R = \sum_{i=0}^{d-1} E^*_{i+1} A E^*_i.$$

*We refer to L, F and R as the **lowering**, the **flat** and the **raising matrix with respect to** x, respectively. Note that $L, F, R \in T$. Moreover, $F = F^\top$, $R = L^\top$ and $A = L + F + R$.*

Observe also that for $y, z \in X$ we have the (z,y)-entry of L equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) - 1$, and 0 otherwise. In addition, the (z,y)-entry of F is equal to 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y)$, and 0 otherwise. Similarly, the (z,y)-entry of R equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) + 1$, and 0 otherwise. Consequently, for $v \in E^*_i V$ $(0 \le i \le d)$ we have

$$Lv \in E^*_{i-1} V, \qquad Fv \in E^*_i V, \qquad Rv \in E^*_{i+1} V. \tag{6.1}$$

By a **T-module** we mean a subspace of V which is B-invariant for every $B \in T$. Let W denote a T-module. Then W is said to be **irreducible** whenever W is nonzero and W contains no T-modules other than 0 and W. Since the algebra T is closed under the conjugate-transposed map, it turns out that any T-module is an orthogonal direct sum of irreducible T-modules.

Let W be an irreducible T-module. We observe that W is an orthogonal direct sum of the nonvanishing subspaces $E^*_i W$ for $0 \le i \le d$. By the **endpoint** of W we mean $\min\{i \mid 0 \le i \le d,\ E^*_i W \ne 0\}$. We say W is **thin** whenever the dimension of $E^*_i W$ is at most 1 for $0 \le i \le d$.

Let W and W' denote two irreducible T-modules. By a **T-isomorphism** from W to W' we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W and W' are said to be **T-isomorphic** (or simply **isomorphic**) whenever there exists a T-isomorphism $\sigma : W \to W'$. We note that isomorphic irreducible T-modules have the same endpoint. It turns out that two non-isomorphic irreducible T-modules are orthogonal.

It is known that T has a unique irreducible T-module with endpoint 0, namely the subspace $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$. We refer to $T\widehat{x}$ as the **trivial T-module**. It was proved in [88] by Terwilliger that the trivial T-module is thin if x is distance-regularized (see also Theorem 3.5.3 and Subsection 3.7.1). In this case vectors s_i $(0 \le i \le d)$, where

$$s_i = \sum_{y \in \Gamma_i(x)} \widehat{y},$$

form a basis of the trivial T-module.

In the rest of this chapter we will study irreducible T-modules with endpoint 1 in the case when Γ is distance-regular around x. By Proposition 4.2.2, there are no irreducible T-modules with endpoint 1 if and only if x is a leaf of Γ, that is, if and only if $|\Gamma(x)| = 1$ (see also [26, Proposition 3.2]). Therefore, we will assume for the rest of this chapter that $|\Gamma(x)| \ge 2$.

We finish this section with a result which will play an important role later in this chapter. In Theorem 5.4.3 (see also [27, Theorem 5.3]), under the assumption that a graph Γ is bipartite, we prove that certain matrices of T are linearly dependent. However, the assumption that Γ is bipartite was never used in the proof of Theorem 5.4.3. Consequently, the next result is true:

Theorem 6.1.2 ([27, Theorem 5.3])**.** *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple, connected graph with vertex set X and edge set $\mathcal{R}$. Fix a distance-regularized vertex $x \in X$ with $|\Gamma(x)| \ge 2$ and eccentricity d. Let $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \le i \le d)$ denote the dual idempotents of Γ with respect to x and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Pick matrices $F_1, F_2, F_3 \in T$ and an integer i $(1 \le i \le d)$. Then the matrices $E_i^* F_1 E_1^*$, $E_i^* F_2 E_1^*$ and $E_i^* F_3 E_1^*$ are linearly dependent.*

Observe that the conclusion of Theorem 6.1.2 is equivalent to the fact that the dimension of $E_i^* T E_1^*$ $(1 \le i \le d)$ is at most 2.

6.2 The intersection diagrams

Throughout this section let $\Gamma = (X, \mathcal{R})$ denote a connected graph. We define a certain partition of X that we will find useful later.

Definition 6.2.1. *Let $\Gamma = (X, \mathcal{R})$ denote a graph with diameter D. Pick $x, y \in X$, such that $y \in \Gamma(x)$. For integers i, j we define sets $D^i_j := D^i_j(x,y)$ as follows:*

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(y).$$

*Observe that $D^i_j = \emptyset$ if $i < 0$ or $j < 0$. Similarly, $D^i_j = \emptyset$ if $i > \epsilon(x)$ or $j > \epsilon(y)$. Furthermore, by the triangle inequality we have that $D^i_j = \emptyset$ if $|i - j| \geq 2$. Note also that if Γ is bipartite, the set D^i_i is empty for $0 \leq i \leq D$. The collection of all the subsets D^i_{i-1} $(1 \leq i \leq \epsilon(x))$, D^i_i $(1 \leq i \leq \min\{\epsilon(x), \epsilon(y)\})$ and D^{i-1}_i $(1 \leq i \leq \epsilon(y))$ is called the **distance partition of Γ with respect to the edge** $\{x, y\}$.*

The proofs of the following lemmas are straightforward and therefore left to the reader.

Lemma 6.2.2. *Let $\Gamma = (X, \mathcal{R})$ denote a connected graph with diameter D. Pick $x, y \in X$, such that $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Then the following (i)–(v) hold for $1 \leq i \leq D$.*

(i) If $w \in D^i_{i-1}$ then $\Gamma(w) \subseteq D^{i-1}_{i-2} \cup D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^{i+1}_i$.

(ii) If $w \in D^i_i$ then $\Gamma(w) \subseteq D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^i_{i+1} \cup D^{i+1}_i \cup D^{i+1}_{i+1}$.

(iii) If $w \in D^{i-1}_i$ then $\Gamma(w) \subseteq D^{i-2}_{i-1} \cup D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^i_{i+1}$.

(iv) $\Gamma_i(x) = D^i_{i-1} \cup D^i_i \cup D^i_{i+1}$ and $\Gamma_i(y) = D^{i-1}_i \cup D^i_i \cup D^{i+1}_i$.

(v) If $D^i_{i+1} \neq \emptyset$ ($D^{i+1}_i \neq \emptyset$, respectively) then $D^j_{j+1} \neq \emptyset$ ($D^{j+1}_j \neq \emptyset$, respectively) for every $0 \leq j \leq i$.

Lemma 6.2.3. *Let $\Gamma = (X, \mathcal{R})$ denote a connected graph with diameter D. Pick $x \in X$ and assume that x is distance-regularized. Pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Then, the following (i), (ii) hold.*

(i) Assume that $z \in D^i_{i-1}$ $(1 \leq i \leq D)$. Then,

(a) $\left|\Gamma(z) \cap D^{i-1}_{i-2}\right| + \left|\Gamma(z) \cap D^{i-1}_{i-1}\right| + \left|\Gamma(z) \cap D^{i-1}_i\right| = c_i(x)$.

(b) $\left|\Gamma(z) \cap D^i_{i-1}\right| + \left|\Gamma(z) \cap D^i_i\right| = a_i(x)$.

(c) $\left|\Gamma(z)\cap D_i^{i+1}\right| = b_i(x)$.

(ii) *Assume that* $z \in D_{i+1}^i$ $(0 \le i \le D)$. *Then,*

(a) $\left|\Gamma(z)\cap D_i^{i+1}\right| + \left|\Gamma(z)\cap D_{i+1}^{i+1}\right| + \left|\Gamma(z)\cap D_{i+2}^{i+1}\right| = b_i(x)$.

(b) $\left|\Gamma(z)\cap D_{i+1}^{i}\right| + \left|\Gamma(z)\cap D_i^{i}\right| = a_i(x)$.

(c) $\left|\Gamma(z)\cap D_i^{i-1}\right| = c_i(x)$.

Below, a graphical representation of a distance partition for the case when the eccentricity of a vertex $y \in \Gamma(x)$ is equal to $\epsilon(x)$ is presented in Figure 6.1. A line between D_j^i and $D_{j'}^{i'}$ indicates the possibility of existence of edges between these two sets. Such a graphical representation of a distance partition is called the **intersection diagram of** Γ **with respect to the edge** $\{x,y\}$.

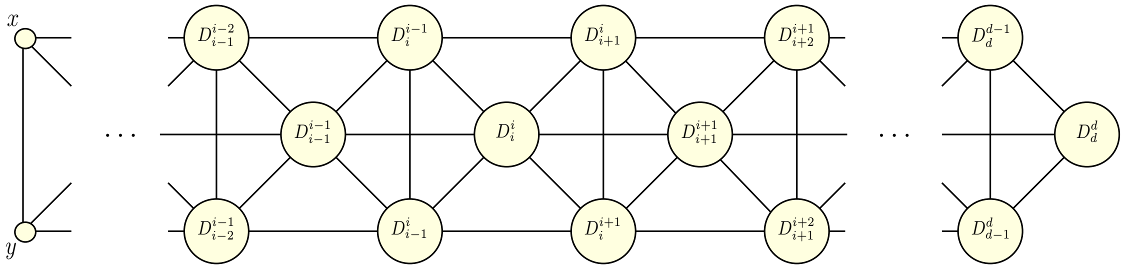

Figure 6.1: The intersection diagram of a connected graph Γ where $\epsilon(y) = \epsilon(x) = d$.

Lemma 6.2.4. *Let* $\Gamma = (X,\mathcal{R})$ *be a graph with diameter* D. *Pick a vertex* $x \in X$ *and assume that* x *is distance-regularized. Pick* $y \in \Gamma(x)$ *and let* $D_j^i = D_j^i(x,y)$. *Assume that* $D_{i+1}^i \neq \emptyset$ *or* $D_i^i \neq \emptyset$, *where* $1 \le i \le D$. *Then* $D_{i-1}^i \neq \emptyset$. *In particular,* $D_{i-1}^i \neq \emptyset$ *for* $1 \le i \le \epsilon(x)$.

Proof. If $i = 1$ then the result holds as $D_0^1 = \{y\}$. Assume therefore that $i \ge 2$ and that $D_{i+1}^i \neq \emptyset$ or $D_i^i \neq \emptyset$. Suppose to the contrary that $D_{i-1}^i = \emptyset$ and let t be the greatest integer such that $D_{t-1}^t \neq \emptyset$. Observe that $b_t(x) = 0$ by Lemma 6.2.3(i), which is impossible as $t < \epsilon(x)$. To prove the last part of the lemma note that $\Gamma_i(x) = D_{i+1}^i \cup D_i^i \cup D_{i-1}^i$ (disjoint union) is nonempty for $0 \le i \le \epsilon(x)$. ∎

Lemma 6.2.5. *Let* $\Gamma = (X,\mathcal{R})$ *denote a connected graph with diameter* D. *Pick* $x,y \in X$, *such that* $y \in \Gamma(x)$ *and let* $D_j^i = D_j^i(x,y)$. *Assume that* $D_i^i \neq \emptyset$ *and* $D_{i-1}^{i-1} = \emptyset$, *where* $1 \le i \le D$. *Then every vertex* $z \in D_i^i$ *has a neighbour in* D_{i-1}^i.

Proof. If $i=1$ then the result holds as $D_0^1=\{y\}$. Assume therefore that $i\geq 2$ and that $D_i^i\neq\emptyset$ and $D_{i-1}^{i-1}=\emptyset$. Pick $z\in D_i^i$. Then, there exists a path $[y=y_0,\cdots,y_{i-1},y_i=z]$ such that $\partial(y,y_{i-1})=i-1$ and $y_{i-1}\in\Gamma(z)$. By Lemma 6.2.2(ii), it follows that $\Gamma(z)\cap\Gamma_{i-1}(y)\subseteq D_{i-1}^{i-1}\cup D_{i-1}^i$. Hence, since D_{i-1}^{i-1} is empty, we have that y_{i-1} is a neighbour of z in D_{i-1}^i. The claim follows. ■

6.3 The main result

Throughout this section let $\Gamma=(X,\mathcal{R})$ denote a connected graph. In this section we state our main result. To do this we need the following definition.

Definition 6.3.1. *Let $\Gamma=(X,\mathcal{R})$ denote a connected graph. Pick $x,y,z\in X$ and let $P=[y=x_0,x_1,\dots,x_j=z]$ denote a yz-walk. The **shape of** P **with respect to** x is a sequence of symbols $t_1t_2\dots t_j$, where $t_i\in\{f,\ell,r\}$, and such that $t_i=r$ if $\partial(x,x_i)=\partial(x,x_{i-1})+1$, $t_i=f$ if $\partial(x,x_i)=\partial(x,x_{i-1})$ and $t_i=\ell$ if $\partial(x,x_i)=\partial(x,x_{i-1})-1$ $(1\leq i\leq j)$. We use exponential notation for shapes containing several consecutive identical symbols. For instance, instead of $rrrrfff\ell\ell r$ we simply write $r^4f^3\ell^2r$. Analogously, $r^0f=f$ and $r^0\ell=\ell$ is also conventional. For a non-negative integer m, let $r^m\ell(y,z)$, $r^mf(y,z)$ and $r^m(y,z)$ respectively denote the number of yz-walks of the shape $r^m\ell$, r^mf and r^m with respect to x where $r^0(y,z)=1$ if $y=z$ and $r^0(y,z)=0$ otherwise. See Figure 6.2 for an example.*

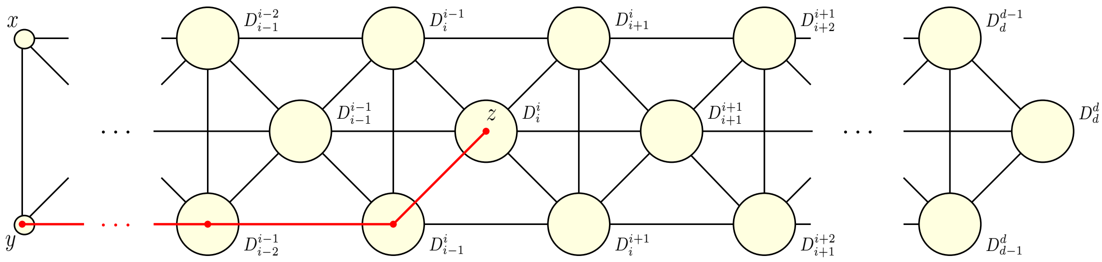

Figure 6.2: A yz-walk of the shape $r^{i-1}f$ for $y\in\Gamma(x)$ and $z\in D_i^i$ in a graph Γ where $\epsilon(y)=\epsilon(x)=d$.

The following observation is straightforward to prove (using elementary matrix multiplication and (6.1)).

Lemma 6.3.2. *Let $\Gamma=(X,\mathcal{R})$ denote a connected graph. Pick $x\in X$ and let $T=T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $L=L(x)$, $F=F(x)$ and $R=R(x)$ denote the lowering, the flat and the raising matrix of T, respectively. Pick $y,z\in X$ and let m be a positive integer. Then the following (i)–(iii) hold:*

(i) *The (z,y)-entry of R^m is equal to the number $r^m(y,z)$ with respect to x.*

(ii) *The (z,y)-entry of LR^m is equal to the number $r^m\ell(y,z)$ with respect to x.*

(iii) *The (z,y)-entry of FR^m is equal to the number $r^m f(y,z)$ with respect to x.*

For the rest of the chapter we adopt the following notation.

Notation 6.3.3. *Let $\Gamma = (X,\mathcal{R})$ denote a finite, simple, connected graph with vertex set X, edge set $\mathcal{R}$ and diameter D. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix a distance-regularized vertex $x \in X$ with $|\Gamma(x)| \geq 2$. Let d denote the eccentricity of x, and let $a_i(x), b_i(x), c_i(x)$ $(0 \leq i \leq d)$ denote the intersection numbers of x. Let $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \leq i \leq d)$ denote the dual idempotents of Γ with respect to x. Let V denote the standard module of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $L = L(x)$, $F = F(x)$ and $R = R(x)$ denote the lowering, the flat and the raising matrix of T, respectively. Let J denote the all 1's matrix in $\mathrm{Mat}_X(\mathbb{C})$. Recall that the unique irreducible T-module with endpoint 0 is thin. We denote this T-module by V_0. For $y \in \Gamma(x)$ and $z \in X$ let the sets $D_j^i = D_j^i(x,y)$ be as defined in Definition 6.2.1, and let the numbers $r^m\ell(y,z)$, $r^m f(y,z)$ and $r^m(y,z)$ be as defined in Definition 6.3.1.*

We are now ready to state our main result.

Theorem 6.3.4. *With reference to Notation 6.3.3, the following (i), (ii) are equivalent:*

(i) *Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin.*

(ii) *For every integer i $(1 \leq i \leq d)$ there exist scalars κ_i, μ_i, θ_i, ρ_i, such that for every $y \in \Gamma(x)$ the following (a), (b) hold:*

(a) *For every vertex $z \in D_{i+1}^i(x,y) \cup D_i^i(x,y)$ we have that $r^i\ell(y,z) = \mu_i$ and $r^{i-1}f(y,z) = \rho_i$. In particular, $r^i\ell(y,z)$ and $r^{i-1}f(y,z)$ do not depend on the choice of y,z.*

(b) *For every $z \in D_{i-1}^i(x,y)$ we have that*

$$\begin{aligned} r^i\ell(y,z) &= \kappa_i\, r^{i-1}(y,z) + \mu_i, \\ r^{i-1}f(y,z) &= \theta_i\, r^{i-1}(y,z) + \rho_i. \end{aligned}$$

Moreover, $\rho_i = 0$ whenever the set $D_{i+1}^i(x,y)$ is nonempty.

We finish this section with the following observation.

Proposition 6.3.5. *With reference to Notation 6.3.3, the following holds for* $0 \le i \le d$*:*

$$\left(E_i^* R^i L E_1^*\right)_{zy} = \begin{cases} \prod_{k=1}^{i} c_k(x) & \text{if } y \in \Gamma(x) \text{ and } z \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is straightforward to check that the (z, y)-entry of $E_i^* R^i L E_1^*$ is zero if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$. It is also straightforward to check that the result is true if $i = 0$. Suppose now that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$ with $i \ge 1$. Then $\left(E_i^* R^i L E_1^*\right)_{zy} = \left(R^i L\right)_{zy}$. By Lemma 6.3.2(*ii*), the (z, y)-entry of $R^i L$ is equal to the number of paths of length i from z to x. Since x is distance-regularized we observe that there are precisely $c_i(x) c_{i-1}(x) \cdots c_1(x)$ such paths. The claim follows. ■

Corollary 6.3.6. *With reference to Notation 6.3.3, the following holds for* $0 \le i \le d$*:*

$$E_i^* R^i L E_1^* = \left(\prod_{k=1}^{i} c_k(x)\right) E_i^* J E_1^*.$$

In particular, $E_i^* J E_1^* \in T$.

Proof. Immediately from Proposition 6.3.5. ■

6.4 Algebraic condition implies combinatorial properties

With reference to Notation 6.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. In this section we prove that in this case combinatorial conditions (a), (b) described in part (ii) of Theorem 6.3.4 hold.

Lemma 6.4.1. *With reference to Notation 6.3.3, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. Then for every* i $(1 \le i \le d)$ *there exist scalars* κ_i, μ_i*, such that*

$$E_i^* L R^i E_1^* = \kappa_i E_i^* R^{i-1} E_1^* + \mu_i E_i^* J E_1^*. \tag{6.2}$$

Proof. Pick i ($1 \le i \le d$) and observe that by Definition 6.1.1 and Corollary 6.3.6, the matrices LR^i, R^{i-1} and $E^*_i J E^*_1$ are elements of algebra T. Therefore, by (eiv) from Section 6.1 and Theorem 6.1.2, there exist scalars $\lambda_j = \lambda^{(i)}_j$ ($1 \le j \le 3$), not all zero, such that

$$\lambda_1 E^*_i LR^i E^*_1 + \lambda_2 E^*_i R^{i-1} E^*_1 + \lambda_3 E^*_i J E^*_1 = 0.$$

Assume for the moment that $\lambda_1 \neq 0$. Then (6.2) holds with $\kappa_i = -\lambda_2/\lambda_1$ and $\mu_i = -\lambda_3/\lambda_1$. Now, assume that $\lambda_1 = 0$. We first claim that in this case we have $D^i_{i+1}(x,y) \cup D^i_i(x,y) = \emptyset$ for every $y \in \Gamma(x)$. Indeed, suppose to the contrary that there exists $y \in \Gamma(x)$ such that the set $D^i_{i+1}(x,y) \cup D^i_i(x,y) \neq \emptyset$. Abbreviate $D^k_j = D^k_j(x,y)$ for $0 \le k, j \le D$, and observe that $D^i_{i-1} \neq \emptyset$ by Lemma 6.2.4. Pick $z \in D^i_{i+1} \cup D^i_i$ and note that it follows from Lemma 6.3.2(i) that the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ is 0, while the (z,y)-entry of $E^*_i J E^*_1$ is 1. This implies that $\lambda_3 = 0$. Pick now $z \in D^i_{i-1}$ and note that it follows from Lemma 6.3.2(i) that the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ is nonzero. This implies $\lambda_2 = 0$, contradicting the fact that the scalars λ_j ($1 \le j \le 3$) are not all zero. This proves our claim.

We next claim that $\lambda_2 \neq 0$. Suppose to the contrary that $\lambda_2 = 0$. Pick $y \in \Gamma(x)$ and abbreviate $D^k_j = D^k_j(x,y)$ for $0 \le k, j \le D$. As $D^i_{i+1} \cup D^i_i = \emptyset$, we clearly have that $D^i_{i-1} \neq \emptyset$. Pick $z \in D^i_{i-1}$ and observe that the (z,y)-entry of $E^*_i J E^*_1$ is equal to 1, which forces $\lambda_3 = 0$, again contradicting the fact that the scalars λ_j ($1 \le j \le 3$) are not all zero. It follows that $\lambda_2 \neq 0$. Therefore, we have that

$$E^*_i R^{i-1} E^*_1 = -\frac{\lambda_3}{\lambda_2} E^*_i J E^*_1, \tag{6.3}$$

and so for any $y \in \Gamma(x)$ and for any $z \in D^i_{i-1}(x,y)$, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ is equal to $-\lambda_3/\lambda_2$. In other words, for any $y \in \Gamma(x)$ and for any $z \in D^i_{i-1}(x,y)$ there are exactly $-\lambda_3/\lambda_2$ walks of the shape r^{i-1} from y to z. Since the set $D^i_{i-1} \neq \emptyset$, this also implies that $\lambda_3 \neq 0$.

Pick again any $y \in \Gamma(x)$. Observe that since x is distance-regularized and since $D^i_{i+1}(x,y) \cup D^i_i(x,y) = \emptyset$, Lemma 6.2.3 implies that every $z \in D^i_{i-1}(x,y)$ has exactly $b_i(x)$ neighbours in $D^{i+1}_i(x,y)$, and that every $z \in D^{i+1}_i(x,y)$ has exactly $c_{i+1}(x)$ neighbours in $D^i_{i-1}(x,y)$. It follows from the above comments that for any vertex $z \in D^i_{i-1}(x,y)$ there are exactly $-b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$ walks of the shape $r^i\ell$ from y to z. We now claim that (6.2) holds for any κ_i, μ_i such that $\lambda_3\kappa_i - \lambda_2\mu_i = b_i(x)c_{i+1}(x)\lambda_3$. For example, we may let either $\kappa_i = b_i(x)c_{i+1}(x)$ and $\mu_i = 0$, or $\kappa_i = 0$ and $\mu_i = -b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$. Indeed, pick any $y, z \in X$. If either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$, then the (z,y)-entry of both sides of (6.2) equals 0. If $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$, then $z \in D^i_{i-1}(x,y)$ as $D^i_{i+1}(x,y) \cup D^i_i(x,y) = \emptyset$. The (z,y)-entry of the left-hand side of (6.2) equals the number of yz-walks of the shape $r^i\ell$, which equals $-b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$ by the above comments. However, it follows from Lemma 6.3.2 and (6.3) that also the (z,y)-entry of the right-hand side of (6.2) equals $-b_i(x)c_{i+1}(x)\lambda_3/\lambda_2$, and the result follows. ■

The proof of the next lemma can be carried out using the same arguments as in the proof of Lemma 6.4.1.

Lemma 6.4.2. *With reference to Notation 6.3.3, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. Then for every* i $(1 \le i \le d)$ *there exist scalars* θ_i, ρ_i*, such that*

$$E_i^* F R^{i-1} E_1^* = \theta_i E_i^* R^{i-1} E_1^* + \rho_i E_i^* J E_1^*. \tag{6.4}$$

Proof. Pick i $(1 \le i \le d)$ and observe that by Definition 6.1.1 and Corollary 6.3.6, the matrices FR^{i-1}, R^{i-1} and $E_i^* J E_1^*$ are elements of algebra T. Therefore, by (eiv) from Section 6.1 and Theorem 6.1.2, there exist scalars $\lambda_j = \lambda_j^{(i)}$ $(1 \le j \le 3)$, not all zero, such that

$$\lambda_1 E_i^* F R^{i-1} E_1^* + \lambda_2 E_i^* R^{i-1} E_1^* + \lambda_3 E_i^* J E_1^* = 0.$$

Assume for the momment that $\lambda_1 \ne 0$. Then (6.4) holds with $\theta_i = -\lambda_2/\lambda_1$ and $\rho_i = -\lambda_3/\lambda_1$. Now, assume that $\lambda_1 = 0$. We first claim that in this case we have $D_{i+1}^i(x,y) \cup D_i^i(x,y) = \emptyset$ for every $y \in \Gamma(x)$. Indeed, suppose to the contrary that there exist $y \in \Gamma(x)$ such that the set $D_{i+1}^i(x,y) \cup D_i^i(x,y) \ne \emptyset$. Abbreviate $D_j^k = D_j^k(x,y)$ for $0 \le k, j \le D$, and observe that $D_{i-1}^i \ne \emptyset$ by Lemma 6.2.4. Pick $z \in D_{i+1}^i \cup D_i^i$ and note that it follows from Lemma 6.3.2(i) that the (z,y)-entry of $E_i^* R^{i-1} E_1^*$ is 0, while the (z,y)-entry of $E_i^* J E_1^*$ is 1. This implies that $\lambda_3 = 0$. Pick now $z \in D_{i-1}^i$ and note that it follows from Lemma 6.3.2(i) that the (z,y)-entry of $E_i^* R^{i-1} E_1^*$ is nonzero. This implies $\lambda_2 = 0$, contradicting the fact that the scalars λ_j $(1 \le j \le 3)$ are not all zero. This proves our claim.

We next claim that $\lambda_2 \ne 0$. Suppose to the contrary that $\lambda_2 = 0$. Pick $y \in \Gamma(x)$ and abbreviate $D_j^k = D_j^k(x,y)$ for $0 \le k, j \le D$. As $D_{i+1}^i \cup D_i^i = \emptyset$, we clearly have that $D_{i-1}^i \ne \emptyset$. Pick $z \in D_{i-1}^i$ and observe that the (z,y)-entry of $E_i^* J E_1^*$ is equal to 1, which forces $\lambda_3 = 0$, again contradicting the fact that the scalars λ_j $(1 \le j \le 3)$ are not all zero. It follows that $\lambda_2 \ne 0$. Therefore, we have that

$$E_i^* R^{i-1} E_1^* = -\frac{\lambda_3}{\lambda_2} E_i^* J E_1^*, \tag{6.5}$$

and so for any $y \in \Gamma(x)$ and for any $z \in D^i_{i-1}(x,y)$, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ is equal to $-\lambda_3/\lambda_2$. In other words, for any $y \in \Gamma(x)$ and for any $z \in D^i_{i-1}(x,y)$ there are exactly $-\lambda_3/\lambda_2$ walks of the shape r^{i-1} from y to z. Since the set $D^i_{i-1} \neq \emptyset$, this also implies that $\lambda_3 \neq 0$.

Pick again any $y \in \Gamma(x)$. Observe that since x is distance-regularized and since we also have that $D^i_{i+1}(x,y) \cup D^i_i(x,y) = \emptyset$, Lemma 6.2.3($i$) implies that every $z \in D^i_{i-1}(x,y)$ has exactly $a_i(x)$ neighbours in $D^i_{i-1}(x,y)$. Hence, it follows from the above comments that for any $z \in D^i_{i-1}(x,y)$ there are exactly $-a_i(x)\lambda_3/\lambda_2$ walks of the shape $r^{i-1}f$ from y to z. We now claim that (6.4) holds for any θ_i, ρ_i such that $\lambda_3\theta_i - \lambda_2\rho_i = a_i(x)\lambda_3$. For example, we may let either $\theta_i = a_i(x)$ and $\rho_i = 0$, or $\theta_i = 0$ and $\rho_i = -a_i(x)\lambda_3/\lambda_2$. Indeed, pick any $y, z \in X$. If either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$, then the (z,y)-entry of both sides of (6.4) equals 0. If $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$, then $z \in D^i_{i-1}(x,y)$ as $D^i_{i+1}(x,y) \cup D^i_i(x,y) = \emptyset$. The (z,y)-entry of the left-hand side of (6.4) equals the number of yz-walks of the shape $r^{i-1}f$, which equals $-a_i(x)\lambda_3/\lambda_2$ by the above comments. However, it follows from Lemma 6.3.2 and (6.5) that also the (z,y)-entry of the right-hand side of (6.4) equals $-a_i(x)\lambda_3/\lambda_2$, and the result follows. ■

We are now ready to prove the main result of this section.

Theorem 6.4.3. *With reference to Notation 6.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. For every integer i $(1 \le i \le d)$ there exist scalars κ_i, μ_i, θ_i, ρ_i, such that for every $y \in \Gamma(x)$ the following (a), (b) hold:*

(a) For every $z \in D^i_{i+1}(x,y) \cup D^i_i(x,y)$ we have that $r^i\ell(y,z) = \mu_i$ and $r^{i-1}f(y,z) = \rho_i$. In particular, $r^i\ell(y,z)$ and $r^{i-1}f(y,z)$ do not depend on the choice of y,z.

(b) For every $z \in D^i_{i-1}(x,y)$ we have that

$$\begin{aligned} r^i\ell(y,z) &= \kappa_i\, r^{i-1}(y,z) + \mu_i, \\ r^{i-1}f(y,z) &= \theta_i\, r^{i-1}(y,z) + \rho_i. \end{aligned}$$

Moreover, $\rho_i = 0$ if the set $D^i_{i+1}(x,y)$ is nonempty for some $y \in \Gamma(x)$.

Proof. Pick an integer i $(1 \le i \le d)$ and recall that by Lemma 6.4.1 and Lemma 6.4.2, equations (6.2) and (6.4) hold. Pick $y \in \Gamma(x)$.

(a) Pick $z \in D^i_{i+1}(x,y) \cup D^i_i(x,y)$ and observe that by Lemma 6.3.2 the (z,y)-entry of the left-hand side of (6.2) equals $r^i\ell(y,z)$ while the (z,y)-entry of the left-hand side of (6.4) equals $r^{i-1}f(y,z)$. On the other hand, again by Lemma 6.3.2, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ equals 0, while the (z,y)-entry of $E^*_i J E^*_1$ is obviously equal to 1. Therefore, the (z,y)-entry of the right-hand side of (6.2) equals μ_i and the (z,y)-entry of the right-hand side of (6.4) equals ρ_i. In particular, $r^i\ell(y,z)$ and $r^{i-1}f(y,z)$ do not depend on the choice of y,z.

(b) Pick now $z \in D^i_{i-1}(x,y)$ and observe that by Lemma 6.3.2 the (z,y)-entry of the left-hand side of (6.2) equals $r^i\ell(y,z)$. Similarly, the (z,y)-entry of the left-hand side of (6.4) equals $r^{i-1}f(y,z)$. On the other hand, again by Lemma 6.3.2, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ equals $r^{i-1}(y,z)$, while the (z,y)-entry of $E^*_i J E^*_1$ is obviously equal to 1. Therefore, the (z,y)-entry of the right-hand side of (6.2) equals $\kappa_i\, r^{i-1}(y,z) + \mu_i$ and the (z,y)-entry of the right-hand side of (6.4) equals $\theta_i\, r^{i-1}(y,z) + \rho_i$.

Moreover, for $z \in D^i_{i+1}(x,y)$ we observe there is no yz-walk of the shape $r^{i-1}f$ and so $\rho_i = 0$ if the set $D^i_{i+1}(x,y)$ is nonempty for some $y \in \Gamma(x)$. The result follows. ■

6.5 Combinatorial properties imply algebraic condition

With reference to Notation 6.3.3, assume that Γ satisfies part (ii) of Theorem 6.3.4. In this section we prove that in this case Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. We also display a basis of this module and the matrix representing the action of the adjacency matrix on this basis.

Proposition 6.5.1. *With reference to Notation 6.3.3, assume that Γ satisfies part (ii) of Theorem 6.3.4. For every integer i $(1 \le i \le d)$, the following equalities hold:*

$$E^*_i L R^i E^*_1 = \kappa_i E^*_i R^{i-1} E^*_1 + \mu_i E^*_i J E^*_1, \tag{6.6}$$
$$E^*_i F R^{i-1} E^*_1 = \theta_i E^*_i R^{i-1} E^*_1 + \rho_i E^*_i J E^*_1. \tag{6.7}$$

Proof. Pick an integer i $(1 \le i \le d)$ and vertices $y,z \in X$. We will show that the (z,y)-entries of both sides of (6.6) and (6.7) agree. Observe first that if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$, then the (z,y)-entry of both sides of (6.6) and (6.7) equals 0. Therefore, assume that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$. Abbreviate $D^k_j(x,y) = D^k_j$ for $0 \le k,j \le D$ and recall that $\Gamma_i(x) = D^i_{i-1} \cup D^i_i \cup D^i_{i+1}$.

Assume first that $z \in D^i_{i+1} \cup D^i_i$, and note that the (z,y)-entry of $E^*_i L R^i E^*_1$ is equal to the number $r^i \ell(y,z)$, while the (z,y)-entries of $E^*_i R^{i-1} E^*_1$ and $E^*_i J E^*_1$ are 0 and 1 respectively. In addition, the (z,y)-entry of $E^*_i F R^{i-1} E^*_1$ is equal to $r^{i-1} f(y,z)$. As $r^i \ell(y,z) = \mu_i$ and $r^{i-1} f(y,z) = \rho_i$ by the assumption, the (z,y)-entries of both sides of (6.6) and (6.7) agree.

Assume next that $z \in D^i_{i-1}$ and note the (z,y)-entry of $E^*_i L R^i E^*_1$, $E^*_i F R^{i-1} E^*_1$ and $E^*_i R^{i-1} E^*_1$ are equal to the numbers $r^i \ell(y,z)$, $r^{i-1} f(y,z)$ and $r^{i-1}(y,z)$ respectively. In addition, the (z,y)-entry of $E^*_i J E^*_1$ is of course equal to 1. By the assumption we have that $r^i \ell(y,z) = \kappa_i \, r^{i-1}(y,z) + \mu_i$ and $r^i f(y,z) = \theta_i \, r^{i-1}(y,z) + \rho_i$. So, the (z,y)-entries of both sides of (6.6) and (6.7) agree. This finishes the proof. ∎

Lemma 6.5.2. *With reference to Notation 6.3.3, assume that Γ satisfies part (ii) of Theorem 6.3.4. Pick $w \in E^*_1 V$, $w \neq 0$, which is orthogonal to s_1. Then the following (a), (b) hold:*

(i) $Lw = 0$ *and* $LR^i w = \kappa_i R^{i-1} w$ $(1 \le i \le d)$.

(ii) $FR^{i-1} w = \theta_i R^{i-1} w$ $(1 \le i \le d)$ *and* $FR^d w = 0$.

Proof. As $w \in E^*_1 V$ we have that $E^*_1 w = w$ and so,

$$\langle \boldsymbol{j}, w \rangle = \langle \boldsymbol{j}, E^*_1 w \rangle = \langle E^*_1 \boldsymbol{j}, w \rangle = \langle s_1, w \rangle = 0,$$

where $\boldsymbol{j}$ denotes the all 1's vector in V. This shows $Jw = 0$. By elementary matrix multiplication it is easy to see $E^*_0 A E^*_1 = E^*_0 J E^*_1$. Therefore, by Definition 6.1.1 and the above comments we have that $Lw = E^*_0 A E^*_1 w = E^*_0 J E^*_1 w = E^*_0 J w = 0$. Observe also that $R^d E^*_1$ is the zero matrix and so $FR^d w = 0$. In addition, by (6.1) and Proposition 6.5.1, for $1 \le i \le d$ we have

$$\begin{aligned} LR^i w &= E^*_i L R^i E^*_1 w = \kappa_i E^*_i R^{i-1} E^*_1 w = \kappa_i R^{i-1} w. \\ FR^{i-1} w &= E^*_i F R^{i-1} E^*_1 w = \theta_i E^*_i R^{i-1} E^*_1 w = \theta_i R^{i-1} w. \end{aligned}$$

The result follows. ∎

Lemma 6.5.3. *With reference to Notation 6.3.3, assume that Γ satisfies part (ii) of Theorem 6.3.4. Pick $w \in E^*_1 V$, $w \neq 0$, which is orthogonal to s_1. Then the following (i)–(iii) hold:*

(i) $\left\| R^i w \right\|^2 = \kappa_i \left\| R^{i-1} w \right\|^2$ $(1 \le i \le d)$.

(ii) $\left\langle R^i w, R^j w \right\rangle = \delta_{ij} \prod_{l=1}^{i} \kappa_l \|w\|^2$ $(0 \le i, j \le d)$.

(iii) *There exists i $(1 \le i \le d)$ such that $\kappa_i = 0$.*

Proof. (i) Pick $1 \le i \le d$. Then by Lemma 6.5.2(i) we have

$$\left\| R^i w \right\|^2 = \left\langle R^i w,\ R^i w \right\rangle = \left\langle L R^i w,\ R^{i-1} w \right\rangle = \kappa_i \left\| R^{i-1} w \right\|^2.$$

(ii) If $i \neq j$, then the result follows from (eii), (eiii) and (eiv) below the definition of the dual idempotents in Section 6.1 and from (6.1). If $i = j$ then the result follows from (i) above by a straightforward induction argument.

(iii) Immediate from (ii) above since by (6.1) we have $R^d w = 0$ and w is a nonzero vector. ■

Theorem 6.5.4. *With reference to Notation 6.3.3, assume that Γ satisfies part (ii) of Theorem 6.3.4. Pick $w \in E_1^* V$, $w \neq 0$, which is orthogonal to s_1. Let W denote the vector subspace of V spanned by the vectors $R^i w$ $(0 \le i \le d)$. Let s $(1 \le s \le d)$ be the least integer such that $\kappa_s = 0$. Then W is a thin irreducible T-module with endpoint* 1 *and the vectors $\{R^{i-1} w \mid 1 \le i \le s\}$ form an orthogonal basis of W. In particular, the dimension of W is s.*

Proof. Observe that by (6.1) and since $RE_d^* = 0$, the subspace W is invariant under the action of the dual idempotents. By construction and since $R^d w = 0$ by (6.1) it is also clear that W is closed under the action of R. Moreover, it follows from Lemma 6.5.2 that W is invariant under the action of L and F. Since $A = L + F + R$, it turns out that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Therefore, W is a T-module. It is clear that W is thin, since by construction, (6.1) and Lemma 6.5.2, the subspace $E_i^* W$ is generated by $R^{i-1} w$.

Now, let us show that W is irreducible. Note that $w \in W$ and so W is non-zero. Recall that W is an orthogonal direct sum of irreducible T-modules. Since $E_0^* W$ is the zero subspace and $E_1^* w = w \neq 0$, there exists an irreducible T-module W', such that the endpoint of W' is 1 and $W' \subseteq W$. Consequently, $E_1^* W' \subseteq E_1^* W$. However, the dimension of $E_1^* W$ is 1, and so $E_1^*W' = E_1^*W$. But now we have

$$W = TE_1^*W = TE_1^*W' \subseteq W',$$

implying that $W = W'$. Hence, W is irreducible and its endpoint equals 1.

Finally, notice that $R^s w = 0$ by Lemma 6.5.3(i). Furthermore, it holds that vectors $\{R^{i-1}w \mid 1 \leq i \leq s\}$ are nonzero and pairwise orthogonal by Lemma 6.5.3(ii) and the definition of number s. The result follows. ■

Theorem 6.5.5. *With reference to Notation 6.3.3, assume that* Γ *satisfies part (ii) of Theorem 6.3.4. Let* W *denote an irreducible* T*-module with endpoint* 1*. Let* s $(1 \leq s \leq d)$ *be the least integer such that* $\kappa_s = 0$*. Pick a vector* $w \in E_1^*W$*,* $w \neq 0$*. Then the vectors* $\{R^{i-1}w \mid 1 \leq i \leq s\}$ *form an orthogonal basis of* W*. In particular,* W *is a thin irreducible module with dimension* s*.*

Proof. Let W' denote the vector subspace of V spanned by the vectors $\{R^{i-1}w \mid 1 \leq i \leq d\}$. Recall that W and the unique irreducible T-module with endpoint 0 are not isomorphic, and so w is orthogonal to s_1. By Theorem 6.5.4, W' is a T-module. Note that W' is nonzero and contained in W. As W is irreducible, we have that $W = W'$. The result now follows from Theorem 6.5.4. ■

Theorem 6.5.6. *With reference to Notation 6.3.3, assume that* Γ *satisfies part (ii) of Theorem 6.3.4. Then there is, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin.*

Proof. Let W and W' be irreducible T-modules with endpoint 1, and pick any nonzero vectors $w \in E_1^*W$ and $w' \in E_1^*W'$. Let s $(1 \leq s \leq d)$ be the least integer such that $\kappa_s = 0$. By Theorem 6.5.5, the vectors

$$\left\{R^{i-1}w \mid 1 \leq i \leq s\right\} \text{ and } \left\{R^{i-1}w' \mid 1 \leq i \leq s\right\}$$

are orthogonal bases of W and W', respectively. Hence, the linear map $\sigma : W \to W'$, defined by $\sigma\left(R^{i-1}w\right) = R^{i-1}w'$ is a vector space isomorphism. It is clear that σ commutes with R. By Lemma 6.5.2 it follows that σ also commutes with L and F. Since $A = L + F + R$, it turns out that σ commutes with A as well. Furthermore, σ is a T-module isomorphism since by (eiv) from Section 6.1, it commutes also with E_i^* $(0 \leq i \leq d)$. Thus W and W' are T-isomorphic. ■

Theorem 6.5.7. *With reference to Notation 6.3.3, assume that* Γ *satisfies part (ii) of Theorem 6.3.4. Let* W *denote an irreducible* T*-module with endpoint* 1. *Pick* $w \in E_1^*W$, $w \neq 0$, *and recall that*

$$\mathcal{B} = \left\{ R^{i-1}w \mid 1 \leq i \leq s \right\}$$

is a basis of W, *where* s *is the least integer such that* $\kappa_s = 0$ $(1 \leq s \leq d)$. *Then the matrix representing the action of* A *on* W *with respect to the (ordered) basis* $\mathcal{B}$ *is given by*

$$\begin{pmatrix} \theta_1 & \kappa_1 & & & & \\ 1 & \theta_2 & \kappa_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \kappa_{s-2} & \\ & & & 1 & \theta_{s-2} & \kappa_{s-1} \\ & & & & 1 & \theta_{s-1} \end{pmatrix}.$$

Proof. Recall that $A = L + F + R$. The result now follows from Lemma 6.5.2. ■

6.6 Comments on the distance partition

Throughout this section let $\Gamma = (X, \mathcal{R})$ denote a connected graph. Let $x \in X$ and let $T = T(x)$. Suppose that Γ is distance-regular around x so that the unique irreducible T-module with endpoint 0 is thin. Assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin. In this section we have some comments about the combinatorial structure of the intersection diagrams of Γ with respect to the edge $\{x, y\}$, for every $y \in \Gamma(x)$. In particular, we will discuss which of the sets $D_j^i(x, y)$ are (non)empty.

Lemma 6.6.1. *With reference to Notation 6.3.3, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1, *and that this module is thin. Pick an integer* i $(1 \leq i \leq d)$ *and assume for some* $y \in \Gamma(x)$, *the set* $D_{i+1}^i(x, y) \neq \emptyset$. *Then, the set* $D_j^j(x, y)$ *is empty for every* j $(1 \leq j \leq i)$ *and for all* $y \in \Gamma(x)$.

Proof. Suppose there exists j $(1 \leq j \leq i)$ and $w \in \Gamma(x)$ such that $D_j^j(x, w)$ is nonempty. Without loss of generality, we may pick j as the least integer such that the set $D_{j-1}^{j-1}(x, w) = \emptyset$ but the set $D_j^j(x, w)$ is nonempty. By Definition 6.1.1 and Corollary 6.3.6, the matrices R^{j-1}, FR^{j-1} and $E_j^*JE_1^*$ are elements of algebra T. Therefore, by (eiv) from Section 6.1

and Theorem 6.1.2, there exist scalars $\lambda_k = \lambda_k^{(j)}$ $(1 \le k \le 3)$, not all zero, such that

$$\lambda_1 E_j^* R^{j-1} E_1^* + \lambda_2 E_j^* F R^{j-1} E_1^* + \lambda_3 E_j^* J E_1^* = 0. \tag{6.8}$$

By Lemma 6.2.2(v), notice that the set $D_{j+1}^j(x,y)$ is nonempty. Pick $z \in D_{j+1}^j(x,y)$ and note that it follows from Lemma 6.3.2$(i),(iv)$ that the (z,y)-entry of $E_j^* R^{j-1} E_1^*$ and $E_j^* F R^{j-1} E_1^*$ are both 0, respectively. This implies that $\lambda_3 = 0$ since the (z,y)-entry of $E_j^* J E_1^*$ equals 1. Pick now $z \in D_j^j(x,w)$. We observe from Lemma 6.3.2(i) that the (z,w)-entry of $E_j^* R^{j-1} E_1^*$ is 0. In addition, as the set $D_{j-1}^{j-1}(x,w)$ is empty, it follows from Lemma 6.2.2(v) and Lemma 6.2.5 that there exists a wz-walk of the shape $r^{j-1}f$ with respect to x. So, by Lemma 6.3.2(iv), the (z,w)-entry of $E_j^* F R^{j-1} E_1^*$ is nonzero. This implies that $\lambda_2 = 0$. So, from equation (6.8) we have that $\lambda_1 E_j^* R^{j-1} E_1^*$ is the zero matrix. Observe that $D_{j-1}^j(x,w)$ is nonempty by Lemma 6.2.4. We now pick $z \in D_{j-1}^j(x,w)$ and note that it follows from Lemma 6.3.2(i) that the (z,w)-entry of $E_j^* R^{j-1} E_1^*$ is nonzero. This implies $\lambda_1 = 0$, contradicting the fact that the scalars λ_k $(1 \le k \le 3)$ are not all zero. The claim follows. ■

The above lemma together with the fact that the set $D_1^0(x,y)$ is nonempty for every $y \in \Gamma(x)$ motivate the next result.

Proposition 6.6.2. *With reference to Notation 6.3.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint* 1*, and that this module is thin. Pick $y \in \Gamma(x)$ and let $D_j^i = D_j^i(x,y)$. Then, there exists an integer $t := t(y)$ $(0 \le t \le d)$ such that the following $(i),(ii)$ hold:*

(i) *For every i $(0 \le i \le t)$ the set D_{i+1}^i is nonempty and the set $D_i^i(x,z)$ is empty for every $z \in \Gamma(x)$.*

(ii) *For every i $(t < i \le d)$ the set D_{i+1}^i is empty.*

Moreover, $\Gamma_i(x) = D_{i+1}^i \cup D_{i-1}^i$ for every $0 \le i \le t$.

Proof. For $y \in \Gamma(x)$, since the set $D_1^0(x,y)$ is nonempty, let us define $t := t(y)$ as the greatest integer i $(1 \le i \le d)$ such that the set $D_{i+1}^i(x,y)$ is nonempty. Then, it is clear that the set D_{i+1}^i is empty for $i > t$ and, by Lemma 6.2.2(v), the set D_{i+1}^i is nonempty for every $0 \le i \le t$. Moreover, by Lemma 6.6.1 the set $D_i^i(x,z)$ is empty for every $z \in \Gamma(x)$ and for every $0 \le i \le t$. The result follows. ■

Proposition 6.6.3. *With reference to Notation 6.3.3, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. Pick* $y \in \Gamma(x)$ *and let* $D^i_j = D^i_j(x,y)$*. If there exists* j $(1 \le j \le d)$ *such that* D^j_j *is nonempty then* D^i_i *is nonempty for every* $t(y) < i \le j$*.*

Proof. Suppose the set D^j_j is nonempty for some j $(1 \le j \le d)$. Then, by Proposition 6.6.2 we have $t(y) < j$. Assume now there exists an integer i $(t(y) < i < j)$ such that D^i_i is empty. Notice that in this case $\Gamma_i(x) = D^i_{i-1}$ which is nonempty by Lemma 6.2.4. Pick now $w \in D^j_j$. We observe every shortest xw-path must pass through a vertex in D^i_{i-1}. This clearly shows $\partial(x,w) \ge i + (j-i+1) = j+1$, a contradiction. The result follows. ■

Propositions 6.6.2 and 6.6.3 help us to understand the combinatorial structure of graphs which have, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin.

We now consider the possible intersection diagrams of Γ with respect to the edge $\{x,y\}$, for every $y \in \Gamma(x)$. Let d denote the eccentricity of x. Then, we observe $\epsilon(y) \in \{d-1, d, d+1\}$. We have two cases.

If $\epsilon(y) > d$ then the set $D^d_{d+1}(x,y)$ is not empty and so the scalar $t(y) = d$. By Proposition 6.6.2, the set $D^i_i(x,y)$ is empty for every $y \in \Gamma(x)$ and for every i $(0 \le i \le d)$. Moreover, notice the sets $D^i_{i+1}(x,y)$ $(0 \le i \le d)$ and $D^i_{i-1}(x,y)$ $(1 \le i \le d)$ are all nonempty. See Figure 6.3 for a graphical representation of the intersection diagram of Γ with respect to the edge $\{x,y\}$ when $\epsilon(y) > \epsilon(x)$.

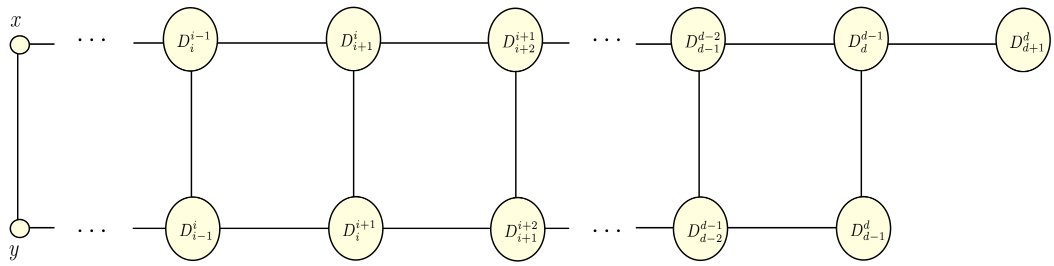

Figure 6.3: Intersection diagram of graph Γ which has, up to isomorphism, exactly one irreducible $T(x)$-module with endpoint 1, and this module is thin: case $\epsilon(y) > \epsilon(x)$.

If $\epsilon(y) \le d$ then the set $D^d_{d+1}(x,y)$ is empty and so, $t := t(y) < d$. Furthermore, in this case the sets $D^i_{i+1}(x,y)$ $(0 \le i \le t)$ and $D^i_{i-1}(x,y)$ $(1 \le i \le d)$ are all nonempty. Moreover, if $D^{t+1}_{t+1}(x,y) \ne \emptyset$, then let u $(1 \le u \le d-1-t)$ denote the greatest positive integer such that $D^{t+u}_{t+u}(x,y) \ne \emptyset$. See Figure 6.4 for a graphical representation of the intersection diagram of Γ with respect to the edge $\{x,y\}$ when $\epsilon(y) \le \epsilon(x)$.

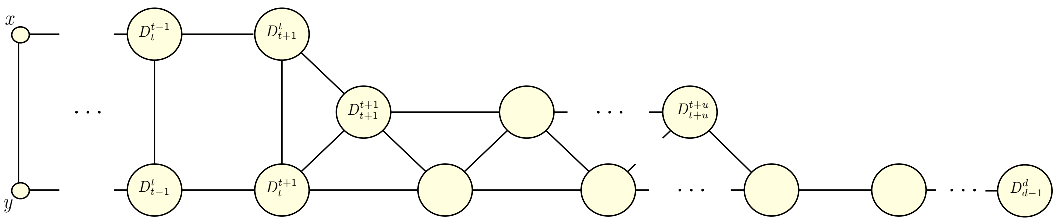

Figure 6.4: Intersection diagram of graph Γ which has, up to isomorphism, exactly one irreducible $T(x)$-module with endpoint 1, and this module is thin: case $\epsilon(y) \leq \epsilon(x)$.

It is easy to see that the integer $t := t(y)$, which Proposition 6.6.2 refers to, is independent of the choice of $y \in \Gamma(x)$ if and only if the next statement is true for each i $(1 \leq i \leq d)$:

$$\text{if for some } y \in \Gamma(x) \text{ the set } D_{i+1}^i(x,y) \neq \emptyset, \text{ then } D_{i+1}^i(x,y) \neq \emptyset \text{ for every } y \in \Gamma(x). \quad (6.9)$$

For $i = 1$, we observe (6.9) immediately follows. However, the proof of the general case seems to need a nontrivial approach. At this point, the next question naturally arises.

Question 6.6.4. *With reference to Notation 6.3.3 and Proposition 6.6.2, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. Does the integer* $t := t(y)$ *depend on the choice of* $y \in \Gamma(x)$*?*

6.7 Examples

In this section we present several examples of graphs for which the equivalent conditions of Theorem 6.3.4 hold for a certain vertex x. Some examples of bipartite graphs where the equivalent conditions of Theorem 6.3.4 hold for a certain vertex x are presented throughout Section 5.7 in Chapter 5. We therefore turn our attention to nonbipartite ones.

A distance-regular graph with diameter D is said to be almost-bipartite if the intersection numbers satisfy $a_i = 0$ $(1 \leq i \leq D-1)$ and $a_D \neq 0$ (see [6] for the definition of distance-regular graphs). In this case it is easy to see, that for any vertex $x \in X$, the partition from Definition 6.2.1 is equitable for every $y \in \Gamma(x)$, and the parameters of this partition do not depend on the choice of $y \in \Gamma(x)$ (see for example [39, Subsection 9.3] for the definition of equitable partitions). Moreover, the set $D_i^i(x,y)$ is empty for every $y \in \Gamma(x)$ and for every integer i $(1 \leq i \leq D-1)$ and the set $D_D^D(x,y)$ is nonempty for every $y \in \Gamma(x)$. It is thus clear that in this case the conditions (a), (b) described in part (ii) of Theorem

6.3.4 are satisfied, and so Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and this module is thin. So, almost-bipartite distance-regular graphs are examples of nonbipartite graphs for which the equivalent conditions of Theorem 6.3.4 hold for any given vertex x.

Our next example shows that there exist graphs which admit vertex x, such that there is, up to isomorphism, a unique irreducible $T(x)$-module of endpoint 1, and this module is thin, but the corresponding partitions from Definition 6.2.1 are not equitable.

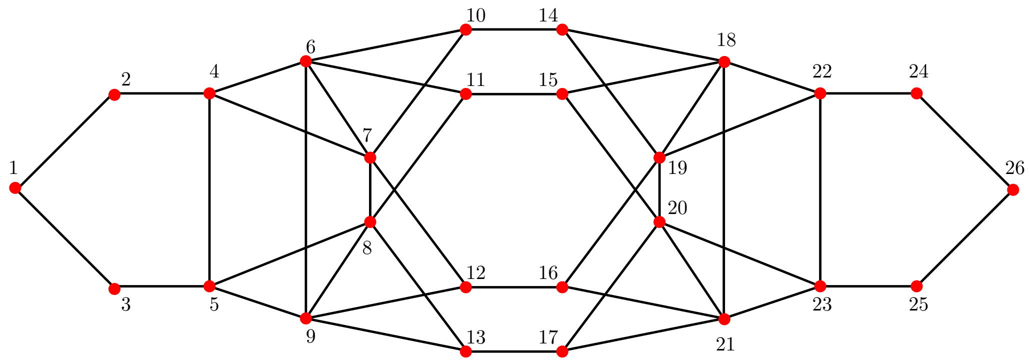

Figure 6.5: Graph Γ which has, up to isomorphism, exactly one irreducible $T(1)$-module with endpoint one, and this module is thin.

Let Γ denote the graph in Figure 6.5 and let $x = 1$. It is easy to check that Γ is non-bipartite and distance-regular around vertex 1. Let $T = T(1)$ be the Terwilliger algebra of Γ with respect to vertex 1.

The intersection diagram for the distance partition with respect to the edge $\{1,2\}$ is presented in Figure 6.6. Given the symmetry fixing vertex 1 and swapping vertices 2 and 3, the intersection diagram for the distance partition with respect to the edge $\{1,3\}$ is similar; see Figure 6.7.

It is now straightforward to check that properties (a), (b) described in part (ii) of Theorem 6.3.4 hold with the values of $\kappa_i, \mu_i, \theta_i, \rho_i$ $(1 \le i \le 9)$ as presented in Table 6.1.

i	1	2	3	4	5	6	7	8	9
κ_i	1	2	2	1	2	2	1	0	0
μ_i	0	0	1	0	2	0	0	8	0
θ_i	0	-1	0	0	0	0	-1	0	0
ρ_i	0	1	1	0	0	4	8	0	0

Table 6.1: Values of scalars κ_i, μ_i, θ_i and ρ_i, $(1 \le i \le 9)$.

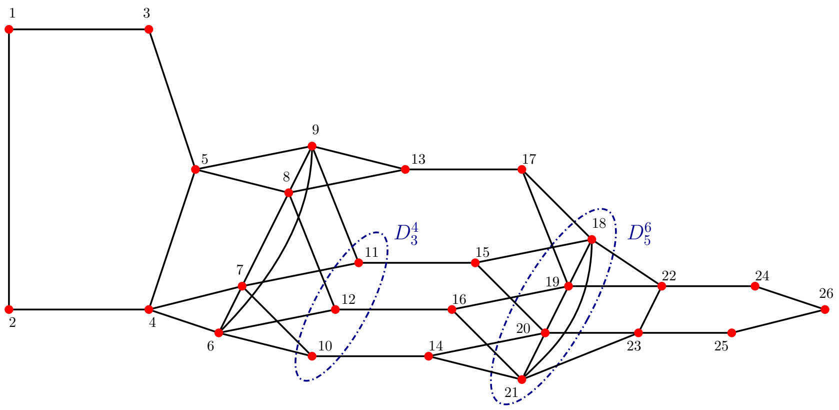

Figure 6.6: Distance partition of Γ with respect to the edge $\{1,2\}$.

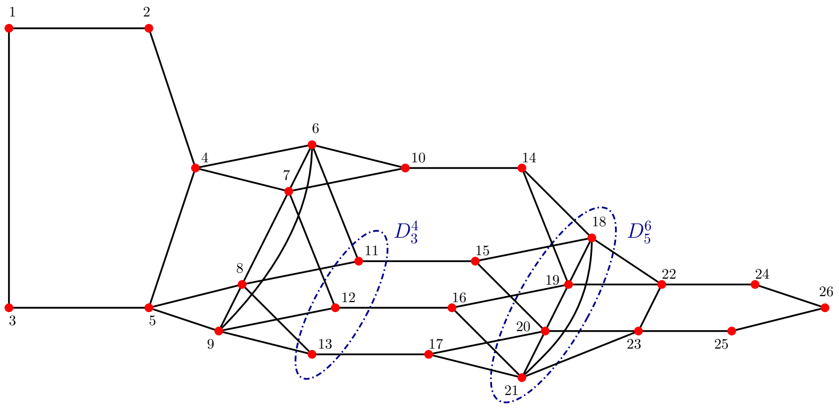

Figure 6.7: Distance partition of Γ with respect to the edge $\{1,3\}$.

Consequently, by Theorem 6.3.4, it holds that Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin. Moreover, this T-module has dimension $s = 8$. Note also that the partitions presented by the intersection diagrams in Figures 6.6 and 6.7 are not equitable.

Chapter 7

Graphs with exactly one irreducible T-module with endpoint 1, which is thin: the pseudo-distance-regularized case

Let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ which is not a leaf and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume that the unique irreducible T-module with endpoint 0 is thin, or equivalently that Γ is pseudo-distance-regular around x. We consider the property that Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and that this T-module is thin. The main result of the chapter is a combinatorial characterization of this property.

The chapter is organized as follows. In Section 7.1 we recall basic definitions and results about Terwilliger algebras that we will find useful later in the chapter. In Section 7.2 we then state our main result in Theorem 7.2.5. In Section 7.3, we prove that certain matrices of the Terwilliger algebra are linearly dependent, and we use this in Sections 7.4 and 7.5 to prove the main result. In Section 7.6, we have some comments about certain distance partitions of graphs which are pseudo-distance-regular around a fixed vertex and also have a unique irreducible T-module (up to isomorphism) with endpoint 1, and this module is thin. We finish the chapter presenting some examples in Section 7.7 and giving some concluding remarks in Section 7.8.

The chapter is based on a solo article which will be submitted for its publication; see [24] for more details.

7.1 Preliminaries

In this section we review some definitions and basic concepts. Throughout this chapter, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops and multiple edges, with vertex set X and edge set $\mathcal{R}$.

Let $x, y \in X$. The **distance** between x and y, denoted by $\partial(x,y)$, is the length of a shortest xy-path. The **eccentricity of** x, denoted by $\epsilon(x)$, is the maximum distance between x and any other vertex of Γ: $\epsilon(x) = \max\{\partial(x,z) \mid z \in X\}$. Let D denote the maximum eccentricity of any vertex in Γ. We call D the **diameter of** Γ. For an integer i we define $\Gamma_i(x)$ by

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}.$$

We will abbreviate $\Gamma(x) = \Gamma_1(x)$. Note that $\Gamma(x)$ is the set of neighbours of x. Observe that $\Gamma_i(x)$ is empty if and only if $i < 0$ or $i > \epsilon(x)$.

We now recall some definitions and basic results concerning a Terwilliger algebra of Γ. Let $\mathbb{C}$ denote the complex number field. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let V denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the **standard module**. We endow V with the Hermitian inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^{\top}\overline{v}$ for $u, v \in V$, where $\top$ denotes transpose and $^{-}$ denotes complex conjugation. For $y \in X$, let $\widehat{y}$ denote the element of V with a 1 in the y-coordinate and 0 in all other coordinates. We observe that $\{\widehat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. That is, the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with entries given as follows:

$$(A)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = 1, \\ 0 & \text{if} \quad \partial(x,y) \neq 1, \end{cases} \qquad (x, y \in X).$$

The **adjacency algebra of** Γ, also called the **Bose-Mesner algebra of** Γ, is the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ.

We now recall the dual idempotents of Γ. To do this fix a vertex $x \in X$ and let $d = \epsilon(x)$. We view x as a *base vertex*. For $0 \le i \le d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in

$\mathrm{Mat}_X(\mathbb{C})$ with (y,y)-entry as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the **i-th dual idempotent of Γ with respect to x** [89, p. 378]. We also observe (ei) $\sum_{i=0}^d E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le d)$; (eiii) $E_i^{*\top} = E_i^*$ $(0 \le i \le d)$; (eiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i,j \le d)$ where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. By these facts, matrices $E_0^*, E_1^*, \ldots, E_d^*$ form a basis for the commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. Note that for $0 \le i \le d$ we have that

$$E_i^* V = \mathrm{Span}\{\hat{y} \mid y \in \Gamma_i(x)\}, \tag{7.1}$$

and that

$$V = E_0^* V + E_1^* V + \cdots + E_d^* V \qquad (\textit{orthogonal direct sum}).$$

We call $E_i^* V$ the **i-th subconstituent of Γ with respect to x**. For convenience we define E_{-1}^* and E_{d+1}^* to be the zero matrix of $\mathrm{Mat}_X(\mathbb{C})$.

We next recall the definition of a Terwilliger algebra of Γ which was first studied in [89]. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the **Terwilliger algebra of Γ with respect to x**. Recall that M is generated by A. So, T is generated by A and the dual idempotents. We observe that T has finite dimension. In addition, since by construction T is generated by real-symmetric matrices, it follows that T is closed under the conjugate-transpose map. For a vector subspace $W \subseteq V$, we denote by TW the subspace $\{Bw \mid B \in T, w \in W\}$.

We now recall the lowering, the flat and the raising matrix of T.

Definition 7.1.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x \in X$. Let $d = \epsilon(x)$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Define $L = L(x)$, $F = F(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by*

$$L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*, \qquad F = \sum_{i=0}^{d} E_i^* A E_i^*, \qquad R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*.$$

*We refer to L, F and R as the **lowering**, the **flat** and the **raising matrix with respect to** x, respectively. Note that $L, F, R \in T$. Moreover, $F = F^\top$, $R = L^\top$ and $A = L + F + R$.*

Observe that for $y,z \in X$ we have that the (z,y)-entry of L equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) - 1$, and 0 otherwise. The (z,y)-entry of F is equal to 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y)$, and 0 otherwise. Similarly, the (z,y)-entry of R equals 1 if $\partial(z,y) = 1$ and $\partial(x,z) = \partial(x,y) + 1$, and 0 otherwise. Consequently, for $v \in E_i^*V$ $(0 \le i \le d)$ we have that

$$Lv \in E_{i-1}^*V, \qquad Fv \in E_i^*V, \qquad Rv \in E_{i+1}^*V. \tag{7.2}$$

By a T**-module** we mean a subspace W of V, such that $TW \subseteq W$. Let W denote a T-module. Then W is said to be **irreducible** whenever W is nonzero and W contains no T-modules other than 0 and W. Since the algebra T is closed under the conjugate-transposed map, it turns out that any T-module is an orthogonal direct sum of irreducible T-modules.

Let W be an irreducible T-module. We observe that W is an orthogonal direct sum of the nonvanishing subspaces E_i^*W for $0 \le i \le d$. By the **endpoint** of W we mean $r := r(W) = \min\{i \mid 0 \le i \le d,\ E_i^*W \ne 0\}$. Define the **diameter** of W by $d' := d'(W) = |\{i \mid 0 \le i \le d,\ E_i^*W \ne 0\}| - 1$. By Proposition 3.1.5, we have that $E_i^*W \ne 0$ if and only if $r \le i \le r + d'$ $(0 \le i \le d)$. We also say that W is **thin** whenever the dimension of E_i^*W is at most 1 for $0 \le i \le d$.

Let W and W' denote two irreducible T-modules. By a T**-isomorphism** from W to W' we mean a vector space isomorphism $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The T-modules W and W' are said to be T**-isomorphic** (or simply **isomorphic**) whenever there exists a T-isomorphism $\sigma : W \to W'$. We note that isomorphic irreducible T-modules have the same endpoint. It turns out that two non-isomorphic irreducible T-modules are orthogonal.

Observe that the subspace $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$ is a T-module. Suppose that W is an irreducible T-module with endpoint 0. Then, $\widehat{x} \in W$, which implies that $T\widehat{x} \subseteq W$. Since W is irreducible, we therefore have $T\widehat{x} = W$. Hence, $T\widehat{x}$ is the unique irreducible T-module with endpoint 0. We refer to $T\widehat{x}$ as the **trivial** T**-module**.

Assume now the trivial T-module is thin. In this case, by Lemma 3.6.1, vectors $R^i\widehat{x}$ $(0 \le i \le d)$ form a basis of the trivial T-module. In the rest of this chapter we will study irreducible T-modules of endpoint 1. Therefore, we will first characterize those vertices x of Γ, for which the corresponding Terwilliger algebra $T = T(x)$ has no irreducible T-modules with endpoint 1.

Proposition 7.1.2. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple, and connected graph. Pick a vertex $x \in X$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. Then, there are no irreducible T-modules with endpoint 1 if and only if $\dim(E_1^* T\widehat{x}) = |\Gamma(x)|$. In particular, if the trivial module is thin, there are no irreducible T-modules with endpoint 1 if and only if $|\Gamma(x)| = 1$.*

Proof. Let V denote the standard module, and let $T\widehat{x}$ denote the trivial T-module. We observe $T\widehat{x} \subseteq V$ and so, $\dim(E_1^* T\widehat{x}) \leq |\Gamma(x)|$.

Assume first that there are no irreducible T-modules with endpoint 1. Since V is orthogonal direct sum of irreducible T-modules and none of these T-modules has endpoint 1 we have $E_1^* V = E_1^* T\widehat{x}$ which implies that $\dim(E_1^* T\widehat{x}) = \dim(E_1^* V) = |\Gamma(x)|$.

Next, we proceed by contraposition. Suppose there exists an irreducible T-module W with endpoint 1. Let V_1 the sum of all irreducible T-modules with endpoint 1. Note that $E_1^* W$ is nonzero and since $E_1^* W \subseteq E^* V_1$, we have that $\dim(E_1^* V_1) > 0$. We also have that $E_1^* V = E_1^* T\widehat{x} + E_1^* V_1$. This shows that

$$|\Gamma(x)| = \dim(E_1^* V) = \dim(E_1^* T\widehat{x}) + \dim(E_1^* V_1) > \dim(E_1^* T\widehat{x}).$$

To prove the second part of our assertion, recall that if $T\widehat{x}$ is thin, by Lemma 3.6.1, the subspace $E_1^* T\widehat{x}$ is spanned by the nonzero vector $R\widehat{x}$. This concludes the proof. ■

In view of Proposition 7.1.2, we will assume that $|\Gamma(x)| \geq 2$ from now on.

7.2 The main result

Throughout this section let $\Gamma = (X, \mathcal{R})$ denote a connected graph. Here we state our main result. To do this we need the following definitions.

We first define a certain partition of X that we will find useful later for the proof of our main result.

Definition 7.2.1. *Let $\Gamma = (X, \mathcal{R})$ denote a graph with diameter D. Pick $x, y \in X$, such that $y \in \Gamma(x)$. For integers i, j we define sets $D_j^i := D_j^i(x, y)$ as follows:*

$$D_j^i = \Gamma_i(x) \cap \Gamma_j(y).$$

Observe that $D^i_j = \emptyset$ if $i < 0$ or $j < 0$. Similarly, $D^i_j = \emptyset$ if $i > \epsilon(x)$ or $j > \epsilon(y)$. Furthermore, by the triangle inequality we have that $D^i_j = \emptyset$ if $|i-j| \geq 2$. Note also that if Γ is bipartite, the set D^i_i is empty for $0 \leq i \leq D$. The collection of all the subsets D^i_{i-1} $(1 \leq i \leq \epsilon(x))$, D^i_i $(1 \leq i \leq \min\{\epsilon(x), \epsilon(y)\})$ and D^{i-1}_i $(1 \leq i \leq \epsilon(y))$ is called the ***distance partition of Γ with respect to the edge*** *$\{x,y\}$.*

A graphical representation of a distance partition for the case when the eccentricity of a vertex $y \in \Gamma(x)$ is equal to $\epsilon(x)$ is presented below in Figure 7.1. A line between D^i_j and $D^{i'}_{j'}$ indicates the possibility of existence of edges between these two sets. Such a graphical representation of a distance partition is called the **intersection diagram of Γ with respect to the edge** $\{x,y\}$.

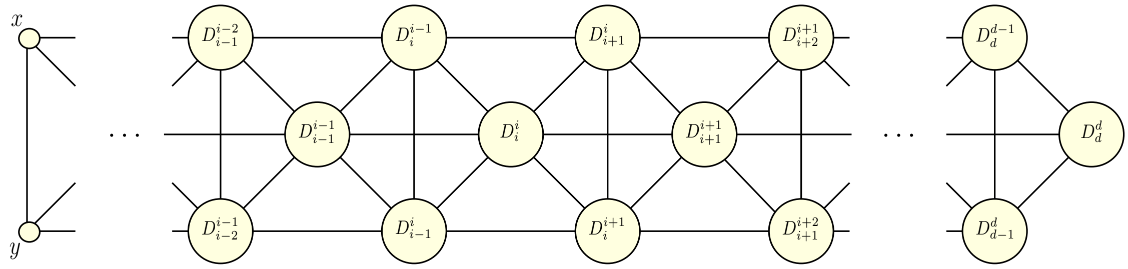

Figure 7.1: The intersection diagram of a connected graph Γ where $\epsilon(y) = \epsilon(x) = d$.

Next, we consider walks of a certain shape with respect to a given vertex in Γ.

Definition 7.2.2. *Let $\Gamma = (X, \mathcal{R})$ denote a connected graph. Pick $x, y, z \in X$ and let $P = [y = x_0, x_1, \ldots, x_j = z]$ denote a yz-walk. The* ***shape of P with respect to x*** *is a sequence of symbols $t_1 t_2 \ldots t_j$, where $t_i \in \{f, \ell, r\}$, and such that $t_i = r$ if $\partial(x, x_i) = \partial(x, x_{i-1}) + 1$, $t_i = f$ if $\partial(x, x_i) = \partial(x, x_{i-1})$ and $t_i = \ell$ if $\partial(x, x_i) = \partial(x, x_{i-1}) - 1$ $(1 \leq i \leq j)$. We use exponential notation for shapes containing several consecutive identical symbols. For instance, instead of $rrrrfff\ell\ell r$ we simply write $r^4 f^3 \ell^2 r$. Analogously, $r^0 f = f$ and $r^0 \ell = \ell r^0 = \ell$ is also conventional. For a non-negative integer m, let $\ell r^m(y,z)$, $r^m \ell(y,z)$, $r^m f(y,z)$ and $r^m(y,z)$ respectively denote the number of yz-walks of the shape ℓr^m, $r^m \ell$, $r^m f$ and r^m with respect to x where $r^0(y,z) = 1$ if $y = z$ and $r^0(y,z) = 0$ otherwise. We abbreviate $r^m \ell(z) = r^m \ell(x,z)$, $r^m f(z) = r^m f(x,z)$ and $r^m(z) = r^m(x,z)$.*

The following observation is straightforward to prove (using elementary matrix multiplication and (7.2)).

Lemma 7.2.3. *Let $\Gamma = (X, \mathcal{R})$ denote a connected graph. Pick $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $L = L(x)$, $F = F(x)$ and $R = R(x)$*

denote the lowering, the flat and the raising matrix of T, respectively. Pick $y,z \in X$ and let m be a non-negative integer. Then the following (i)–(iv) hold:

(i) The (z,y)-entry of R^m is equal to the number $r^m(y,z)$ with respect to x.

(ii) The (z,y)-entry of LR^m is equal to the number $r^m\ell(y,z)$ with respect to x.

(iii) The (z,y)-entry of R^mL is equal to the number $\ell r^m(y,z)$ with respect to x.

(iv) The (z,y)-entry of FR^m is equal to the number $r^m f(y,z)$ with respect to x.

For the rest of the paper we adopt the following notation.

Notation 7.2.4. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple, connected graph with vertex set X, edge set $\mathcal{R}$ and diameter D. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix a vertex $x \in X$ with $|\Gamma(x)| \geq 2$. Let d denote the eccentricity of x. Let $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \leq i \leq d)$ denote the dual idempotents of Γ with respect to x. Let V denote the standard module of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $L = L(x)$, $F = F(x)$ and $R = R(x)$ denote the lowering, the flat and the raising matrix of T, respectively. Assume that the unique irreducible T-module with endpoint 0 is thin. We denote this T-module by $T\widehat{x}$. For $y \in \Gamma(x)$ let the sets $D_j^i = D_j^i(x,y)$ be as defined in Definition 7.2.1. For $w,z \in X$ let the numbers $r^m\ell(w,z)$, $r^m f(w,z)$ and $r^m(w,z)$ be as defined in Definition 7.2.2.*

We are now ready to state our main result.

Theorem 7.2.5. *With reference to Notation 7.2.4, the following (i), (ii) are equivalent:*

(i) Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin.

(ii) For every integer i $(1 \leq i \leq d)$ there exist scalars $\kappa_i, \mu_i, \theta_i, \rho_i$, such that for every $y \in \Gamma(x)$ the following (a), (b) hold:

(a) For every $z \in D_{i+1}^i(x,y) \cup D_i^i(x,y)$ we have that

$$\begin{aligned} r^i\ell(y,z) &= \mu_i\, \ell r^i(y,z), \\ r^{i-1}f(y,z) &= \rho_i\, \ell r^i(y,z). \end{aligned}$$

(b) *For every* $z \in D^i_{i-1}(x,y)$ *we have that*

$$\begin{aligned} r^i\ell(y,z) &= \kappa_i\, r^{i-1}(y,z) + \mu_i\, \ell r^i(y,z), \\ r^{i-1}f(y,z) &= \theta_i\, r^{i-1}(y,z) + \rho_i\, \ell r^i(y,z). \end{aligned}$$

Moreover, $\rho_i = 0$ *whenever the set* $D^i_{i+1}(x,y)$ *is nonempty for some* $y \in \Gamma(x)$.

We will prove Theorem 7.2.5 in Sections 7.4 and 7.5. We next give a direct consequence of this result under the assumption that Γ is bipartite.

Corollary 7.2.6. *With reference to Notation 7.2.4, assume that* Γ *is bipartite. The following (i), (ii) are equivalent:*

(i) Γ *has, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin.*

(ii) *For every integer* i $(1 \le i \le d)$ *there exist scalars* κ_i, μ_i *such that for every* $y \in \Gamma(x)$ *the following (a), (b) hold:*

(a) *For every* $z \in D^i_{i+1}(x,y)$ *we have that* $r^i\ell(y,z) = \mu_i\, \ell r^i(y,z)$.

(b) *For every* $z \in D^i_{i-1}(x,y)$ *we have that* $r^i\ell(y,z) = \kappa_i\, r^{i-1}(y,z) + \mu_i\, \ell r^i(y,z)$.

Proof. Since Γ is bipartite, we observe the matrix $F = 0$ and the sets $D^i_i(x,y)$ are empty for every $y \in \Gamma(x)$ and for every integer i $(1 \le i \le d)$. Now, the result immediately follows from Theorem 7.2.5. ■

With reference to Notation 7.2.4, assume that Γ is distance-regular around x and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. In this case, it was proved in Chapter 3 (see also [28, 88]) that the unique irreducible T-module with endpoint 0 is thin. In addition, for an integer i $(1 \le i \le d)$ and vertices $y \in \Gamma(x), z \in \Gamma_i(x)$, we observe the number of yz-walks of the shape ℓr^i with respect to x is equal to the number of paths of length i from z to x. Since x is distance-regularized, there are precisely $c_i(x)c_{i-1}(x)\cdots c_1(x)$ such paths. Consequently, $\ell r^i(y,z) = c_i(x)c_{i-1}(x)\cdots c_1(x)$ and so, $\ell r^i(y,z)$ is independent of the choice of y and z. Therefore, Theorem 5.3.4 and Theorem 6.3.4 immediately follow from Theorem 7.2.5 and the above comments.

We finish this section with the following observations which will be needed later for the proof of Theorem 7.2.5.

Proposition 7.2.7. *With reference to Notation 7.2.4, the following holds for* $0 \le i \le d$*:*

$$\left(E_i^* R^i L E_1^*\right)_{zy} = \begin{cases} \ell r^i(y,z) & \text{if } y \in \Gamma(x) \text{ and } z \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $E_i^* R^i L E_1^*$ *is nonzero.*

Proof. It is straightforward to check that the (z,y)-entry of $E_i^* R^i L E_1^*$ is zero if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$. It is also straightforward to check that the result is true if $i = 0$. Suppose now that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$ with $i \ge 1$. Then $\left(E_i^* R^i L E_1^*\right)_{zy} = \left(R^i L\right)_{zy}$ and the result follows from Lemma 7.2.3. Note also that in this case we have that $\ell r^i(y,z) > 0$ and so, $E_i^* R^i L E_1^*$ is nonzero. ■

Proposition 7.2.8. *With reference to Notation 7.2.4, the following holds for* $1 \le i \le d$*:*

$$\left(E_i^* R^{i-1} E_1^*\right)_{zy} = \begin{cases} r^{i-1}(y,z) & \text{if } y \in \Gamma(x) \text{ and } z \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $E_i^* R^{i-1} E_1^*$ *is nonzero.*

Proof. It is easy to see that the (z,y)-entry of $E_i^* R^{i-1} E_1^*$ is zero if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$. It is also straightforward to check that the result is true if $i = 1$. Suppose now that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$ with $i > 1$. Then $\left(E_i^* R^{i-1} E_1^*\right)_{zy} = \left(R^{i-1}\right)_{zy}$ and the result follows from Lemma 7.2.3. Note also that in this case we have that $r^{i-1}(y,z) > 0$ for some $y \in \Gamma(x)$ and so, $E_i^* R^{i-1} E_1^*$ is nonzero. ■

7.3 Linear dependency

With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. In this section we show that certain matrices in T are linearly dependent.

Theorem 7.3.1. *With reference to Notation 7.2.4, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin with diameter* d'*. Pick matrices* $F_1, F_2, F_3 \in T$ *and an integer* i $(1 \le i \le d)$*. Then the following (i), (ii) hold:*

(i) *For every integer* i $(1 \le i \le d'+1)$ *the matrices* $E_i^* F_1 E_1^*$, $E_i^* F_2 E_1^*$ *and* $E_i^* F_3 E_1^*$ *are linearly dependent.*

(ii) *For every integer* i $(d'+1 < i \le d)$ *the matrices* $E_i^* F_1 E_1^*$ *and* $E_i^* F_2 E_1^*$ *are linearly dependent.*

Proof. Recall that $T\widehat{x}$ is thin and by Lemma 3.6.1, the subspace $E_1^* T\widehat{x}$ is spanned by the nonzero vector $R\widehat{x}$ and so, $\dim(E_1^* T\widehat{x}) = 1$.

Let W be a thin irreducible T-module with endpoint 1 and diameter d'. Firstly, we observe that $d'+1 \le d$ and so, (i) immediately follows from Theorem 5.4.3. We would like to point out that the same conclusions of Theorem 5.4.3 are true without assuming that Γ is bipartite and distance-regular around x. Namely, in the proof of Theorem 5.4.3, the hypothesis that Γ is bipartite was never applied and local distance-regularity around x was used to conclude that $\dim(E_1^* T\widehat{x}) = 1$, which is also true in our case.

We now proceed to prove the second assertion. To do that, pick an integer i $(d'+1 < i \le d)$. We claim that there exist scalars λ_1, λ_2, not both zero, such that $\lambda_1 E_i^* F_1 E_1^* v + \lambda_2 E_i^* F_2 E_1^* v = 0$ for every $v \in E_1^* T\widehat{x}$. To see this, pick nonzero vectors $v_0 \in E_1^* T\widehat{x}$ and $v_1 \in E_1^* W$. Let u_0 be an arbitrary nonzero vector of $E_i^* T\widehat{x}$. As the trivial module is thin, there exist scalars $r_{0,1}$, $r_{0,2}$ such that

$$E_i^* F_1 E_1^* v_0 = r_{0,1}\, u_0 \qquad \text{and} \qquad E_i^* F_2 E_1^* v_0 = r_{0,2}\, u_0. \tag{7.3}$$

It is clear that the linear equation $r_{0,1}\, x_1 + r_{0,2}\, x_2 = 0$ with unknowns x_1, x_2 has a nontrivial solution, and so there exist scalars λ_1, λ_2, not both zero, such that

$$r_{0,1}\, \lambda_1 + r_{0,2}\, \lambda_2 = 0. \tag{7.4}$$

Pick a vector $v \in E_1^* T\widehat{x}$. Since the trivial T-module is thin, there exists a scalar λ such that $v = \lambda v_0$. Therefore, by (7.3) and (7.4) we have that

$$\begin{aligned} \lambda_1 E_i^* F_1 E_1^* v + \lambda_2 E_i^* F_2 E_1^* v &= \lambda\,(\lambda_1 E_i^* F_1 E_1^* v_0 + \lambda_2 E_i^* F_2 E_1^* v_0) \\ &= \lambda\,(\lambda_1\, r_{0,1} u_0 + \lambda_2\, r_{0,2} u_0) \\ &= \lambda\,(r_{0,1}\, \lambda_1 + r_{0,2}\, \lambda_2)\, u_0 = 0. \end{aligned}$$

This proves our claim. Let V_1 denote the sum of all irreducible T-modules with endpoint 1 and let $\{W^t \mid t \in \mathcal{I}\}$ be the set of all irreducible T-modules with endpoint 1, where $\mathcal{I}$ is an index set. Pick a vector $v \in E_1^* V_1$. Observe that v can be written as a sum

$$v = \sum_{t \in \mathcal{I}} v_t, \tag{7.5}$$

where $v_t \in E_1^* W^t$ for every $t \in \mathcal{I}$. Pick now a T-module W^s, $s \in \mathcal{I}$. As any two irreducible T-modules with endpoint 1 are isomorphic, it follows that $d'(W^s) = d'(W) = d'$. So, we observe that in this case $E_i^* W^s$ is zero. In addition, for every $t \in \mathcal{I}$ there exists a T-isomorphism $\sigma_t : W^s \to W^t$. Let $w_t \in W^s$ be such that $v_t = \sigma_t(w_t)$. Then, we notice that for every $t \in \mathcal{I}$,

$$E_i^* F_j E_1^* v_t = E_i^* F_j E_1^* \sigma_t(w_t) = \sigma_t \left(E_i^* F_j E_1^* w_t \right) = 0.$$

Hence, by (7.5) we have that $E_i^* F_j E_1^* v = 0$ for every $v \in E_1^* V_1$.

To conclude the proof, pick now an arbitrary vector $w \in V$ and observe that $E_1^* w = w_0 + w_1$ for some $w_0 \in T\widehat{x}$ and $w_1 \in V_1$. It follows from the above comments that there exist scalars λ_1, λ_2, not both zero, such that

$$\lambda_1 E_i^* F_1 E_1^* w + \lambda_2 E_i^* F_2 E_1^* w \quad = \quad \lambda_1 E_i^* F_1 E_1^* (w_0 + w_1) + \lambda_2 E_i^* F_2 E_1^* (w_0 + w_1) = 0.$$

As w was arbitrary, the result follows. ■

Observe that the conclusion of Theorem 7.3.1 is equivalent to the fact that the dimension of $E_i^* T E_1^*$ $(1 \le i \le d' + 1)$ is at most 2 and that the dimension of $E_i^* T E_1^*$ $(d' + 1 < i \le d)$ is at most 1.

7.4 Algebraic condition implies combinatorial properties

With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. In this section we prove that in this case combinatorial conditions $(a), (b)$ described in part (ii) of Theorem 7.2.5 hold.

Lemma 7.4.1. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint* 1, *and that this module is thin. Then for* *every i $(1 \le i \le d)$ there exist scalars $\kappa_i, \mu_i, \theta_i, \rho_i$, such that*

$$E_i^* L R^i E_1^* = \kappa_i E_i^* R^{i-1} E_1^* + \mu_i E_i^* R^i L E_1^*, \tag{7.6}$$

$$E_i^* F R^{i-1} E_1^* = \theta_i E_i^* R^{i-1} E_1^* + \rho_i E_i^* R^i L E_1^*. \tag{7.7}$$

Proof. Pick i $(1 \le i \le d)$ and observe that by Definition 7.1.1, the matrices LR^i, R^{i-1}, FR^{i-1} and R^iL are elements of algebra T. Consequently, by Theorem 7.3.1, there exist scalars $\alpha_j^{(i)}$ $(1 \le j \le 3)$, not all zero, and $\beta_j^{(i)}$ $(1 \le j \le 3)$, not all zero, such that

$$\alpha_1^{(i)} E_i^* L R^i E_1^* + \alpha_2^{(i)} E_i^* R^{i-1} E_1^* + \alpha_3^{(i)} E_i^* R^i L E_1^* = 0, \tag{7.8}$$

$$\beta_1^{(i)} E_i^* F R^{i-1} E_1^* + \beta_2^{(i)} E_i^* R^{i-1} E_1^* + \beta_3^{(i)} E_i^* R^i L E_1^* = 0. \tag{7.9}$$

Assume for the moment that $\alpha_1^{(i)}\beta_1^{(i)} \ne 0$. Then (7.6) and (7.7) hold with $\kappa_i = -\alpha_2^{(i)}/\alpha_1^{(i)}$, $\mu_i = -\alpha_3^{(i)}/\alpha_1^{(i)}$, $\theta_i = -\beta_2^{(i)}/\beta_1^{(i)}$, and $\rho_i = -\beta_3^{(i)}/\beta_1^{(i)}$.

Now, assume that $\alpha_1^{(i)}\beta_1^{(i)} = 0$. Let W denote an irreducible T-module with endpoint 1. Let k denote the least integer such that $\alpha_1^{(k)}\beta_1^{(k)} = 0$. We observe that $k \le i$. Assume for a moment that $k = 1$. Without loss of generality assume that $\alpha_1^{(1)} = 0$. Pick $y, z \in \Gamma(x)$, $y \ne z$. As the (z, y)-entries of E_1^* and $E_1^* R L E_1^*$ are 0 and 1 respectively, (7.8) implies that $\alpha_3^{(1)} = 0$. As E_1^* is nonzero, we get that $\alpha_2^{(1)} = 0$ as well, a contradiction. Therefore, $k \ge 2$. Pick a nonzero vector $w \in E_1^* W$ and let W' denote the vector subspace of V spanned by the vectors $R^i w$ $(0 \le i \le d)$. Note that W' is nonzero and $W' \subseteq W$. Observe also that by (7.2) and by (eiv) from Section 7.1, the subspace W' is invariant under the action of the dual idempotents. Since $\alpha_1^{(k)}\beta_1^{(k)} = 0$ and by Proposition 7.2.7 the matrix $E_k^* R^k L E_1^*$ is nonzero, it follows from (7.8) and (7.9) that there exists $\gamma \in \mathbb{C}$ such that $E_i^* R^{k-1} E_1^* = \gamma E_k^* R^k L E_1^*$. Now, from (7.2) we notice that $Lw = 0$ and so, $R^{k-1} w = 0$. This implies $FR^j w = LR^j w = R^j w = 0$ for $k - 1 \le j \le d$. Therefore, by construction and by (7.2), it is also clear that W' is closed under the action of R. Moreover, for every $1 \le j \le k-1$ the scalar $\alpha_1^{(j)}\beta_1^{(j)}$ is nonzero. Therefore, from (7.8) and (7.9), we have that (7.6) and (7.7) hold for $1 \le j \le k-1$ with $\kappa_j = -\alpha_2^{(j)}/\alpha_1^{(j)}$, $\mu_j = -\alpha_3^{(j)}/\alpha_1^{(j)}$, $\theta_j = -\beta_2^{(j)}/\beta_1^{(j)}$, and $\rho_j = -\beta_3^{(j)}/\beta_1^{(j)}$. So, $LR^j w = \kappa_j R^{j-1} w$ and $FR^{j-1} w = \theta_j R^{j-1} w$ for $1 \le j \le k-1$. This implies that W' is invariant under the action of L and F. Since $A = L + F + R$, it turns out that W' is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Therefore, W' is a T-module and $W' = W$ as W is irreducible. Notice that by construction and (7.2), the subspace $E_i^* W$ is generated by $R^{i-1} w$. This shows $E_i^* W = 0$ since $k \le i$. We thus have that $d' + 1 < i \le d$ where d' denotes the diameter of W. Hence, by Theorem 7.3.1(ii), any two matrices in $E_i^* T E_1^*$ are linearly dependent. Consequently, there exist scalars α, β (not both zero) and α', β' (not both zero), such that

$$\alpha E_i^* L R^i E_1^* + \beta E_i^* R^{i-1} E_1^* = 0, \tag{7.10}$$

$$\alpha' E_i^* F R^{i-1} E_1^* + \beta' E_i^* R^{i-1} E_1^* = 0. \tag{7.11}$$

If α (α', respectively) is zero, then β (β', respectively) is also zero by Proposition 7.2.8, a contradiction. This shows that $E_i^* L R^i E_1^* = -\frac{\beta}{\alpha} E_i^* R^{i-1} E_1^*$ and $E_i^* F R^{i-1} E_1^* = -\frac{\beta'}{\alpha'} E_i^* R^{i-1} E_1^*$. Similarly we show that $E_i^* R^i L E_1^* = \lambda E_i^* R^{i-1} E_1^*$ for some nonzero scalar $\lambda \in \mathbb{C}$. It is now clear that (7.6) and (7.7) hold for any κ_i, μ_i, θ_i, ρ_i satisfying $\kappa_i + \lambda \mu_i = -\beta/\alpha$ and $\theta_i + \lambda \rho_i = -\beta'/\alpha'$. This finishes the proof. ■

We are now ready to prove the main result of this section.

Theorem 7.4.2. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. For every integer i $(1 \le i \le d)$ there exist scalars κ_i, μ_i, θ_i, ρ_i, such that for every $y \in \Gamma(x)$ the following (a), (b) hold:*

(a) For every $z \in D^i_{i+1}(x,y) \cup D^i_i(x,y)$ we have that

$$\begin{aligned} r^i \ell(y,z) &= \mu_i \, \ell r^i(y,z), \\ r^{i-1} f(y,z) &= \rho_i \, \ell r^i(y,z). \end{aligned}$$

(b) For every $z \in D^i_{i-1}(x,y)$ we have that

$$\begin{aligned} r^i \ell(y,z) &= \kappa_i \, r^{i-1}(y,z) + \mu_i \, \ell r^i(y,z), \\ r^{i-1} f(y,z) &= \theta_i \, r^{i-1}(y,z) + \rho_i \, \ell r^i(y,z). \end{aligned}$$

Moreover, $\rho_i = 0$ if the set $D^i_{i+1}(x,y)$ is nonempty for some $y \in \Gamma(x)$.

Proof. Pick an integer i $(1 \le i \le d)$ and recall that by Lemma 7.4.1 equations (7.6) and (7.7) hold. Pick $y \in \Gamma(x)$.

(a) Pick $z \in D^i_{i+1}(x,y) \cup D^i_i(x,y)$ and observe that by Lemma 7.2.3 the (z,y)-entry of the left-hand side of (7.6) ((7.7), respectively) equals $r^i \ell(y,z)$ ($r^{i-1} f(y,z)$, respectively). On the other hand, again by Lemma 7.2.3, the (z,y)-entry of $E_i^* R^{i-1} E_1^*$ ($E_i^* R^i L E_1^*$, respectively) equals 0 ($\ell r^i(y,z)$, respectively). Therefore, the (z,y)-entry of the right-hand side of (7.6) ((7.7), respectively) equals $\mu_i\, \ell r^i(y,z)$ ($\rho_i\, \ell r^i(y,z)$, respectively).

(b) Pick now $z \in D^i_{i-1}(x,y)$ and observe that by Lemma 7.2.3 the (z,y)-entry of the left-hand side of (7.6) ((7.7), respectively) equals $r^i\ell(y,z)$ ($r^{i-1}f(y,z)$, respectively). On the other hand, again by Lemma 7.2.3, the (z,y)-entry of $E^*_i R^{i-1} E^*_1$ ($E^*_i R^i L E^*_1$, respectively) equals $r^{i-1}(y,z)$ ($\ell r^i(y,z)$, respectively). Therefore, the (z,y)-entry of the right-hand side of (7.6) ((7.7), respectively) equals $\kappa_i\, r^{i-1}(y,z) + \mu_i\, \ell r^i(y,z)$ ($\theta_i\, r^{i-1}(y,z) + \rho_i\, \ell r^i(y,z)$, respectively).

Moreover, for $z \in D^i_{i+1}(x,y)$ we observe that there is no yz-walk of the shape $r^{i-1}f$ and so $\rho_i = 0$ if the set $D^i_{i+1}(x,y)$ is nonempty for some $y \in \Gamma(x)$ as $\ell r^i(y,z) > 0$. The result follows. ■

7.5 Combinatorial properties imply algebraic condition

With reference to Notation 7.2.4, assume that Γ satisfies part (ii) of Theorem 7.2.5. In this section we prove that in this case Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. We also display a basis of this module and the matrix representing the action of the adjacency matrix on this basis.

Proposition 7.5.1. *With reference to Notation 7.2.4, assume that Γ satisfies part (ii) of Theorem 7.2.5. For every integer i $(1 \le i \le d)$, the following equalities hold:*

$$E^*_i L R^i E^*_1 = \kappa_i E^*_i R^{i-1} E^*_1 + \mu_i E^*_i R^i L E^*_1, \tag{7.12}$$

$$E^*_i F R^{i-1} E^*_1 = \theta_i E^*_i R^{i-1} E^*_1 + \rho_i E^*_i R^i L E^*_1. \tag{7.13}$$

Proof. Pick an integer i $(1 \le i \le d)$ and vertices $y, z \in X$. We will show that the (z,y)-entries of both sides of (7.12) and (7.13) agree. Observe first that if either $y \notin \Gamma(x)$ or $z \notin \Gamma_i(x)$, then the (z,y)-entry of both sides of (7.12) and (7.13) equals 0. Therefore, assume that $y \in \Gamma(x)$ and $z \in \Gamma_i(x)$. Abbreviate $D^k_j(x,y) = D^k_j$ for $0 \le k, j \le D$ and recall that $\Gamma_i(x) = D^i_{i-1} \cup D^i_i \cup D^i_{i+1}$.

Assume first that $z \in D^i_{i+1} \cup D^i_i$, and note that the (z,y)-entry of $E^*_i L R^i E^*_1$ is equal to the number $r^i\ell(y,z)$, while the (z,y)-entries of $E^*_i R^{i-1} E^*_1$ and $E^*_i R^i L E^*_1$ are 0 and $\ell r^i(y,z)$ respectively. In addition, the (z,y)-entry of $E^*_i F R^{i-1} E^*_1$ is equal to $r^{i-1}f(y,z)$.

As $r^i\ell(y,z) = \mu_i\,\ell r^i(y,z)$ and $r^{i-1}f(y,z) = \rho_i\,\ell r^i(y,z)$ by the assumption, the (z,y)-entries of both sides of (7.12) and (7.13) agree.

Assume next that $z \in D^i_{i-1}$ and note that the (z,y)-entry of $E^*_i LR^i E^*_1$, $E^*_i FR^{i-1}E^*_1$, $E^*_i R^{i-1}E^*_1$ and $E^*_i R^i LE^*_1$ are equal to the numbers $r^i\ell(y,z)$, $r^{i-1}f(y,z)$, $r^{i-1}(y,z)$ and $\ell r^i(y,z)$ respectively. By the assumption we have that $r^i\ell(y,z) = \kappa_i\, r^{i-1}(y,z) + \mu_i\,\ell r^i(y,z)$ and $r^i f(y,z) = \theta_i\, r^{i-1}(y,z) + \rho_i\,\ell r^i(y,z)$. So, the (z,y)-entries of both sides of (7.12) and (7.13) agree. This finishes the proof. ■

Lemma 7.5.2. *With reference to Notation 7.2.4, assume that Γ satisfies part (ii) of Theorem 7.2.5. Pick $w \in E^*_1V$, $w \neq 0$, which is orthogonal to $s_1 = R\hat{x}$. Then the following $(i),(ii)$ hold:*

(i) $Lw = 0$ *and* $LR^iw = \kappa_i R^{i-1}w$ $(1 \le i \le d)$.

(ii) $FR^{i-1}w = \theta_i R^{i-1}w$ $(1 \le i \le d)$ *and* $FR^d w = 0$.

Proof. Let J denote the all 1's matrix in $\mathrm{Mat}_X(\mathbb{C})$. As $w \in E^*_1V$ we have that $E^*_1w = w$ and so,

$$\langle \boldsymbol{j}, w\rangle = \langle \boldsymbol{j}, E^*_1 w\rangle = \langle E^*_1\boldsymbol{j}, w\rangle = \langle s_1, w\rangle = 0,$$

where $\boldsymbol{j}$ denotes the all 1's vector in V. This shows $Jw = 0$. By elementary matrix multiplication it is easy to see $E^*_0AE^*_1 = E^*_0JE^*_1$. Therefore, by Definition 7.1.1 and the above comments we have that $Lw = E^*_0AE^*_1w = E^*_0JE^*_1w = E^*_0Jw = 0$. Moreover, we also have that $E^*_iR^iLE^*_1w = R^iLw = 0$ for $1 \le i \le d$. In addition, by (7.2) and Proposition 7.5.1, for $1 \le i \le d$ it holds that

$$\begin{aligned} LR^iw &= E^*_iLR^iE^*_1w = \kappa_i E^*_iR^{i-1}E^*_1w = \kappa_i R^{i-1}w, \\ FR^{i-1}w &= E^*_iFR^{i-1}E^*_1w = \theta_i E^*_iR^{i-1}E^*_1w = \theta_i R^{i-1}w. \end{aligned}$$

Observe also that $R^dE^*_1$ is the zero matrix and so $FR^dw = 0$. The result follows. ■

Lemma 7.5.3. *With reference to Notation 7.2.4, assume that Γ satisfies part (ii) of Theorem 7.2.5. Pick $w \in E^*_1V$, $w \neq 0$, which is orthogonal to $s_1 = R\hat{x}$. Then the following (i)–(iii) hold:*

(i) $\left\|R^iw\right\|^2 = \kappa_i \left\|R^{i-1}w\right\|^2$ $(1 \le i \le d)$.

(ii) $\left\langle R^iw, R^jw\right\rangle = \delta_{ij}\prod_{l=1}^{i}\kappa_l\,\|w\|^2$ $(0 \le i,j \le d)$.

(iii) *There exists* i $(1 \le i \le d)$ *such that* $\kappa_i = 0$.

Proof. (i) Pick $1 \le i \le d$. Then by Lemma 7.5.2(i) we have that

$$\left\|R^i w\right\|^2 = \left\langle R^i w,\ R^i w\right\rangle = \left\langle LR^i w,\ R^{i-1} w\right\rangle = \kappa_i \left\|R^{i-1} w\right\|^2.$$

(ii) If $i \neq j$, then the result follows from (eii), (eiii) and (eiv) below the definition of the dual idempotents in Section 7.1 and from (7.2). If $i = j$ then the result follows from (i) above by a straightforward induction argument.

(iii) Immediate from (ii) above since by (7.2) we have that $R^d w = 0$ and w is a nonzero vector. ■

Theorem 7.5.4. *With reference to Notation 7.2.4, assume that* Γ *satisfies part (ii) of Theorem 7.2.5. Pick* $w \in E_1^* V$, $w \neq 0$, *which is orthogonal to* $s_1 = R\hat{x}$. *Let* W *denote the vector subspace of* V *spanned by the vectors* $R^i w$ $(0 \le i \le d)$. *Let* s $(1 \le s \le d)$ *be the least integer such that* $\kappa_s = 0$. *Then* W *is a thin irreducible* T*-module with endpoint* 1 *and the vectors* $\{R^{i-1} w \mid 1 \le i \le s\}$ *form an orthogonal basis of* W. *In particular, the dimension of* W *is* s.

Proof. Let W denote the vector subspace of V spanned by the vectors $\{R^i w \mid 0 \le i \le d\}$. Observe that by (7.2) and by (eiv) from Section 7.1, the subspace W is invariant under the action of the dual idempotents. By construction and since $R^d w = 0$ by (7.2), it is also clear that W is closed under the action of R. Moreover, it follows from Lemma 7.5.2 that W is invariant under the action of L and F. Since $A = L + F + R$, it turns out that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Therefore, W is a T-module. It is also clear that W is thin, since by construction, (7.2) and Lemma 7.5.2, the subspace $E_i^* W$ is generated by $R^{i-1} w$.

Now, let us show that W is irreducible. Note that $w \in W$ and so W is non-zero. Recall that W is an orthogonal direct sum of irreducible T-modules. Since $E_0^* W$ is the zero subspace and $E_1^* w = w \neq 0$, there exists an irreducible T-module W', such that the endpoint of W' is 1 and $W' \subseteq W$. Consequently, $E_1^* W' \subseteq E_1^* W$. However, the dimension of $E_1^* W$ is 1, and so $E_1^* W' = E_1^* W$. But now we have

$$W = TE_1^* W = TE_1^* W' \subseteq W',$$

implying that $W = W'$. Hence, W is irreducible and its endpoint equals 1.

Finally, notice that $R^s w = 0$ by Lemma 7.5.3(i). Furthermore, it holds that vectors $\{R^{i-1}w \mid 1 \leq i \leq s\}$ are nonzero and pairwise orthogonal by Lemma 7.5.3(ii) and the definition of number s. The result follows. ∎

Theorem 7.5.5. *With reference to Notation 7.2.4, assume that* Γ *satisfies part (ii) of Theorem 7.2.5. Let* W *denote an irreducible* T*-module with endpoint* 1*. Let* s $(1 \leq s \leq d)$ *be the least integer such that* $\kappa_s = 0$*. Pick* $w \in E_1^* W$*,* $w \neq 0$*. Then, it follows that the vectors* $\{R^{i-1}w \mid 1 \leq i \leq s\}$ *form an orthogonal basis of* W*. In particular,* W *is thin with dimension* s*.*

Proof. Let W' denote the vector subspace of V spanned by the vectors $\{R^{i-1}w \mid 1 \leq i \leq d\}$. Recall that W and the unique irreducible T-module with endpoint 0 are not isomorphic, and so w is orthogonal to s_1. By Theorem 7.5.4, W' is a T-module. Note that W' is nonzero and contained in W. As W is irreducible, we have that $W = W'$. The result now follows from Theorem 7.5.4. ∎

Theorem 7.5.6. *With reference to Notation 7.2.4, assume that* Γ *satisfies part* (ii) *of Theorem 7.2.5. Then there is, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin.*

Proof. Let W and W' be irreducible T-modules with endpoint 1, and pick any nonzero vectors $w \in E_1^* W$ and $w' \in E_1^* W'$. Let s $(1 \leq s \leq d)$ be the least integer such that $\kappa_s = 0$. By Theorem 7.5.5, the vectors

$$\left\{R^{i-1}w \mid 1 \leq i \leq s\right\} \text{ and } \left\{R^{i-1}w' \mid 1 \leq i \leq s\right\}$$

are orthogonal bases of W and W', respectively. Hence, the linear map $\sigma : W \to W'$, defined by $\sigma\left(R^{i-1}w\right) = R^{i-1}w'$ is a vector space isomorphism. It is clear that σ commutes with R. By Lemma 7.5.2 it follows that σ also commutes with L and F. Since $A = L + F + R$, it turns out that σ commutes with A as well. Furthermore, σ is a T-module isomorphism since by (eiv) from Section 7.1, it commutes also with E_i^* $(0 \leq i \leq d)$. Thus W and W' are T-isomorphic. ∎

Theorem 7.5.7. *With reference to Notation 7.2.4, assume that* Γ *satisfies part (ii) of Theorem 7.2.5. Let* W *denote an irreducible* T*-module with endpoint* 1*. Pick* $w \in E_1^* W$*,* $w \neq 0$*, and recall that*

$$\mathcal{B} = \left\{R^{i-1}w \mid 1 \leq i \leq s\right\}$$

is a basis of W, where s is the least integer such that $\kappa_s = 0$ $(1 \le s \le d)$. Then the matrix representing the action of A on W with respect to the (ordered) basis $\mathcal{B}$ is given by

$$\begin{pmatrix} \theta_1 & \kappa_1 & & & & \\ 1 & \theta_2 & \kappa_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \kappa_{s-2} & \\ & & & 1 & \theta_{s-2} & \kappa_{s-1} \\ & & & & 1 & \theta_{s-1} \end{pmatrix}.$$

Proof. Recall that $A = L + F + R$. The result now follows from Lemma 7.5.2. ■

7.6 The distance partition

Throughout this section let $\Gamma = (X, \mathcal{R})$ denote a connected graph. Let $x \in X$ and let $T = T(x)$. Suppose that the unique irreducible T-module with endpoint 0 is thin. Assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin. In this section we have some comments about the combinatorial structure of the intersection diagrams of Γ with respect to the edge $\{x, y\}$, for every $y \in \Gamma(x)$. In particular, we will discuss which of the sets $D^i_j(x, y)$ are (non)empty.

Lemma 7.6.1. *With reference to Notation 7.2.4, pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x, y)$. Then, the set $D^i_{i-1}(x, y)$ is nonempty for every i $(1 \le i \le d)$ and for all $y \in \Gamma(x)$.*

Proof. Suppose there exist i $(1 \le i \le d)$ and $y \in \Gamma(x)$ such that the set $D^i_{i-1}(x, y)$ is empty. Since $D^1_0 = \{y\}$ we observe that $i \ge 2$. Moreover, we notice that $D^i_{i+1} \ne \emptyset$ or $D^i_i \ne \emptyset$, as otherwise, the set $\Gamma_i(x) = D^i_{i+1} \cup D^i_i \cup D^i_{i-1}$ is empty, contradicting that the eccentricity of x equals d. Let k be the greatest integer such that $D^k_{k-1} \ne \emptyset$. Note that $1 \le k \le i-1$. Since the set $D^i_{i+1} \cup D^i_i \ne \emptyset$ then it is easy to see that there exists a vertex $z \in D^k_{k+1} \cup D^k_k$ and so, that the numbers $r^{k+1}\ell(z) > 0$ and $r^k(z) > 0$. Moreover, for $w \in D^k_{k-1}$ we observe $r^{k+1}\ell(w) = 0$ and $r^k(w) > 0$. This contradicts with Theorem 3.5.3(*iii*) and so, with the assumption that the trivial module is thin. The result follows. ■

Lemma 7.6.2. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Pick an integer i $(1 \le i \le d)$ and assume for some $y \in \Gamma(x)$, the set $D^i_{i+1}(x, y) \ne \emptyset$. Then, the set $D^j_j(x, y)$ is empty for every j $(1 \le j \le i)$ and for all $y \in \Gamma(x)$.*

Proof. Suppose there exists j $(1 \le j \le i)$ and $w \in \Gamma(x)$ such that $D^j_j(x,w)$ is nonempty. Without loss of generality, we may pick j as the least integer such that the set $D^{j-1}_{j-1}(x,w) = \emptyset$ but the set $D^j_j(x,w)$ is nonempty. By Definition 7.1.1, the matrices R^{j-1}, FR^{j-1} and R^jL are elements of algebra T. Therefore, by Theorem 7.3.1, there exist scalars $\lambda_k = \lambda_k^{(j)}$ $(1 \le k \le 3)$, not all zero, such that

$$\lambda_1 E^*_j R^{j-1} E^*_1 + \lambda_2 E^*_j F R^{j-1} E^*_1 + \lambda_3 E^*_j R^j L E^*_1 = 0. \tag{7.14}$$

By Lemma 6.2.2(v), notice that the set $D^j_{j+1}(x,y)$ is nonempty. Pick $z \in D^j_{j+1}(x,y)$ and note that it follows from Lemma 7.2.3(i),(iv) that the (z,y)-entry of $E^*_j R^{j-1} E^*_1$ and $E^*_j F R^{j-1} E^*_1$ are both 0, respectively. This implies that $\lambda_3 = 0$ since the (z,y)-entry of $E^*_j R^j L E^*_1$ equals $\ell r^j(y,z) > 0$ by Lemma 7.2.3(iii) and Proposition 7.2.7. Pick now $z \in D^j_j(x,w)$. We observe from Lemma 7.2.3(i) that the (z,w)-entry of $E^*_j R^{j-1} E^*_1$ is 0. In addition, as the set $D^{j-1}_{j-1}(x,w)$ is empty, it follows from Lemma 6.2.2(v) and Lemma 6.2.5 that there exists a wz-walk of the shape $r^{j-1}f$ with respect to x. So, by Lemma 7.2.3(iv), the (z,w)-entry of $E^*_j F R^{j-1} E^*_1$ is nonzero. This implies that $\lambda_2 = 0$. So, from equation (7.14) we have that $\lambda_1 E^*_j R^{j-1} E^*_1$ is the zero matrix. By Proposition 7.2.8, we observe that $E^*_j R^{j-1} E^*_1$ is nonzero. This implies $\lambda_1 = 0$, contradicting the fact that the scalars λ_k $(1 \le k \le 3)$ are not all zero. The claim follows. ■

The above lemma together with the fact that the set $D^0_1(x,y)$ is nonempty for every $y \in \Gamma(x)$ motivate the next result.

Proposition 7.6.3. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint* 1*, and that this module is thin. Pick $y \in \Gamma(x)$ and let $D^i_j = D^i_j(x,y)$. Then, there exists an integer $t := t(y)$ $(0 \le t \le d)$ such that the following* (i), (ii) *hold:*

- *(i) For every i $(0 \le i \le t)$ the set D^i_{i+1} is nonempty and the set $D^i_i(x,z)$ is empty for every $z \in \Gamma(x)$.*
- *(ii) For every i $(t < i \le d)$ the set D^i_{i+1} is empty.*

Moreover, $\Gamma_i(x) = D^i_{i+1} \cup D^i_{i-1}$ for every $0 \le i \le t$.

Proof. For $y \in \Gamma(x)$, since the set $D^0_1(x,y)$ is nonempty, let us define $t := t(y)$ as the greatest integer i $(1 \le i \le d)$ such that the set $D^i_{i+1}(x,y)$ is nonempty. Then, it is clear that the set D^i_{i+1} is empty for $i > t$ and, by Lemma 6.2.2(v), the set D^i_{i+1} is nonempty for every $0 \leq i \leq t$. Moreover, by Lemma 7.6.2 the set $D^i_i(x,z)$ is empty for every $z \in \Gamma(x)$ and for every $0 \leq i \leq t$. The result follows. ■

Proposition 7.6.4. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Pick $y \in \Gamma(x)$. Let the sets $D^i_j = D^i_j(x,y)$ and let $t(y)$ be as in Proposition 7.6.3. If there exists j $(1 \leq j \leq d)$ such that D^j_j is nonempty then D^i_i is nonempty for every $t(y) < i \leq j$.*

Proof. Suppose the set D^j_j is nonempty for some j $(1 \leq j \leq d)$. Then, by Proposition 7.6.3 we have $t(y) < j$. Assume now there exists an integer i $(t(y) < i < j)$ such that D^i_i is empty. Notice that in this case $\Gamma_i(x) = D^i_{i-1}$ which is nonempty as $i < d$. Pick now $w \in D^j_j$. We observe every shortest xw-path must pass through a vertex in D^i_{i-1}. This clearly shows $\partial(x,w) \geq i + (j-i+1) = j+1$, a contradiction. The result follows. ■

Propositions 7.6.3 and 7.6.4 help us to understand the combinatorial structure of graphs which have, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin.

We now consider the possible intersection diagrams of Γ with respect to the edge $\{x,y\}$, for every $y \in \Gamma(x)$. Let d denote the eccentricity of vertex x. Then, we observe that $\epsilon(y) \in \{d-1, d, d+1\}$. We have two cases.

If $\epsilon(y) > d$ then the set $D^d_{d+1}(x,y)$ is not empty and so the scalar $t(y) = d$. By Proposition 7.6.3, the set $D^i_i(x,y)$ is empty for every $y \in \Gamma(x)$ and for every i $(0 \leq i \leq d)$. Moreover, notice that the sets $D^i_{i+1}(x,y) \neq \emptyset$ $(0 \leq i \leq d)$ and by Lemma 7.6.1, the sets $D^i_{i-1}(x,y)$ $(1 \leq i \leq d)$ are all nonempty as well. See Figure 7.2 for a graphical representation of the intersection diagram of Γ with respect to the edge $\{x,y\}$ when $\epsilon(y) > \epsilon(x)$.

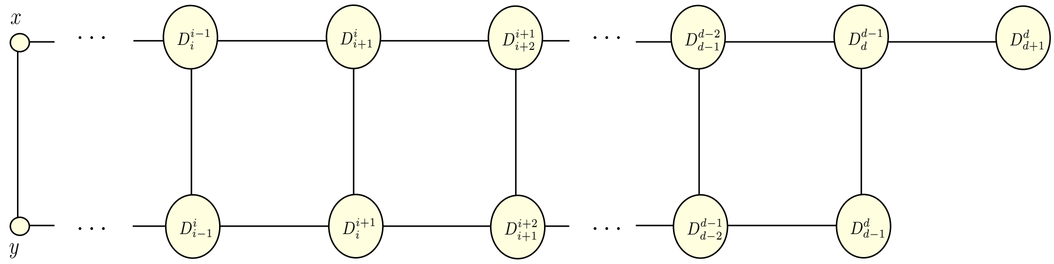

Figure 7.2: Intersection diagram of graph Γ which has, up to isomorphism, exactly one irreducible $T(x)$-module with endpoint 1, and this module is thin: case $\epsilon(y) > \epsilon(x)$.

If $\epsilon(y) \leq d$ then the set $D^d_{d+1}(x,y)$ is empty and so, $t := t(y) < d$. Furthermore, in this case the sets $D^i_{i+1}(x,y) \neq \emptyset$ $(0 \leq i \leq t)$ and, by Lemma 7.6.1, the sets $D^i_{i-1}(x,y)$ $(1 \leq i \leq d)$ are all nonempty as well. Moreover, if $D^{t+1}_{t+1}(x,y) \neq \emptyset$, then let u $(1 \leq u \leq d-1-t)$ denote the greatest positive integer such that $D^{t+u}_{t+u}(x,y) \neq \emptyset$. See Figure 7.3 for a graphical representation of the intersection diagram of Γ with respect to the edge $\{x,y\}$ when $\epsilon(y) \leq \epsilon(x)$.

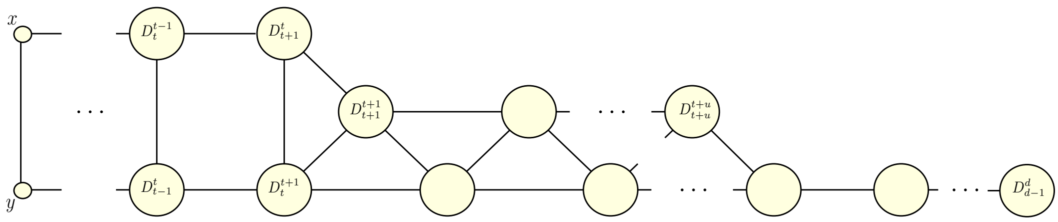

Figure 7.3: Intersection diagram of graph Γ which has, up to isomorphism, exactly one irreducible $T(x)$-module with endpoint 1, and this module is thin: case $\epsilon(y) \leq \epsilon(x)$.

With reference to Proposition 7.6.3, it is easy to see the following (i)-(ii) are equivalent:

(i) The integer $t := t(y)$ is independent of the choice of $y \in \Gamma(x)$.

(ii) For each i $(1 \leq i \leq d)$, if for some $y \in \Gamma(x)$ the set $D^i_{i+1}(x,y) \neq \emptyset$ then for every $y \in \Gamma(x)$ the set $D^i_{i+1}(x,y) \neq \emptyset$.

At this point, the next question naturally arises.

Question 7.6.5. *With reference to Notation 7.2.4 and Proposition 7.6.3, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Does the integer $t := t(y)$ depend on the choice of $y \in \Gamma(x)$?*

The following results partially answer the above question. However, a proof of the general case seems to need a nontrivial approach.

Proposition 7.6.6. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. For $y \in \Gamma(x)$, let $t(y)$ be as in Proposition 7.6.3. If for some $z \in \Gamma(x)$ the set $D^1_1(x,z)$ is nonempty then the integer $t := t(y)$ does not depend on the choice of $y \in \Gamma(x)$.*

Proof. Suppose for some $z \in \Gamma(x)$ the set $D^1_1(x,z)$ is nonempty. Then, by Lemma 7.6.2, the set $D^1_2(x,y)$ is empty for every $y \in \Gamma(x)$. This shows that $t(y) = 0$ for every $y \in \Gamma(x)$. The result follows. ∎

Proposition 7.6.7. *With reference to Notation 7.2.4, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. For* $y \in \Gamma(x)$*, let* $t(y)$ *be as in Proposition 7.6.3. If for every* $y \in \Gamma(x)$ *there exists an integer* i $(1 \leq i \leq d)$ *such that the set* $D^i_i(x,y)$ *is nonempty then the integer* $t := t(y)$ *does not depend on the choice of* $y \in \Gamma(x)$.

Proof. Pick $w \in \Gamma(x)$ such that $t(w) = \min\{t(y) \mid y \in \Gamma(x)\}$. Then, by the choice of $w \in \Gamma(x)$, we have that $t(w) \leq t(y)$ for all $y \in \Gamma(x)$. Let k be the least integer such that $D^k_k(x,w) \neq \emptyset$. We assert that $t(w) = k-1$. To prove our claim, we first observe that, by Lemma 7.6.2, we have that $D^k_{k+1}(x,w) = \emptyset$. This shows that $t(w) \leq k-1$. Suppose now that $t(w) < k-1$. Then, $t(w)+1 < k$ and, by the choice of k, $D^{t(w)+1}_{t(w)+1}(x,w) = \emptyset$, contradicting Proposition 7.6.4. Therefore, we have that $t(w) = k-1$. Moreover, by Lemma 7.6.2, the set $D^k_{k+1}(x,y) = \emptyset$ for all $y \in \Gamma(x)$. This yields that $t(y) \leq t(w)$ for all $y \in \Gamma(x)$. Consequently, $t(y) = t(w)$ for all $y \in \Gamma(x)$. The result follows. ■

Proposition 7.6.8. *With reference to Notation 7.2.4, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. For* $y \in \Gamma(x)$*, let* $t(y)$ *be as in Proposition 7.6.3. If* Γ *is a tree then the integer* $t := t(y)$ *does not depend on the choice of* $y \in \Gamma(x)$.

Proof. Pick $y \in \Gamma(x)$. Suppose there exists an integer i $(1 \leq i \leq d)$ such that the set $D^i_{i+1}(x,y)$ is empty. Let k be the least integer such that $D^k_{k+1}(x,y)$ is empty. Since Γ is bipartite and x has valency at least 2, we observe $D^1_2(x,y)$ is not empty. This implies that $k \geq 2$. By the choice of k, we have that the set $D^{k-1}_k(x,y)$ is nonempty. Then, since Γ has no cycles, for a vertex $z \in D^{k-1}_k(x,y)$ we have that $b_{k-1}(x,z) = 0$. By Lemma 7.6.1, the set $D^j_{j-1}(x,y)$ is nonempty for every j $(1 \leq j \leq d)$ and so, for $w \in D^{k-1}_{k-2}(x,y)$, the scalar $b_{k-1}(x,w) > 0$. This shows that Γ is not distance-regular around x. Therefore, by Corollary 3.7.4 the trivial module $T\widehat{x}$ is not thin, a contradiction. Hence, for every integer i $(1 \leq i \leq d)$ the set $D^i_{i+1}(x,y)$ is not empty. This yields $t(y) = d$. The result follows. ■

Proposition 7.6.9. *With reference to Notation 7.2.4, assume that* Γ *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, and that this module is thin. For* $y \in \Gamma(x)$*, let* $t(y)$ *be as in Proposition 7.6.3. With reference to Definition 5.7.1, assume also that* Γ *is* 1*-homogeneous with respect to* x *(in the sense of Curtin and Nomura). Then, the integer* $t := t(y)$ *does not depend on the choice of* $y \in \Gamma(x)$.

Proof. For an integer i, and for vertices $y \in \Gamma(x)$ and $z \in D^i_{i+1}(x,y)$, let $\gamma^{i,i+1}_{i+1,i+2}(x,y,z)$ denote the number of neighbours of $z \in D^i_{i+1}(x,y)$ in the set $D^{i+1}_{i+2}(x,y)$. Pick $u,v \in \Gamma(x)$.

Assume to the contrary that $t(u) \neq t(v)$. Without loss of generality, we may assume that $t(u) < t(v)$. We observe that the set $D^{t(u)}_{t(u)+1}(x,u)$ is nonempty but the set $D^{t(u)+1}_{t(u)+2}(x,u) = \emptyset$ by the definition of $t(u)$. This shows that $\gamma^{t(u),t(u)+1}_{t(u)+1,t(u)+2}(x,u,z) = 0$ for $z \in D^{t(u)}_{t(u)+1}(x,u)$. Similarly, by the definition of $t(v)$, the set $D^{t(v)}_{t(v)+1}(x,v)$ is nonempty and, by Lemma 6.2.2(v), the set $D^{t(u)}_{t(u)+1}(x,v)$ is also nonempty. Furthermore, for a vertex $w \in D^{t(v)}_{t(v)+1}(x,v)$, there exists an xw-path of length $t(v)$ passing through a vertex $z \in D^{t(u)}_{t(u)+1}(x,v)$. Notice that z has a neighbour in $D^{t(u)+1}_{t(u)+2}(x,v)$ and so, the scalar $\gamma^{t(u),t(u)+1}_{t(u)+1,t(u)+2}(x,v,z) > 0$, contradicting that Γ is 1-homogeneous with respect to x. Consequently, $t(u) = t(v)$ for every $u,v \in \Gamma(x)$. The result follows. ■

Proposition 7.6.10. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. For $y \in \Gamma(x)$, let $t(y)$ be as in Proposition 7.6.3. If Γ is distance-regularized (distance-regular or distance-biregular) then the integer $t := t(y)$ does not depend on the choice of $y \in \Gamma(x)$.*

Proof. Since Γ is distance-regularized then every vertex is distance-regularized. Therefore, for $x \in X$ and $y \in \Gamma(x)$, it is easy to see that for $1 \leq i \leq \epsilon(y) - 1$ we have that

$$|D^i_{i+1}(x,y)| = \prod_{i=1}^{i} \frac{b_i(y)}{c_i(x)}.$$

In particular, the sets $D^i_{i+1}(x,y)$ $(1 \leq i \leq \epsilon(y) - 1)$ are nonempty and have the same cardinality for every $y \in \Gamma(x)$. This implies that $t(y) = \epsilon(y) - 1$ for every $y \in \Gamma(x)$. The claim now immediately follows as Γ is distance-regularized and so, in this case, the eccentricity $\epsilon(y)$ does not depend on the choice of $y \in \Gamma(x)$. ■

Remark 7.6.11. *Proposition 7.6.10 is also an immediate consequence of Proposition 7.6.9 since it is not hard to see that every distance-regularized graph (distance-regular or distance-biregular) is 1-homogeneous (in the sense of Curtin and Nomura) with respect to any of its vertices.*

A graph Γ is called *strongly distance-balanced* (SDB for short) if $|D^i_{i-1}(x,y)| = |D^{i-1}_i(x,y)|$ holds for every $i \geq 1$ and every edge xy in Γ.

Proposition 7.6.12. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module* *is thin. For $y \in \Gamma(x)$, let $t(y)$ be as in Proposition 7.6.3. If Γ is strongly distance-balanced then the integer $t := t(y)$ does not depend on the choice of $y \in \Gamma(x)$.*

Proof. By Lemma 7.6.1 the set $D^i_{i-1}(x,y)$ is nonempty for every i $(1 \le i \le d)$ and for all $y \in \Gamma(x)$. Since Γ is SDB we have $D^{i-1}_i(x,y)$ is nonempty for every i $(1 \le i \le d)$ and for all $y \in \Gamma(x)$. This shows that $t(y) = d-1$ for all $y \in \Gamma(x)$. The claim follows. ■

Remark 7.6.13. *We checked, using program package* MAGMA*, Question 7.6.5 against the list of all connected graphs of order at most 9 which have a pseudo-distance-regularized vertex x and $T(x)$ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin. For all such graphs, the integer $t := t(y)$ which Question 7.6.5 refers to, does not depend on the choice of $y \in \Gamma(x)$.*

With reference to Notation 7.2.4, we let $\Delta = \Delta(x)$ denote the subgraph of Γ induced by the neighbourhood of x. Namely, the graph $\Delta = \Delta(x) = (X', \mathcal{R}')$, with vertex set $X' = \{y \in X \mid \partial(x,y) = 1\}$ and edges $\mathcal{R}' = \{yz \mid y,z \in X', yz \in \mathcal{R}\}$. We end this section presenting some result about the local graph Δ.

Proposition 7.6.14. *With reference to Notation 7.2.4, assume that Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Then, the subgraph $\Delta = \Delta(x)$ of Γ induced by the neighbourhood of x is either isomorphic to an empty graph or a complete graph.*

Proof. Pick $x \in X$ and let $k := |\Gamma(x)|$. Assume first that for some $z \in \Gamma(x)$ the set $D^1_2(x,z)$ is not empty. Then, by Lemma 7.6.2, the set $D^1_1(x,y)$ is empty for all $y \in \Gamma(x)$. This shows that x is not contained in any triangle and so, that there are no edges of Γ in $D^1_2(x,y)$ for all $y \in \Gamma(x)$. Therefore, in this case, Δ is isomorphic to the empty graph S_k of k vertices. Assume now that for all $y \in \Gamma(x)$ the set $D^1_2(x,y)$ is empty. Then, it holds that $|D^1_1(x,y)| = k-1$ for all $y \in \Gamma(x)$. Since x has valency at least 2, we have that the set $D^1_1(x,y)$ is nonempty for all $y \in \Gamma(x)$. If $k = 2$ then it is easy to see that Δ is isomorphic to the complete graph K_2 of 2 vertices. Suppose that $k > 2$ and pick $y \in \Gamma(x)$. We claim that any two vertices in $D^1_1(x,y)$ are adjacent. To prove this claim, assume that $z,w \in D^1_1(x,y)$ are not adjacent. Then, we observe w is a neighbour of x which is at distance 2 from z. That is, $w \in D^1_2(x,z)$, contradicting Lemma 7.6.2. Hence, for any $y \in \Gamma(x)$, we have that any two vertices in $D^1_1(x,y)$ are adjacent. Therefore, it follows that Δ is isomorphic to the complete graph K_k of k vertices. This finishes the proof. ■

7.7 Examples

In this section we present some examples of graphs for which the equivalent conditions of Theorem 7.2.5 hold for a certain vertex x. Several examples of such graphs where x is distance-regularized, are presented in Sections 5.7 and 6.7; see also [23, 27]. We therefore turn our attention to the case when x is not necessarily distance-regularized. Recall that we are still refering to Definition 7.2.2 and Notation 7.2.4 throughout this section.

Example 7.7.1. *Let Γ be the connected graph with vertex set $X = \{1,2,3,4,5,6\}$ and edge set $\mathcal{R} = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{2,5\},\{3,5\},\{3,6\}\}$. See also Figure 10.3 and observe that Γ is not bipartite. Fix vertex $1 \in X$ and note that $\epsilon(1) = 2$. Notice that Γ is not distance-regular around 1. Consider the Terwilliger algebra of Γ with respect to vertex 1. It is now easy to verify that for every integer i $(0 \le i \le 2)$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ the following hold:*

$$r^{i+1}\ell(y) = \alpha_i\, r^i(y), \qquad r^i f(y) = \beta_i\, r^i(y),$$

with the values of α_i, β_i $(0 \le i \le 2)$ as presented in Table 7.1.

i	0	1	2
α_i	2	3	0
β_i	0	1	0

Table 7.1: Values of scalars α_i and β_i, $(0 \le i \le 2)$.

Therefore, by Theorem 3.5.3 the trivial T-module is thin. Moreover, properties $(a),(b)$ described in part (ii) of Theorem 7.2.5 are satisfied with the values of $\kappa_i, \mu_i, \theta_i, \rho_i$ $(1 \le i \le 2)$ as presented in Table 7.2. Consequently, by Theorem 7.2.5, it holds that Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin. Moreover, since $\dim(E_1^*V) = |\Gamma(x)| = 2$, it is easy to see that there is actually only one irreducible T-module with endpoint 1. This T-module has dimension $s = 2$ and is spanned by $w = \widehat{3} - \widehat{2}$ and $Rw = \widehat{6} - \widehat{4}$. Note also that the partitions given by the intersection diagrams of Γ with respect to the edges $\{1,2\}$ and $\{1,3\}$ are not equitable.

We next give another example of a non-bipartite graph where the equivalent conditions of Theorem 7.2.5 hold for a non-distance-regularized vertex x.

i	1	2
κ_i	1	0
μ_i	1	0
θ_i	-1	0
ρ_i	1	0

Table 7.2: Values of scalars κ_i, μ_i, θ_i and ρ_i, $(1 \leq i \leq 2)$.

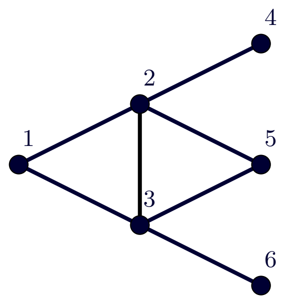

Figure 7.4: Graph Γ from Example 7.7.1.

Example 7.7.2. *Let Γ be the connected graph with vertex set $X = \{n \in \mathbb{N} \mid 1 \leq n \leq 12\}$ given in Figure 7.5. Observe that Γ is not bipartite. Fix vertex $x = 1 \in X$ and note that $\epsilon(1) = 5$. Notice that Γ is not distance-regular around 1. Consider the Terwilliger algebra of Γ with respect to vertex 1. It is now easy to check that for every integer i $(0 \leq i \leq 5)$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ the following hold:*

$$r^{i+1}\ell(y) = \alpha_i\, r^i(y), \qquad r^i f(y) = \beta_i\, r^i(y),$$

with the values of α_i, β_i $(0 \leq i \leq 5)$ as presented in Table 7.3. Therefore, by Theorem 3.5.3

i	0	1	2	3	4	5
α_i	2	3	1	3	2	0
β_i	0	1	0	0	1	0

Table 7.3: Values of scalars α_i and β_i, $(0 \leq i \leq 5)$.

the trivial T-module is thin. Moreover, properties $(a),(b)$ described in part (ii) of Theorem 7.2.5 hold with the values of $\kappa_i, \mu_i, \theta_i, \rho_i$ $(1 \leq i \leq 5)$ as presented in Table 7.4. Consequently,

i	1	2	3	4	5
κ_i	1	1	1	0	0
μ_i	1	0	1	1	0
θ_i	-1	0	0	-1	0
ρ_i	1	0	0	1	0

Table 7.4: Values of scalars κ_i, μ_i, θ_i and ρ_i, $(1 \leq i \leq 5)$.

by Theorem 7.2.5, it holds that Γ *has, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin. Moreover, since* $\dim(E_1^*V) = |\Gamma(x)| = 2$*, it is easy to see that there is actually only one irreducible* T*-module with endpoint* 1*. This* T*-module has dimension* $s = 4$ *and is spanned by the vectors* $w = \widehat{3} - \widehat{2}$*,* $Rw = \widehat{6} - \widehat{4}$*,* $R^2w = \widehat{9} - \widehat{7}$*,* $R^3w = \widehat{11} - \widehat{10}$*. Note also that the partitions presented by the intersection diagrams of* Γ *with respect to the edges* $\{1,2\}$ *and* $\{1,3\}$ *are not equitable.*

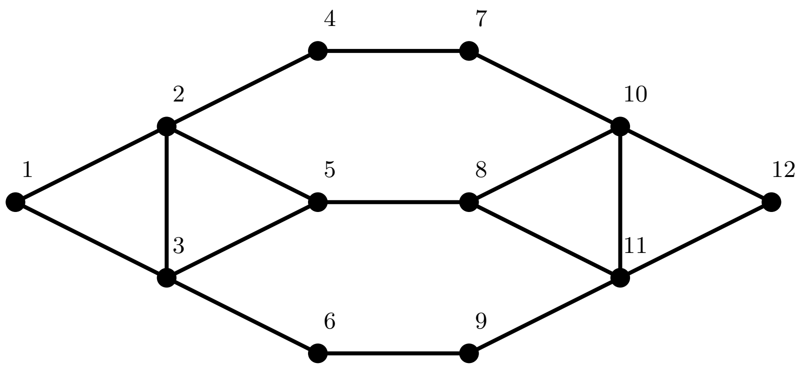

Figure 7.5: Graph Γ from Example 7.7.2.

7.7.1 A construction

Our next goal is to focus on the construction of infinitely many new graphs, that satisfy the equivalent conditions of Theorem 7.2.5 for a certain vertex. To do this, we will need the folowing notation.

Notation 7.7.3. *Let* Γ *and* Σ *denote finite, simple graphs with vertex set* X *and* Y*, respectively. Assume that* Γ *is a connected graph which is pseudo-distance-regular around a vertex* $x \in X$*. Assume also that* Σ *is regular with order at least* 2*. Consider the Cartesian product* $\Gamma \square \Sigma$*. Namely, the graph with vertex set* $X \times Y$ *where two vertices* (x,y) *and* (x',y') *are adjacent if and only if* $x = x'$ *and* y *is adjacent to* y'*, or* $y = y'$ *and* x *is adjacent to* x'*. Let* $H = H(\Gamma,\Sigma)$ *denote the graph obtained by adding a new vertex* w *to the graph* $\Gamma \square \Sigma$*, and connecting this new vertex* w *with all vertices* (x,y)*, where* y *is an arbitrary vertex of* Σ*; see for example Figures 7.6 and 7.7.*

With reference to Notation 7.7.3, we observe that for an arbitrary vertex (x',y') of H different from w, the distance between w and (x',y') satisfies $\partial_H(w,(x',y')) = \partial_\Gamma(x,x')+1$.

It thus follows that $d_H = d+1$, where d_H is the eccentricity of w in H and d is the eccentricity of x in Γ. Moreover, for $1 \le i \le d_H$ we have that

$$H_i(w) = \Gamma_{i-1}(x) \times Y = \{(u,y) \mid u \in \Gamma_{i-1}(x), y \in Y\}.$$

In addition, it is easy to see that H is distance-regular around w if and only if Γ is distance-regular around x.

We are now ready to give some constructions of infinitely many graphs, that satisfy the equivalent conditions of Theorem 7.2.5 for a certain vertex.

Proposition 7.7.4. *With reference to Notation 7.7.3, pick vertex w in H and consider the Terwilliger algebra $T = T(w)$. Then, the trivial T-module is thin.*

Proof. Immediate from Section 3.7.6. ■

With reference to Notation 7.7.3, in what follows, we use subscripts to distinguish the number of walks of a particular shape in H and in Γ. For example, for $x' \in \Gamma_i(x)$, we denote the number of walks from x to x' of the shape $r^{i+1}\ell$ with respect to x by $r^{i+1}\ell_\Gamma(x')$. For $(x',y') \in H_i(w)$, we denote the number of walks from w to (x',y') of the shape $r^{i+1}\ell$ with respect to w by $r^{i+1}\ell_H((x',y'))$. We next study the instances when Σ is either an empty or a complete graph.

Proposition 7.7.5. *With reference to Notation 7.7.3, pick vertex w in H and consider the Terwilliger algebra $T = T(w)$. If Σ is isomorphic to the empty graph S_n $(n \ge 2)$ then graph H has, up to isomorphism, exactly one irreducible T-module with endpoint* 1*, which is thin.*

Proof. By Proposition 7.7.4, we first observe that the trivial module is thin. We will next show that H satisfies the combinatorial conditions of Theorem 7.2.5. Suppose that Σ is isomorphic to the empty graph S_n $(n \ge 2)$. Pick $(x,y) \in H(w)$ and consider the sets $D^i_j = D^i_j(w,(x,y))$. Since the eccentricity of x equals d it is easy to see that the sets D^j_{j+1} $(0 \le j \le d_H)$ and D^j_{j-1} $(1 \le j \le d_H)$ are all nonempty for all $(x,y) \in H(w)$. Consequently, by Lemma 7.6.2, the set D^j_j is empty for every j $(1 \le j \le d_H)$ and for all $(x,y) \in H(w)$. In addition, we also notice

$$\begin{aligned} D^j_{j+1}(w,(x,y)) &= \Gamma_{j-1}(x) \times (Y \setminus \{y\}) = \{(u,y') \mid u \in \Gamma_{j-1}(x), y' \in Y, y' \neq y\}, \\ D^j_{j-1}(w,(x,y)) &= \Gamma_{j-1}(x) \times \{y\} = \{(u,y) \mid u \in \Gamma_{j-1}(x)\}. \end{aligned}$$

Pick $(x',y') \in H_i(w)$ for $1 \le i \le d_H$. We observe that

$$\ell r^i_H \Big((x,y),(x',y')\Big) = r^{i-1}_\Gamma(x'), \tag{7.15}$$

which is a positive integer since $\partial_\Gamma(x,x') = i-1$ implies $r^{i-1}_\Gamma(x') > 0$. Moreover, for $(x',y') \in D^i_{i+1}$ $(1 \le i \le d_H)$ we have that

$$r^i \ell_H \Big((x,y),(x',y')\Big) = r^{i-1} f_H \Big((x,y),(x',y')\Big) = 0. \tag{7.16}$$

Similarly, for $(x',y') \in D^i_{i-1}$ $(1 \le i \le d_H)$ we have that

$$r^i \ell_H \Big((x,y),(x',y')\Big) = r^i \ell_\Gamma(x'), \tag{7.17}$$

$$r^{i-1} f_H \Big((x,y),(x',y')\Big) = r^{i-1} f_\Gamma(x'), \tag{7.18}$$

$$r^{i-1}_H \Big((x,y),(x',y')\Big) = r^{i-1}_\Gamma(x'). \tag{7.19}$$

Since vertex x is pseudo-distance-regularized, by Theorem 3.5.3, we know that for every integer i $(0 \le i \le d)$ there exist scalars α_i, β_i, such that for every $z \in \Gamma_i(x)$ the following hold:

$$r^{i+1}\ell_\Gamma(z) = \alpha_i\, r^i_\Gamma(z), \qquad r^i f_\Gamma(z) = \beta_i\, r^i_\Gamma(z). \tag{7.20}$$

It follows from (7.17), (7.18), (7.19) and (7.20) that for $1 \le i \le d_H$ and for every vertex $(x',y') \in D^i_{i-1}$ we have that

$$\begin{aligned} r^i \ell_H \Big((x,y),(x',y')\Big) &= r^i \ell_\Gamma(x') \\ &= \alpha_{i-1} r^{i-1}_\Gamma(x') \\ &= \alpha_{i-1} r^{i-1}_H \Big((x,y),(x',y')\Big), \end{aligned} \tag{7.21}$$

$$\begin{aligned} r^{i-1} f_H \Big((x,y),(x',y')\Big) &= r^{i-1} f_\Gamma(x') \\ &= \beta_{i-1} r^{i-1}_\Gamma(x') \\ &= \beta_{i-1} r^{i-1}_H \Big((x,y),(x',y')\Big). \end{aligned} \tag{7.22}$$

Therefore, from (7.15), (7.16), (7.21) and (7.22), we see that vertex w of H satisfies the combinatorial conditions of Theorem 7.2.5 with the values of $\kappa_i = \alpha_{i-1}$, $\mu_i = 0$, $\theta_i = \beta_{i-1}$, $\rho_i = 0$ for every integer i $(1 \le i \le d_H)$. Consequently, H has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin. ■

Example 7.7.6. *Let* Γ *be the connected graph presented in Example 7.7.1 and let* S_n

denote the empty graph of n *vertices, for some integer* $n \geq 2$*. Let* $H = H(\Gamma, S_n)$*; see for example Figure 7.6 for the case* $n = 2$*. Consider the Terwilliger algebra* $T = T(w)$ *of* H *with respect to* w*. Notice that* H *is not distance-regular around* w *since* Γ *is not distance-regular around* x*. However, the trivial module is thin by Proposition 7.7.4. It follows from Table 7.1 and the above comments that the properties* $(a), (b)$ *described in part* (ii) *of Theorem 7.2.5 hold with the values of* $\kappa_i, \mu_i, \theta_i, \rho_i$ $(1 \leq i \leq 3)$ *as presented in Table 7.5. Consequently, by Theorem 7.2.5, it holds that* H *has, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin. Moreover, since* $\dim(E_1^* V) = |H(w)| = n$*, it is easy to see that there are actually* $n-1$ *irreducible* T*-modules with endpoint* 1 *and these isomorphic* T*-modules have dimension* $s = 3$*.*

i	1	2	3
κ_i	2	3	0
μ_i	0	0	0
θ_i	0	1	0
ρ_i	0	0	0

Table 7.5: Values of scalars κ_i, μ_i, θ_i and ρ_i, $(1 \leq i \leq 3)$.

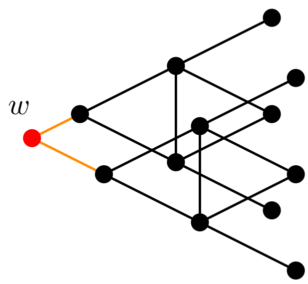

Figure 7.6: Graph H obtained from the Cartesian product $\Gamma \square S_2$ where Γ is the graph from Example 7.7.1 and S_2 denotes the empty graph on 2 vertices.

Proposition 7.7.7. *With reference to Notation 7.7.3, pick vertex* w *in* H *and consider the Terwilliger algebra* $T = T(w)$*. If* Σ *is isomorphic to the complete graph* K_n $(n \geq 2)$ *then graph* H *has, up to isomorphism, exactly one irreducible* T*-module with endpoint* 1*, which is thin.*

Proof. By Proposition 7.7.4, we first observe that the trivial module is thin. We will next show that H satisfies the combinatorial conditions of Theorem 7.2.5. Suppose that Σ is isomorphic to the complete graph K_n $(n \geq 2)$. Pick $(x,y) \in H(w)$ and consider the sets $D^i_j = D^i_j(w,(x,y))$. Since the eccentricity of x equals d it is easy to see that the sets D^j_j $(1 \leq j \leq d_H)$ and D^j_{j-1} $(1 \leq j \leq d_H)$ are all nonempty for all $(x,y) \in H(w)$. Consequently, by Lemma 7.6.2 the set D^j_{j+1} is empty for every j $(1 \leq j \leq d_H)$ and for all

$(x, y) \in H(w)$. In addition, we also notice that

$$\begin{aligned} D^j_j(w,(x,y)) &= \Gamma_{j-1}(x) \times (Y \setminus \{y\}) \\ &= \{(u,y') \mid u \in \Gamma_{j-1}(x), y' \in Y \setminus \{y\}\}, \end{aligned} \tag{7.23}$$

$$\begin{aligned} D^j_{j-1}(w,(x,y)) &= \Gamma_{j-1}(x) \times \{y\} \\ &= \{(u,y) \mid u \in \Gamma_{j-1}(x)\}. \end{aligned} \tag{7.24}$$

Pick $(x', y') \in H_i(w)$ for $1 \le i \le d_H$. We observe that

$$\ell r^i_H \Big((x,y),(x',y')\Big) = r^{i-1}_\Gamma(x'), \tag{7.25}$$

which is a positive integer since $\partial_\Gamma(x, x') = i-1$ implies $r^{i-1}_\Gamma(x') > 0$. Moreover, since every vertex in D^i_i has no neighbours in D^{i+1}_i, for $(x', y') \in D^i_i$ $(1 \le i \le d_H)$, it follows that

$$r^i \ell_H \Big((x,y),(x',y')\Big) = 0. \tag{7.26}$$

In addition, from the definition of H, (7.23) and (7.24), it is easy to see every vertex $(x', y') \in D^i_i$ $(1 \le i \le d_H)$ has exactly one neighbour in D^i_{i-1} which is the vertex (x', y). This implies that the number of walks from (x, y) to (x', y') of the shape $r^{i-1}f$ with respect to w is equal to the number of walks from x to x' of the shape r^{i-1} with respect to x. Therefore, from the above comments and (7.25), for $(x', y') \in D^i_i$ $(1 \le i \le d_H)$,

$$r^{i-1} f_H \Big((x,y),(x',y')\Big) = r^{i-1}_\Gamma(x') = \ell r^i_H \Big((x,y),(x',y')\Big). \tag{7.27}$$

Similarly, for $(x', y') \in D^i_{i-1}$ $(1 \le i \le d_H)$ we have that

$$r^i \ell_H \Big((x,y),(x',y')\Big) = r^i \ell_\Gamma(x'), \tag{7.28}$$

$$r^{i-1} f_H \Big((x,y),(x',y')\Big) = r^{i-1} f_\Gamma(x'), \tag{7.29}$$

$$r^{i-1}_H \Big((x,y),(x',y')\Big) = r^{i-1}_\Gamma(x'). \tag{7.30}$$

Since vertex x is pseudo-distance-regularized, by Theorem 3.5.3, we know that for every integer i $(0 \le i \le d)$ there exist scalars α_i, β_i, such that for every $z \in \Gamma_i(x)$ the following hold:

$$r^{i+1}\ell_\Gamma(z) = \alpha_i \, r^i_\Gamma(z), \qquad r^i f_\Gamma(z) = \beta_i \, r^i_\Gamma(z). \tag{7.31}$$

It follows from (7.28), (7.29), (7.30) and (7.31) that for $1 \leq i \leq d_H$ and for every vertex $(x', y') \in D^i_{i-1}$ we have that

$$\begin{aligned} r^i \ell_H\Big((x,y),(x',y')\Big) &= r^i \ell_\Gamma(x') \\ &= \alpha_{i-1} r^{i-1}_\Gamma(x') \\ &= \alpha_{i-1} r^{i-1}_H\Big((x,y),(x',y')\Big), \end{aligned} \tag{7.32}$$

$$\begin{aligned} r^{i-1} f_H\Big((x,y),(x',y')\Big) &= r^{i-1} f_\Gamma(x') \\ &= \beta_{i-1} r^{i-1}_\Gamma(x') \\ &= \beta_{i-1} r^{i-1}_H\Big((x,y),(x',y')\Big). \end{aligned} \tag{7.33}$$

Therefore, from (7.25), (7.26), (7.27), (7.32) and (7.33), we see that vertex w of H satisfies the combinatorial conditions of Theorem 7.2.5 with the values of $\kappa_i = \alpha_{i-1}$, $\mu_i = 0$, $\theta_i = \beta_{i-1} - 1$, $\rho_i = 1$ for every integer i $(1 \leq i \leq d_H)$. Consequently, H has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin. ■

Example 7.7.8. *Let Γ be the connected graph presented in Example 7.7.1 and let K_n denote the complete graph of n vertices, for some integer $n \geq 2$. Let $H = H(\Gamma, K_n)$; see for example Figure 7.7 for the case $n = 2$. Consider the Terwilliger algebra $T = T(w)$ of H with respect to w. Notice that H is not distance-regular around w since Γ is not distance-regular around x. However, the trivial module is thin by Proposition 7.7.4. It follows from Table 7.1 and the above comments that the properties $(a), (b)$ described in part (ii) of Theorem 7.2.5 hold with the values of $\kappa_i, \mu_i, \theta_i, \rho_i$ $(1 \leq i \leq 3)$ as presented in Table 7.6. Consequently, by Theorem 7.2.5, it holds that H has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin. Moreover, since $\dim(E^*_1 V) = |H(w)| = n$, it is easy to see that there are actually $n-1$ irreducible T-modules with endpoint 1 and these isomorphic T-modules have dimension $s = 3$.*

We are now ready to prove the main result of this subsection.

Theorem 7.7.9. *With reference to Notation 7.7.3, pick vertex w in H and consider the Terwilliger algebra $T = T(w)$. Graph H has, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin if and only if Σ is either isomorphic to the empty graph S_n $(n \geq 2)$ or to the complete graph K_n $(n \geq 2)$.*

Proof. By Proposition 7.7.4, we observe that the trivial module is thin. Assume first that H has, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is

i	1	2	3
κ_i	2	3	0
μ_i	0	0	0
θ_i	-1	0	-1
ρ_i	1	1	1

Table 7.6: Values of scalars κ_i, μ_i, θ_i and ρ_i, $(1 \leq i \leq 3)$.

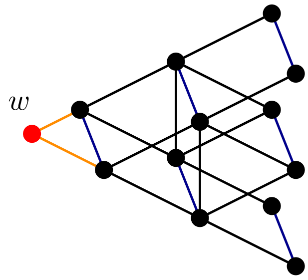

Figure 7.7: Graph H obtained from the Cartesian product $\Gamma \square K_2$ where Γ is the graph from Example 7.7.1 and K_2 denotes the complete graph on 2 vertices.

thin. We next claim that Σ is either isomorphic to the empty graph S_n $(n \geq 2)$ or to the complete graph K_n $(n \geq 2)$. Let Y denote the vertex set of Σ. If $|Y| = 2$ then the statement trivially follows. So, to prove this assertion, assume that $|Y| > 2$. Given any three vertices $y, y', y'' \in Y$, suppose there exist both a pair of adjacent vertices and a pair of nonadjacent vertices in Σ. Without loss of generality we could assume that y is adjacent to y' but not to y''. Since y and y' are adjacent, we thus have that (x, y') is a common neighbour of both w and (x, y) in H. Moreover, note that $\partial_H(w, (x, y'')) = 1$ and since y and y'' are not adjacent, $\partial_H((x, y), (x, y'')) = 2$. Hence, the sets $D^1_2(w, (x, y))$ and $D^1_1(w, (x, y))$ are both nonempty, contradicting Lemma 7.6.2. Consequently, any three vertices in Y either form a stable set or a clique. This clearly implies that Σ is either isomorphic to the empty graph S_n $(n \geq 2)$ or to the complete graph K_n $(n \geq 2)$, which proves our claim. Notice also that the second part of the result immediately follows from Proposition 7.7.5 and Proposition 7.7.7. This finishes the proof. ■

7.8 Concluding remarks

We conclude this chapter with some remarks about conditions (i), (ii) of Theorem 7.2.5.

In this chapter we study irreducible T-modules with endpoint 1 in the case when the trivial T-module is thin. We observe, there are no irreducible T-modules with endpoint 1 if and only if x is a leaf of Γ, that is, if and only if $|\Gamma(x)| = \dim(E^*_1 T\hat{x}) = 1$. Therefore, we assume that $|\Gamma(x)| \geq 2$. These arguments were used throughtout Section 7.5 to prove that certain combinatorial conditions imply some algebraic property. Namely, with reference to

Notation 7.2.4, if we assume that Γ satisfies part (ii) of Theorem 7.2.5 then it follows that in this case Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, and that this module is thin. Although it was assumed that the trivial module is thin, this hypothesis was only used to claim that $\dim(E_1^*T\widehat{x}) = 1$ and so, to guarantee the existence of irreducible T-modules with endpoint 1 as $|\Gamma(x)| > 1$.

However, by Proposition 7.1.2, there are no irreducible T-modules with endpoint 1 if and only if $\dim(E_1^*T\widehat{x}) = |\Gamma(x)|$. Consequently, if we would like to explore a more general situation when the trivial T-module is not necessarily thin, we will need that $\dim(E_1^*T\widehat{x}) < |\Gamma(x)|$. Moreover, keeping that in mind and following the same arguments used in the proofs given in Section 7.5, condition (ii) of Theorem 7.2.5 implies condition (i). Namely, the next result is true:

Theorem 7.8.1. *Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple, connected graph with vertex set X and edge set $\mathcal{R}$. Fix a vertex $x \in X$ and let d denote the eccentricity of x. Let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let V_0 denote the trivial module and assume $\dim(E_1^*V_0) < |\Gamma(x)|$. For $y \in \Gamma(x)$ and $z \in X$ let the sets $D^i_j = D^i_j(x,y)$ be as defined in Definition 7.2.1, and let the numbers $r^m\ell(y,z)$, $r^m f(y,z)$ and $r^m(y,z)$ be as defined in Definition 7.2.2. Assume for every integer i $(1 \le i \le d)$ there exist scalars κ_i, μ_i, θ_i, ρ_i, such that for every $y \in \Gamma(x)$ the following (a), (b) hold:*

(a) *For every $z \in D^i_{i+1}(x,y) \cup D^i_i(x,y)$ we have that*

$$\begin{aligned} r^i\ell(y,z) &= \mu_i\, \ell r^i(y,z), \\ r^{i-1}f(y,z) &= \rho_i\, \ell r^i(y,z). \end{aligned}$$

(b) *For every $z \in D^i_{i-1}(x,y)$ we have that*

$$\begin{aligned} r^i\ell(y,z) &= \kappa_i\, r^{i-1}(y,z) + \mu_i\, \ell r^i(y,z), \\ r^{i-1}f(y,z) &= \theta_i\, r^{i-1}(y,z) + \rho_i\, \ell r^i(y,z). \end{aligned}$$

Then, Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and this module is thin.

However, the assumption that the trivial T-module is thin is neccesary to prove that conditions (i), (ii) of Theorem 7.2.5 are equivalent. In particular, if this condition on the trivial T-module is not assumed, condition (i) of Theorem 7.2.5 does not imply condition (ii), as we will see below.

Example 7.8.2. *Let Γ be the connected graph with vertex set $X = \{1,2,3,4,5,6,7\}$ and edge set $\mathcal{R} = \{\{1,2\},\{1,3\},\{2,4\},\{2,5\},\{3,5\},\{3,6\},\{4,7\},\{5,7\},\{6,7\}\}$; see also Figure 7.8. Observe that Γ is bipartite. Fix vertex $1 \in X$ and note that $\epsilon(1) = 3$. Observe that Γ is not distance-regular around vertex 1. Namely, vertex $4 \in \Gamma_2(1)$ has only one neighbour in $\Gamma(1)$, while vertex $5 \in \Gamma_2(1)$ has two neighbours in $\Gamma(1)$. Let A denote the adjacency matrix of Γ and let $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \le i \le 3)$ denote the dual idempotents of Γ with respect to 1. Let V denote the standard module of Γ and let $T = T(1)$ denote the Terwilliger algebra of Γ with respect to 1. Let L and R denote the lowering and the raising matrix of T, respectively. For $y \in \Gamma(x)$ and $z \in X$ let the sets $D_j^i = D_j^i(x,y)$ be as defined in Definition 7.2.1, and let the numbers $r^m\ell(y,z)$ and $r^m(y,z)$ be as defined in Definition 7.2.2.*

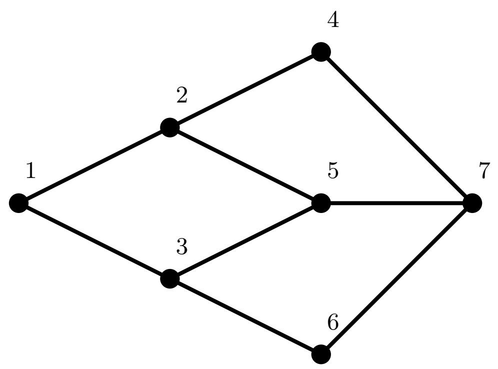

Figure 7.8: Graph Γ from Example 7.8.2.

Claim 7.8.3. *With reference to Example 7.8.2, the trivial T-module is not thin. Moreover, the set $\{\hat{1}, R\hat{1}, R^2\hat{1}, R^3\hat{1}, LR^3\hat{1}\}$ is a basis of the trivial T-module.*

Proof. Let V denote the standard module and let $T\hat{1}$ denote the unique irreducible module with endpoint 0. Let S be the subspace of V spanned by the vectors $R^i\hat{1}$ $(0 \le i \le 3)$ and $LR^3\hat{1}$. It is straightforward to check that $LR^i\hat{1} \in S$ $(0 \le i \le 3)$. Moreover, it holds that $L^2R^3\hat{1} = 8 \cdot R\hat{1}$ and $RLR^3\hat{1} = 3 \cdot R^3\hat{1}$. This yields that S is invariant under the action of L and R. Since the adjacency matrix A of Γ can be written as $A = L + R$, it follows that S is A-invariant. Observe that by (7.2) and by (eiv) from Section 7.1, the subspace S is invariant under the action of the dual idempotents. We thus have S is a T-module. Since $\hat{1} \in S$ and $S \subseteq T\hat{1}$, it must be $S = T\hat{1}$ as the trivial module is irreducible. By Proposition 3.5.5 and (7.2), vectors $R^i\hat{1}$ $(0 \le i \le 3)$ are nonzero and orthogonal. Moreover, by (7.2), we have $R^2\hat{1}$ and $LR^3\hat{1}$ belong to $E_2^*(T\hat{1})$. Since $R^2\hat{1} = \hat{4} + 2 \cdot \hat{5} + \hat{6}$ and $LR^3\hat{1} = 4 \cdot (\hat{4} + \hat{5} + \hat{6})$ the vectors are linearly independent. We therefore have $\dim(E_2^*(T\hat{1})) = 2$ which implies that the trivial T-module is not thin. This finishes the proof. ■

Claim 7.8.4. *With reference to Example 7.8.2, pick $w \in E_1^*V$, $w \neq 0$, which is orthogonal to $s_1 = R\hat{1}$. Let W denote the vector subspace of V spanned by the vectors $R^i w$ $(0 \le i \le 3)$. Then W is a thin irreducible T-module with endpoint 1 and the set $\{w, Rw\}$ forms an orthogonal basis of W. In particular, the dimension of W is 2.*

Proof. Let W denote the vector subspace of V spanned by the vectors $\{R^i w \mid 0 \le i \le 3\}$. Observe that by (7.2) and by (eiv) from Section 7.1, the subspace W is invariant under the action of the dual idempotents. By construction and since $R^3 w = 0$ by (7.2), it is also clear that W is closed under the action of R. Let J denote the all 1's matrix in $\mathrm{Mat}_X(\mathbb{C})$. As $w \in E_1^*V$ we have that $E_1^* w = w$ and so,

$$\langle \boldsymbol{j}, w\rangle = \langle \boldsymbol{j}, E_1^* w\rangle = \langle E_1^* \boldsymbol{j}, w\rangle = \langle s_1, w\rangle = 0,$$

where $\boldsymbol{j}$ denotes the all 1's vector in V. This shows $Jw = 0$. By elementary matrix multiplication it is easy to see $E_0^* A E_1^* = E_0^* J E_1^*$. Therefore, by Definition 7.1.1 and the above comments we have that $Lw = E_0^* A E_1^* w = E_0^* J E_1^* w = E_0^* J w = 0$. Moreover, it is easy to see that the following equations are true:

$$E_1^* L R E_1^* = E_1^* + E_1^* R L E_1^*, \tag{7.34}$$

$$E_2^* L R^2 E_1^* = 2 E_2^* J E_1^*. \tag{7.35}$$

It follows from (7.34), (7.35) and the above comments that $LRw = w$ and $LR^2 w = 0$. Since $LR^3 w = 0$, this implies that W is invariant under the action of L. Since $A = L + R$, it turns out that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents. Therefore, W is a T-module. It is also clear that W is thin, since by construction and (7.2), the subspace $E_i^* W$ is generated by $R^{i-1} w$. Next, we show that W is irreducible. Note that $w \in W$ and so W is non-zero. Recall that W is an orthogonal direct sum of irreducible T-modules. Since $E_0^* W$ is the zero subspace and $E_1^* w = w \neq 0$, there exists an irreducible T-module W', such that the endpoint of W' is 1 and $W' \subseteq W$. Consequently, $E_1^* W' \subseteq E_1^* W$. However, the dimension of $E_1^* W$ is 1, and so $E_1^* W' = E_1^* W$. But now we have $W = T E_1^* W = T E_1^* W' \subseteq W'$, implying that $W = W'$. Hence, W is irreducible and its endpoint equals 1. Finally, taking norm, it is easy to see $\|Rw\| = \|w\|$ and $\left\|R^2 w\right\| = 0$. Furthermore, it holds that vectors w and Rw are nonzero and orthogonal. The result follows. ∎

Claim 7.8.5. *With reference to Example 7.8.2, pick $w \in E_1^*V$, $w \neq 0$, which is orthogonal to $s_1 = R\hat{1}$. Let W denote an irreducible T-module with endpoint 1. Pick $w \in E_1^* W$, $w \neq 0$.*

Then the vectors $\{R^{i-1}w \mid 1 \le i \le 2\}$ *form an orthogonal basis of* W*. In particular,* W *is thin with dimension* 2*.*

Proof. Let W' denote the vector subspace of V spanned by the vectors $\{R^{i-1}w \mid 1 \le i \le 3\}$. Recall that W and the unique irreducible T-module with endpoint 0 are not isomorphic, and so w is orthogonal to s_1. By Claim 7.8.4, W' is a T-module. Note that W' is nonzero and contained in W. As W is irreducible, we have that $W = W'$. The result now follows from Claim 7.8.4. ■

Claim 7.8.6. *With reference to Example 7.8.2, graph* Γ *has, up to isomorphism, a unique irreducible* T*-module with endpoint* 1*, and this module is thin.*

Proof. Let W and W' be irreducible T-modules with endpoint 1, and pick any nonzero vectors $w \in E_1^*W$ and $w' \in E_1^*W'$. By Claim 7.8.5, the vectors

$$\left\{R^{i-1}w \mid 1 \le i \le 2\right\} \text{ and } \left\{R^{i-1}w' \mid 1 \le i \le 2\right\}$$

are orthogonal bases of W and W', respectively. Hence, the linear map $\sigma : W \to W'$, defined by $\sigma\left(R^{i-1}w\right) = R^{i-1}w'$ is a vector space isomorphism. It is clear that σ commutes with L and R. Since $A = L+R$, it turns out that σ commutes with A as well. Furthermore, σ is a T-module isomorphism since by (eiv) from Section 7.1, it commutes also with E_i^* $(0 \le i \le 3)$. Thus W and W' are T-isomorphic. ■

Claim 7.8.7. *With reference to Example 7.8.2, condition (ii) of Theorem 7.2.5 does not hold.*

Proof. Pick $1 \in X$ and consider the distance partition of Γ with respect to the edge $\{1,2\}$. We observe the sets $D_1^2(1,2) = \{4,5\}$ and $D_3^2(1,2) = \{6\}$. Suppose to the contrary that Γ satisfies condition (ii) of Theorem 7.2.5. Then, there exist scalars κ_2, μ_2 such that for $6 \in D_3^2(1,2)$ we have $r^2\ell(2,6) = \mu_2\, \ell r^2(2,6)$. This implies $\mu_2 = 2$. Moreover, for $z \in D_1^2(1,2)$ we have that $r^2\ell(2,z) = 2$, $r(2,z) = 1$ and so, $\kappa_2 + 2\,\ell r^2(2,z) = 2$. If $z = 4$ we get that $\kappa_2 = 0$ while if $z = 5$, we have that $\kappa_2 = -2$, a contradiction as κ_2 does not depend on the choice of z. The claim follows. ■

With reference to Example 7.8.2, we would like to point out that, by Claim 7.8.6, graph Γ has, up to isomorphism, exactly one irreducible T-module with endpoint 1, which is thin but, by Claim 7.8.7, condition (ii) of Theorem 7.2.5 does not hold. Notice however that, by Claim 7.8.3, the unique module with endpoint 0 is not thin.

Part B

On distance-balanced graphs

Chapter 8

Overview

Let Γ be a finite, undirected, connected graph with diameter d, and let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively. For $u, v \in V(\Gamma)$, let $\Gamma(u)$ be the set of neighbours of u, and let $\partial(u, v) = d_\Gamma(u, v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of Γ we denote

$$W_{u,v} = \{x \in V(\Gamma) \mid \partial(x, u) < \partial(x, v)\}.$$

We say that Γ is *distance–balanced* (DB for short) whenever for an arbitrary pair of adjacent vertices u and v of Γ we have that

$$|W_{u,v}| = |W_{v,u}|.$$

We refer the reader to Chapters 9 and 10 for further details and formal definitions about this family of graphs and some of its subclasses.

The investigation of distance-balanced graphs was initiated in 1999 by Handa [45], who considered distance-balanced partial cubes. The term itself was introduced by Jerebic, Klavžar and Rall in [52], who gave some basic properties and characterized Cartesian and lexicographic products of distance-balanced graphs. The family of distance-balanced graphs is very rich and its study is interesting from various purely graph-theoretic aspects where one focuses on particular properties of such graphs such as symmetry [55, 56, 98], connectivity [45, 75], or complexity aspects of algorithms related to such graphs [8]. However, the balancedness property of these graphs also makes them very appealing in areas such as mathematical chemistry and communication networks. For instance, the investigation of such graphs is highly related to the well-studied Wiener index and Szeged index (see [2, 52, 50, 87]) and they present very desirable models in various real-life situations related to (communication) networks [2]. Recently, the relations between distance-balanced graphs and the traveling salesman problem were studied in [12]. It turns out that these graphs can be characterized by properties that at first glance do not seem to have much in common with the original definition from [52]. For example, in [3] it was shown that the distance-balanced graphs coincide with the *self-median* graphs, that is, graphs for which the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex. Other such examples are *equal opportunity graphs* (see [2] for the definition). In [2] it is shown that distance-balanced graphs of even order are also equal to opportunity graphs. Finally, let us also mention that various generalizations of the distance-balanced property were defined and studied in the literature; see, for example, [1, 36, 49, 53, 76].

The notion of nicely distance-balanced graphs appears quite naturally in the context of DB graphs. We say that Γ is *nicely distance–balanced* (NDB for short) whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for an arbitrary pair of adjacent vertices u and v of Γ we have that

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds. Clearly, every NDB graph is also DB, but the opposite is not necessarily true. For example, if $n \geq 3$ is an odd positive integer, then the prism graph on $2n$ vertices is DB, but not NDB.

Assume now that Γ is NDB. Let us denote the diameter of Γ by d. In [57], where these graphs were first defined, it was proved that $d \leq \gamma$ and NDB graphs with $d = \gamma$ were classified. It turns out that Γ is NDB with $d = \gamma$ if and only if Γ is either isomorphic to a complete graph on $n \geq 2$ vertices, a complete multipartite graph with parts of cardinality 2 or to a cycle on $2d$ or $2d+1$ vertices. In Chapter 9 we study regular NDB graphs for which $\gamma = d+1$ (see also [29]). The situation in this case is much more complex than in the case $\gamma = d$. We show that the only regular NDB graphs with valency k, diameter d and $\gamma = d+1$ are the Petersen graph (with $k = 3$ and $d = 2$), the complement of the Petersen graph (with $k = 6$ and $d = 2$), the complete multipartite graph $K_{t\times 3}$ with t parts of cardinality 3, $t \geq 2$ (with $k = 3(t-1)$ and $d = 2$), the Möbius ladder graph on 8 vertices (with $k = 3$ and $d = 2$), the Paley graph on 9 vertices (with $k = 4$ and $d = 2$), the 3-dimensional hypercube Q_3 (with $k = 3$ and $d = 3$), the line graph of the 3-dimensional hypercube Q_3 (with $k = 4$ and $d = 3$), and the icosahedron (with $k = 5$ and $d = 3$).

Another concept closely related to the concept of distance-balanced graphs is the one of strongly distance-balanced graphs. For an arbitrary pair of adjacent vertices u and v of a given graph Γ, and any two non-negative integers i, j, we let

$$D_j^i(u,v) = \{x \in V(\Gamma) \mid \partial(u,x) = i \text{ and } \partial(v,x) = j\}.$$

A graph Γ is called *strongly distance-balanced* (SDB for short) if $|D_{i-1}^i(u,v)| = |D_i^{i-1}(u,v)|$ holds for every $i \geq 1$ and every pair of adjacent vertices u and v in Γ. It is easy to see that a strongly distance-balanced graph is also distance-balanced, but the converse is not true in general (see [55]). For more results on this and related concepts see, for example, [3, 8, 50, 57, 75].

In Chapter 10, we solve an open problem posed by Kutnar and Miklavič [57] by constructing several infinite families of nonbipartite nicely distance-balanced graphs which are not strongly distance-balanced. We also disprove a conjecture regarding the characterization of strongly distance-balanced graphs posed by Balakrishnan et al. [3] by providing infinitely many counterexamples, and answer a question posed by Kutnar et al. in [55] regarding the existence of semisymmetric distance-balanced graphs which are not strongly distance-balanced by providing an infinite family of such examples. We also show that for a graph Γ with n vertices and m edges it can be checked in $O(mn)$ time if Γ is strongly distance-balanced and if Γ is nicely distance-balanced.

Chapter 9

On certain regular nicely distance-balanced graphs

A connected graph Γ is called *nicely distance–balanced* (NDB for short), whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for any two adjacent vertices u, v of Γ there are exactly γ vertices of Γ which are closer to u than to v, and exactly γ vertices of Γ which are closer to v than to u. Let d denote the diameter of Γ. It is known that $d \leq \gamma$, and that nicely distance-balanced graphs with $\gamma = d$ are precisely complete graphs, complete multipartite graphs with parts of cardinality 2, and cycles of length $2d$ or $2d+1$. In this chapter we classify regular nicely distance-balanced graphs with $\gamma = d+1$.

The chapter is organized as follows. After some preliminaries in Section 9.1 we prove certain structural results about NDB graphs with $\gamma = d+1$ in Section 9.2. In Section 9.3 we show that if Γ is a regular NDB graph with $\gamma = d+1$, then $d \leq 5$ and the valency of Γ is either 3, 4 or 5. In Sections 9.4, 9.5 and 9.6 we consider each of these three cases separately. In Section 9.7 we prove our main result.

The chapter is based on joint work with Štefko Miklavič and Safet Penjić. Our main results will be published in Revista de la Unión Matemática Argentina; see [29] for more details.

9.1 Preliminaries

In this section we recall some preliminary results that we will find useful later in the chapter. Let Γ be a finite, simple, connected graph with vertex set $V(\Gamma)$, and edge set $E(\Gamma)$. If $u, v \in V(\Gamma)$ are adjacent then we simply write $u \sim v$ and we denote the corresponding edge by uv with an understanding that $uv = vu$. For $u \in V(\Gamma)$ and an integer i we let $\Gamma_i(u)$ denote the set of vertices of $V(\Gamma)$ that are at distance i from u. We abbreviate $\Gamma(u) = \Gamma_1(u)$. We set $\epsilon(u) = \max\{\partial(u,z) \mid z \in V(\Gamma)\}$ and we call $\epsilon(u)$ the *eccentricity* of u. Let $d = \max\{\epsilon(u) \mid u \in V(\Gamma)\}$ denote the *diameter* of Γ. Pick adjacent vertices u, v of Γ. For any two non-negative integers i, j we let

$$D^i_j(u,v) = \Gamma_i(u) \cap \Gamma_j(v).$$

By the triangle inequality we observe only the sets $D^{i-1}_i(u,v)$, $D^i_i(u,v)$ and $D^i_{i-1}(u,v)$ $(1 \le i \le d)$ can be nonempty. Moreover, the next result holds.

Lemma 9.1.1. *With the above notation, abbreviate $D^i_j = D^i_j(u,v)$. Then the following (i)–(iv) hold for $1 \le i \le d$.*

- *(i) If $w \in D^i_{i-1}$ then $\Gamma(w) \subseteq D^{i-1}_{i-2} \cup D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^{i+1}_i$.*
- *(ii) If $w \in D^i_i$ then $\Gamma(w) \subseteq D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^i_{i+1} \cup D^{i+1}_i \cup D^{i+1}_{i+1}$.*
- *(iii) If $w \in D^{i-1}_i$ then $\Gamma(w) \subseteq D^{i-2}_{i-1} \cup D^{i-1}_{i-1} \cup D^{i-1}_i \cup D^i_{i-1} \cup D^i_i \cup D^i_{i+1}$.*
- *(iv) If $D^i_{i+1} \neq \emptyset$ ($D^{i+1}_i \neq \emptyset$, respectively) then $D^j_{j+1} \neq \emptyset$ ($D^{j+1}_j \neq \emptyset$, respectively) for every $0 \le j \le i$.*

Proof. Straightforward (see also Figure 9.1). ■

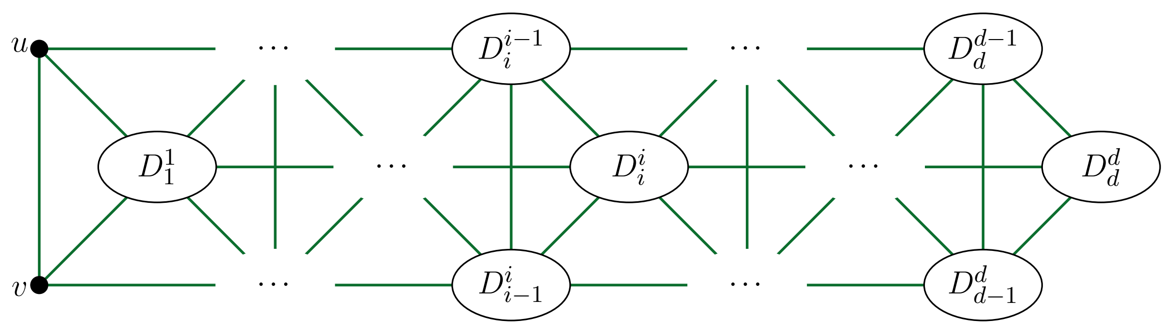

Figure 9.1: Graphical representation of the sets $D^i_j(u,v)$. The line between D^i_j and D^n_m indicates possible edges between vertices of D^i_j and D^n_m.

Let us recall the definition of nicely distance-balanced graphs. For an edge uv of Γ we denote

$$W_{u,v} = \{x \in V(\Gamma) \mid \partial(x,u) < \partial(x,v)\}.$$

We say that Γ is *nicely distance–balanced* (NDB for short) whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for any edge uv of Γ,

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds. One can easily see that Γ is NDB if and only if for every edge $uv \in E(\Gamma)$ we have that

$$\sum_{i=1}^{d} |D_{i-1}^{i}(u,v)| = \sum_{i=1}^{d} |D_{i}^{i-1}(u,v)| = \gamma. \tag{9.1}$$

Pick adjacent vertices u, v of Γ. For the purposes of this chapter we say that the edge uv is $(d+1)$-*balanced*, if (9.1) holds for vertices u, v with $\gamma = d+1$.

Graph Γ is said to be *regular*, if there exists a non-negative integer k, such that $|\Gamma(u)| = k$ for every vertex $u \in V(\Gamma)$. In this case we also say that Γ is regular with *valency* k (or k-regular for short). The following simple observation about regular graphs will be very useful in the rest of the chapter.

Lemma 9.1.2. *Let Γ be a connected regular graph. Then for every edge uv of Γ we have that*

$$|D_2^1(u,v)| = |D_1^2(u,v)|.$$

Proof. Note that $\Gamma(u) = \{v\} \cup D_1^1(u,v) \cup D_2^1(u,v)$ and $\Gamma(v) = \{u\} \cup D_1^1(u,v) \cup D_1^2(u,v)$. As Γ is regular, the claim follows. ∎

Assume that Γ is regular with valency k. If there exists a non-negative integer λ, such that every pair u, v of adjacent vertices of Γ has exactly λ common neighbours (that is, if $|D_1^1(u,v)| = \lambda$), then we say that Γ is *edge-regular* (with parameter λ). Before we start with our study of regular NDB graphs with $\gamma = d+1$ we have a remark.

Remark 9.1.3. *Let Γ be a regular NDB graph with diameter d and $\gamma = d+1$. Observe first that $d \geq 2$. Moreover, if $d = 2$ then it follows from [57, Theorem 5.2] that Γ is one of the following graphs:*

1. *the Petersen graph,*
2. *the complement of the Petersen graph,*
3. *the complete multipartite graph $K_{t\times 3}$ with t parts of cardinality 3 ($t \geq 2$),*
4. *the Möbius ladder graph on 8 vertices,*
5. *the Paley graph on 9 vertices.*

In what follows we will therefore assume that $d \geq 3$.

Let Γ be a NDB graph with diameter $d \geq 3$ and with $\gamma = \gamma(\Gamma) = d+1$. Pick vertices x_0, x_d of Γ such that $\partial(x_0, x_d) = d$, and let $x_0, x_1, \ldots, x_d$ be a shortest path between x_0 and x_d. Consider the edge x_0x_1 and note that

$$\{x_1, x_2, \ldots, x_d\} \subseteq W_{x_1,x_0}.$$

It follows that there is a unique vertex $u \in W_{x_1,x_0} \setminus \{x_1, x_2, \ldots, x_d\}$. Let $\ell = \ell(x_0, x_1)$ $(2 \leq \ell \leq d)$ be such that $u \in D_\ell^{\ell-1}(x_1, x_0)$, and so $D_\ell^{\ell-1}(x_1, x_0) = \{u, x_\ell\}$ and $D_i^{i-1}(x_1, x_0) = \{x_i\}$ for $2 \leq i \leq d, i \neq \ell$.

9.2 Some structural results

Let Γ be a NDB graph with diameter $d \geq 3$ and $\gamma = \gamma(\Gamma) = d+1$. In this section we prove certain structural results about Γ. To do this, let us pick arbitrary vertices x_0, x_d of Γ with $\partial(x_0, x_d) = d$, and let us pick a shortest path $x_0, x_1, \ldots, x_d$ between x_0 and x_d. Set $D_j^i = D_j^i(x_1, x_0)$ and $\ell = \ell(x_0, x_1)$. Recall that the unique vertex $u \in W_{x_1,x_0} \setminus \{x_1, x_2, \ldots, x_d\}$ is contained in $D_\ell^{\ell-1}$. Observe that

$$\{x_0, x_1, \ldots, x_{d-1}\} \subseteq W_{x_{d-1},x_d} \tag{9.2}$$

and

$$\{x_2, x_3, \ldots, x_d\} \subseteq W_{x_2,x_1}. \tag{9.3}$$

Note that if $\ell \geq 3$, then also $u \in W_{x_2,x_1}$. In addition, we will use the following abbreviations:

$$A = \bigcup_{i=2}^{d} (\Gamma(x_i) \cap D_i^i),$$

$$B = (\Gamma(x_2) \cap D_1^2) \cup (\Gamma(x_d) \cap D_{d-1}^d).$$

Proposition 9.2.1. *With the notation above, the following (i), (ii) hold.*

(i) *There are no edges between x_i and $D_{i-1}^i \cup D_{i-1}^{i-1}$ for $3 \leq i \leq d-1$.*

(ii) $|\Gamma(x_2) \cap (D_1^1 \cup D_1^2)| \leq 1$.

Proof. (*i*) Assume that for some $3 \leq i \leq d-1$ we have that z is a neighbour of x_i contained in $D_{i-1}^i \cup D_{i-1}^{i-1}$. Let $x_0, y_1, \ldots, y_{i-2}, z$ be a shortest path between x_0 and z. Observe that $\{y_1, \ldots, y_{i-2}, z\} \cap \{x_0, x_1, \ldots, x_{d-1}\} = \emptyset$ and that $\{y_1, \ldots, y_{i-2}, z\} \subseteq W_{x_{d-1},x_d}$. These comments, together with (9.2), yield $|W_{x_{d-1},x_d}| \geq d+2$, which contradicts the fact that $\gamma = d+1$.

(ii) Let $z_1, z_2 \in \Gamma(x_2) \cap (D_1^1 \cup D_1^2)$, $z_1 \neq z_2$. Then $z_1, z_2 \in W_{x_{d-1},x_d}$. This, together with (9.2), contradicts the fact that $\gamma = d+1$. ■

Proposition 9.2.2. *With the notation above, the following (i), (ii) hold.*

(i) $|A \cup B| \leq 2$.

(ii) *If* $\ell \geq 3$, *then* $|A \cup B \cup (\Gamma(u) \cap (D_\ell^\ell \cup D_{\ell-1}^\ell))| = 1$.

Proof. (i) Note that $A \cup B \subseteq W_{x_2,x_1}$ and that $(A \cup B) \cap \{x_2, \ldots, x_d\} = \emptyset$. This, together with (9.3), forces $|A \cup B| \leq 2$.

(ii) Note that in this case we have that $u \in W_{x_2,x_1}$. The proof that $|A \cup B \cup (\Gamma(u) \cap (D_\ell^\ell \cup D_{\ell-1}^\ell))| \leq 1$ is now similar to the proof of (i) above. On the other hand, if $|A \cup B \cup (\Gamma(u) \cap (D_\ell^\ell \cup D_{\ell-1}^\ell))| = 0$, then $|W_{x_2,x_1}| = d$, contradicting the fact that $\gamma = d+1$. ■

9.3 Regular NDB graphs with $\gamma = d+1$

Let Γ be a regular NDB graph with valency k, diameter $d \geq 3$ and $\gamma = \gamma(\Gamma) = d+1$. In this section we use the results from Section 9.2 to find bounds on k and d. As in the previous section, let us pick arbitrary vertices x_0, x_d of Γ with $\partial(x_0, x_d) = d$, and let us pick a shortest path $x_0, x_1, \ldots, x_d$ between x_0 and x_d. Set $D_j^i = D_j^i(x_1, x_0)$ and $\ell = \ell(x_0, x_1)$.

Proposition 9.3.1. *Let* Γ *be a regular NDB graph with valency* k, *diameter* $d = 3$ *and* $\gamma = 4$. *Then for every* $x \in V(\Gamma)$ *we have eccentricity* $\epsilon(x) = 3$.

Proof. Since $d = 3$ there exists $y \in V(\Gamma)$ such that $\epsilon(y) = 3$. Pick $x \in \Gamma(y)$. By the triangle inequality we also observe that $\epsilon(x) \in \{2, 3\}$. Suppose that $\epsilon(x) = 2$. Then, the sets $D_2^3(x,y)$ and $D_3^3(x,y)$ are both empty. Recall that $\gamma = 4$, and so by Lemma 9.1.2 we thus have $|D_2^1(x,y)| = |D_1^2(x,y)| = 3$, which implies $D_3^2(x,y) = \emptyset$, contradicting that $\epsilon(y) = 3$. Therefore, $\epsilon(x) = 3$ for every $x \in \Gamma(y)$. Since Γ is connected, this finishes the proof as every neighbour of a vertex of eccentricity 3 has also eccentricity 3. ■

Proposition 9.3.2. *There exists no regular NDB graph with valency* $k = 6$, *diameter* $d = 3$ *and* $\gamma = 4$.

Proof. Suppose to the contrary that there exists a regular NDB graph Γ with valency $k = 6$, diameter $d = 3$ and $\gamma = 4$. Then, by Proposition 9.3.1, every vertex $x \in V(\Gamma)$ has eccentricity $\epsilon(x) = 3$.

Let us pick an edge $xy \in E(\Gamma)$. By Lemma 9.1.2 we have that $|D_2^1(x,y)| = |D_1^2(x,y)|$, and so it follows from (9.1) that $|D_3^2(x,y)| = |D_2^3(x,y)|$ as well. We will prove that the sets $D_3^2(x,y)$ and $D_2^3(x,y)$ are nonempty.

Assume to the contrary that the sets $D_2^3(x,y)$ and $D_3^2(x,y)$ are empty. As $\gamma = d+1 = 4$ we have that $|D_2^1(x,y)| = |D_1^2(x,y)| = 3$. Moreover, by Proposition 9.3.1 the set $D_3^3(x,y)$ is nonempty. Pick $z \in D_3^3(x,y)$ and note that there exists a vertex $w \in \Gamma(z) \cap D_2^2(x,y)$. Pick $x_1 \in D_2^1(x,y)$ and observe that $\partial(x_1,z) \in \{2,3\}$. We first claim that $\partial(x_1,z) = 3$. Suppose to the contrary that $\partial(x_1,z) = 2$. Without loss of generality, we could assume that w and x_1 are adjacent. Notice that there exists a neighbour v of w in $D_1^1(x,y) \cup D_1^2(x,y)$ since $\partial(w,y) = 2$. Therefore, we have $\{x,y,x_1,v,w\} \subseteq W_{w,z}$, contradicting that $\gamma = 4$. This yields that $\partial(x_1,z) = 3$, and so there exists a shortest path x_1,v_1,w_1,z between x_1 and z of length 3. Note that by the above claim we have that $w_1 \in D_2^2$, and so $\{x,y,x_1,v_1,w_1\} \subseteq W_{w_1,z}$. As $x_1 \notin \{x,y\}$, this yields a contradiction with $\gamma = 4$. This shows that the sets $D_3^2(x,y)$ and $D_2^3(x,y)$ are nonempty.

Assume for the moment that $|D_3^2(x,y)| = 2$. Since $\gamma = 4$, it follows from (9.1) that $|D_2^1(x,y)| = 1$. Let x_2 denote the unique vertex of Γ in $D_2^1(x,y)$ and let x_3 be a neighbour of x_2 which is in $D_3^2(x,y)$. Since the edge xx_2 is 4-balanced and $D_3^2(x,y) \cup \{x_2\} \subseteq W_{x_2,x}$ we have that x_2 has at most one neighbour in $D_2^2(x,y) \cup D_1^2(x,y)$. However, as $k = 6$, this shows that x_2 has at least two neighbours in $D_1^1(x,y)$ and so the edge x_2x_3 is not 4-balanced. Consequently, for every edge $xy \in E(\Gamma)$ we have that $|D_3^2(x,y)| = |D_2^3(x,y)| = 1$.

It follows from the above comments and (9.1) that $|D_2^1(x,y)| = |D_1^2(x,y)| = 2$ for every edge $xy \in E(\Gamma)$. This implies that $|D_1^1(x,y)| = 3$ for every edge $xy \in E(\Gamma)$ and so, Γ is edge-regular with $\lambda = 3$.

Pick an edge $xy \in E(\Gamma)$. Let $D_2^1(x,y) = \{x_2,u\}$ and let x_3 be a neighbour of x_2 in $D_3^2(x,y)$. We observe that the three common neighbours of x_2 and x_3 are not all in $D_2^2(x,y)$, since the edge xx_2 is 4-balanced. Then, u is a common neighbour of x_2 and x_3 and there exist two common neighbours of x_2 and x_3 in $D_2^2(x,y)$. Moreover, since the edge xx_2 is 4-balanced, x_2 has no neighbours in $D_1^2(x,y)$. Furthermore, as $k = 6$ we have that x_2 has a neighbour, say z, in $D_1^1(x,y)$. It now follows that $\Gamma(x) \cap \Gamma(x_2) = \{u,z\}$, contradicting that $\lambda = 3$. ■

Theorem 9.3.3. *Let Γ be a regular NDB graph with valency k, diameter $d \geq 3$ and $\gamma = d+1$. Then $k \in \{3,4,5\}$.*

Proof. First note that a cycle C_n $(n \geq 3)$ is NDB with $\gamma(C_n)$ equal to the diameter of C_n. Therefore, $k \geq 3$.

Assume first that $\ell = 2$ and recall that in this case the set $D_2^1 = \{x_2,u\}$. We observe that x_1 and x_3 are the only neighbours of x_2 in the set $D_1^0 \cup D_3^2$. Furthermore, by Proposition 9.2.1(*ii*), x_2 has at most one neighbour in $D_1^1 \cup D_1^2$ and by Proposition 9.2.2(*i*), x_2 has at most two neighbours in D_2^2. Moreover, since $\ell = 2$, we also notice that x_2 has at most one neighbour in D_2^1. If x_2 and u are not adjacent, then $k \leq 5$. Assume next that x_2 and u are adjacent. We consider the cases $d \geq 4$ and $d = 3$ separately. If $d \geq 4$, we also have that $u \in W_{x_{d-1},x_d}$, and so $W_{x_{d-1},x_d} = \{x_0, x_1, \ldots, x_{d-1}, u\}$ (recall that $\gamma = d+1$). If $w \in D_1^1 \cup D_1^2$ is adjacent to x_2, then we have that $w \in W_{x_{d-1},x_d}$, a contradiction. Therefore, x_2 has no neighbours in $D_1^1 \cup D_1^2$. As x_2 has at most 2 neighbours in D_2^2, it follows that $k \leq 5$. If x_2 and u are adjacent and $d = 3$, then $k \leq 6$. However, by Proposition 9.3.2, there exists no regular NDB graph with valency $k = 6$, diameter $d = 3$ and $\gamma = 4$. This shows that $k \leq 5$.

Assume next that $\ell \geq 3$. By Propositions 9.2.1(ii) and 9.2.2(ii), x_2 has at most one neighbour in $D_1^1 \cup D_1^2$, and at most one neighbour in D_2^2. Since x_2 has at most two neighbours in D_3^2 (namely x_3 and u), it follows that $k \leq 5$. This concludes the proof. ■

Theorem 9.3.4. *Let Γ be a regular NDB graph with valency k, diameter $d \geq 3$ and $\gamma = d+1$. Then the following (i)–(iii) hold.*

(i) If $k = 3$, then $d \in \{3, 4, 5\}$.

(ii) If $k = 4$, then $d \in \{3, 4\}$.

(iii) If $k = 5$, then $d = 3$.

Proof. (i) Assume that $d \geq 6$ and consider first the case $\ell = 2$. Note that by Proposition 9.2.1(i) x_4 and x_5 have a neighbour in D_4^4 and D_5^5 respectively. If x_3 has a neighbour in D_3^3 then this contradicts Proposition 9.2.2(i). Therefore, x_3 and u are adjacent and so $u \in W_{x_{d-1},x_d}$. This and (9.2) implies that x_2 has no neighbours in $D_1^1 \cup D_1^2$. If x_2 and u are adjacent, then we have that $|W_{u,x_2}| = |W_{x_2,u}| = 1$, contradicting $\gamma = d+1$. Therefore, x_2 has a neighbour in D_2^2, contradicting Proposition 9.2.2(i).

If $\ell = 3$, then by Proposition 9.2.1(i) vertex x_5 has a neighbour in D_5^5. By Proposition 9.2.1(i) and Proposition 9.2.2(ii), x_3 and x_4 are both adjacent with u. But then $|W_{u,x_3}| = |W_{x_3,u}| = 1$, contradicting $\gamma = d+1$.

If $\ell = d-1$, then by Proposition 9.2.1(i) vertex x_3 has a neighbour in D_3^3. Proposition 9.2.1(i) and Proposition 9.2.2(ii) now force that x_2 has a neighbour in D_1^1 and that x_{d-1} and u are adjacent. As $|W_{x_{d-1},x_d}| = d+1$ we have that also x_d and u are adjacent (otherwise $u \in W_{x_{d-1},x_d}$). But now $|W_{u,x_{d-1}}| = |W_{x_{d-1},u}| = 1$, contradicting $\gamma = d+1$.

If $\ell = d$, then x_3 and x_4 both have a neighbour in D_3^3 and D_4^4 respectively, contradicting Proposition 9.2.2(ii).

Assume finally that $4 \leq \ell \leq d-2$. Similarly as above we see that x_ℓ and $x_{\ell+1}$ are not both adjacent to u, so either x_ℓ has a neighbour in D_ℓ^ℓ or $x_{\ell+1}$ has a neighbour in $D_{\ell+1}^{\ell+1}$ (but not both).

Therefore we have that $u \in W_{x_{d-1},x_d}$, and so x_2 has no neighbours in $D_1^1 \cup D_1^2$. Consequently, x_2 has a neighbour in D_2^2, contradicting Proposition 9.2.2(ii).

(ii) Assume $d \geq 5$. If $\ell = 2$, then by Proposition 9.2.1(i) vertex x_3 has at least one neighbour in D_3^3, while vertex x_4 has two neighbours in D_4^4. However, this contradicts Proposition 9.2.2(i).

If $\ell \geq 3$, then again by Proposition 9.2.1(i) vertex x_3 (vertex x_4, respectively) has at least one neighbour in D_3^3 (D_4^4, respectively), contradicting Proposition 9.2.2(ii).

(iii) Assume $d \geq 4$. It follows from the proof of Theorem 9.3.3 that in this case $\ell \in \{2,3\}$ holds. If $\ell = 2$, then by Proposition 9.2.1(ii) and since $k = 5$, vertex x_2 has at least one neighbour in D_2^2, while vertex x_3 has at least two neighbours in D_3^3. However, this contradicts Proposition 9.2.2(i).

If $\ell \geq 3$, then by Proposition 9.2.1(i) vertex x_3 has at least two neighbours in D_3^3, again contradicting Proposition 9.2.2(ii). This shows that $d = 3$. ■

Proposition 9.3.5. *Let* Γ *be a regular NDB graph with valency* k*, diameter* $d = 3$ *and* $\gamma = 4$*. Then for every edge* $xy \in E(\Gamma)$ *we have that* $|D_3^2(x,y)| = |D_2^3(x,y)| \neq 0$.

Proof. Let us pick an edge $xy \in E(\Gamma)$. Recall that by Lemma 9.1.2, we have that $|D_2^1(x,y)| = |D_1^2(x,y)|$, and so it follows from (9.1) that $|D_3^2(x,y)| = |D_2^3(x,y)|$ as well. Therefore, it remains to prove that the sets $D_3^2(x,y)$ and $D_2^3(x,y)$ are nonempty.

Assume to the contrary that the sets $D_2^3(x,y)$ and $D_3^2(x,y)$ are empty. As $\gamma = d+1 = 4$ we have that $|D_2^1(x,y)| = |D_1^2(x,y)| = 3$. In view of Theorem 9.3.3 we therefore have $k \in \{4,5\}$. Moreover, by Proposition 9.3.1 the set $D_3^3(x,y)$ is nonempty. Pick $z \in D_3^3(x,y)$ and note that there exists a vertex $w \in \Gamma(z) \cap D_2^2(x,y)$.

Assume first that $k = 4$. Then the set $D_1^1(x,y)$ is empty. Hence, there exist vertices $u \in D_2^1(x,y)$ and $v \in D_1^2(x,y)$ which are neighbours of w. We thus have $\{u,v,w,x,y\} \subseteq W_{w,z}$, contradicting $\gamma = 4$.

Assume next that $k = 5$. Note that in this case $|D_1^1(x,y)| = 1$. Let us denote the unique vertex of $D_1^1(x,y)$ by u. If w and u are not adjacent, then a similar argument as in the previous paragraph shows that $|W_{w,z}| \geq 5$, a contradiction. Therefore, w and u are adjacent, and so $W_{w,z} = \{x,y,u,w\}$. It follows that the remaining three neighbours of w (let us denote these neighbours by v_1, v_2, v_3) are also adjacent to z. As $\{u,w,z\} \subseteq W_{u,x}$, at least two of these three common neighbours (say v_1 and v_2) are in D_2^2 (recall D_3^2 and D_2^3 are empty). By the same argument as above (that is $\Gamma(v_1) \cap (D_2^1 \cup D_1^2) = \emptyset$ and $\Gamma(v_2) \cap (D_2^1 \cup D_1^2) = \emptyset$), v_1 and v_2 are adjacent to u, and so $\{u,w,v_1,v_2,z\} \subseteq W_{u,x}$, a contradiction. This shows that $D_3^2(x,y)$ and $D_2^3(x,y)$ are both nonempty. ■

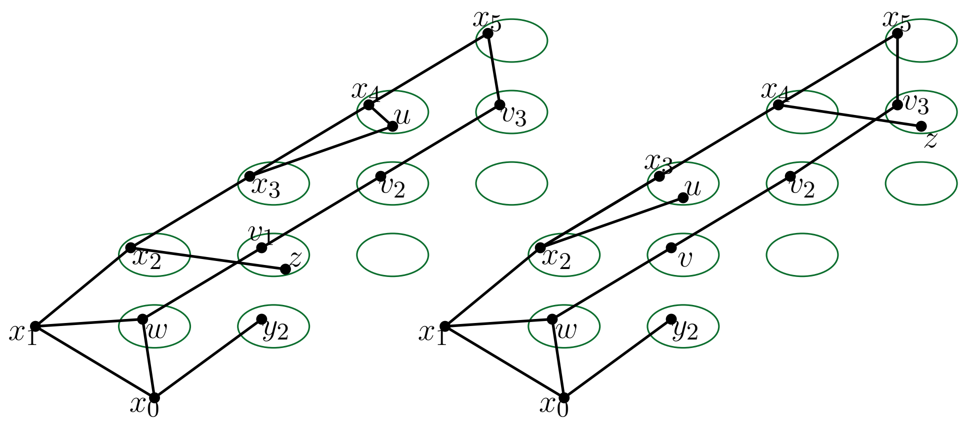

Figure 9.2: (a) Case $d=5$, $k=3$ and $\ell=4$ (left). (b) Case $d=5$, $k=3$ and $\ell=3$ (right).

9.4 Case $k=3$

Let Γ be a regular NDB graph with valency $k=3$, diameter $d\geq 3$ and $\gamma=\gamma(\Gamma)=d+1$. Recall that by Theorem 9.3.4(i) we have $d\in\{3,4,5\}$. In this section we first show that in fact $d=4$ or $d=5$ is not possible, and then classify NDB graphs with $k=d=3$. We start with a proposition which claims that $d\neq 5$. Although the proof of this proposition is rather tedious and lengthy, it is in fact pretty straightforward.

Proposition 9.4.1. *Let Γ be a regular NDB graph with valency $k=3$, diameter $d\geq 3$ and $\gamma=\gamma(\Gamma)=d+1$. Then $d\neq 5$.*

Proof. Assume to the contrary that $d=5$. Pick vertices x_0,x_5 of Γ such that $\partial(x_0,x_5)=5$. Pick also a shortest path x_0,x_1,x_2,x_3,x_4,x_5 from x_0 to x_5 in Γ. Let $D^i_j=D^i_j(x_1,x_0)$, let $\ell=\ell(x_0,x_1)$ and recall that $2\leq\ell\leq 5$. Observe that if $\ell\geq 3$, then there is a unique vertex $w\in D^1_1$ and a unique vertex $y_2\in D^2_1$. In this case x_2 and w are not adjacent, otherwise edge wx_1 is not 6-balanced. Similarly we could prove that w and y_2 are not adjacent, and so w has a neighbour v in D^2_2.

Assume first that $\ell=5$. Then by Proposition 9.2.1(i) vertex x_3 has exactly one neighbour in D^3_3. Now vertex x_2 has a neighbour in $D^2_1\cup D^2_2$, contradicting Proposition 9.2.2(ii).

Assume $\ell=4$. As x_2 has a neighbour in $D^2_1\cup D^2_2$, Propositions 9.2.1(i) and 9.2.2(ii) imply that x_4 is adjacent to u. If x_5 is adjacent to u, then $W_{u,x_4}=\{u\}$, a contradiction. Therefore, x_5 and u are not adjacent, and so $W_{x_4,x_5}=\{x_4,x_3,x_2,x_1,x_0,u\}$. Consequently, $w\notin W_{x_4,x_5}$, which implies $\partial(x_5,w)=4$. It follows that there exists a path w,v_1,v_2,v_3,x_5 of length 4, and it is easy to see that $v_1=v$, $v_2\in D^3_3$ and $v_3\in D^4_4$, see Figure 9.2(a).

If x_2 is adjacent with y_2, then $y_2\in W_{x_4,x_5}$, a contradiction. Therefore, x_2 has a neighbour

$z \in D_2^2$. If $z = v$, then $\{x_2, x_3, x_4, x_5, u, v, v_2, v_3\} \subseteq W_{x_2,x_1}$, a contradiction. Therefore $z \neq v$, $W_{x_2,x_1} = \{x_2, x_3, x_4, x_5, u, z\}$, and z is adjacent to y_2 (recall that z must be at distance 2 from x_0 and that y is not adjacent with x_1 and v). If z has a neighbour in $D_2^3 \cup D_3^3$, then this neighbour would be another vertex in W_{x_2,x_1}, which is not possible. The only other possible neighbour of z is v, and so z and v are adjacent. It is now clear that $W_{w,v} = \{w, x_0, x_1\}$, contradicting $\gamma = 6$.

Assume $\ell = 3$. By Proposition 9.2.1(i), we have that either x_4 is adjacent to u, or that x_4 has a neighbour in D_4^4. Let us first consider the case when x_4 and u are adjacent. If also x_3 and u are adjacent, then ux_3 is clearly not 6-balanced, and so Propositions 9.2.1(i) and 9.2.2(ii) imply that u and x_3 have a common neighbour v_2 in D_3^3. Since x_4x_5 is 6-balanced, v_2 must be at distance 2 from x_5, which implies that v_2 and x_5 have a common neighbour $v_3 \in D_4^4$. But now $\{x_2, x_3, x_4, x_5, u, v_2, v_3\} \subseteq W_{x_2,x_1}$, a contradiction. Therefore x_4 is not adjacent to u, and so x_4 has a neighbour z in D_4^4. Propositions 9.2.1(i) and 9.2.2(ii) imply that x_3 has no neighbours in $D_2^2 \cup D_2^3 \cup D_3^3$, and so x_3 is adjacent to u. This implies that z and x_5 are adjacent, as otherwise x_4x_5 is not 6-balanced. Similarly, by Proposition 9.2.2(ii) u has no neighbours in $D_2^3 \cup D_3^3$, and so u is adjacent to v (note that v is the unique vertex of D_2^2). As in the previous paragraph (since $w \notin W_{x_4,x_5} = \{x_4, x_3, x_2, x_1, x_0, u\}$) we obtain that there exists a path w, v, v_2, v_3, x_5 of length 4, and that $v_2 \in D_3^3$, $v_3 \in D_4^4$ (note that it could happen that $z = v_3$). Note that u and x_3 have no neighbours in D_3^3, and that the only neighbour of v in D_3^3 is v_2. Therefore, as $k = 3$, this implies that v_2 is the unique vertex of D_3^3. Let us now examine the cardinality of D_4^4. By Proposition 9.2.2(ii), both neighbours of x_5, different from x_4, are in D_4^4, and so $|D_4^4| \geq 2$. On the other hand, if v_2 has two neighbours in D_4^4, then wx_0 is not 6-balanced, and so v_3 is the unique neighbour of v_2 in D_4^4. As x_4 has exactly one neighbour in D_4^4 (namely z), this shows that $|D_4^4| = 2$ and that $v_3 \neq z$. But as Γ is a cubic graph, it must have an even order. Then, there exists a vertex t in D_5^5. Note that t is not adjacent to x_5, and so it must be adjacent to at least one of z, v_3. However, if t is adjacent to z, then x_2x_1 is not 6-balanced, while if it is adjacent to v_3, then wx_0 is not 6-balanced. This shows that $\ell \neq 3$

Assume finally that $\ell = 2$. By Proposition 9.2.1(i), vertex x_4 has a neighbour $z \in D_4^4$. Also by Proposition 9.2.1(i), vertex x_3 either has a neighbour in D_3^3, or is adjacent with u. Assume first that x_3 is adjacent with u. Note that in this case $x_2 \not\sim u$ (otherwise edge x_2u is not 6-balanced) and $\{x_4, x_3, x_2, x_1, x_0, u\} = W_{x_4,x_5}$. It follows that x_2 cannot have a neighbour in D_1^2 (otherwise the edge x_4x_5 is not 6-balanced) and so x_2 has a neighbour $v \in D_2^2$. Now if v has a neighbour $v_2 \in D_3^3$, then $\{x_2, x_3, x_4, x_5, z, v, v_2\} \subseteq W_{x_2,x_1}$, a contradiction. Therefore v has no neighbours in D_3^3, implying that $\partial(x_5, v) = 4$. But this forces $v \in W_{x_4,x_5}$, a contradiction. Thus $x_3 \not\sim u$, and it follows that x_3 has a neighbour $v_2 \in D_3^3$. As $\{x_2, x_3, x_4, x_5, v_2, z\} = W_{x_2,x_1}$, vertex x_2 has no neighbours in $D_1^2 \cup D_2^2$, implying that x_2 is adjacent to u. Since $W_{x_4,x_5} = \{x_4, x_3, x_2, x_1, x_0, u\}$, vertex z is adjacent to x_5, and vertices v_2 and x_5 have a common neighbour in D_4^4. Now, since x_1x_2 is 6-balanced we have that this common neighbour is in fact z, and so z is adjacent to v_2. Now consider the edge v_2z. Note that $\{x_1, x_2, x_3, v_2\} \subseteq W_{v_2,z}$. As $\partial(x_0, v_2) = 3$, there exist vertices y_1, y_2, such that x_0, y_1, y_2, v_2 is a path of length 3 between x_0 and v_2. Observe that $\{x_0, y_1, y_2, v_2\} \subseteq W_{v_2,z}$. As $\{x_1, x_2, x_3\} \cap \{x_0, y_1, y_2\} = \emptyset$, we have that $|W_{v_2,z}| \geq 7$, a contradiction. ∎

9.4.1 Case $d = 4$ is not possible

Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$ and $\gamma = \gamma(\Gamma) = d + 1$. We now consider the case $d = 4$. Our main result in this subsection is to prove that this case is not possible. For the rest of this subsection pick arbitrary vertices x_0, x_4 of Γ such that $\partial(x_0, x_4) = 4$. Pick a shortest path x_0, x_1, x_2, x_3, x_4 between x_0 and x_4. Let $D^i_j = D^i_j(x_1, x_0)$ and let $\ell = \ell(x_0, x_1)$. Let u denote the unique vertex of $D^{\ell-1}_{\ell} \setminus \{x_\ell\}$.

Proposition 9.4.2. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 4$ and $\gamma = \gamma(\Gamma) = d + 1 = 5$. With the notation above, we have that $\ell \neq 4$.*

Proof. Assume to the contrary that $\ell = 4$. Note that in this case since $k = 3$ and $|D^1_2| = |D^2_1| = 1$ we have that $|D^1_1| = 1$. Let w denote the unique vertex of D^1_1, and let z denote the neighbour of x_2, different from x_1 and x_3. Observe that $z \neq w$, as otherwise $x_1 w$ is not 5-balanced. Similarly, w is not adjacent to the unique vertex y_2 of D^2_1. Observe also that $\{x_0, x_1, x_2, x_3\} \subseteq W_{x_3,u}$. We claim that $u \in \Gamma(x_4)$. To prove this, suppose that x_4 and u are not adjacent. Then $x_4 \in W_{x_3,u}$, and so z is contained in D^2_2. Observe that $\partial(z, u) = 2$, otherwise $x_3 u$ is not 5-balanced. Therefore, u and z must have a common neighbour z_1 and it is clear that $z_1 \in D^3_3$. But now $\{x_2, x_3, x_4, u, z, z_1\} \subseteq W_{x_2,x_1}$, a contradiction. This proves our claim that $u \sim z$.

Suppose now that $z = y_2$. Then $D^3_2 \cup D^4_3 \cup \{u, x_2, x_3, x_4, y_2\} \subseteq W_{x_2,x_1}$. Note that by the NDB condition we have $|D^3_2 \cup D^4_3| = 3$, and so $x_2 x_1$ is not 5-balanced, a contradiction. We therefore have that $z \in D^2_2$.

By Proposition 9.2.2(ii) it follows that u and x_4 have a neighbour z_1 and z_2 in D^3_3, respectively. We observe $z_1 \neq z_2$, as otherwise $x_4 u$ is not 5-balanced. Note that z has no neighbours in D^3_3, as otherwise $x_2 x_1$ is not 5-balanced. Therefore, z is not adjacent to any of z_1, z_2, which gives us $W_{x_3,x_4} = W_{x_3,u} = \{x_3, x_2, x_1, x_0, z\}$. Consequently, $\partial(w, u) = \partial(w, x_4) = 3$, and so the (unique) neighbour of w in D^2_2 is adjacent to both z_1 and z_2. But this implies that $w x_0$ is not 5-balanced, a contradiction. ∎

Proposition 9.4.3. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 4$ and $\gamma = \gamma(\Gamma) = d + 1 = 5$. With the notation above, we have that $\ell \neq 3$.*

Proof. Suppose that $\ell = 3$. By Lemma 9.1.2 we have $|D^2_1| = 1$, and since $k = 3$ also $|D^1_1| = 1$. Let w and y_2 denote the unique vertex of D^1_1 and D^2_1, respectively. Since $\gamma = 5$, y_2 has at least one neighbour y_3 in D^3_2, and $|D^4_3| \leq 2$. If $D^4_3 = \emptyset$, then there are three vertices in D^3_2, which are all adjacent to y_2, contradicting $k = 3$. By Proposition 9.4.2 we have that $|D_3^4| \neq 2$, and so $|D_3^4| = 1$, $|D_2^3| = 2$. Let y_4 denote the unique element of D_3^4 and let u_1 denote the unique element of $D_2^3 \setminus \{y_3\}$. Without loss of generality assume that y_4 and y_3 are adjacent. Observe that $\Gamma(y_2) = \{x_0, y_3, u_1\}$, and so w has a neighbour $v \in D_2^2$, and it is easy to see that v is the unique vertex of D_2^2 (see Figure 9.3(a)). By Proposition 9.2.1(i) we find that either $x_3 \in \Gamma(u)$, or x_3 has a neighbour in D_3^3.

Case 1: there exists $z \in \Gamma(x_3) \cap D_3^3$. Note that in this case we have $W_{x_2,x_1} = \{x_2, x_3, x_4, u, z\}$. We split our analysis into two subcases.

Subcase 1.1: vertices u and x_4 are not adjacent. As x_2x_1 is 5-balanced and as v is the unique vertex of D_2^2, this forces that u is adjacent with v and z. As every vertex in D_3^3 is at distance 3 from x_1 and as vertices u, x_3 already have three neighbours each, this implies that beside z there is at most one more vertex in D_3^3 (which must be adjacent with v). But this shows that x_4 could have at most one neighbour in D_3^3 (observe that z could not be adjacent with x_4, as otherwise z is not at distance 3 from x_0), and consequently x_4 has at least one neighbour in $D_4^4 \cup D_3^4$. But now x_2x_1 is not 5-balanced, a contradiction.

Subcase 1.2: vertices u and x_4 are adjacent. By Proposition 9.2.2(ii), vertex u is either adjacent to $v \in D_2^2$ or to $z \in D_3^3$. If u is adjacent to v, then $\{x_0, x_1, x_2, u, v, w\} \subseteq W_{u,x_4}$, a contradiction. This shows that $u \sim z$. Note that the third neighbour of z is one of the vertices v, y_3, u_1, and so z and x_4 are not adjacent. Consequently, $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, z\}$, and so w must be at distance 3 from x_4. Therefore, v and x_4 have a common neighbour $v_1 \in D_3^3$. Note that $v_1 \neq z$ as z and x_4 are not adjacent. Every vertex in D_3^3, different from z and v_1, must be adjacent with v in order to be at distance 3 from x_1. This shows that $|D_3^3| \leq 3$. If there exists vertex $v_2 \in D_3^3$, which is different from z and v_1, then there must be a vertex $t \in D_4^4$ (recall that Γ is of even order). As t could not be adjacent with x_4, it must be adjacent with at least one of v_1, v_2. However, this is not possible (note that in this case $\{w, v, v_1, v_2, x_4, t\} \subseteq W_{w,x_0}$, a contradiction). Therefore, $D_3^3 = \{z, v_1\}$ and $D_4^4 = \emptyset$. It follows that y_4 is adjacent with v_1 and u_1. If z and v are adjacent, then $W_{x_1,w} = \{x_1, x_2, u, x_3\}$, contradicting $\gamma = 5$. Therefore, z is adjacent to either y_3 or u_1. This shows that either y_3 or u_1 is contained in $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, z\}$, a contradiction.

Case 2: x_3 and u are adjacent. Observe that $x_4 \notin \Gamma(u)$, otherwise ux_3 is not 5-balanced. It follows that $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, u\}$, and so $\partial(w, x_4) = 3$. Therefore there exists a common neighbour z of x_4 and v, and note that $z \in D_3^3$. Reversing the roles of the paths x_0, x_1, x_2, x_3, x_4 and x_1, x_0, y_2, y_3, y_4, we get that u_1 and y_3 are adjacent, and that $y_4 \notin \Gamma(u_1)$. As $|W_{x_1,w}| = 5$, vertex u must have a neighbour, which is at distance 3 from x_1 and at distance 4 from w. As x_4, y_3 and u_1 ar all at distance 3 from w, this implies that u has a neighbour $z_1 \in D_3^3$, which is not adjacent with v (and is therefore different from z). Note that since z_1 is at distance 3 from x_0, it is adjacent with u_1. As $k = 3$, v has a neighbour $z_2 \neq z$ in D_3^3. Pick now a vertex $t \in D_4^4$ (observe that $D_4^4 \neq \emptyset$ as Γ has even order). If t is adjacent with x_4 or with z_1, then $t \in W_{x_2,x_1} = \{x_2, x_3, x_4, u, z_1\}$, a

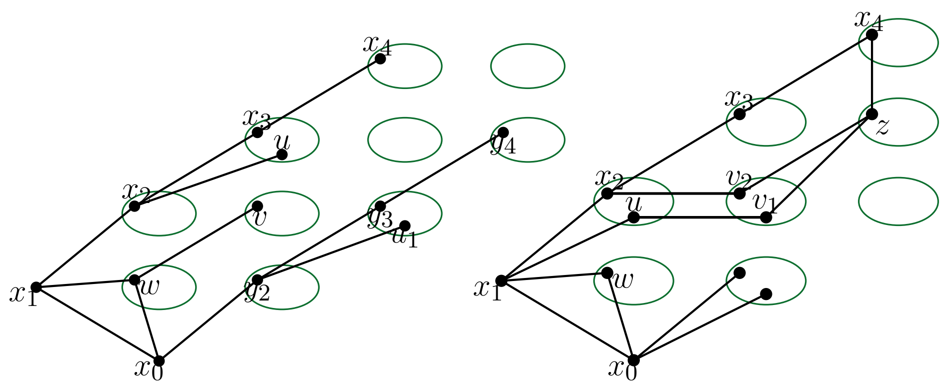

Figure 9.3: (a) Case $d = 4$, $k = 3$ and $\ell = 3$ (left). (b) Case $d = 4$, $k = 4$ and $\ell = 2$ (right).

contradiction. If t is adjacent with z or z_2, then $t \in W_{w,x_0} = \{w, v, z, z_2, x_4\}$, a contradiction. This finally proves that $\ell \neq 3$. ■

Proposition 9.4.4. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 4$ and $\gamma = \gamma(\Gamma) = d+1 = 5$. With the notation above, Γ is triangle-free.*

Proof. Pick an edge $xy \in E(\Gamma)$ and let $D^i_j = D^i_j(x,y)$. If either D^4_3 or D^3_4 is nonempty, then Propositions 9.4.2 and 9.4.3 together with Lemma 9.1.2 imply that $|D^1_2| = |D^2_1| = 2$. As Γ is 3-regular, the set D^1_1 is empty, and so xy is not contained in any triangle.

Assume next that $D^4_3 = D^3_4 = \emptyset$. If the edge xy is contained in a triangle, then D^1_2 and D^2_1 both contain at most one vertex, and so D^2_3 and D^3_2 could contain at most two vertices as Γ is 3-regular. We thus have $|W_{x,y}| \leq 4$, contradicting $\gamma = 5$. The result follows. ■

Proposition 9.4.5. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$ and $\gamma = \gamma(\Gamma) = d+1$. Then $d \neq 4$.*

Proof. Towards a contradiction suppose that $d = 4$, and so $\gamma = 5$. Assume the notation from the first paragraph of this subsection, and note that Propositions 9.4.2 and 9.4.3 imply that $\ell = 2$. By Lemma 9.1.2 we have $|D^2_1| = 2$. Let u_1, y_2 denote the vertices of D^2_1. Note that D^1_1 is empty. We also observe that by Proposition 9.2.1(i) either $u \in \Gamma(x_3)$, or x_3 has a neighbour in D^3_3. We consider these two cases separately.

Case 1: u and x_3 are adjacent. Then $\{x_0, x_1, x_2, x_3, u\} = W_{x_3,x_4}$, and so neither x_2 nor u have neighbours in D^2_1. Since Γ is triangle-free, there exists $w \in \Gamma(x_2) \cap D^2_2$, and w has a neighbour in D^2_1 (by definition of the set D^2_2). We may assume without loss of generality that $w \in \Gamma(y_2)$. Note that $\partial(w, x_3) = 2$, and so $\partial(w, x_4) = 2$ as well, as otherwise x_3x_4 is not 5-balanced. It follows that there exists a common neighbour z of w and x_4, and it is clear that $z \in D^3_3$.

Similarly we find that u has a neighbour $w_1 \in D_2^2$, and as $k=3$, we have that $w_1 \neq w$. Note that $\{x_2, x_1, x_0, w, y_2\} = W_{x_2,x_3}$, and so $\partial(x_3, u_1) = 3$ (otherwise $u_1 \in W_{x_2,x_3}$, a contradiction). Note however that $\partial(x_3, u_1) = 3$ is only possible if w_1 and u_1 are adjacent. A similar argument as above shows that w_1 and x_4 must have a common neighbour $z_1 \in D_3^3$. If $z_1 = z$, then $\{z, w, w_1, y_2, u_1, x_0\} \subseteq W_{z,x_4}$, a contradiction. Therefore $z_1 \neq z$, and it is now clear that $D_2^2 = \{w, w_1\}$, $D_3^3 = \{z, z_1\}$. If there exists $t \in D_4^4$, then t is adjacent to either z or z_1, but none of these two possible edges is 5-balanced, and so $D_4^4 = \emptyset$. If z (z_1, respectively) has a neighbour in D_3^4, then x_2x_1 (ux_1, respectively) is not 5-balanced, a contradiction. As Γ is triangle-free, z and z_1 both have a neighbour in D_2^3. Assume now for a moment that there exists a vertex $y_4 \in D_3^4$. In this case $\gamma = 5$ forces that there is a unique vertex in D_2^3, which is therefore adjacent to both z and z_1, to y_4 and to at least one of y_2, u_1, contradicting $k=3$. It follows that $D_3^4 = \emptyset$. Let us denote the neighbours of z and z_1 in D_2^3 by v and v_1 respectively. Note that as zx_4 and z_1x_4 are 5-balanced, we have that $W_{z,x_4} = \{z, w, v, y_2, x_0\}$ and $W_{z_1,x_4} = \{z_1, w_1, v_1, u_2, x_0\}$. It follows that v and v_1 must be adjacent to y_2 and u_1, respectively, and so $v \neq v_1$. As $k=3$, also v and v_1 are adjacent. It is now easy to see that Γ is not NDB with $\gamma = 5$ (for example, edge x_1u is not 5-balanced). This shows that u and x_3 are not adjacent.

Case 2: x_3 has a neighbour w in D_3^3. As Γ is triangle-free, x_2 has a neighbour z in $D_1^2 \cup D_2^2$, and $w \not\sim x_4$. If $z \in D_1^2$, then $\{x_0, x_1, x_2, x_3, z, w\} \subseteq W_{x_3,x_4}$, a contradiction. This yields that $z \in D_2^2$. If $\partial(z, x_4) \geq 3$, then again $\{x_0, x_1, x_2, x_3, z, w\} \subseteq W_{x_3,x_4}$, a contradiction. Therefore, z and x_4 have a common neighbour $w_1 \in D_3^3$, and $w_1 \neq w$ as $w \not\sim x_4$. But now $\{x_2, x_3, x_4, z, w, w_1\} \subseteq W_{x_2,x_1}$, a contradiction. This finishes the proof. ∎

9.4.2 Case $d = 3$

In this subsection we consider the case $d=3$. We start with the following proposition.

Proposition 9.4.6. *Let Γ be a regular NDB graph with valency $k=3$, diameter $d=3$ and $\gamma=4$. Then for every edge x_0x_1 of Γ we have that $|D_2^1(x_1,x_0)| = |D_1^2(x_1,x_0)| = 2$.*

Proof. Pick an edge x_0x_1 of Γ and let $D_j^i = D_j^i(x_1,x_0)$. Observe first that $|D_2^1| \leq 2$ as $k=3$. By Proposition 9.3.5 we have that $D_3^2 \neq \emptyset$, and so pick $x_3 \in D_3^2$. Note that x_1 and x_3 have a common neighbour $x_2 \in D_2^1$. Assume to the contrary that $|D_2^1| = 1$, and so $|D_3^2| = 2$, $|D_1^1| = 1 = |D_1^2|$. Let us denote the unique vertex of D_1^2 by y_2 (note that y_2 has two neighbors, say y_3 and u_1 in D_2^3), the unique vertex of D_1^1 by w, and the unique vertex of $D_3^2 \setminus \{x_3\}$ by u (note that $\Gamma(x_2) = \{x_1, x_3, u\}$). Note that w has a neighbour v in D_2^2, and that $D_2^2 = \{v\}$.

Assume first that u and x_3 are not adjacent. Then $W_{x_2,x_3} = \{x_2, u, x_1, x_0\}$, and so w is at distance 2 from x_3 (otherwise $w \in W_{x_2,x_3}$). It follows that x_3 is adjacent with v. Similarly we show that

u is adjacent with v. As none of the neighbours of v is contained in D_3^3, every vertex from D_3^3 must be adjacent to either u or x_3, and so $D_3^3 \cup \{x_2, x_3, u\} \subseteq W_{x_2,x_1}$. It follows that $|D_3^3| \leq 1$. As Γ is a cubic graph, it must have an even order, which gives us $D_3^3 = \emptyset$. This shows that both u and x_3 have a neighbour in D_2^3. But now $\{y_2, y_3, u_1, x_3, u\} \cup D_2^3 \subseteq W_{y_2,x_0}$, a contradiction.

Therefore, u and x_3 must be adjacent, and they have a common neighbour x_2. Let z_1 and z_2 denote the third neighbour of u and x_3, respectively. If $z_1 = z_2$ then ux_3 is not 4-balanced, and so we have that $z_1 \neq z_2$. Furthermore, as $\{x_2, x_3, u\} \subseteq W_{x_2,x_1}$, not both of z_1, z_2 are contained in $D_3^3 \cup D_2^3$. Therefore, either z_1 or z_2 is equal to v. Without loss of generality assume that $z_1 = v$. But then $d = 3$ forces $W_{x_2,u} = \{x_2, x_1, x_0\}$, a contradiction. This shows that $|D_2^1| = 2$, and by Lemma 9.1.2 also $|D_1^2| = 2$. ■

Corollary 9.4.7. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d = 3$ and $\gamma = 4$. Then Γ is triangle-free and $D_3^3(x,y) = \emptyset$ for every edge xy of Γ.*

Proof. Pick an arbitrary edge xy of Γ and let $D_j^i = D_j^i(x,y)$. By Proposition 9.3.5 we get that the sets D_2^1, D_1^2, D_3^2 and D_2^3 are all nonempty. Furthermore, by Proposition 9.4.6 and Lemma 9.1.2 we have that $|D_2^1| = |D_1^2| = 2$ and $|D_2^3| = |D_3^2| = 1$ (recall that $\gamma = 4$). Since $k = 3$, it follows that $D_1^1 = \emptyset$. This shows that Γ is triangle-free.

We next assert the set D_3^3 is empty. Suppose to the contrary there exists $z \in D_3^3$ and let w denote a neighbour of z. Assume first that $w \in D_2^2$. Since $D_1^1 = \emptyset$, there exist vertices $u \in D_2^1$ and $v \in D_1^2$ which are neighbours of w. We thus have $\{u, v, w, x, y\} \subseteq W_{w,z}$, contradicting $\gamma = 4$. This shows that $w \notin D_2^2$. Therefore z is adjacent to both vertices which are in D_2^3 and D_3^2. As z has three neighbours, none of which is in D_2^2, and as $|D_3^2| = |D_2^3| = 1$, it follows that z has a neighbour $w' \in D_3^3$. But by the same argument as above, w' must be adjacent to both vertices in D_2^3 and D_3^2, contradicting the fact that Γ is triangle-free. ■

Theorem 9.4.8. *Let Γ be a regular NDB graph with valency $k = 3$, diameter $d \geq 3$ and $\gamma = d+1$. Then Γ is isomorphic to the 3-dimensional hypercube Q_3.*

Proof. By Theorem 9.3.4(i), Proposition 9.4.1 and Proposition 9.4.5 we have that $d = 3$. Pick an edge xy of Γ and let $D_j^i = D_j^i(x,y)$. Observe that Γ is triangle-free and $D_3^3 = \emptyset$ by Corollary 9.4.7. We first show that $D_2^2 = \emptyset$ as well. Observe that as $D_1^1 = \emptyset$, every vertex of D_2^2 must have a neighbour in both D_2^1 and D_1^2. This shows $|D_2^2| \in \{1,2,3\}$, and so $|V(\Gamma)| \in \{9,10,11\}$. However, since Γ is regular with $k = 3$, we have $|V(\Gamma)| = 10$ and $|D_2^2| = 2$. In [7], it is shown that the number of connected 3-regular graphs with 10 vertices is 19, but only 5 of them have diameter $d = 3$ and girth $g \geq 4$. Out of these five graphs, only four of them have all vertices with eccentricity 3, see Figure 9.4. It is easy to see that none of these graphs is NDB with $\gamma = 4$. This shows that $D_2^2 = \emptyset$, and so $|V(\Gamma)| = 8$. But it is well-known (and also easy to see) that Q_3 is the only cubic triangle-free graph with eight vertices and diameter $d = 3$. ■

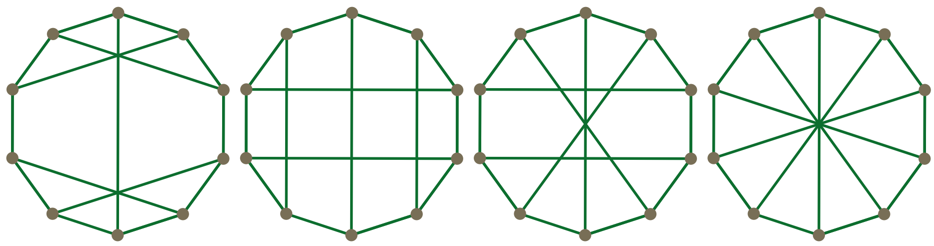

Figure 9.4: Connected 3-regular graphs of order 10 with diameter $d=3$, girth $g\geq 4$ and with all vertices with eccentricity 3.

9.5 Case $k=4$

Let Γ be a regular NDB graph with valency $k=4$, diameter $d\geq 3$ and $\gamma=\gamma(\Gamma)=d+1$. Recall that by Theorem 9.3.4(ii) we have $d\in\{3,4\}$. In this section we first show that case $d=4$ is not possible, and then classify regular NDB graphs with $k=4$ and $d=3$. We start with the following lemma.

Lemma 9.5.1. *Let Γ be a regular NDB graph with valency $k=4$, diameter $d=4$ and $\gamma=\gamma(\Gamma)=d+1$. Pick vertices x_0,x_4 of Γ such that $\partial(x_0,x_4)=4$, and pick a shortest path x_0,x_1,x_2,x_3,x_4 between x_0 and x_4. Let $\ell=\ell(x_0,x_1)$, $D_j^i=D_j^i(x_1,x_0)$ and $D_\ell^{\ell-1}=\{x_\ell,u\}$. Then $\ell=2$. Moreover, $u\sim x_2$ and $u\sim x_3$.*

Proof. Assume first that $\ell=4$. By Proposition 9.2.1(i), vertex x_3 has a neighbour z in D_3^3. Now $W_{x_2,x_1}=\{x_2,x_3,x_4,u,z\}$, and so x_2 has no neighbours in $D_2^2\cup D_1^2$. Consequently, x_2 has two neighbours in D_1^1, contradicting Proposition 9.2.1(ii).

Assume now that $\ell=3$. By Proposition 9.2.1(i) x_3 does not have neighbours in $D_2^3\cup D_2^2$, and so by Proposition 9.2.2(ii) we get that x_3 and u are adjacent, and that x_3 has a neighbour z in D_3^3. By Proposition 9.2.2(ii) vertex x_2 has no neighbours in $D_2^2\cup D_1^2$, and so x_2 has a neighbour w in D_1^1. Now $\{x_3,x_2,x_1,x_0,w\}\subseteq W_{x_3,x_4}$, implying that x_4 is adjacent to both u and z. Similarly, $\{u,x_2,x_1,x_0,w\}\subseteq W_{u,x_4}$, and so u has no neighbours in $D_2^2\cup D_2^3$. It follows that u has a neighbour in D_3^3, and by Proposition 9.2.2(ii), this neighbour is z. But now the edge x_3u is not 5-balanced, a contradiction.

This shows that $\ell=2$. By Proposition 9.2.1(i), vertex x_3 has either one or two neighbours in D_3^3. If x_3 has two neighbours in D_3^3, then by Proposition 9.2.2(i) vertex x_2 has no neighbours in $D_2^2\cup D_1^2$. Therefore, x_2 is adjacent to the unique vertex $w\in D_1^1$, and is also adjacent to u. But now we have that $\{x_3,x_2,x_1,x_0,u,w\}\subseteq W_{x_3,x_4}$, a contradiction.

Therefore, x_3 has exactly one neighbour in D_3^3. As by Proposition 9.2.1(i) vertex x_3 has no neighbours in $D_2^2 \cup D_2^3$, we have that $x_3 \sim u$. Consequently $\{x_3, x_2, x_1, x_0, u\} \subseteq W_{x_3,x_4}$, and so x_2 and u have no neighbours in $D_1^1 \cup D_1^2$. Since $k = 4$ and since edges x_2x_1 and ux_1 are 5-balanced, it follows that both of x_2 and u have exactly one neighbour in D_2^2, and that $x_2 \sim u$. ■

Proposition 9.5.2. *Let* Γ *be a regular NDB graph with valency* $k = 4$*, diameter* $d \geq 3$ *and* $\gamma = \gamma(\Gamma) = d + 1$*. Then* $d \neq 4$*.*

Proof. Assume on the contrary that $d = 4$. Pick vertices x_0, x_4 of Γ such that $\partial(x_0, x_4) = 4$. Pick a shortest path x_0, x_1, x_2, x_3, x_4 between x_0 and x_4. Let $D_j^i = D_j^i(x_1, x_0)$, let $\ell = \ell(x_0, x_1)$ and let $D_\ell^{\ell-1} = \{x_\ell, u\}$. Recall that by Lemma 9.5.1 we have that $\ell = 2$ and that vertex u is adjacent with x_2 and x_3. Let z denote a neighbour of x_3 in D_3^3 (note that by Proposition 9.2.1(i) vertex x_3 has no neighbours in $D_2^2 \cup D_2^3$).

Since $W_{x_3,x_4} = \{x_3, x_2, x_1, x_0, u\}$, vertices x_2 and u have no neighbours in $D_1^1 \cup D_1^2$. Let us denote the neighbours of u and x_2 in D_2^2 by v_1, v_2, respectively. Note that $v_1 \neq v_2$, otherwise edge ux_2 is not 5-balanced. Furthermore, $\{x_3, x_2, x_1, x_0, u\} = W_{x_3,x_4}$ implies that x_4 and z are adjacent, and that x_4 is at distance 2 from both v_1 and v_2. Consequently, v_1 and v_2 both have a common neighbour, say z_1 and z_2 respectively, with x_4, and these common neighbours must be in D_3^3. But as edges x_2x_1 and ux_1 are 5-balanced, this implies that $z_1 = z = z_2$ (see Figure 9.3(b)).

Note that v_1 and v_2 both have at least one neighbour in $D_1^1 \cup D_1^2$. Let us denote a neighbour of v_1 (v_2, respectively) in $D_1^1 \cup D_1^2$ by w_1 (w_2, respectively). If $w_1 \neq w_2$, then $\{z, v_1, v_2, w_1, w_2, x_0\} \subseteq W_{z,x_4}$, contradicting $\gamma = 5$. Therefore $w_1 = w_2$ and by applying Lemma 9.5.1 to the path x_0, w_1, v_1, z, x_4 we get that vertices v_1 and v_2 are adjacent. But now it is easy to see that $W_{u,x_2} = \{u, v_1\}$, a contradiction. This finishes the proof. ■

Proposition 9.5.3. *Let* Γ *be a regular NDB graph with valency* $k = 4$*, diameter* $d = 3$ *and* $\gamma = \gamma(\Gamma) = 4$*. Then for every edge* x_0x_1 *of* Γ *we have that* $|D_2^1(x_1, x_0)| = |D_1^2(x_1, x_0)| = 2$*.*

Proof. Pick an edge x_0x_1 of Γ and let $D_j^i = D_j^i(x_1, x_0)$. By Proposition 9.3.5 we have that $D_3^2 \neq \emptyset$, and so $\gamma = 4$ implies $|D_2^1| \leq 2$. Assume to the contrary that $|D_2^1| = 1$, and so $|D_3^2| = 2$, $|D_1^1| = 2$ and $|D_1^2| = 1$. Let x_3, u be vertices of D_3^2, and let x_2 be the unique vertex of D_2^1. Let z denote the neighbour of x_2, different from x_1, x_3, u, and note that $z \in D_2^2 \cup D_1^2 \cup D_1^1$. In each of these three cases we derive a contradiction.

Assume first that $z \in D_2^2$. Then $D_2^1(x_2, x_1) = \{x_3, u, z\}$, and $\gamma = 4$ forces $D_3^2(x_2, x_1) = \emptyset$, contradicting Proposition 9.3.5.

Assume next that $z \in D_1^2$ (note that z is the unique vertex in D_1^2). Then $\{x_2, z, x_3, u\} \cup D_2^3 \subseteq W_{x_2,x_1}$. As $D_2^3 \neq \emptyset$ by Proposition 9.3.5, this contradicts $\gamma = 4$.

Assume finally that $z \in D_1^1$. Recall that $|D_1^1| = 2$ and denote the other vertex of D_1^1 by w. If z and w are adjacent, then $W_{x_1,z} = \{x_1\}$, a contradiction. If z has a neighbour $v \in D_2^2$, then $\{z, v, x_2, u, x_3\} \subseteq W_{z,x_0}$, a contradiction. This shows that z is adjacent to the unique vertex of D_1^2. Let us denote this vertex by y_2. As $W_{x_2,x_3} = W_{x_2,u} = \{x_2, x_1, x_0, z\}$, vertices x_3 and u are both at distance 2 from y_2. But this shows that $W_{z,y_2} = \{x_1, z, x_2\}$, a contradiction. ■

Theorem 9.5.4. *Let Γ be a regular NDB graph with valency $k = 4$, diameter $d \geq 3$ and $\gamma = \gamma(\Gamma) = d + 1$. Then Γ is isomorphic to the line graph of the 3-dimensional hypercube Q_3.*

Proof. By Theorem 9.3.4(ii) and Proposition 9.5.2 we have that $d = 3$. Pick an arbitrary edge xy of Γ. By Proposition 9.5.3 we have that $|D_2^1(x,y)| = |D_1^2(x,y)| = 2$. Consequently $|D_1^1(x,y)| = 1$, and so Γ is an edge-regular graph with $\lambda = 1$. Observe that $\gamma = 4$ also implies that $|D_3^2(x,y)| = |D_2^3(x,y)| = 1$. Observe that Γ contains $|V(\Gamma)|k/6 = 2|V(\Gamma)|/3$ triangles, and so $|V(\Gamma)|$ is divisible by 3.

Pick vertices x_0, x_3 of Γ at distance 3 and let x_0, x_1, x_2, x_3 be a shortest path from x_0 to x_3. Abbreviate $D_j^i = D_j^i(x_1, x_0)$. Obviously $D_3^2 = \{x_3\}$ and $x_2 \in D_2^1$. Let us denote the other vertex of D_2^1 by u, the vertices of D_1^2 by y_2, v, the vertex of D_2^3 by y_3 and the vertex of D_1^1 by w. Without loss of generality we may assume that y_2 and y_3 are adjacent. Since Γ is edge-regular with $\lambda = 1$, we also obtain that x_2 and u are adjacent, that y_2 and v are adjacent, and that w has two neighbours, say z_1 and z_2, in D_2^2, and that z_1, z_2 are also adjacent. As $W_{x_2,x_3} = \{x_2, x_1, x_0, u\}$, x_3 is at distance 2 from w, and so x_3 is adjacent to exactly one of z_1, z_2. Without loss of generality we could assume that x_3 and z_1 are adjacent.

Note that $\Gamma(w) = \{x_0, x_1, z_1, z_2\}$, and so x_2 and w are not adjacent. Vertex x_2 is also not adjacent to y_2, as otherwise edge x_2y_2 is not contained in a triangle. If $x_2 \sim v$ then $v \sim u$ and the edge ux_2 is contained in two triangles, contradicting $\lambda = 1$. It follows that x_2 has no neighbours in D_1^2. Therefore, x_2 has a neighbour in D_2^2. Consequently, by Proposition 9.2.2(i), x_3 could have at most one neighbour in $D_3^3 \cup D_2^3$.

We now show that $D_3^3 = \emptyset$. Assume to the contrary that there exists $t \in D_3^3$. If t is adjacent to z_1 or z_2, then $\{w, z_1, z_2, x_3, t\} \subseteq W_{w,x_0}$, a contradiction. If t is adjacent with $z \in D_2^2 \setminus \{z_1, z_2\}$, then z has a neighbour in D_2^1 and a neighbour in D_1^2, implying that $|W_{z,t}| \geq 5$, a contradiction. It follows that t has no neighbours in D_2^2, and so t is adjacent with x_3 (and with y_3). Now the unique common neighbour of x_3 and t must be contained in $D_3^3 \cup D_2^3$, contradicting the fact that x_3 could have at most one neighbour in $D_3^3 \cup D_2^3$. This shows that $D_3^3 = \emptyset$.

Let us now estimate the cardinality of D_2^2. Observe that each $z \in D_2^2 \setminus \{z_1, z_2\}$ has a neighbour in D_2^1. But u could have at most two neighbours in D_2^2, while x_2 has exactly one neighbour in D_2^2. It follows that $2 \leq |D_2^2| \leq 5$, and so $11 \leq |V(\Gamma)| \leq 14$. As $|V(\Gamma)|$ is divisible by 3, we have that $|V(\Gamma)| = 12$. By [42, Corollary 6], there are just two edge-regular graphs on 12 vertices with

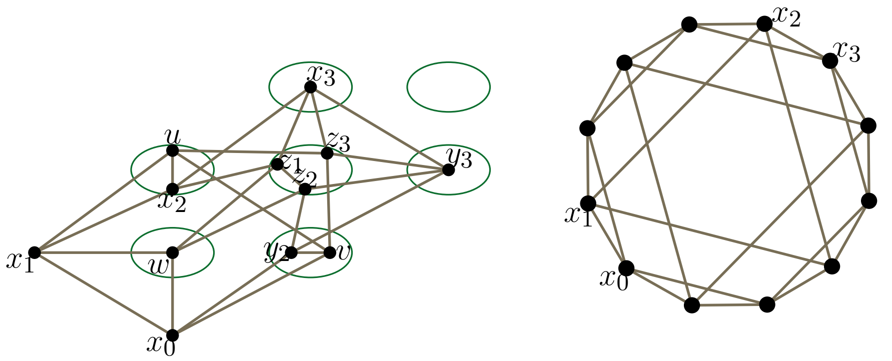

Figure 9.5: The line graph of Q_3, drawn in two different ways.

$\lambda = 1$, namely the line graph of 3-dimensional hypercube (see Figure 9.5), and the line graph of the Möbius ladder graph on eight vertices. It is easy to see that the latter one is not even distance-balanced. ■

9.6 Case $k = 5$

Let Γ be a regular NDB graph with valency $k = 5$, diameter $d \geq 3$ and $\gamma = \gamma(\Gamma) = d+1$. Recall that by Theorem 9.3.4 we have that $d = 3$, and so $\gamma = 4$. In this section we classify such NDB graphs. We first show that in this case we have that $|D_2^1(x_1, x_0)| = |D_1^2(x_1, x_0)| = 2$ for every edge x_1x_0 of Γ.

Proposition 9.6.1. *Let Γ be a regular NDB graph with valency $k = 5$, diameter $d = 3$ and $\gamma = 4$. Then for every edge x_0x_1 of Γ we have that $|D_2^1(x_1, x_0)| = |D_1^2(x_1, x_0)| = 2$.*

Proof. Pick an edge x_0x_1 of Γ and let $D_j^i = D_j^i(x_1, x_0)$. By Proposition 9.3.5 we have that $D_3^2 \neq \emptyset$, and so $\gamma = 4$ implies $|D_2^1| \leq 2$. Assume to the contrary that $|D_2^1| = 1$, and so $|D_3^2| = 2$, $|D_1^1| = 3$ and $|D_1^2| = 1$. Let x_3, u be vertices of D_3^2, and let x_2 be the unique vertex of D_2^1. Let us denote the unique vertex of D_1^2 by y_2, and the vertices of D_1^1 by z_1, z_2, z_3. Note that also $|D_2^3| = 2$, and let us denote these two vertices by y_3, u_1. Clearly we have that x_2 is adjacent to both x_3 and u, and y_2 is adjacent to both y_3 and u_1, see the diagram on the left side of Figure 9.6.

Observe that each edge xy of Γ is contained in at least one triangle; otherwise $|W_{x,y}| \geq 5 > \gamma$, a contradiction. Therefore, x_2 and y_2 both have at least one neighbour in D_1^1. On the other hand, these two vertices could not have more than one neighbour in D_1^1, as otherwise $|W_{x_2,x_3}| \geq 5$ ($|W_{y_2,y_3}| \geq 5$, respectively), a contradiction. Without loss of generality we could assume that z_1 is the unique neighbour of x_2 in D_1^1. Note that it follows from Proposition 9.2.1(ii) that

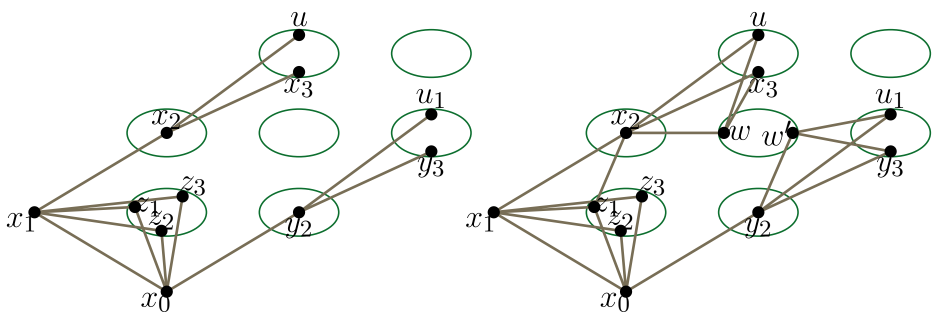

Figure 9.6: Graph Γ from Proposition 9.6.1.

x_2 and y_2 are not adjacent. This shows that x_2 has a unique neighbour (say w) in D_2^2. As $W_{x_2,x_3} = W_{x_2,u} = \{x_2, x_1, x_0, z_1\}$, vertex w is adjacent to both u and x_3. Similarly we prove that also y_2 has a unique neighbour in D_2^2, say w', and that w' is adjacent to both u_1 and y_3. If $w = w'$, then the degree of w is at least 6, a contradiction. Therefore, $w \neq w'$, see the diagram on the right side of Figure 9.6.

Note that $W_{x_2,x_1} = \{x_2, x_3, u, w\}$, and so both y_3 and u_1 are at distance 3 from x_2. Similarly, $W_{x_1,x_2} = \{x_1, x_0, z_2, z_3\}$, and so y_2 is at distance 2 from x_2. Therefore y_2 and x_2 have a common neighbour, and by the above comments the only possible common neighbour is z_1. It follows that z_1 and y_2 are adjacent. But now $\{y_2, x_0, x_1, z_1, x_2\} \subseteq W_{y_2,y_3}$ (recall that $\partial(x_2, y_3) = 3$), a contradiction. This shows that $|D_2^1| = 2$. By Lemma 9.1.2 we obtain that $|D_1^2| = 2$ as well. ■

Theorem 9.6.2. *Let Γ be a regular NDB graph with valency $k = 5$, diameter $d \geq 3$ and $\gamma = d+1$. Then Γ is isomorphic to the icosahedron.*

Proof. First recall that by Theorem 9.3.4 we have $d = 3$, and so $\gamma = 4$. We will first show that Γ is edge-regular with $\lambda = 2$. Pick an arbitrary edge xy and observe that by Proposition 9.6.1 we obtain $|D_2^1(x,y)| = 2$, which forces $|D_1^1(x,y)| = 2$. This shows that Γ is edge-regular with $\lambda = 2$. It follows that for every vertex x of Γ, the subgraph of Γ which is induced on $\Gamma(x)$, is isomorphic to the five-cycle C_5. By [6, Proposition 1.1.4], Γ is isomorphic to the icosahedron. ■

9.7 Proof of the main result

The main result of this chapter is the following theorem.

Theorem 9.7.1. *Let Γ be a regular NDB graph with valency k and diameter d. Then $\gamma = d+1$ if and only if Γ is isomorphic to one of the following graphs:*

1. *the Petersen graph (with* $k = 3$ *and* $d = 2$*);*

2. *the complement of the Petersen graph (with* $k = 6$ *and* $d = 2$*);*

3. *the complete multipartite graph* $K_{t\times 3}$ *with* t *parts of cardinality* 3*,* $t \geq 2$ *(with* $k = 3(t-1)$ *and* $d = 2$*);*

4. *the Möbius ladder graph on* 8 *vertices (with* $k = 3$ *and* $d = 2$*);*

5. *the Paley graph on* 9 *vertices (with* $k = 4$ *and* $d = 2$*);*

6. *the* 3*-dimensional hypercube* Q_3 *(with* $k = 3$ *and* $d = 3$*);*

7. *the line graph of the* 3*-dimensional hypercube* Q_3 *(with* $k = 4$ *and* $d = 3$*);*

8. *the icosahedron (with* $k = 5$ *and* $d = 3$*).*

Proof. It is straightforward to see that all graphs from Theorem 9.7.1 are regular NDB graphs with $\gamma = d+1$. Assume now that Γ is a regular NDB graph with valency k, diameter d and $\gamma = d+1$. If $d = 2$, then it follows from Remark 9.1.3 that Γ is isomorphic either to the Petersen graph, the complement of the Petersen graph, the complete multipartite graph $K_{t\times 3}$ with t parts of cardinality 3 ($t \geq 2$), the Möbius ladder graph on eight vertices, or the Paley graph on 9 vertices. If $d \geq 3$, then it follows from Theorem 9.3.4 that $k \in \{3,4,5\}$. If $k = 3$, then Γ is isomorphic to the 3-dimensional hypercube Q_3 by Theorem 9.4.8. If $k = 4$ then Γ is isomorphic to the line graph of Q_3 by Theorem 9.5.4. If $k = 5$, then Γ is isomorphic to the icosahedron by Theorem 9.6.2. ∎

Chapter 10

On some problems regarding distance-balanced graphs

A graph Γ is said to be *distance-balanced* if for any edge3 uv of Γ, the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u, and it is called *nicely distance-balanced* if in addition this number is independent of the chosen edge uv. A graph Γ is said to be *strongly distance-balanced* if for any edge uv of Γ and any integer k, the number of vertices at distance k from u and at distance $k+1$ from v is equal to the number of vertices at distance $k+1$ from u and at distance k from v.

In this chapter we solve an open problem posed by Kutnar and Miklavič [57] regarding the existence of nonbipartite nicely distance-balanced graphs which are not strongly distance-balanced. We construct several infinite families of such graphs, see Proposition 10.2.7 and Corollary 10.2.8 for a construction of regular examples, and Proposition 10.2.16 for a construction of non-regular examples. In Section 10.3 we provide an infinite family of counterexamples to a conjecture regarding the characterization of strongly distance-balanced graphs posed by Balakrishnan et al. [3]. In Section 10.4 we answer a question posed by Kutnar et al. in [55] regarding the existence of semisymmetric distance-balanced graphs which are not strongly distance-balanced and provide an infinite family of such examples. In Section 10.5 we show that for a graph Γ with n vertices and m edges it can be checked in $O(mn)$ time if Γ is strongly distance-balanced and if Γ is nicely distance-balanced.

The chapter is based on joint work with Ademir Hujdurović. Our main results are currently published in European Journal of Combinatorics (2022); see [25] for more details.

10.1 Preliminaries

In this section we recall some preliminary results that we will find useful later in the chapter. Let Γ denote a finite, simple, connected graph with vertex set $V(\Gamma)$, and edge set $E(\Gamma)$. If $u,v \in V(\Gamma)$ are adjacent then we simply write $u \sim v$ and we denote the corresponding edge by uv with an understanding that $uv = vu$. For $u \in V(\Gamma)$ and an integer i we let $S_i(u)$ denote the set of vertices of $V(\Gamma)$ that are at distance i from u. We abbreviate $S(u) = S_1(u)$. We set $\epsilon(u) = \max\{\partial(u,z) \mid z \in V(\Gamma)\}$ and we call $\epsilon(u)$ the *eccentricity* of u. Let $d = \max\{\epsilon(u) \mid u \in V(\Gamma)\}$ denote the *diameter* of Γ. Pick adjacent vertices u,v of Γ. For any two non-negative integers i,j we let

$$D^i_j(u,v) = S_i(u) \cap S_j(v).$$

By the triangle inequality we observe only the sets $D^{i-1}_i(u,v)$, $D^i_i(u,v)$ and $D^i_{i-1}(u,v)$ $(1 \le i \le d)$ can be nonempty (see also Figure 10.1).

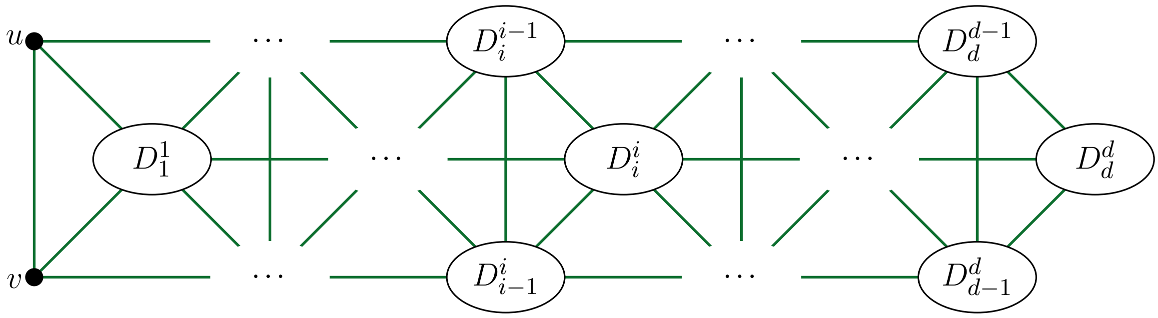

Figure 10.1: Graphical representation of the sets $D^i_j(u,v)$. The line between D^i_j and D^ℓ_m indicates possible edges between vertices of D^i_j and D^ℓ_m.

Let us recall the definition of nicely distance-balanced graphs. For an edge uv of Γ we denote

$$W_{u,v} = \{x \in V(\Gamma) \mid \partial(x,u) < \partial(x,v)\}.$$

We say that Γ is *nicely distance–balanced* (NDB for short) whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for any edge uv of Γ,

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds. One can easily see that Γ is NDB if and only if for every edge $uv \in E(\Gamma)$ we have that

$$\sum_{i=1}^{d} |D^i_{i-1}(u,v)| = \sum_{i=1}^{d} |D^{i-1}_i(u,v)| = \gamma.$$

Pick adjacent vertices u, v of Γ. For the purposes of this chapter we say that the edge uv is *balanced*, if $|W_{u,v}| = |W_{v,u}|$ holds for vertices u, v.

Another concept closely related to the concept of distance-balanced graphs is the one of strongly distance-balanced graphs. A graph Γ is called *strongly distance-balanced* (SDB for short) if $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$ holds for every $i \geq 1$ and every edge uv in Γ. Please note SDB graphs are also called *distance-degree regular* and were first studied in [46]. It is easy to see that a strongly distance-balanced graph is also distance-balanced, but the converse is not true in general (see [55]).

Kutnar et al. gave the following characterization of strongly distance-balanced graphs.

Proposition 10.1.1 ([55, Proposition 2.1])**.** *Let Γ be a graph with diameter d. Then Γ is strongly distance-balanced if and only if $|S_i(u)| = |S_i(v)|$ holds for every edge $uv \in E(\Gamma)$ and every $i \in \{0, \ldots, d\}$.*

We say that an edge uv of a graph Γ is *strongly distance-balanced* if $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$ holds for every $i \geq 1$. From the proof of [55, Proposition 2.1] the following result can be obtained. We include the proof here for the sake of completeness.

Lemma 10.1.2. *Let Γ be a graph with diameter d, and uv an arbitrary edge of Γ. Then the edge uv is strongly distance-balanced if and only if $|S_i(u)| = |S_i(v)|$ for every $i \in \{1, \ldots, d\}$.*

Proof. Assume first the edge uv of Γ is strongly distance-balanced. Then, by definition, we have $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$ for every $i \geq 1$. However, since $S_i(u) = D^i_{i+1}(u,v) \cup D^i_i(u,v) \cup D^i_{i-1}(u,v)$ (disjoint union) and $S_i(v) = D^{i-1}_i(u,v) \cup D^i_i(u,v) \cup D^{i+1}_i(u,v)$ (disjoint union), we have also that $|S_i(u)| = |S_i(v)|$ for every $i \in \{1, \ldots, d\}$.

Next assume that $|S_i(u)| = |S_i(v)|$ holds for every $i \in \{1, \ldots, d\}$. Using induction we show that $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$ holds for every $i \in \{1, \ldots, d\}$. Obviously, $|D^1_0(u,v)| = |D^0_1(u,v)| = 1$. Suppose now that $|D^k_{k-1}(u,v)| = |D^{k-1}_k(u,v)|$ holds for $1 \leq k \leq d$. We observe

$$|D^k_{k+1}(u,v)| \quad = \quad |S_k(u)| - |D^k_k(u,v)| - |D^k_{k-1}(u,v)| \tag{10.1}$$

$$|D^{k+1}_k(u,v)| \quad = \quad |S_k(v)| - |D^k_k(u,v)| - |D^{k-1}_k(u,v)| \tag{10.2}$$

Since $|S_k(u)| = |S_k(v)|$ and in view of the induction hypothesis, $|D^k_{k-1}(u,v)| = |D^{k-1}_k(u,v)|$, it follows from (10.1) and (10.2) that $|D^k_{k+1}(u,v)| = |D^{k+1}_k(u,v)|$. This finishes the proof. ■

An *automorphism* of a graph is a permutation of its vertex set that preserves the adjacency relation of the graph. The set of all automorphisms of a graph Γ is called *the automorphism group* and denoted by $\text{Aut}(\Gamma)$. A graph is *vertex-transitive* if its automorphism group acts transitively on the vertex-set, and it is called *edge-transitive* if its automorphism group acts transitively on the edge set. Kutnar et al. [55] used Proposition 10.1.1 to prove that vertex-transitive graphs are strongly distance-balanced. Lemma 10.1.2 implies that in order to check if a given graph is strongly distance-balanced, one only needs to check the pairs of adjacent vertices that belong to different orbits under the action of the automorphism group of the graph.

10.2 Constructions of nonbipartite NDB graphs that are not SDB

Nicely distance-balanced graphs were studied in [57], where it is proved that in the class of bipartite graphs, the families of DB graphs and NDB graphs coincide, while there are examples of bipartite NDB graphs that are not SDB given by Handa [45]. In [57] examples of nonbipartite SDB graphs that are not NDB were constructed and the following problem was posed.

Problem 10.2.1 ([57, Problem 3.3])**.** *Find a nonbipartite NDB graph which is not SDB.*

In this section we will construct several infinite families of nonbipartite NDB graphs which are not SDB and so, solve Problem 10.2.1. To do this, we first study the Cartesian product of graphs. NDB graphs in the framework of the Cartesian graph product were studied in [57]. We start this section with the definition of this product.

Let G and H denote connected graphs. The *Cartesian product of* G *and* H, denoted by $G\Box H$, is the graph with vertex set $V(G)\times V(H)$ where two vertices (g_1,h_1) and (g_2,h_2) are adjacent if and only if $g_1=g_2$ and $h_1\sim h_2$ in H, or $h_1=h_2$ and $g_1\sim g_2$ in G. We observe that the Cartesian product is commutative and that

$$\partial_{G\Box H}((g_1,h_1),(g_2,h_2))=\partial_G(g_1,g_2)+\partial_H(h_1,h_2).$$

The next result is a direct consequence of [57, Theorem 4.1].

Lemma 10.2.2. *Let* G *and* H *denote connected NDB graphs with* $|V(H)|\cdot\gamma_G=|V(G)|\cdot\gamma_H$*. Then, the Cartesian product* $G\Box H$ *is NDB with* $\gamma_{G\Box H}=|V(H)|\cdot\gamma_G=|V(G)|\cdot\gamma_H$*. In particular, the Cartesian product of* n*-copies of* G *is NDB with* $\gamma=|V(G)|^{n-1}\cdot\gamma_G$*.*

Proof. Immediate from [57, Theorem 4.1] and a straightforward induction argument. ∎

It was proved by Kutnar et al. in [55, Theorem 3.3] that the Cartesian product of graphs is SDB if and only if both factors are SDB. Similarly, the Cartesian product of graphs is bipartite if and only if both factors are bipartite. Therefore the next results holds:

Lemma 10.2.3. *Let G and H denote connected graphs. Then, the Cartesian product $G \square H$ is SDB if and only if both G and H are SDB. In particular, the Cartesian product of n-copies of G is SDB if and only if G is SDB.*

Lemma 10.2.4. *Let G and H denote connected graphs. Then, the Cartesian product $G \square H$ is bipartite if and only if both G and H are bipartite. In particular, the Cartesian product of n-copies of G is bipartite if and only if G is bipartite.*

We now show how the above results can be used to construct infinitely many examples of nonbipartite NDB graphs which are not SDB, provided that at least one such example exists.

Proposition 10.2.5. *Let G denote a nonbipartite NDB graph which is not SDB. If H is a NDB graph and $|V(H)| \cdot \gamma_G = |V(G)| \cdot \gamma_H$ then the Cartesian product $G \square H$ is a nonbipartite NDB graph with $\gamma_{G \square H} = |V(H)| \cdot \gamma_G = |V(G)| \cdot \gamma_H$ which is not SDB. In particular, the Cartesian product of n-copies of G is a nonbipartite NDB graph with $\gamma = |V(G)|^{n-1} \cdot \gamma_G$ that is not SDB.*

Proof. Immediate from Lemmas 10.2.2, 10.2.3 and 10.2.4. ■

We will now construct an example of a nonbipartite NDB graph which is not SDB.

Definition 10.2.6. *Let Γ be the graph with vertex set $V = \{0,1,2\} \times \mathbb{Z}_{10}$ where the adjacencies are $(0,j) \sim (1,j+1)$, $(0,j) \sim (1,j+4)$, $(0,j) \sim (2,j+1)$, $(0,j) \sim (2,j+4)$, $(1,j) \sim (1,j+4)$ and $(2,j) \sim (2,j+4)$ for every $j \in \mathbb{Z}_{10}$ with all the computations in the second component performed modulo 10. A graphical representation of Γ is shown in Figure 10.2.*

Keeping in mind the graph Γ defined in Definition 10.2.6, we now consider certain maps on $V(\Gamma)$.

Let ρ, τ and φ the functions such that for every $j \in \mathbb{Z}_{10}$,

$$\begin{aligned}
\rho(0,j) &= (0,j+7), \quad \rho(1,j) = (1,j+7), \quad \rho(2,j) = (2,j+7),\\
\tau(0,j) &= (0,7-j), \quad \tau(1,j) = (1,2-j), \quad \tau(2,j) = (2,2-j),\\
\varphi(0,j) &= (0,j), \qquad \varphi(1,j) = (2,j), \qquad \varphi(2,j) = (1,j),
\end{aligned}$$

with all the computations in the second component performed modulo 10. It is easy to see that these maps are automorphisms of Γ. Moreover, we observe ρ is a rotation, τ is a reflection and φ swaps vertices with 1 and 2 as first coordinate and fixes all the others.

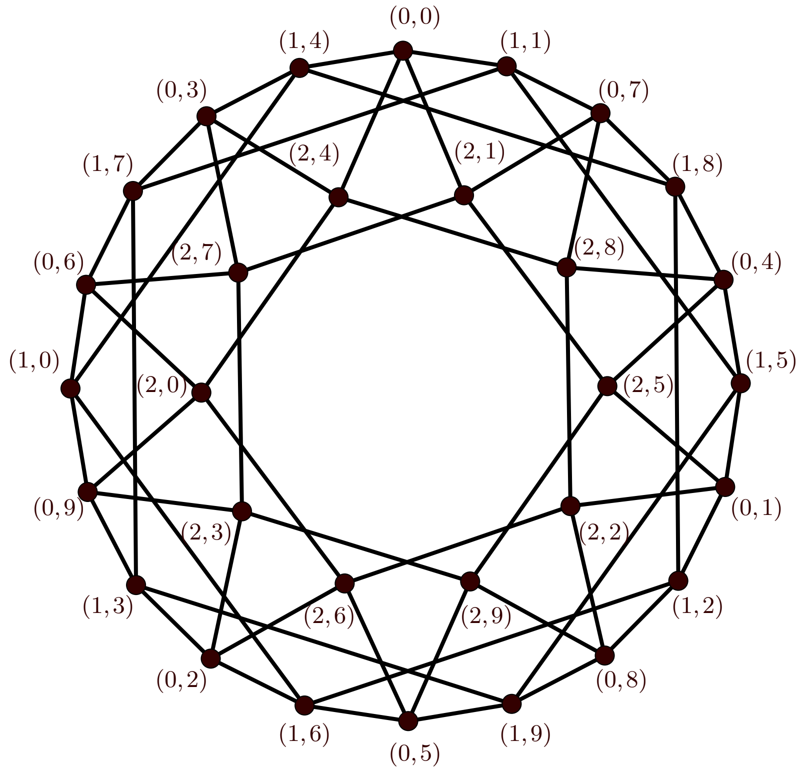

Figure 10.2: A regular nonbipartite NDB graph Γ that is not SDB.

Proposition 10.2.7. *Let the graph Γ be as defined in Definition 10.2.6. Then, Γ is a regular nonbipartite NDB graph that is not SDB.*

Proof. Let the graph Γ be as defined in Definition 10.2.6. See also Figure 10.2. Notice that Γ has diameter 4. By construction we observe that every vertex in Γ has valency 4 and that Γ has odd cycles. Therefore, Γ is a regular nonbipartite graph. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ. For $\alpha \in \mathrm{Aut}(\Gamma)$ and every pair of adjacent vertices $u, v \in V(\Gamma)$ we have that $\alpha(W_{u,v}) = W_{\alpha(u),\alpha(v)}$ and since α is a bijection, $|W_{\alpha(u),\alpha(v)}| = |W_{u,v}|$. Pick now the edge $(0,0)(1,1)$ and note the following hold:

$$\begin{aligned}
W_{(0,0),(1,1)} &= \{(0,0),(2,1),(2,4),(1,4),(1,0),(2,0),(2,5),(2,7),(1,6),(2,3),(2,9),(2,6)\},\\
W_{(1,1),(0,0)} &= \{(1,1),(1,7),(1,5),(0,7),(1,3),(0,4),(1,9),(0,6),(0,1),(0,2),(0,5),(0,8)\}.
\end{aligned}$$

Then, the edge $(0,0)(1,1)$ is balanced and $|W_{(0,0),(1,1)}| = |W_{(1,1),(0,0)}| = 12$. Furthermore, the automorphism τ maps the edge $(0,0)(1,1)$ to the edge $(0,7)(1,1)$ and so, $(0,7)(1,1)$ is balanced and $|W_{(0,7),(1,1)}| = 12$. Considering $\varphi \in \mathrm{Aut}(\Gamma)$ we also observe the edges $(0,0)(1,1)$ and $(0,7)(1,1)$ are respectively mapped to the edges $(0,0)(2,1)$ and $(0,7)(2,1)$ which shows the edges $(0,0)(2,1)$ and $(0,7)(2,1)$ are balanced and $|W_{(0,0),(2,1)}| = |W_{(0,7),(2,1)}| = 12$. Therefore, since ρ is an automorphism of Γ, it follows from the above comments that all the edges $(0,j)(1,j+1)$, $(0,j)(1,j+4)$, $(0,j)(2,j+1)$, $(0,j)(2,j+4)$ are all balanced and

$$|W_{(0,j),(1,j+1)}| = |W_{(0,j),(1,j+4)}| = |W_{(0,j),(2,j+1)}| = |W_{(0,j),(2,j+4)}| = 12$$

for every $j \in \mathbb{Z}_{10}$. Pick now the edge $(1,1)(1,5)$ and note that

$$\begin{aligned} W_{(1,1),(1,5)} &= \{(1,1),(1,7),(0,0),(0,7),(0,3),(2,4),(2,1),(1,4),(0,6),(1,0),(2,0),(2,7)\}, \\ W_{(1,5),(1,1)} &= \{(1,5),(0,4),(1,9),(0,1),(2,2),(1,2),(0,5),(2,5),(0,8),(1,6),(2,9),(2,6)\}, \end{aligned}$$

which shows this edge is balanced and $|W_{(1,1),(1,5)}| = |W_{(1,5),(1,1)}| = 12$. Since $\rho \in \text{Aut}(\Gamma)$, it is easy to see there exists an automorphism of Γ that maps the edge $(1,1)(1,5)$ to the edge $(1,j)(1,j+4)$ and as $\varphi \in \text{Aut}(\Gamma)$ swaps vertices with 1 and 2 as first coordinate and fixes all the others, that there exists an automorphism of Γ that maps the edge $(1,1)(1,5)$ to the edge $(2,j)(2,j+4)$. We thus have the edges $(1,j)(1,j+4)$ and $(2,j)(2,j+4)$ are all balanced and $|W_{(1,j),(1,j+4)}| = |W_{(2,j),(2,j+4)}| = 12$. Hence, Γ is NDB with $\gamma = 12$. We also notice

$$\begin{aligned} D_3^2((1,1),(0,0)) &= \{(1,3),(0,4),(1,9),(0,6),(0,1)\}, \\ D_2^3((1,1),(0,0)) &= \{(1,0),(2,0),(2,5),(2,7)\}. \end{aligned}$$

This yields that Γ is not SDB. The result follows. ■

The graph given in Definition 10.2.6 can be used to construct an infinite family of regular nonbipartite NDB graphs which are not SDB.

Corollary 10.2.8. *There exists infinitely many regular nonbipartite NDB graphs which are not SDB.*

Proof. Let the graph Γ be as defined in Definition 10.2.6 and consider the Cartesian product of n copies of Γ. The result now is a straightforward consequence of Propositions 10.2.5 and 10.2.7. ■

Corollary 10.2.8 provides an infinite family of nonbipartite regular NDB graphs which are not SDB. We next give a construction of a nonregular infinite family.

Definition 10.2.9. *Let $k \geq 3$ be an integer. Let $\Gamma^{(k)}$ denote the graph of order $12k+6$ with vertex set $V_k = \{x_i \mid i \in \mathbb{Z}_{8k+4}\} \cup \{y_i \mid i \in \mathbb{Z}_{4k+2}\}$ where $x_i \sim x_{i+1}$ and $y_i \sim x_{i+m}$ with*

$$m \in \{0, 2k-1, 2k+1, 4k+2, 6k+1, 6k+3\}.$$

All the computations in the index of x_j are performed modulo $8k+4$ while all the computations in the index of y_j are performed modulo $4k+2$.

Throughout this section we will need the following notation.

Notation 10.2.10. *With reference to Definition 10.2.9, for an integer $k \geq 3$, any subset $X \subseteq V_k$ will be identified with a pair of sets (A,B) where A is the set of indexes of x_i vertices that belong to X, while B is the set of indexes of y_i vertices that belong to X, that is $A = \{i \in \mathbb{Z}_{8k+4} \mid x_i \in X\}$ and $B = \{i \in \mathbb{Z}_{4k+2} \mid y_i \in X\}$. Let $\ell \in \{4k+2, 8k+4\}$ and let $H \subseteq \mathbb{Z}_\ell$. For any integer j, we denote $j+H = \{j+h \mid h \in H\}$ where the computations are performed modulo ℓ. Moreover, for $h \in H$ we denote $\langle h \rangle = \{nh : n \in \mathbb{Z}\}$ and $\langle h \rangle^* = \langle h \rangle \setminus \{0\}$.*

The following results will be very useful in the rest of the chapter.

Lemma 10.2.11. *For an integer $k \geq 3$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9. Let $K = \{0, 2k+1, 2k+3\}$ and $M = \{0, 2k-1, 2k+1, 4k+2, 6k+1, 6k+3\}$. The following holds:*

(i) $S_0(x_j) = (\{j\}, \emptyset)$ *and* $S_1(x_j) = (\{j \pm 1\}, j+K)$ *for* $x_j \in V_k$. *In particular,* $|S_0(x_j)| = 1$ *and* $|S_1(x_j)| = 5$.

(ii) $S_0(y_j) = (\emptyset, \{j\})$ *and* $S_1(y_j) = (j+M, \emptyset)$ *for* $y_j \in V_k$. *In particular,* $|S_0(y_j)| = 1$ *and* $|S_1(y_j)| = 6$.

Proof. Pick $x_j, y_j \in V_k$. It is clear that $S_0(x_j) = \{x_j\}$ and $S_0(y_j) = \{y_j\}$. By Definition 10.2.9 we observe that $\{x_{j-1}, x_{j+1}\} \subseteq S_1(x_j)$ and $x_j \sim y_{j+m}$ with $m \in \{0, 2k+1, 2k+3\}$. Similarly, vertex $y_j \sim x_{j+m}$ with $m \in \{0, 2k-1, 2k+1, 4k+2, 6k+1, 6k+3\}$. The result follows. ■

Lemma 10.2.12. *For an integer $k \geq 5$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9. For $x_j \in V_k$ the following hold:*

(i) $S_2(x_j) = (\pm 2 + j + \langle 2k+1 \rangle \cup j + \langle 2k+1 \rangle^*, \pm 1 + j + K)$,

(ii) $S_3(x_j) = (\pm 3 + j + \langle 2k+1 \rangle \cup \pm 1 + j + \langle 2k+1 \rangle^*, \pm 3 + j + 1 + \langle 2k+1 \rangle \cup \{j+2\})$,

(iii) $S_4(x_j) = (\pm 4 + j + \langle 2k+1 \rangle, \pm 4 + j + 1 + \langle 2k+1 \rangle \cup \{j+3\})$,

(iv) $S_i(x_j) = (\pm i + j + \langle 2k+1 \rangle, \pm i + j + 1 + \langle 2k+1 \rangle)$, *for every* $i \in \{5, \ldots, k\}$,

(v) $|S_2(x_j)| = 16$, $|S_3(x_j)| = 19$, $|S_4(y_j)| = 13$ *and* $|S_i(x_j)| = 12$ *for every* $5 \leq i \leq k$. *Moreover, the eccentricity of* x_j *equals* k.

Proof. Pick a vertex $x_j \in V_k$. Assume for a moment that $z \in S_i(x_j)$ for some $0 \leq i \leq \epsilon(x_j)$ and let w be a neighbour of z. Then, by the triangle inequality, $\partial(x_j, w) \in \{i-1, i, i+1\}$

and so $w \in S_{i-1}(x_j) \cup S_i(x_j) \cup S_{i+1}(x_j)$. Therefore, $S_{i+1}(x_j)$ consists of all the neighbours of vertices in $S_i(x_j)$ which are not in $S_{i-1}(x_j)$ nor $S_i(x_j)$. Now, (i)–(iii) immediately follow from Lemma 10.2.11 and the above comments after a careful inspection of the neighbours' sets of vertices in $S_i(x_j)$. We now prove part (iv) by induction. Similarly as above we see that (iv) holds for $i \in \{5,6\}$. Let us now assume that (iv) holds for $i-1$ and i, where $i \geq 6$. Hence, we have that

$$\begin{aligned} S_{i-1}(x_j) &= (\pm(i-1)+j+\langle 2k+1\rangle, \pm(i-1)+j+1+\langle 2k+1\rangle), \\ S_i(x_j) &= (\pm i+j+\langle 2k+1\rangle, \pm i+j+1+\langle 2k+1\rangle). \end{aligned} \tag{10.3}$$

Next, we compute the neighbours of the vertices belonging to the set $S_i(x_j)$. By Lemma 10.2.11 and equation (10.3), we get that

$$S((\pm i+j+\langle 2k+1\rangle, \emptyset)) = (\pm i \pm 1+j+\langle 2k+1\rangle, \pm i+j+\langle 2k+1\rangle+K), \tag{10.4}$$

$$S((\emptyset, \pm i+j+1+\langle 2k+1\rangle)) = (\pm i+j+1+\{0,2k+1\}+M, \emptyset), \tag{10.5}$$

where K and M are the sets as defined in Lemma 10.2.11. Observe that

$$\langle 2k+1\rangle + K = \langle 2k+1\rangle \cup (2+\langle 2k+1\rangle), \tag{10.6}$$

where the operations are performed modulo $4k+2$. Similarly, we have that

$$\{0,2k+1\}+M = (-2+\langle 2k+1\rangle) \cup \langle 2k+1\rangle, \tag{10.7}$$

where the operations are performed modulo $8k+4$. Therefore, from (10.4)–(10.7) it turns out that the set of all neighbours of the vertices which are in $S_i(x_j)$ is given as follows:

$$S(S_i(x_j)) = (\pm i \pm 1+j+\langle 2k+1\rangle, \pm i \pm 1+j+1+\langle 2k+1\rangle).$$

We thus have that

$$\begin{aligned} S_{i+1}(x_j) &= S(S_i(x_j)) \setminus (S_{i-1}(x_j) \cup S_i(x_j)) \\ &= (\pm(i+1)+j+\langle 2k+1\rangle, \pm(i+1)+j+1+\langle 2k+1\rangle), \end{aligned}$$

proving the claim (iv).

Let us now prove (v). The first part of the statement immediately holds from (i)–(iv) above. To prove the second part, let ℓ denote the eccentricity of x_j. From Lemma 10.2.11 and (i)–(iv) above, the sets $S_i(x_j)$ $(0 \leq i \leq k)$ are nonempty and so, $\ell \geq k$. Observe that $\sum_{i=0}^{k} |S_i(x_j)| = 12k+6 = |V_k|$. Since the collection of all the sets $S_i(x_j)$ $(0 \leq i \leq \ell)$ is a partition of the vertex set it follows that the sets $S_i(x_j)$ are empty for $i > k$. Then, $\ell \leq k$ and the result follows. ■

The proof of the next result can be done in a similar way to that of Lemma 10.2.12 above and is therefore omitted and left to the reader.

Lemma 10.2.13. *For an integer $k \geq 5$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9. For $y_j \in V_k$ the following hold:*

(i) $S_2(y_j) = (\pm 1 + j + \langle 2k+1 \rangle \cup j + \{2k-2, 6k\}, \pm 2 + j + \langle 2k+1 \rangle \cup j + \{2k+1\})$.

(ii) $S_3(y_j) = (\pm 3 + j - 1 + \langle 2k+1 \rangle \cup -2 + j + \langle 4k+2 \rangle, \pm 3 + j + \langle 2k+1 \rangle \cup \pm 1 + j + \langle 2k+1 \rangle)$.

(iii) $S_4(y_j) = (\pm 4 + j - 1 + \langle 2k+1 \rangle \cup -3 + j + \langle 4k+2 \rangle, \pm 4 + j + \langle 2k+1 \rangle)$.

(iv) *For every $5 \leq i \leq k$, the set $S_i(y_j) = (\pm i + j - 1 + \langle 2k+1 \rangle, \pm i + j + \langle 2k+1 \rangle)$.*

(v) $|S_2(y_j)| = 15$*,* $|S_3(y_j)| = 18$*,* $|S_4(y_j)| = 14$ *and* $|S_i(y_j)| = 12$ *for every $5 \leq i \leq k$. Moreover, the eccentricity of y_j equals k.*

For an integer $k \geq 5$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9. We next show that some edges of $\Gamma^{(k)}$ are balanced.

Lemma 10.2.14. *For an integer $k \geq 5$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9. For the edge $x_j x_{j+1}$ the following hold:*

(i) $|D_0^1(x_j, x_{j+1})| = |D_1^0(x_j, x_{j+1})| = 1$.

(ii) $|D_1^2(x_j, x_{j+1})| = |D_2^1(x_j, x_{j+1})| = 4$.

(iii) $|D_2^3(x_j, x_{j+1})| = |D_3^2(x_j, x_{j+1})| = 12$.

(iv) $|D_3^4(x_j, x_{j+1})| = |D_4^3(x_j, x_{j+1})| = 7$.

(v) $|D_\ell^{\ell+1}(x_j, x_{j+1})| = |D_{\ell+1}^\ell(x_j, x_{j+1})| = 6$ *for all $4 \leq \ell \leq k-1$.*

(vi) $|D_k^k(x_j, x_{j+1})| = 6$.

(vii) *The edge $x_j x_{j+1}$ is balanced and the sets $D_i^i(x_j, x_{j+1})$ $(1 \leq i \leq k-1)$ are all empty.*

Proof. Pick $j \in \mathbb{Z}_{8k+4}$ and consider the edge $x_j x_{j+1}$. By Lemma 10.2.12 and Lemma 10.2.13 we first observe that $\Gamma^{(k)}$ has diameter k. Now, (i)–(vi) immediately follows from Lemma 10.2.12. Let us now prove (vii). From (i)–(v) above, we notice

$$|W_{x_j, x_{j+1}}| = \sum_{i=0}^{k-1} |D_i^{i+1}(x_j, x_{j+1})| = 6k = \sum_{i=0}^{k-1} |D_{i+1}^i(x_j, x_{j+1})| = |W_{x_{j+1}, x_j}|.$$

Hence, the edge x_jx_{j+1} is balanced. Moreover, by (vi) above we also notice that

$$\sum_{i=1}^{k-1} |D_i^i(x_j,x_{j+1})| = |V_k| - 2|W_{x_j,x_{j+1}}| - |D_k^k(x_j,x_{j+1})| = 0.$$

The result follows. ■

The proof of the next result is omitted as it can be carried out using the same arguments as the proof of Lemma 10.2.14.

Lemma 10.2.15. *For an integer $k \geq 5$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9 and let $K = \{0, 2k+1, 2k+3\}$. For every $\ell \in K$ and for every edge x_jy_ℓ the following hold:*

(i) $|D_0^1(x_j,y_\ell)| = |D_1^0(x_j,y_\ell)| = 1.$

(ii) $|D_1^2(x_j,y_\ell)| = 5$ *and* $|D_2^1(x_j,y_\ell)| = 4.$

(iii) $|D_2^3(x_j,y_\ell)| = |D_3^2(x_j,y_\ell)| = 11.$

(iv) $|D_3^4(x_j,y_\ell)| = 7$ *and* $|D_4^3(x_j,y_\ell)| = 8.$

(v) $|D_i^{i+1}(x_j,y_\ell)| = |D_{i+1}^i(x_j,y_\ell)| = 6$ *for all* $4 \leq i \leq k-1.$

(vi) $|D_k^k(x_j,y_\ell)| = 6.$

(vii) The edge x_jy_ℓ is balanced and the sets $D_i^i(x_j,y_\ell)$ $(1 \leq i \leq k-1)$ are all empty.

We are now ready to provide an infinite family of nonbipartite and nonregular NDB graphs which are not SDB.

Proposition 10.2.16. *For an integer $k \geq 5$, let the graph $\Gamma^{(k)}$ be as defined in Definition 10.2.9. Then, $\Gamma^{(k)}$ is a nonbipartite NDB graph which is not SDB nor regular.*

Proof. By Definition 10.2.9 and Lemma 10.2.11, it is clear that $\Gamma^{(k)}$ is not regular. This implies that $\Gamma^{(k)}$ is not SDB since for at least one edge uv the corresponding sets $D_2^1(u,v)$ and $D_1^2(u,v)$ will not be of the same cardinality. Pick $j \in \mathbb{Z}_{8k+4}$. Recall that $\{x_{j-1}, x_{j+1}\} \subseteq S_1(x_j)$ and $x_j \sim y_{j+m}$ with $m \in \{0, 2k+1, 2k+3\}$. It now follows from Lemma 10.2.15 that the edges x_jx_{j+1}, x_jy_j, x_jy_{2k+1+j} and x_jy_{2k+3+j} are all balanced. Moreover, it turns out that

$$|W_{x_j,x_{j+1}}| = |W_{x_j,y_j}| = |W_{x_j,y_{2k+j+1}}| = |W_{x_j,x_{2k+3+j}}| = 6k.$$

In addition, for $i, i' \in \mathbb{Z}_{4k+2}$ we observe vertices y_i and $y_{i'}$ are not adjacent. Since j is arbitrary, we thus have all the edges of $\Gamma^{(k)}$ are balanced. Consequently, it follows from the above comments that $\Gamma^{(k)}$ is NDB with $\gamma = 6k$. We also notice that $\Gamma^{(k)}$ is nonbipartite as the set $D^k_k(x_j, y_j)$ is nonempty by Lemma 10.2.15. This concludes the proof. ■

We end this section with the following two remarks.

Remark 10.2.17. *Graphs $\Gamma^{(3)}$ and $\Gamma^{(4)}$ are also nonbipartite NDB graphs which are not SDB, with $\gamma = 18$ and $\gamma = 24$ respectively, but we considered only the case when $k \geq 5$ for the simplicity of proofs.*

Remark 10.2.18. *Graphs $\Gamma^{(k)}$ defined in Definition 10.2.9 are prime with respect to the Cartesian product of graphs (cannot be obtained as a Cartesian product of two non-trivial graphs). Suppose $\Gamma^{(k)} \cong G \square H$ for some graphs G and H. Observe that the edge $x_i x_{i+1}$ lies on exactly 2 cycles of length 4 in $\Gamma^{(k)}$ for every $i \in \mathbb{Z}_{8k+4}$. Since the vertices of $\Gamma^{(k)}$ have degree 5 or 6, without loss of generality we may assume that the minimum degree in G is at least 3. It follows that the edge $x_i x_{i+1}$ must belong to the H-layers in the Cartesian product $G \square H$, since it lies only on 2 cycles of length 4. Then, it holds that all of the x_i vertices belong to the same H-layer, implying that H has at least $8k+4$ vertices. Since $|V(\Gamma^{(k)})| = 12k+6 = |V(G)| \cdot |V(H)|$, it follows that G is the graph with one vertex.*

10.3 Counterexamples to a conjecture regarding SDB graphs

Let Γ be a graph, and let S be a subset of its vertex set. For a vertex v of Γ we define

$$\partial(v, S) = \sum_{x \in S} \partial(v, x).$$

Balakrishnan et al. [3] proved that a connected graph Γ is distance-balanced if and only if $\partial(v, V(\Gamma)) = \partial(u, V(\Gamma)))$ for all $u, v \in V(\Gamma)$. They posed the following conjecture regarding a similar characterization of strongly distance-balanced graphs.

Conjecture 10.3.1 ([3, Conjecture 3.2])**.** *A graph Γ is strongly distance-balanced if and only if $\partial(u, W_{u,v}) = \partial(v, W_{v,u})$ holds for every pair of adjacent vertices u, v of Γ.*

It is clear that strongly distance-balanced graphs satisfy the above condition, but the question was if the converse also holds. We will now provide an infinite family of counterexamples to Conjecture 10.3.1.

Let k and l be positive integers. Let $C_6(k, l)$ denote the graph obtained from the 6-cycle by replacing every vertex in one bipartition set of C_6 with k pairwise non-adjacent vertices, and replacing every vertex in the other bipartition set of C_6 with l pairwise non-adjacent vertices, see Figure 10.3 for an example. To be more precise, let $\{x_0,x_1,x_2,x_3,x_4,x_5\}$ be the vertex set of the 6-cycle, and let the vertex-set of $C_6(k,l)$ be $(\{x_0,x_2,x_4\}\times\mathbb{Z}_k)\cup(\{x_1,x_3,x_5\}\times\mathbb{Z}_l)$, and adjacencies given by $(x_{2i},r)\sim(x_{2i\pm1},s)$ for every $i\in\{0,1,2\}$ and every $r\in\mathbb{Z}_k$, $s\in\mathbb{Z}_l$. Observe that any permutation of vertices inside sets $\{x_{2i}\}\times\mathbb{Z}_k$ and $\{x_{2i+1}\}\times\mathbb{Z}_l$, preserves all the edges. Hence, it is an automorphism. Observe also that the 2-step rotation, function mapping (x_i,j) into (x_{i+2},j) it is also an automorphism of $C_6(k,l)$. It follows that the graph $C_6(k,l)$ is edge-transitive. Observe that $C_6(k,l)$ is vertex-transitive if and only if $k=l$.

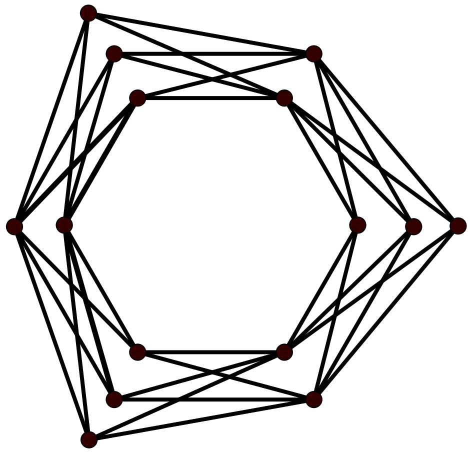

Figure 10.3: Graph $C_6(2,3)$.

The following proposition shows that graph $C_6(k,l)$ with $k\neq l$ is a counterexample to Conjecture 10.3.1.

Proposition 10.3.2. *Let k and l be positive integers, and let the graph $C_6(k,l)$. Then $C_6(k,l)$ is strongly-distance balanced if and only if $k=l$, while $\partial(u,W_{u,v})=\partial(v,W_{v,u})$ holds for every pair of adjacent vertices u,v of $C_6(k,l)$.*

Proof. Observe that $C_6(k,l)$ is regular if and only if $k=l$. It follows that for $k\neq l$, the graph $C_6(k,l)$ is not strongly-distance-balanced. Moreover, for $k=l$, the graph $C_6(k,l)$ is vertex-transitive, and since every vertex-transitive graph is strongly-distance-balanced it follows that $C_6(k,l)$ is SDB if and only if $k=l$.

Let $u = (x_0, 0)$ and $v = (x_1, 0)$. Observe that

$$\begin{aligned}
D_2^1(u,v) &= (\{x_1\} \times (\mathbb{Z}_l \setminus \{0\})) \cup (\{x_5\} \times \mathbb{Z}_l),\\
D_1^2(u,v) &= (\{x_0\} \times (\mathbb{Z}_k \setminus \{0\})) \cup (\{x_2\} \times \mathbb{Z}_k),\\
D_3^2(u,v) &= (\{x_4\} \times \mathbb{Z}_k),\\
D_2^3(u,v) &= (\{x_3\} \times \mathbb{Z}_l).
\end{aligned}$$

It follows that $\partial(u, W_{u,v}) = |D_2^1(u,v)| + 2 \cdot |D_3^2(u,v)| = (2l-1) + 2k = 2k + 2l - 1$. Similarly we have that $\partial(v, W_{v,u}) = |D_1^2(u,v)| + 2 \cdot |D_2^3(u,v)| = (2k-1) + 2l = 2k + 2l - 1$. We conclude that $\partial(u, W_{u,v}) = \partial(v, W_{v,u})$. Since the graph $C_6(k,l)$ is edge-transitive, it follows that the same holds for any pair of adjacent vertices. This concludes the proof. ■

10.4 Distance-balanced property in semisymmetric graphs

The main goal for this section is to answer a question by Kutnar et al. from [55].

Symmetry is perhaps one of those purely mathematical concepts that has found wide applications in several other branches of science and in many of these problems, symmetry conditions are naturally blended with certain metric properties of the underlying graphs. Kutnar et al. explored a purely metric property of being (strongly) distance-balanced in the context of graphs enjoying certain special symmetry conditions. They showed that vertex-transitive graphs are not only distance-balanced, they are also strongly distance-balanced (see [55]). Furthermore, since being vertex-transitive is not a necessary condition for a graph to be distance-balanced, it was therefore natural for the authors to explore the property of being distance-balanced within the class of *s*emisymmetric graphs; a class of objects which are as close to vertex-transitive graphs as one can possibly get, that is, regular edge-transitive graphs which are not vertex-transitive. The smallest semisymmetric graph has 20 vertices and its discovery is due to Folkman [35], the initiator of this topic of research.

A semisymmetric graph is necessarily bipartite, with the two sets of bipartition coinciding with the two orbits of the automorphism group. Consequently, semisymmetric graphs have no automorphisms which switch adjacent vertices, and therefore, may arguably be considered as good candidates for graphs which are not distance-balanced. Indeed, Kutnar et al. proved there are infinitely many semisymmetric graphs which are not distance-balanced, but there are also infinitely many semisymmetric graphs which are distance-balanced. They also wondered the following question.

Question 10.4.1 ([55, Question 4.6]). *Is it true that a distance-balanced semisymmetric graph is also strongly distance-balanced?*

We next answer this question negatively by giving a construction of an infinite family of semisymmetric DB graphs which are not SDB. Before embarking on the corresponding construction, we make the following observations about the distance-balanced property in semisymmetric graphs using certain graph product.

Let G and H denote graphs. The *lexicographic product of* G *and* H, denoted by $G[H]$, is the graph with vertex set $V(G) \times V(H)$ where two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if $g_1 \sim g_2$, or $g_1 = g_2$ and $h_1 \sim h_2$. It turns out that the lexicographic product $G[H]$ is connected if and only if G is connected.

Necessary and sufficient conditions under which the lexicographic product give rises to a distance-balanced graph are given in [52].

Lemma 10.4.2 ([52, Theorem 4.2]). *Let* G *and* H *be connected graphs. Then, the lexicographic product* $G[H]$ *is distance-balanced if and only if* G *is distance-balanced and* H *is regular.*

Kutnar et al. also investigated the strongly distance-balanced property of lexicographic graph products.

Lemma 10.4.3 ([55, Theorem 3.4]). *Let* G *and* H *be graphs such that* $G[H]$ *is connected. Then, the lexicographic product* $G[H]$ *is strongly distance-balanced if and only if* G *is strongly distance-balanced and* H *is regular.*

For constructions of several infinite families of semisymmetric distance-balanced graphs the following result will be useful:

Lemma 10.4.4 ([55, Proposition 4.3]). *Let* Γ *be a semisymmetric graph. Then for every positive integer* n*, the lexicographic product* $\Gamma[nK_1]$ *is semisymmetric, where* nK_1 *denotes the empty graph of* n *vertices.*

With these results in mind, we would like to point out that the desired construction can be given provided we find at least one connected distance-balanced semisymmetric graph which is not strongly distance-balanced. Namely, let Γ be such a graph. Then combining together Lemma 10.4.2 and Lemma 10.4.4, we have that $\Gamma[nK_1]$ is a distance-balanced semisymmetric graph for every positive integer n. Additionally, since Γ is a connected graph which is not SDB, it follows from Lemma 10.4.3 that $\Gamma[nK_1]$ is not SDB. For every positive integer n, we thus have that the lexicographic product $\Gamma[nK_1]$ is a DB semisymmetric graph which is not SDB. Kutnar et al. checked the list of all semisymmetric connected cubic graphs of order up to 768 [14], and there are exactly 11 distance-balanced graphs in this list, all of them are also strongly distance-balanced. They also checked the list of all connected semisymmetric tetravalent graphs of order up to 100 from the list of Potočnik and Wilson, and there are 26 distance-balanced graphs in this list, all of which are also strongly distance-balanced. In the meantime, Potočnik and Wilson extended their list of connected tetravalent edge-transitive graphs up to 512 vertices [82], and using this extended list we were able to find examples of semisymmetric graphs which are distance-balanced but not strongly distance-balanced.

Example 10.4.5. *Graphs* $C4[150,9]$, $C4[240,60]$, $C4[240,61]$, $C4[240,105]$, $C4[240,168]$, $C4[288,145]$, $C4[288,171]$, $C4[288,246]$, $C4[312,40]$, $C4[336,46]$, $C4[336,49]$, $C4[336,107]$, $C4[336,129]$, $C4[336,135]$, $C4[336,157]$, $C4[336,166]$, $C4[360,177]$, $C4[384,81]$, $C4[384,85]$, $C4[384,341]$, $C4[384,380]$, $C4[384,462]$, $C4[384,499]$, $C4[400,44]$, $C4[432,163]$, $C4[432,164]$, $C4[432,198]$, $C4[432,229]$, $C4[432,241]$, $C4[432,253]$, $C4[432,274]$, $C4[432,282]$, $C4[480,126]$, $C4[480,131]$, $C4[480,300]$, $C4[480,359]$, $C4[480,453]$, $C4[480,461]$, $C4[480,520]$, $C4[480,523]$, $C4[486,68]$, $C4[486,69]$, $C4[486,74]$, $C4[504,154]$, $C4[504,155]$ *defined in [82] are connected semisymmetric graphs of valency 4 which are distance-balanced but not strongly distance-balanced. (The parameter* n *in* $C4[n,i]$ *denotes the order of the corresponding graph). Using the distance-orbit chart given in [82] (where the sizes of orbits of the stabilizer* $\mathrm{Aut}(\Gamma)_u$ *of a vertex* u *at distances* $0,1,\ldots,d$ *from* u *are shown) one can easily check the distance-balanced and strongly distance-balanced properties of the graph under consideration (the orbit sizes are given for representatives of bipartition sets). For example, the distance-orbit chart of the graph* $C4[150,9]$ *is presented below in Table 10.1.*

Distance	0	1	2	3	4	5	6	7	8
White vertex	1	4	$2,4^2$	$2^2,4^4$	$2,4^7$	$2^2,4^9$	$2^3,4^7$	$1,2,4^2$	
Black vertex	1	4	$2,4^2$	$2^2,4^4$	$2^2,4^7$	$2^2,4^9$	$2,4^7$	$1,2,4^2$	2

Table 10.1: The distance-orbit chart of the graph $C4[150,9]$.

This means that there are 4 vertices at distance 1 from a white vertex, 10 vertices at distance two (one orbit of size 2 and two orbits of size 4), 20 vertices at distance 3 (two orbits of size 2 and 4 orbits of size 4), and so on. By the result of Balakrishnan et al. [3], a graph is distance-balanced if and only if the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex, which can easily be verified from the distance-orbit chart. Similarly, a graph is strongly distance-balanced if and only if the number of vertices at distance i from a given vertex is independent of the chosen vertex, which can also easily be read from the distance-orbit chart.

Corollary 10.4.6. *There exist infinite families of distance-balanced semisymmetric graphs which are not strongly distance-balanced.*

10.5 Recognition of SDB and NDB graphs

Let Γ be a graph with n vertices and m edges. In [3] it is proved that it can be verified in $O(mn)$ time if Γ is distance-balanced. We will now prove that the same result holds for strongly distance-balanced graphs and nicely distance-balanced graphs.

Proposition 10.5.1. *Let Γ be a connected graph with n vertices and m edges. It can be checked in $O(mn)$ time if Γ is strongly distance-balanced.*

Proof. By Proposition 10.1.1 it follows that Γ is strongly distance-balanced if and only if $|S_i(u)|$ does not depend on the choice of vertex u, for any $i \in \{1, \ldots, d\}$ where d is the diameter of Γ. Using BFS algorithm, the sizes of sets $|S_i(u)|$ can be determined in $O(m)$ time, for any fixed vertex u. Calculating these numbers for every vertex of Γ can then be done in $O(mn)$ time. ■

Proposition 10.5.2. *Let Γ be a connected graph with m edges. It can be checked in $O(mn)$ time if Γ is nicely distance-balanced.*

Proof. Using the BFS algorithm, computing the distance from each vertex to all other vertices can be done in $O(mn)$ time, and this information can be stored, for example in a distance matrix. For a fixed edge uv, iterating over each vertex w and checking whether $\partial(u, w)$ is smaller, larger or equal than $\partial(v, w)$, we can compute the sizes of $W_{u,v}$ and $W_{v,u}$, which can be done in $O(n)$ time (for a single edge). Calculating the values of $W_{u,v}$ and $W_{v,u}$ can then be done in $O(mn)$ time. ■

Conclusion

Chapter 11

Final remarks on Terwilliger algebras

The contributions to algebraic combinatorics within this dissertation can be roughly divided into two different but interrelated categories: the study of Terwilliger algebras of certain graphs and the resolution of some problems related to distance-balanced graphs.

All the original results presented in this Ph.D. dissertation about Terwilliger algebras of graphs are contained in research papers which are/will be published in specialized SCI journals; see [23, 24, 26, 27, 28] for more details.

Terwilliger algebras, originally known as *subconstituent algebras*, are introduced in [89] for association schemes and their representations are extensively studied for (P and Q)-polynomial association schemes in [90, 91]. Subconstituent algebras of any arbitrary finite, simple and connected graph are considered in [88] and their studies for distance-regular graphs have received considerable attention since then. However, the state of the art regarding Terwilliger algebras of graphs, which are not distance-regular, is not as intense.

The research for this Ph.D. dissertation broadens our knowledge of Terwilliger algebras of graphs that are not necessarily distance-regular. Specifically, our research is concentrated around thin irreducible T-modules with endpoint 0 and with endpoint 1 (with respect to a fixed vertex) of general graphs, not necessarily distance-regular. Certain combinatorial conditions in a graph are shown to hold if and only if some algebraic properties of the corresponding Terwilliger algebra are satisfied. As a result, we contribute to a common effort of the mathematical community to understand the Terwilliger algebra of a graph (with respect to a fixed vertex) and the interplay of combinatorial properties of this graph and algebraic properties of its corresponding Terwilliger algebra.

Let us discuss these contributions briefly and make some suggestions for future research.

To begin our investigation, we provided a purely combinatorial characterization of the property that the unique irreducible T-module with endpoint 0 is thin in Chapter 3. The number of walks of a certain shape between vertex x and vertices at some fixed distance from x is used in this characterization.

The study of irreducible T-modules with endpoint 1 of certain graphs that are not necessarily distance-regular follows naturally. Thus, we characterized those vertices x of a graph Γ, for which the corresponding Terwilliger algebra $T = T(x)$ has no irreducible T-modules with endpoint 1. We proved that there are irreducible T-modules with endpoint 1 if and only if x is not a leaf. Hence, we assumed the valency of x is at least 2 from that moment on.

The study of Terwilliger algebras in general appears to be overly complicated at the moment. Therefore, we concentrated on some cases where the irreducible modules with endpoint at most 1 are thin. We assumed that the unique irreducible T-module with endpoint 0 is thin, or equivalently, that x is pseudo-distance-regularized. Our next goal was to find a combinatorial characterization of graphs, which also have a unique irreducible T-module of endpoint 1 (up to isomorphism), and this module is thin. We anticipated that this problem would be too difficult to solve in this dissertation. Instead, we began by laying the groundwork for dealing with this issue by solving other problems which were, of course, closely related to our main goal and which we thought were easier to solve.

According to [21, Theorem 1.3], when the graph is distance-regular, the previously described situation occurs if and only if the graph is bipartite or almost-bipartite. As it seems bipartite distance-regular graphs and distance-biregular graphs are closely related, a natural way to explore the desired situation and get results involving Terwilliger algebras of non-distance-regular graphs was to study the case when the graph is distance-biregular. Consequently, in Chapter 4 we showed that if Γ is distance-biregular, then, again, Γ has (up to isomorphism) a unique irreducible T-module with endpoint 1, and this module is thin.

Bipartite distance-regular graphs and distance-biregular graphs are connected bipartite graphs in which the so-called local distance-regularity holds for each of their vertices. Accordingly, an obvious step forward was to consider the case where the graph is bipartite and the local distance-regularity property holds for the base vertex but not necessarily for all the others. We dealt with this situation in Chapter 5 where we found certain combinatorial consequences of the above algebraic condition.

Our next problem concerned non-bipartite graphs. In other words, even if the graph Γ is not bipartite, the unique irreducible T-module with endpoint 0 is thin if Γ is distance-regular around the base vertex x. Therefore, in Chapter 6 we extended the results from Chapter 5. As a result, when the base vertex x is distance-regularized, certain combinatorial consequences of the above algebraic conditions were given for this more general situation.

We emphasize that by solving the problems listed above, we gained the insight required to attack our main problem and pursue our goal. We thus generalized the above results to the case when

Γ is not necessarily distance-regular around x in Chapter 7. The main result of this Ph.D. thesis is a combinatorial characterization of graphs which are pseudo-distance-regular around x and also have a unique irreducible T-module (up to isomorphism) with endpoint 1, and this module is thin. This characterization of such graphs involves the number of some walks of a particular shape. Last but not least, we gave precise examples to construct many graphs which possess these properties from our general solution.

From the above comments, a natural continuation of this research is to study similar problems in the case when the trivial T-module is not thin. The generalization of these problems under the assumption that the unique irreducible module with endpoint 0 is not thin, in our opinion, may be too difficult to handle using the techniques demonstrated in this dissertation. We thus propose studying certain graphs, for which finding such a combinatorial characterization seems to be achievable.

The following couple of problems we describe below concern graphs where for a certain vertex the trivial module is close to being thin. Let us now define that the trivial module $T\widehat{x}$ is *almost thin*, if the dimension of $E_i^*(T\widehat{x})$ is at most 2 for every $0 \le i \le \epsilon(x)$.

Problem 1. *Let Γ be a graph with vertex set X. Fix $x \in X$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. Find a purely combinatorial condition which is equivalent to the property that the unique irreducible T-module with endpoint 0 is almost thin.*

We also propose to study irreducible T-modules with endpoint 1 in the case when the trivial T-module is not thin. It turns out that there are no irreducible T-modules with endpoint 1 if and only if $\dim(E_1^*T\widehat{x}) = |\Gamma(x)|$. Consequently, if we would like to explore this general situation when the trivial T-module is almost thin, we will need that $\dim(E_1^*T\widehat{x}) < |\Gamma(x)|$.

Problem 2. *Let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Assume that the unique irreducible T-module with endpoint 0 is not thin and that $\dim(E_1^*T\widehat{x}) < |\Gamma(x)|$. Consider the property that Γ has, up to isomorphism, a unique irreducible T-module with endpoint 1, and that this T-module is thin. Find combinatorial consequences of this algebraic condition. Characterize graphs where the above conditions hold.*

It is our impression that the results in the case when the trivial module is almost thin could be of a similar flavor to the results of this dissertation. Nevertheless, the situation will become slightly more complicated.

Chapter 12

Final remarks on distance-balanced graphs

The contributions to algebraic combinatorics within this dissertation can be roughly divided into two different but interrelated categories: the study of Terwilliger algebras of certain graphs and the resolution of some problems related to distance-balanced graphs.

All the original results presented in this Ph.D. dissertation about distance-balanced graphs are contained in research papers which are published in specialized SCI journals; see [25, 29] for more details.

Let $\Gamma = (X, \mathcal{R})$ be a simple, finite, and connected graph and let X and $\mathcal{R}$ denote the vertex set and the edge set of Γ, respectively. For $u, v \in X$, let $\partial(u,v) = \partial_\Gamma(u,v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of Γ we denote

$$W_{u,v} = \{x \in X \mid \partial(x,u) < \partial(x,v)\}.$$

We say that Γ is *distance–balanced* (DB for short) if for an arbitrary pair of adjacent vertices u and v of Γ we have that

$$|W_{u,v}| = |W_{v,u}|.$$

Although Handa began researching distance-balanced graphs in [45], the term itself was coined by Jerebic, Klavžar and Rall in [52]. The family of distance-balanced graphs is very rich, and its study is interesting not only from various purely graph-theoretic perspectives, but also because the balancedness property of these graphs makes them appealing in many research areas.

With the research undertaken for the completion of this Ph.D. dissertation, we provide certain methods and techniques that allow us to not only classify certain DB graphs, but also to construct some infinite families of them which are of interest in this area of research.

Let us briefly discuss these contributions and suggest possible paths for future research.

The notion of nicely distance-balanced graphs appears quite naturally in the context of DB graphs. We say that Γ is *nicely distance–balanced* (NDB for short) whenever there exists a positive integer $\gamma = \gamma(\Gamma)$, such that for an arbitrary pair of adjacent vertices u and v of Γ,

$$|W_{u,v}| = |W_{v,u}| = \gamma$$

holds.

Assume now that Γ is NDB. Let us denote the diameter of Γ by d (the *diameter* of a graph is the maximum distance between two vertices). In [57], where these graphs were first defined, it was proved that $d \leq \gamma$ and NDB graphs with $d = \gamma$ were classified. It turns out that Γ is NDB with $d = \gamma$ if and only if Γ is either isomorphic to a complete graph on $n \geq 2$ vertices, to a complete multipartite graph $K_{t\times 2}$ $(t \geq 2)$ with t parts of cardinality 2, or to a cycle on $2d$ or $2d+1$ vertices. Therefore, we concentrated our study on the class of regular NDB graphs with $\gamma = d+1$ in Chapter 9. The main result is shown in Theorem 9.7.1 where the classification of such graphs is given.

From the above comments, some continuations of this research naturally arise. Therefore, we would like to propose some problems which will be described below.

Problem 1. *Classify NDB graphs with diameter d and $\gamma \in \{d+1, d+2\}$.*

We expect that the situation in these cases is much more complex than in the case $\gamma = d$. Following the techniques we used in Chapter 9, we also propose the following problem which we believe is easier to solve.

Problem 2. *Classify (edge-)regular NDB graphs with diameter d and $\gamma = d+2$.*

One possible way to attack the classification problem for NDB graphs is to try to classify NDB graphs Γ with $\gamma = k$ for a fixed positive integer k. Observe that Γ is NDB with $\gamma = 1$ if and only if Γ is a complete graph. In [57], Kutnar and Miklavič classify NDB graphs Γ with $\gamma \in \{1, 2, 3\}$. Although, for larger integers k the classification quickly becomes very complicated, it is obvious to consider the next situation as well.

Problem 3. *Classify NDB graphs with $\gamma = 4$.*

Another concept closely related to the concept of distance-balanced graphs is the one of strongly distance-balanced graphs. For an arbitrary edge uv of a given graph Γ, and any two nonnegative integers i, j, we let

$$D^i_j(u,v) = \{x \in X \mid \partial(u,x) = i \text{ and } \partial(v,x) = j\}.$$

A graph Γ is called *strongly distance-balanced* (SDB for short) if $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$ holds for every $i \geq 1$ and every edge uv in Γ.

Throughout Chapter 10, we focused our attention on some problems about distance-balanced graphs, especially on the construction of certain families of DB graphs, which seem to be of interest in this area of research.

Our first construction was related to certain NDB graphs which are not SDB. Nicely distance-balanced graphs were studied in [57], where it is proved that in the class of bipartite graphs, the family of DB graphs and NDB graphs coincide, while there are examples of bipartite NDB graphs that are not SDB given by Handa [45]. Moreover, in [57], examples of nonbipartite SDB graphs that are not NDB were constructed. In Chapter 10 we solved [57, Problem 3.3] posed by Kutnar and Miklavič regarding the existence of nonbipartite NDB graphs which are not SDB. We proved there exist infinitely many (regular) nonbipartite NDB graphs which are not SDB.

Our second construction was related with a conjecture by Balakrishnan et al. about a characterization of SDB graphs. Let Γ be a graph, and let S be a subset of its vertex set. For a vertex v of Γ we define

$$\partial(v,S) = \sum_{x \in S} \partial(v,x).$$

Balakrishnan et al. [3] proved that a connected graph Γ is distance-balanced if and only if $\partial(v,X) = \partial(u,X)$ for all $u,v \in X$. Moreover, they conjectured that a graph Γ is strongly distance-balanced if and only if $\partial(u, W_{u,v}) = \partial(v, W_{v,u})$ holds for every pair of adjacent vertices u, v of Γ. It is clear that strongly distance-balanced graphs satisfy the above condition, but the question was if the converse also holds. In Chapter 10 we disproved [3, Conjecture 3.2] by providing infinitely many counterexamples.

Our third construction dealt with the property of being (strongly) distance-balanced in the context of graphs enjoying certain special symmetry conditions. Kutnar et al. showed that vertex-transitive graphs are not only distance-balanced, they are also strongly distance-balanced (see [55]). Furthermore, since being vertex-transitive is not a necessary condition for a graph to be distance-balanced, it was therefore natural for the authors to explore the property of being distance-balanced within the class of semisymmetric graphs. Indeed, Kutnar et al. proved there are infinitely many semisymmetric graphs which are not distance-balanced, but there are also infinitely many semisymmetric graphs which are distance-balanced. In Chapter 10 we also answered [55, Question 4.6] posed by Kutnar et al. regarding the existence of semisymmetric DB graphs which are not SDB. We proved there exist infinite families of distance-balanced semisymmetric graphs which are not strongly distance-balanced.

Let Γ be a graph with n vertices and m edges. In [3] it is proved that it can be verified in $O(mn)$ time if Γ is distance-balanced. We concluded Chapter 10 by showing that for a graph Γ with n vertices and m edges it can be checked in $O(mn)$ time if Γ is strongly distance-balanced and if Γ is nicely distance-balanced.

For a graph Γ and a vertex v, one can construct the sets $S_i(v)$ of all vertices in Γ which are at distance i from v. By Proposition 10.1.1, we observe that Γ is SDB if and only if the sizes of the sets $S_i(v)$ do not depend on the choice of v. In [3], Balakrishnan et al. showed that a graph is distance-balanced if and only if the sum of the distances from a given vertex to all other vertices is independent of the chosen vertex. Namely, Γ is DB if and only if $\sum_i i|S_i(v)|$ is constant. Therefore, the following question naturally arises.

Problem 4. *Does there exist a characterization of NDB graphs in terms of the sets $S_i(v)$?*

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Povzetek v slovenskem jeziku

V naši raziskavi se bomo ukvarjali s kombinatoričnimi objekti, ki jim pravimo grafi. Graf $\Gamma = (X, \mathcal{R})$ je matematičen objekt, ki je sestavljen iz končne množice vozlišč X in množice povezav (oziroma neurejenih parov vozlišč) $\mathcal{R}$. Ponavadi vsako vozlišče $x \in X$ prestavimo s točko v ravnini, povezavo $e = \{x, y\} \in \mathcal{R}$ pa predstavimo s črto, ki povezuje vozlišči x in y.

Teorija grafov spada v kombinatoriko. To je del matematike, ki proučuje strukturo in preštevanje diskretnih objektov. Na nasprotnem polu matematike je matematična analiza, ki proučuje zvezne objekte. Konkretneje, teorija grafov je uporabna za proučevanje kakršnegakoli sistema, v katerem obstajajo nekakšni odnosi med pari elementov tega sistema. Ti odnosi so ponavadi opredeljeni z neko binarno relacijo. Zato ni presenetljivo, da so bili številni problemi in rezultati teorije grafov prvotno formulirani v kontekstu odnosov med ljudmi. Prav tako je tudi številne druge matematične koncepte mogoče opredeliti z uporabo pojmov teorije grafov.

V tej disertaciji je interakcija med grafi in določenimi algebraičnimi objekti še posebej intenzivna in pomembna. V tej disertaciji se bomo namreč ukvarjali s proučevanjem Terwilligerjevih algeber določenih grafov, ter z nekaterimi problemi znotraj razreda razdaljno-uravnoteženih grafov. Zato smo opise znanstvenega ozadja disertacije in njenega akademskega doprinosa razdelili v dva dela. V prvem delu obravnavamo Terwilligerjeve algebre, v drugem pa razdaljno-uravnotežene grafe. Prav tako bomo privzeli, da je bralec seznanjen z osnovnimi definicijami teorije grafov in algebraične kombinatorike. Za nadaljne definicije ter notacijske konvencije iz teh dveh področij priporočamo monografije [6, 39, 40, 96].

Terwilligerjeva algebra grafa

Naj bo Γ graf in naj bo G nek algebraični objekt, ki je prirejen grafu Γ. Ena glavnih motivacij pri našem raziskovanju je naslednje vprašanje: *kaj lahko rečemo o kombinatoričnih lastnostih grafa Γ, če vemo, da ima objekt G določene algebraične lastnosti?* In seveda obratno: *kaj lahko povemo o algebričnih lastnostih objekta G, če vemo, da ima graf Γ določene kombinatorične lastnosti?* Morda najbolj znan primer te interakcije med kombinatoriko in algebro dobimo, če za objekt G vzamemo grupo avtomorfizmov grafa Γ. V tem primeru je znanih veliko povezav med kombinatoričnimi lastnostmi grafa Γ in algebraičnimi lastnostmi grupe G. Na primer, če grupa G deluje tranzitivno na množici vozlišč grafa Γ, potem je Γ regularen graf, v smislu, da ima vsako vozlišče grafa Γ enako število sosedov. V literaturi lahko najdemo še veliko primerov takšnih medsebojnih povezav med kombinatoričnimi lastnostmi grafa Γ in algebraičnimi lastnostmi njegove grupe avtomorfizmov.

V tej disertaciji algebraični objekt, ki bo prirejen grafu Γ, ne bo njegova grupa avtomorfizmov, temveč matrična algebra, imenovana *Terwilligerjeva algebra grafa* Γ. Glavna motivacija pa seveda ostaja enaka: *kaj lahko povemo o kombinatoričnih lastnostih grafa* Γ*, če vemo, da ima pripadajoča Terwilligerjeva algebra določene algebraične lastnosti?* In obratno: *kaj lahko povemo o algebraičnih lastnostih pripadajoče Terwilligerjeve algebre grafa* Γ*, če vemo, da ima graf* Γ *določene kombinatorične lastnosti?*

Terwilligerjeve algebre asociativnih shem je definiral Terwilliger v [89, Definicija 3.3]. Terwilligerjeva algebra grafa je nekomutativna matrična algebra, ki jo generira matrika sosednosti grafa, skupaj z nekaterimi diagonalnimi matrikami, ki vsebujejo lokalne informacije o strukturi grafa glede na neko fiksno vozlišče. Od takrat so bili objavljeni številni članki, v katerih je bila Terwilligerjeva algebra uspešno uporabljena za študij komutativnih asociativnih shem in razdaljno-regularnih grafov; glej [43, 44, 60, 65, 68, 78, 79, 81, 84, 86] za najnovejše rezultate na to temo.

Algebra T je bila v glavnem uporabljena za proučevanje razdaljno-regularnih grafov (glej na primer [6] za definicijo razdaljno-regularnih grafov). Ta algebra je bila uporabljena tudi za proučevanje Q-polinomskih razdaljno-regularnih grafov [9, 11, 38, 47, 58, 72, 71] (glej [6, stran 135] za definicijo Q-polinomski-razdaljno regularnih grafov), dvodelnih razdaljno-regularnih grafov, skoraj dvodelnih razdaljno-regularnih grafov [13], asociativnih sheme grup [4, 5], krepko regularnih grafov [13], Doobovih shem [85] (glej [6, stran 27] za definicijo Doobove sheme) in asociativnih shem nad Galois-evimi kolobarji karakteristike štiri [51]. Uporabljena je bila celo v teoriji kodiranja [37, 83].

Čeprav se lahko definicijo Terwilligerjeve algebre zlahka posploši na poljuben končen, enostaven in povezan graf, ne obstaja veliko rezultatov o Terwilligerjevih algebrah grafov, ki niso razdaljno-regularni. V člankih [54, 61] je bila preučevana Terwilligerjeva algebra incidenčnega grafa tako imenovane Johnsonove geometrije. V članku [94] je avtor preučeval Terwilligerjevo algebro incidenčnega grafa Hammingovega grafa. V članku [93] je bila proučena povezava med Terwilligerjevo algebro grafa Γ in še eno matrično algebro, ki je povezana z grafom Γ, in sicer tako imenovano *kvantno sosednostno algebro grafa* Γ. V člankih [59, 97] pa so avtorji proučevali strukturo nekaterih T-algeber končnih dreves. Omenjeni rezultati so najnovejši rezultati v tej smeri.

V tem poglavju naj bo Γ končen, enostaven in povezan graf. Izberimo si vozlišče x grafa Γ, ki ni list, in naj bo $T = T(x)$ pripadajoča Terwilligerjeve algebra. Algebra T je zaprta za konjugiranje in transponiranje. Zato se v mnogih primerih ta algebra učinkovito proučuje preko njenih nerazcepnih modulov.

Predpostavimo sedaj za trenutek, da je graf Γ razdaljno-regularen. Izkaže se, da je v tem primeru enolično določen nerazcepni T-modul s krajiščem 0 tanek. Predpostavimo tudi, da je Γ dvodelen. Izkaže se, da ima algebra T, do izomorfizma natančno, enolično določen nerazcepen T-modul s krajiščem 1, in da je ta modul prav tako tanek. Prav zaradi tega so bili v tem primeru v literaturi intenzivno proučevani nerazcepni T-moduli s krajiščem 2; glej na primer [9, 11, 15, 16, 17, 18, 19, 20, 38, 62, 63, 66, 67, 69, 70, 81]. Po drugi strani, če Γ ni dvodelen, je struktura nerazcepnih T-modulov s krajiščem 1 veliko bolj zapletena kot za dvodelne grafe. Za tovrstne rezultate glej na primer [21, 47, 71, 72, 92].

Naša raziskava se bo osredotočala na nerazcepne T-module s krajiščem 0 ali 1 splošnih grafov, ki niso nujno razdaljno-regularni.

Kot smo že omenili, je bilo do sedaj mnogo raziskav Terwilligerjevih algeber namenjeno raziskovanju razdaljno-regularnih grafov, katerih Terwilligerjeva algebra (glede na neko njihovo vozlišče) ima, do izomorfizma natančno, relativno malo nerazcepnih modulov z danim krajiščem, ter so vsi ti moduli (ne)tanki. Kot primer glej [63, 64, 65, 66, 67, 68, 74, 81]. V teh raziskavah raziskovalci ponavadi želijo pokazati, da je ta algebraičen pogoj izpolnjen če in samo če graf premore določene kombinatorične lastnosti. Naravno nadaljevanje teh raziskav so raziskave Terwilligerjevih algeber grafov, ki niso nujno razdaljno-regularni. Te raziskave so predstavljene v prvem delu te doktorske disertacije.

Izkaže se, da obstaja enolično določen nerazcepen T-modul s krajiščem 0. Že v [88] je Terwilliger pokazal, da je ta modul tanek, če je graf Γ razdaljno-regularen glede na vozlišče x. Če pa je nerazcepen T-modul s krajiščem 0 tanek, potem ne drži nujno, da je Γ razdaljno-regularen glede na x. Fiol in Garriga [33] sta kasneje vpeljala pojem *pseudo-razdaljne-regularnosti* okoli vozlišča x, ki temelji na priredbi uteži vozliščem, kjer te uteži ustrezajo komponentam (normaliziranega) pozitivnega lastnega vektorja. Pokazala sta, da je enolično določen nerazcepen T-modul s krajiščem 0 tanek natanko takrat, ko je graf Γ pseudo-razdaljno-regularen glede na vozlišče x (glej tudi [30, Izrek 3.1]). V poglavju 3 podamo povsem kombinatorično karakterizacijo lastnosti, da je ta T-modul tanek. V tej karakterizaciji nastopa število sprehodov (ki imajo določeno v naprej predpisano obliko) v grafu Γ med vozliščem x ter vozlišči na določeni fiksni razdalji od vozlišča x.

V nadaljevanju potem privzamemo, da je natančno določen nerazcepen T-modul s krajiščem 0 tanek (oziroma ekvivalentno, da je Γ pseudo-razdaljno-regularen glede na vozlišče x). Naš naslednji cilj je podati kombinatorično karakterizacijo grafov, ki imajo tudi, do izomorfizma natančno, enolično določen nerazcepen T-modul s krajiščem 1, ter je ta T-modul tanek. Če je Γ razdaljno-regularen, potem ima, do izomorfizma natančno, enolično določen nerazcepen T-modul s krajiščem 1 (in je le-ta modul tanek) natanko takrat, ko je Γ dvodelen ali skoraj dvodelen [21, Izrek 1.3]. V poglavju 4 pokažemo, da imajo tudi razdaljno-biregularni grafi do izomorfizma natančno enolično določen nerazcepen T-modul s krajiščem 1 (in je le-ta modul tanek). Primer, ko je graf Γ razdaljno-regularen glede na vozlišče x, ne pa nujno razdalno-regularen ali razdaljno-biregularen, obravnavamo v poglavju 5 in poglavju 6. V poglavju 7 zgornji rezultat posplošimo na primer, ko graf Γ ni nujno razdaljno-regularen glede na vozlišče x. Glavni rezultat tega dela disertacije je kombinatorična karakterizacija takih grafov. Tudi v tem primeru v karakterizaciji nastopa število sprehodov grafa Γ, ki so določene oblike. Pripomnimo, da so ti rezultati posplošitev prejšnjih prizadevanj raziskovalcev, da bi razumeli in klasificirali grafe, ki so pseudo-razdaljno-regularni glede na neko vozlišče, in imajo tudi, do izomorfizma natančno, enolično določen nerazcepen T-modul s krajiščem 1, ter je ta T-modul tanek, glej [13, 16, 21]. Podali bomo tudi konstrukcijo neskončne družine grafov, ki imajo zgoraj opisano lastnost.

Razdaljno-uravnoteženi grafi

Naj bo $\Gamma = (X, \mathcal{R})$ končen, neusmerjen, povezan graf, kjer je X množica njegovih vozlišč, $\mathcal{R}$ pa množica njegovih povezav. Za poljubni vozlišči $u, v \in X$ označimo z $\partial(u,v) = d_\Gamma(u,v)$ dolžino najkrajše poti med u in v. Za par sosednjih vozlišč u, v v grafu Γ definirajmo

$$W_{u,v} = \{x \in X \mid \partial(x,u) < \partial(x,v)\}.$$

Rečemo, da je Γ *razdaljno-uravnotežen*, kadar za poljuben par sosednjih vozlišč u in v v Γ velja

$$|W_{u,v}| = |W_{v,u}|.$$

Z raziskavami razdaljno-uravnoteženih grafov je leta 1999 pričel Handa, ki je v članku [45] proučeval razdaljno-uravnotežene delne kocke. Samo ime *razdaljno-uravnoteženi grafi* pa so vpeljali Jerebic, Klavžar in Rall v članku [52]. V tem članku so dokazali nekatere osnovne lastnosti razdaljno-uravnoteženih grafov, ter karakterizirali kartezične in leksikografske produkte razdaljno-uravnoteženih grafov, ki so razdaljno-uravnoteženi. Družina razdaljno-uravnoteženih grafov je zelo bogata. Študij te družine je zanimiv iz različnih povsem teoretičnih vidikov, kjer se osredotočimo na določeno lastnost teh grafov, kot recimo simetričnost oziroma grupa avtomorfizmov [55, 56, 98], povezanost [45, 75], ali kompleksnost algoritmov, ki so povezani s temi grafi [8]. Vsekakor pa ni presenetljivo, da so ti grafi zaradi svojih lastnosti zanimivi tudi na drugih raziskovalnih področjih, kot so recimo matematična kemija in komunikacijska omrežja. Na primer, raziskave razdaljno-uravnoteženih grafov so močno povezane z raziskavami dobro znanih Wiener-jevega in Szeged-ovega indeksa (glej [2, 52, 50, 87]). Dalje, razdaljno-uravnoteženi grafi predstavljajo zelo zaželjene modele raznih komunikacijskih omrežij [2]. Nedovno so bile v članku [12] proučevane povezave med razdaljno-uravnoteženimi grafi ter problemom trgovskega potnika.

Izkaže se, da se dajo razdaljno-uravnoteženi grafi karakterizirati z lastnostmi, ki nimajo na prvi pogled nič skupnega z njihovo originalno definicijo iz [52]. Na primer, v [3] je bilo pokazano, da razdaljno-uravnoteženi grafi sovpadajo s tako imenovanimi *self-median* grafi; to so grafi, pri katerih je vsota razdalj od izbranega vozlišča x do vseh ostalih vozlišč grafa neodvisna od izbire vozlišča x. Drug tak primer so tako-imenovani *grafi enakih možnosti* (glej [2] za njihovo definicijo). V [2] so avtorji pokazali, da razdaljno-uravnoteženi grafi s sodo mnogo vozlišči sovpadajo z grafi enakih možnosti. Naj omenimo še, da so bile v literaturi definirane in študirane tudi razne posplošitve razreda razdaljno-uravnoteženih grafov, glej na primer [1, 36, 49, 53, 76].

Pojem lepo razdaljno-uravnoteženih grafov se v kontekstu razdaljno-uravnoteženih grafov pojavi povsem naravno. Pravimo, da je Γ *lepo razdaljno-uravnotežen*, kadar obstaja tako naravno število $\gamma = \gamma(\Gamma)$, da za poljuben par sosednjih vozlišč u in v v grafu Γ velja

$$|W_{u,v}| = |W_{v,u}| = \gamma.$$

Jasno je, da je vsak lepo razdaljno-uravnotežen graf tudi razdaljno-uravnotežen, vendar pa nasprotno ni nujno res. Na primer, če je $n \geq 3$ poljubno liho naravno število, je graf prizme na $2n$ vozliščih razdaljno-uravnotežen, ne pa tudi lepo razdaljno-uravnotežen.

Predpostavimo sedaj, da je Γ lepo razdaljno-uravnotežen. Z d označimo premer grafa Γ (*premer* grafa je največja razdalja med dvema vozliščema). V [57], kjer so bili ti grafi prvič definirani, je bilo dokazano, da je $d \leq \gamma$. Poleg tega je bila podana klasifikacija vseh lepo razdaljno-uravnoteženih grafov, za katere velja $d = \gamma$. Izkaže se, da je graf Γ lepo razdaljno-uravnotežen z $d = \gamma$, če in samo če je Γ izomorfen bodisi polnemu grafu na $n \geq 2$ vozliščih, bodisi polnemu večdelnemu grafu $K_{t\times 2}$ $(t \geq 2)$, ali pa ciklu na $2d$ oz. $2d+1$ vozliščih. V tej disertaciji študiramo lepo razdaljno-uravnotežene grafe, za katere je $\gamma = d+1$. Izkaže se, da je situacija v tem primeru bistveno bolj zapletena kot v primeru, ko je $\gamma = d$. Zato smo se osredotočili na študij regularnih razdaljno-uravnoteženih grafov, za katere je $\gamma = d+1$. V poglavju 9 podamo popolno klasifikacijo takšnih grafov, glej Izrek 9.7.1.

Drug koncept, ki je tesno povezan s konceptom razdaljno-uravnoteženih grafov, je koncept krepko razdaljno-uravnoteženih grafov. Za poljubno povezavo uv danega grafa Γ in za katerikoli dve nenegativni celi števili i,j naj bo

$$D^i_j(u,v) = \{x \in X \mid \partial(u,x) = i \text{ in } \partial(v,x) = j\}.$$

Graf Γ se imenuje *krepko razdaljno-uravnotežen*, če za vsak $i \geq 1$ in vsako povezavo uv v Γ velja $|D^i_{i-1}(u,v)| = |D^{i-1}_i(u,v)|$. Preprosto je videti, da je krepko razdaljno-uravnotežen graf tudi razdaljno-uravnotežen, vendar obratno v splošnem ne drži (glej [55]). Za več rezultatov o krepko razdaljno-uravnoteženih grafih in o sorodnih konceptih glej [3, 8, 50, 57, 75].

V poglavju 10 disertacije se osredotočimo na konstrukcije nekaterih družin razdaljno-uravnoteženih grafov, ki so izjemno zanimive na tem področju raziskovanja.

Prva konstrukcija je konstrukcija nedvodelnih lepo razdaljno-uravnoteženih grafov, ki niso krepko razdaljno-uravnoteženi. Namreč, lepo razdaljno-uravnoteženi grafi so bili študirani v [57], kjer je bilo tudi dokazano, da znotraj razreda dvodelnih grafov, razreda razdaljno-uravnoteženih grafov in lepo razdaljno-uravnoteženih grafov sovpadata. Po drugi strani pa obstajajo primeri dvodelnih razdaljno-uravnoteženih grafov, ki niso krepko razdaljno-uravnoteženi [45]. Dalje, v [57] so bili predstavljeni primeri nedvodelnih krepko razdaljno-uravnoteženih grafov, ki niso lepo razdaljno-uravnoteženi. V poglavju 10 tako razrešimo [57, Problem 3.3] glede obstoja nedvodelnih lepo razdaljno-uravnoteženih grafov, ki niso krepko razdaljno-uravnoteženi, ki sta ga postavila Kutnar in Miklavič. Problem rešimo s konstrukcijo neskončne družine takšnih grafov.

Naša druga konstrukcija v tej disertaciji je v povezavi z domnevo o karakterizaciji krepko razdaljno-uravnoteženih grafov, ki so jo postavili Balakrishnan in ostali v [3]. Naj bo Γ graf in naj bo S podmnožica njegove množice vozlišč. Za poljubno vozlišče v grafa Γ definiramo

$$\partial(v, S) = \sum_{x \in S} \partial(v, x).$$

Balakrishnan in ostali [3] so dokazali, da je povezan graf Γ razdaljno-uravnotežen, če in samo če je $\partial(v, X) = \partial(u, X)$ za vse $u, v \in X$. Postavili so naslednjo domnevo glede podobne karakterizacije krepko razdaljno-uravnoteženih grafov: graf Γ je krepko razdaljno-uravnotežen, če in samo če za vsak par sosednjih vozlišč u, v grafa Γ velja $\partial(u, W_{u,v}) = \partial(v, W_{v,u})$. Jasno je, da krepko razdaljno-uravnoteženi grafi izpolnjujejo zgornji pogoj, vendar je še vedno odprto vprašanje, ali velja tudi obratno. V poglavju 10 pokažemo, da domneva [3, Conjecture 3.2] ne drži. To dokažemo s konstrukcijo neskončne družine protiprimerov za to domnevo.

Naša tretja konstrukcija je v povezavi z lastnostjo (krepke) razdaljne-uravnoteženosti v kontekstu grafov, ki premorejo določeno stopnjo simetrije. Kutnar in ostali so pokazali, da vozliščno-tranzitivni grafi niso le razdaljno-uravnoteženi, ampak tudi krepko razdaljno-uravnoteženi (glej [55]). Ker vozliščna tranzitivnost ni nujen pogoj, da je graf razdaljno-uravnotežen, je bilo torej naravno, da so avtorji raziskali lastnost razdaljne-uravnoteženosti znotraj razreda tako imenovanih *semisimetričnih grafov*; to je razred grafov, ki so kolikor je le mogoče blizu vozliščno-tranzitivnim grafom. Ti grafi so torej regularni povezavno-tranzitivni grafi, ki pa niso vozliščno-tranzitivni. Najmanjši semisimetrični graf ima 20 oglišč. Za njegovo odkritje je zaslužen Folkman [35], ki velja tudi za začetnika te veje raziskovanja.

Semisimetrični graf je nujno dvodelen, pri čemer dvodelni množici sovpadata z orbitama njegove grupe avtomorfizmov na množici vozlišč grafa. Posledično semisimetrični grafi ne premorejo avtomorfizmov, ki bi zamenjali par sosednjih vozlišč. Zato se naravno pojavijo kot dobri kandidati za grafe, ki niso razdaljno-uravnoteženi. Kutnar in ostali so dokazali, da obstaja neskončno semisimetričnih grafov, ki niso razdaljno-uravnoteženi, vendar obstaja tudi neskončno semisimetričnih grafov, ki so razdaljno-uravnoteženi. V poglavju 10 odgovorimo na vprašanje [55, Question 4.6], ki so ga postavili Kutnar in ostali: ali obstoja semisimetričen razdaljno-uravnotežen graf, ki ni krepko razdaljno-uravnotežen? Na to vprašanje ogovorimo s konstrukcijo neskončne družine takšnih grafov.

Poglavje 10 zaključimo z rezultatom, da za graf Γ z n vozlišči in m povezavami obstaja algoritem, ki v času $O(mn)$ preveri, ali je graf Γ krepko razdaljno-uravnotežen oziroma lepo razdaljno-uravnotežen.

Declaration

I declare that this thesis does not contain any materials previously published or written by another person except where due reference is made in the text.

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