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(Magistrsko delo)

Canonical double covers of graphs and their automorphisms

(Kanonični dvojni krovi grafov in njihovi avtomorfizmi)

Ime in priimek: *Đorđe Mitrović*
Študijski program: *Matematične znanosti, 2. stopnja*
Mentor: *izr. prof. dr. Ademir Hujdurović*

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Izvleček: Kanonični dvojni krov grafa X je definiran kot graf $BX \coloneqq X \times K_2$. Grupa $\mathrm{Aut}(BX)$ vsebuje podgrupo, izomorfno $\mathrm{Aut}(X) \times S_2$, ki jo generirajo avtomorfizmi faktorjev X in K_2 dvojnega krova BX. V primeru $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X) \times S_2$, se graf X imenuje stabilen. V nasprotnem primeru je graf X nestabilen. Če je X dodatno povezan, ni dvodelen in imajo različne točke različne množice sosedov, rečemo da je graf X netrivialno nestabilen. V tem magistrskem delu obravnavamo nekatere najpomembnejše rezultate o stabilnosti grafov in avtomorfizmih njihovih kanoničnih dvojnih krovov. Večina rezultatov, o katerih bomo razpravljali, spada v eno od naslednjih kategorij: rezultati, ki implicirajo (ne)stabilnost grafov, rezultati, ki določajo netrivialno nestabilne člane izbrane družine točkovno-trazitivnih grafov, ali rezultati za konstrukcijo nestabilnih grafov z različnimi predpisanimi lastnostmi. V vsaki od teh kategorij, bomo predstavili nove rezultate.

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Abstract: The canonical bipartite double cover of a graph X is defined to be the graph $BX := X \times K_2$. The group $\mathrm{Aut}(BX)$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times S_2$, which is generated by the automorphisms coming from the factors X and K_2 of the double cover BX. In case $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X) \times S_2$, the graph X is said to be stable. Otherwise, X is unstable. If X is additionally connected, non-bipartite and distinct vertices have distinct sets of neighbours, it is called non-trivially unstable. In this thesis, we examine some of the most important results on stability of graphs and automorphisms of their canonical double covers. The majority of the results we discuss fall into one of the following categories: results implying (in)stability of graphs, results characterizing non-trivially unstable members of a particular family of vertex-transitive graphs, or results for constructing unstable graphs with various prescribed properties. We present original results in each of these categories.

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1 INTRODUCTION

1.1 MOTIVATION AND RELATED WORK

The canonical bipartite double cover BX of a graph X is defined to be the graph $X \times K_2$, where $\times$ denotes the direct product of graphs and K_2 is the complete graph on two vertices. It is immediate from its definition that BX is a bipartite graph with the vertex set $V(BX) = V(X) \times \{0, 1\}$ and a bipartition given by the sets $V(X) \times \{0\}$ and $V(X) \times \{1\}$, which we will often refer to as colour classes of BX. Moreover, each edge $\{x, y\}$ of X induces two edges of BX, namely $\{(x, 0), (y, 1)\}$ and $\{(x, 1), (y, 0)\}$ for $x, y \in V(X)$.

This concept has been first introduced by Marušič, Scapellato and Zagaglia Salvi in their work on compatible $(0, 1)$-matrices [21]. The authors have studied the canonical double covers in terms of their adjacency matrices. Since then, canonical double covers have proven to play an important role in algebraic graph theory and have been studied by multiple groups of authors from a variety of perspectives [6, 16, 24, 32, 36].

Canonical double covers are fundamental in the study of symmetries of the direct product of graphs. In [9], Hammack and Imrich prove that a direct product $X \times Y$ of a non-bipartite graph X and a bipartite graph Y is vertex-transitive if and only if BX and Y are vertex-transitive.

One of the consequences of how we construct the canonical double cover is that $\mathrm{Aut}(BX)$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times S_2$. This subgroup is generated by the automorphisms $(x, i) \mapsto (\varphi(x), i)$ for $x \in V(X), i \in \{0, 1\}$ with $\varphi \in \mathrm{Aut}(X)$ and the automorphism τ swapping pairs of vertices $(x, 0)$ and $(x, 1)$ for each $x \in V(X)$.

In [34], Wilson refers to the subgroup $\mathrm{Aut}(X) \times S_2$ as the subgroup of expected automorphisms of BX. In the previously cited article [21], Marušič, Scapellato and Zagaglia Salvi refer to the graphs for which all automorphisms of BX are expected, that is, $\mathrm{Aut}(BX)$ is isomorphic to $\mathrm{Aut}(X) \times S_2$, as stable graphs. If a graph fails to satisfy this condition i.e., the automorphism group of the double cover contains additional automorphisms, called unexpected automorphisms, the graph is said to be unstable. Stable graphs include odd cycles and complete graphs K_n with $n \geq 3$, while even cycles are unstable.

Stability of a graph is an important property. For instance, Morris [23] explains that for a connected non-bipartite graph X and a bipartite graph Y (satisfying some mild conditions), it holds that $\mathrm{Aut}(X \times Y) \cong \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$, if X is stable. In [24], the authors provide a method for obtaining all orientable regular maps which are embeddings of the graph BX, given that X is a stable graph and that all of its regular embeddings are known.

It is not difficult to show that graphs which are disconnected, bipartite (with a non-trivial automorphism group) or contain distinct vertices with the same sets of neighbours (such vertices are called "twins") are unstable. This is why Wilson [34] defines an unstable graph to be non-trivially unstable if it is connected, non-bipartite and twin-free.

Non-trivially unstable graphs have received considerable attention. In [17], a close connection between two-fold automorphisms of graphs and unstable graphs is established. The search for non-trivially unstable graphs has lead to the introduction of Generalized Cayley graphs in [22] and extended Generalized Cayley graphs in [11].

In this thesis, we will discuss several natural questions regarding canonical double covers and their (unexpected) automorphisms. We will provide an overview of the results available in the literature that provide at least partial answers to these questions. All original results, excluding the ones that have been discovered over the course of working on this thesis, can be found in the following two articles on double covers of circulant graphs [12] and [13], that the student and supervisor are co-authors of together with Prof. Dave Witte Morris.

The differences between stable and unstable graphs will be studied and exploited when we will be considering a number of criteria for establishing stability or instability of graphs. Once sufficiently many results have been obtained, we will attempt to characterize non-trivially unstable members of various graph families. Often, we will be working with different subfamilies of circulant graphs, which are probably the most well-studied family of graphs from the point of view of stability theory. We will discuss the classifications of Cayley graphs of abelian groups of odd order and arc-transitive circulants as well as the newly obtained classifications of non-trivially unstable circulant graphs of order $2p$ (obtained in [13]) and circulant graphs of low valency (obtained in [12]). We will also consider strongly regular graphs, Andrásfai graphs, Kneser graphs and Johnson graphs and investigate their stability properties.

Similarly, we will be interested in the differences between expected and unexpected automorphisms. In order to describe what an unexpected automorphism can look like, we will study the characterization of non-trivially unstable graphs obtained by Wilson in [34].

In order to understand the relation between instability and other graph theoretic and symmetry properties, we will work with several constructions, each producing a family of non-trivially unstable graphs with surprising properties. These families were often constructed as counterexamples to various conjectures in the field. In [13], we constructed an infinite family of non-trivially unstable circulant graphs with no Wilson type (this shows that Wilson's conjecture from [34] is false).

1.2 STRUCTURE OF THE THESIS

In Chapter 2, we recall results and terminology from permutation group theory and algebraic graph theory that will be used throughout the thesis. We introduce several notions of graph products. We define Cayley graphs and discuss normal Cayley graphs and Paley graphs. We close the section with a selection of results on twin-free graphs and Cartesian skeletons that will be used in the following sections.

In Chapter 3, we introduce our main objects of study: canonical bipartite double covers, stable and unstable graphs, expected and unexpected automorphisms as well as the index of instability. We prove that if a graph is disconnected, bipartite (with a non-trivial automorphism group) or not twin-free, it is necessarily unstable. This leads to the definition of non-trivially unstable graphs. In Example 3.23 we describe the smallest non-trivially unstable graph, namely the Bowtie graph $\mathcal{W}$.

In Chapter 4, we collect stability criteria coming from various sources in the literature. We show that all, but the first, Andrásfai graphs are stable. Towards the end of the section, we discuss two results of Surowski [30]. Proposition 4.15 is a correction of his first result, while Proposition 4.18 is a generalization of the second. Using these results we are able to classify all unstable Johnson graphs in Theorem 4.30. We also show that infinitely many Kneser graphs are stable (see Corollary 4.23 and Corollary 4.24).

In Chapter 5, we introduce four criteria for instability of general graphs formulated by Wilson in [34]. These are described in Theorem 5.5, Theorem 5.14, Theorem 5.20 and Theorem 5.28. The main result of Wilson's article [34] is that every non-trivially unstable graph satisfies at least one of the mentioned conditions (see Theorem 5.35). Applying these results to the family of circulant graphs, one obtains the four Wilson types described in Theorem 5.48. We conclude the section by discussing generalizations of Wilson types, namely Theorem 5.52, Proposition 5.56 and Proposition 5.58, that we have introduced in [13] together with Prof. Dave Witte Morris.

In Chapter 6, we study the results on stability of Cayley graphs of abelian groups. We will cover two results of Qin, Xia and Zhou from [26]: Theorem 6.1 showing that there are no non-trivially unstable circulants of prime order, and Theorem 6.22 showing that there are no non-trivially unstable arc-transitive circulants. We then prove the analogous result of Hujdurović and Fernandez from [7] for circulants of odd order in Theorem 6.8. The result of Morris from [23], given in Theorem 6.11, is the final generalization of the previously mentioned results to Cayley graphs of abelian groups of odd order. We conclude the section with Theorem 6.18, where we characterize all non-trivially unstable circulants of order $2p$, with p a prime, using Wilson types.

Chapter 7 is dedicated to the classification of non-trivially unstable circulant graphs of valency at most 7, which we have obtained in [12] together with Prof. Dave Witte Morris. For each valency, an explicit list of all non-trivially unstable graphs is given. Moreover, instability of each obtained graph is explained by one of the original four Wilson types. The corresponding results are Proposition 7.6 (valency 3), Theorem 7.7 (valency 4), Theorem 7.8 (valency 5), Theorem 7.11 (valency 6) and Theorem 7.12 (valency 7).

In Chapter 8, we discuss several constructions, each producing a family of non-trivially unstable graphs with different additional properties. In Section 8.1, we describe the Swift graph SG (see Example 8.3), constructed by Wilson in [34], and an infinite family of non-trivially unstable asymmetric graphs $\mathcal{U}_k$ (see Example 8.10), first constructed by Lauri, Mizzi and Scapellato in [18] using TF-automorphisms. In Section 8.2, we discuss the infinite family of vertex-intransitive unstable graphs $\mathcal{X}(n)$ (see Theorem 8.22), introduced in [22], whose canonical double cover is a Cayley graph. In Section 8.3, we discuss the infinite family of non-trivially unstable arc-transitive graphs constructed by Surowski in [30] as double graphs of Paley graphs (see Theorem 8.34). We also briefly mention another construction introduced in the same article, which produces a non-trivially unstable family with an arbitrarily large index of instability. Finally, in Section 8.4, we discuss Example 8.41 from [13], which describes an infinite family of non-trivially unstable circulants with no Wilson type.

2 PRELIMINARIES

2.1 BASICS OF PERMUTATION GROUP THEORY

As we will mostly be concerned with graphs and their symmetries, majority of our arguments and results are best expressed in the language of group actions and permutation group theory. In this section, we recall some of the basics of this theory, establish notation and terminology. Standard reference is [3].

Definition 2.1. Let X be a non-empty set.

1. A bijection $f\colon X \to X$ is called a *permutation of* X.
2. The *symmetric group of* X, denoted $\mathrm{Sym}(X)$, is the group of all permutations on X together with function composition. If X is finite with n elements, we will sometimes denote this group by S_n and call it the *symmetric group on* n *letters.*
3. A *permutation group on* X is any subgroup of the group $\mathrm{Sym}(X)$.

Definition 2.2. Let G be a group and X a non-empty set. We say that G *acts* on the set X (from the left) if there exists a function

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto x^g \end{aligned}$$

such that

1. $x^1 = x$ for all $x \in X$ (where 1 denotes the identity element of the group G),
2. $(x^g)^h = x^{gh}$ for all $g, h \in G, x \in X$.

If X is a set and $G \leq \mathrm{Sym}(X)$ is a permutation group on X, then the *natural action* of G of X is given by evaluating elements of G (the permutations) at elements of X (the points).

The following objects are of fundamental importance in the study of permutation groups.

Definition 2.3. Let G be a group acting on a non-empty set X. Let $x \in X$ be a point.

1. The subgroup $G_x \coloneqq \{g \in G \mid x^g = x\}$ of G is called the *stabilizer of the point x*. It consists of all elements $g \in G$ that fix the point x.

2. The subset $x^G \coloneqq \{x^g \mid g \in G\}$ is called *the orbit of the point x* with respect to the action of G. It consists of all elements $y \in X$ for which there exists a $g \in G$ such that $x^g = y$. We will sometimes denote the orbit of x by $\mathcal{O}_G(x)$, or just by $\mathcal{O}(x)$, if the group G is clear from context.

We state the following observations without proof.

Observation 2.4. *Let G be a group acting on a non-empty set X.*

1. *Orbits of G form a partition of X.*

2. *If $x, y \in G$ lie in the same orbit of G i.e., if there exists a $g \in G$ such that $y = x^g$, then the corresponding point stabilizers G_x and G_y are conjugate subgroups of G. In particular, $G_y = gG_xg^{-1}$.*

Now that we have introduced orbits and stabilizers, we can talk about particularly nice group actions that will play a prominent role in our study of graph automorphisms (especially when we will be talking about Cayley graphs).

Definition 2.5. Let G be a group acting on a non-empty set X. Then the action of G is called

1. *transitive* - if G has only one orbit on X, that is, for every $x, y \in X$ there exists a $g \in G$ such that $x^g = y$.

2. *semi-regular* - if for all $x \in X$, the corresponding point stabilizer G_x is trivial. This means that if $g \in G$ fixes a point of X, it is automatically trivial.

3. *regular* - if it is both transitive and semi-regular. Note that, in this case, given $x, y \in X$, there exists a unique element $g \in G$ such that $x^g = y$.

So far, we have not made any assumptions on the set X. When X is finite, a number of nice results becomes available right away. The following fundamental result, which follows by a simple double-counting argument, establishes a connection between point stabilizers and orbits of a finite group acting on a finite set.

Lemma 2.6 (Orbit-Stabilizer lemma, [3, Theorem 1.4A(iii)]). *Let G be a finite group acting on a finite set X. Let $x \in X$ be arbitrary. Then the following equality holds*

$$|G| = |x^G||G_x|.$$

The following concept is one of the main tools in the study of permutation groups and their actions, as it provides a way of reducing an action of a group to an action of the same group on a smaller set, which might be easier to understand.

Definition 2.7 ([3, pp. 12–13])**.** Let G be a group acting on a non-empty set X. Let $B \subseteq X$ be a non-empty subset of X. Then B is called *a block for the action of* G if, for every $g \in G$, we have that

$$\text{either } g(B) = B \text{ or } g(B) \cap B = \emptyset.$$

The following observations can be derived directly from the definition of a block.

Observation 2.8. *Let G be a group acting on a non-empty set X. Let $B \subseteq X$ be a block for the action of G.*

1. *For every $g \in G$, $g(B)$ is a block for the action of G. Two blocks B and B' related by an element of $g \in G$ such that $B' = g(B)$ are called conjugate blocks.*

2. *Suppose that G is transitive in its action on X. The set $\mathcal{P} = \{g(B) \mid g \in G\}$ of all conjugates of B under G is a partition of X. Such a partition is called a block system for the action of G.*

3. *The group G has an obvious action on $\mathcal{P}$, given by $(g, B) \mapsto g(B)$. This defines a group homomorphism $\gamma\colon G \to \mathrm{Sym}(\mathcal{P})$. The kernel $\ker\gamma$ consists of all $g \in G$ that fix every block B in $\mathcal{P}$ set-wise.*

2.2 BASICS OF ALGEBRAIC GRAPH THEORY

In this section, we recall some of the basic definitions and results from algebraic graph theory. For the concepts from graph theory not defined here, we suggest the following references [8, 33].

Unless stated otherwise, all graphs are finite, simple (no loops or multiple edges) and undirected. The notation we will be using is standard. When the group G is clear from context, the identity element 1_G of G will often be denoted by 1.

Graphs will usually be denoted by capital letters X, Y, Z. The *vertex set*, *edge set* and the *automorphism group* of a graph X will be denoted by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$, respectively. *Automorphisms* of graphs will usually be denoted by Greek letters such as α, β, γ and φ.

The *neighbourhood of a vertex* x in a graph X will be denoted by $N_X(x)$. The *complement* of X is the graph $\overline{X}$ with $V(\overline{X}) \coloneqq V(X)$ and $x, y \in V(X)$ adjacent in $\overline{X}$ if and only if they are not adjacent in X.

The distance between x and y, that is, the length of the shortest path from x to y in X, will be denoted by $d_X(x, y)$ or just $d(x, y)$ if the graph X is clear from context. For a finite graph X, a positive integer $i \geq 0$ and a vertex $x \in V(X)$, $X_i(x) \coloneqq \{y \in V(X) \mid d_X(x, y) = i\}$ denotes the *distance sets of X with respect to i*. Recall that these sets form the *distance partition of X* with respect to x.

A lot of our work will have to do with graphs of different types and degrees of symmetry. Here we define some of the most common types of symmetric graphs we will be encountering.

Definition 2.9. Let X be a graph and $\mathrm{Aut}(X)$ its automorphism group. Note that $\mathrm{Aut}(X)$ is a permutation group on the vertex set $V(X)$. We will call X

1. *Vertex-transitive* - if $\mathrm{Aut}(X)$ is transitive in its natural action on the set of vertices $V(X)$ of X,

2. *Edge-transitive* - if $\mathrm{Aut}(X)$ is transitive in its induced action on the edge set $E(X)$ of X, where $\alpha(\{x,y\}) \coloneqq \{\alpha(x), \alpha(y)\}$ for $x, y \in V(X), \alpha \in \mathrm{Aut}(X)$,

3. *Arc-transitive* - if $\mathrm{Aut}(X)$ is transitive in its induced action on the set of arcs of X. By an *arc* of X, we mean an order pair of adjacent vertices (x, y) with $x, y \in V(X)$. The induced action is given by $\alpha(x, y) \coloneqq (\alpha(x), \alpha(y))$ for all $x, y \in V(X), \alpha \in \mathrm{Aut}(X)$.

4. *Distance-transitive* - $\mathrm{Aut}(X)$ is transitive on the set of pairs of equidistant vertices. This means that for all vertices $x, y, z, w \in V(X)$ such that $d(x, y) = d(z, w)$, there exists an $\alpha \in \mathrm{Aut}(X)$ such that $\alpha(x) = z$ and $\alpha(y) = w$. Note that this implies that for all $x \in V(X)$, the stabilizer $\mathrm{Aut}(X)_x$ is transitive on the distance set $X_i(x)$ for all $i \geq 1$.

Remark 2.10. *It is easy to see that arc-transitivity implies both vertex and edge-transitivity. The converse fails in general, as explained in [8, p.36-37].*

The following characterization of arc-transitivity is often useful.

Lemma 2.11. *Let X be a vertex-transitive graph. Then X is arc-transitive if and only if the point stabilizer $\mathrm{Aut}(X)_x$ is transitive on the set of neighbours $N_X(x)$ of x for some (equivalently, every) $x \in V(X)$.*

By noting that for a graph X and its vertex $x \in V(X)$, it holds that $N_X(x) = X_1(x)$, we obtain the following corollary of Lemma 2.11.

Corollary 2.12. *Distance-transitive graphs are arc-transitive.*

Recall the definitions of a block and a block system from the previous section (see Definition 2.7 and Observation 2.8).

Lemma 2.13. *Let X be a bipartite graph. If X is connected, then its bipartition is unique. Consequently, if $V(X) = A \cup B$ is the unique bipartition of X and $\alpha \in \mathrm{Aut}(X)$ is an automorphism of X, then*

$$\textit{either } \alpha(A) = A, \alpha(B) = B \textit{ or } \alpha(A) = B, \alpha(B) = A.$$

Moreover, if X is vertex-transitive, A and B are conjugate blocks and $\{A, B\}$ is a block system for the action of $\mathrm{Aut}(X)$.

Lemma 2.14. *Let X be a connected vertex-transitive graph of odd order. Then X is non-bipartite.*

Proof. Assume for contradiction that X is bipartite with bipartition $V(X) = A \cup B$. Then by Lemma 2.13, it follows that A and B are conjugate blocks with respect to the action of $\mathrm{Aut}(X)$. In particular, $B = \alpha(A)$ for some $\alpha \in \mathrm{Aut}(X)$. Hence, $|A| = |B|$ and $|V(X)| = 2|A|$. This is a contradiction with X being a graph of odd order. □

We will often be working with quotient graphs. We give a precise definition that we will be using below.

Definition 2.15. Let X be a graph. Let $\mathcal{P}$ be a partition of its vertex set. The *quotient graph of X with respect to $\mathcal{P}$* is the simple (so loop-less) graph $X/\mathcal{P}$ with

- $V(X/\mathcal{P}) \coloneqq \mathcal{P}$,
- $E(X/\mathcal{P})$ consisting of $\{B_1, B_2\}$ with $B_1, B_2 \in \mathcal{P}$ such that there exist $x \in B_1, y \in B_2$ with $\{x, y\} \in E(X)$.

A particular instance of this construction that we will encounter is when the partition $\mathcal{P}$ is formed by the orbits of a group $\langle\gamma\rangle$ with $\gamma \in \mathrm{Aut}(X)$. In this case, the corresponding quotient graph will be denoted by X/γ.

2.3 GRAPH PRODUCTS

The following products will be of great importance in the following sections.

Definition 2.16 ([10, p. 35, 36, and 43], [7, p. 53])**.** Let X and Y be graphs.

1. The *direct product* $X \times Y$ is the graph with $V(X \times Y) = V(X) \times V(Y)$, such that (x_1, y_1) is adjacent to (x_2, y_2) if and only if
$$(x_1, x_2) \in E(X) \ \text{ and } \ (y_1, y_2) \in E(Y).$$
2. The *Cartesian product* $X \square Y$ is the graph with $V(X \times Y) = V(X) \times V(Y)$, such that (x_1, y_1) is adjacent to (x_2, y_2) if and only if either
   - $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$, or
   - $y_1 = y_2$ and $(x_1, x_2) \in E(X)$.
3. The *wreath product* $X \wr Y$ is the graph with $V(X \wr Y) = V(X) \times V(Y)$ such that (x_1, y_1) and (x_2, y_2) are adjacent if and only if either

- $\{x_1, x_2\} \in E(X)$, or
- $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$.

Note that $X \wr Y$ can be obtained by replacing each vertex of X with a copy of Y. (Vertices in two different copies of Y are adjacent in $X \wr Y$ if and only if the corresponding vertices of X are adjacent in X.) This product is sometimes called the *lexicographic product* (see [10, p. 43]).

4. The *deleted wreath product of* X *and* $\overline{K_n}$ is the graph $X \wr_d \overline{K_n}$ with $V(X \wr_d \overline{K_n}) = V(X) \times V(\overline{K_n})$, such that (x, i) and (y, j) are adjacent if and only if $\{x, y\} \in E(X)$ and $i \neq j$. Note that $X \wr_d \overline{K_n}$ can be obtained from $X \wr \overline{K_n}$ by removing n vertex-disjoint copies of X. This is why we sometimes write $X \wr_d \overline{K_n} = X \wr \overline{K_n} - nX$.

Remark 2.17 ([10, p. 36])**.** *In the literature, the direct product appears under various other names, including "tensor product," "Kronecker product," "cardinal product," and "conjunction."*

There is a number of results about interactions of these products and their automorphism groups. A standard reference is [10]. The results coming from other sources in the literature, that will be useful to us, are listed here.

The following is a direct consequence of [4, Theorem 5.3] derived in [7]. We define what it means for a graph to be "twin-free" in Definition 2.41.

Lemma 2.18 ([7, Lemma 2.9], [4, Theorem 5.3])**.** *Let $d \geq 3$. Let Y be a twin-free, vertex-transitive graph whose order is not divisible by d. Then it holds that*

$$\mathrm{Aut}(Y \wr_d \overline{K_d}) \cong \mathrm{Aut}(Y) \times S_d.$$

Lemma 2.19 (Dobson-Miklavič-Šparl [4, Proposition 4.5])**.** *Let X be a graph and let $m, n \geq 2$ be integers. Then it holds that*

$$(X \wr K_m) \wr_d K_n \cong (X \wr_d K_n) \wr K_m.$$

Lemma 2.20 (Qin-Xia-Zhou [26, Example 2.1])**.** *Let X be a graph and $d > 1$ an integer. Then it holds that*

$$X \times K_d \cong X \wr \overline{K_d} - dX \cong X \wr_d \overline{K_d}.$$

2.4 CAYLEY GRAPHS

We now introduce Cayley graphs, a class of vertex-transitive graphs of pivotal importance. Most problems we will be considering, we will also consider in the context of Cayley graphs or particular families of Cayley graphs.

Definition 2.21. Let G be a group. A subset of $S \subseteq G$ is called *inverse-closed* if $s \in S$ implies that $s^{-1} \in S$ for all $s \in S$

Definition 2.22. Let G be a group with identity element 1. Let S be an inverse-closed subset of G and assume that $1 \notin S$. The *Cayley graph of G with respect to S* is the graph $\mathrm{Cay}(G, S)$ with

- $V(\mathrm{Cay}(G, S)) \coloneqq G$,
- $E(\mathrm{Cay}(G, S)) \coloneqq \{\{g, gs\} \mid g \in G, s \in S\}$.

The definition of the edge set of $\mathrm{Cay}(G, S)$ implies that $g, h \in G$ are adjacent if and only if $g^{-1}h \in S$ (or equivalently, as S is inverse-closed, $h^{-1}g \in S$).

We refer to S as the *connection set* of $\mathrm{Cay}(G, S)$.

Definition 2.23. (M.-Y. Xu [35, Definition 1.4]) For each $g \in G$, define the *left translation by g* to be the map $g_L \colon G \to G$ given by $x \mapsto gx$ for $x \in G$.

It is clear that this is a permutation of G, with $(g_L)^{-1} = (g^{-1})_L$. The set

$$G_L = \{\, g_L \mid g \in G \,\}$$

is a subgroup of $\mathrm{Sym}(G)$ and it is often called the (*left*) *regular representation* of G. Note that $g \mapsto g_L$ is a group isomorphism between G and G_L.

Definition 2.24. Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph of a group of G. We define

$$\mathrm{Aut}(G, S) \coloneqq \{\varphi \in \mathrm{Aut}(G) \mid \varphi(S) = S\}.$$

Cayley graphs are well-studied in the literature, so all of the following properties are well known. We make a concrete list of the properties that we will be using in the rest of the thesis for easy reference.

Proposition 2.25. *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G with a connection set S. Then X has the following properties.*

1. *The left regular representation $G_L = \{g_L \mid g \in G\}$ is a regular subgroup of $\mathrm{Aut}(X)$ isomorphic to G.*
2. *X is vertex-transitive.*
3. *X is regular with valency $|S|$. Moreover, for $g \in G$, it holds that $N_X(g) = gS \coloneqq \{gs \mid s \in S\}$.*
4. *The group $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(X)_1$.*
5. *X is connected if and only if S generates G.*

6. *The complement of* X *is given by* $\overline{X} = \mathrm{Cay}(G, (G \setminus \{1\}) \setminus S)$.

Proof. The first five items are obvious. For part (6), note that it follows from definitions of X and the graph complement that $x, y \in V(X), x \neq y$ are adjacent in $\overline{X}$ if and only if they are not adjacent in X, which happens if and only if $x^{-1}y \notin S$. Then $x^{-1}y \in G \setminus S$, but $x^{-1}y$ is not the identity, as x and y were assumed to be distinct. Hence, $x^{-1}y \in (G \setminus \{1\}) \setminus S$. Finally, note that as S is inverse-closed and 1 is self-inverse, the set $(G \setminus 1) \setminus S$ is also inverse-closed. □

As we have seen in Proposition 2.25(1), there is a strong connection between the automorphism group of a Cayley graph and the group it is constructed from. As a matter of a fact, this property characterizes Cayley graphs and the result is known as the Sabidussi's theorem.

Theorem 2.26 (Sabidussi [27]). *A graph* X *is a Cayley graph of a group* G *if and only if* $\mathrm{Aut}(X)$ *contains a regular subgroup isomorphic to* G.

We now derive the following properties of Cayley graphs using the results we already considered in Section 2.1.

Corollary 2.27. *Let* $X = \mathrm{Cay}(G, S)$ *be a connected Cayley graph. If* $|G|$ *is odd, then* X *is non-bipartite.*

The blocks and block systems for the natural action of the automorphism group of a Cayley graph of an abelian group can be described in the following two ways.

Lemma 2.28 ([12, Definition 2.7]). *Let* G *be an abelian group and* $X = \mathrm{Cay}(G, S)$ *a Cayley graph of* G. *Let* $B \subseteq G$ *be a block for the natural action of* $\mathrm{Aut}(X)$ *and denote by* $\mathcal{P}$ *the block system consisting of conjugate blocks of* B. *Then there exists a subgroup* H *of* G *such that*

1. B *is a coset of* H *in* G. *Moreover, the partition of* G *into cosets of* H *coincides with the partition* $\mathcal{P}$.

2. B *is an orbit of the subgroup* H_L *of* $G_L \leq \mathrm{Aut}(X)$. *Moreover, the partition of* G *into orbits of* H_L *coincides with the partition* $\mathcal{P}$.

3. *The quotient graph* $X/\mathcal{P}$ *can be identified with* $\mathrm{Cay}(G/H, \{sH \mid s \in S\})$.

Proof. (1) Let H denote the conjugate block of B that contains the identity 1 of G. This shows that $h \in H$ if and only if $h_L(H) = H$. As $(gh)_L = g_L h_L$, this shows that H is closed under product operation of G. As G is abelian, the inversion map $\iota \colon g \mapsto g^{-1}$ is an automorphism of X that fixes the identity. It follows that $\iota(H) = H$, so H is closed under taking inverses. Hence, H is a subgroup of G.

Moreover, note that $H \in \mathcal{P}$, so some block is a conjugate block of B if and only if it is a conjugate block of H. Therefore, as the group G_L is transitive, every block in $\mathcal{P}$ can be obtained as $g_L(H) = gH$ for some $g \in G$.

(2) As $H_L \leq G_L$ acts on X by translations by elements of H, it is clear that the orbit of $g \in G$ under H_L is just gH. We have already shown that $\mathcal{P}$ coincides with the set of cosets of H, so the conclusion follows.

(3) By (1), we can identify the vertex set $\mathcal{P}$ of the quotient graph $X/\mathcal{P}$ with the quotient group G/H (which is just the set of cosets of H in G). Note that two cosets xH and yH are adjacent in $X/\mathcal{P}$ (after possibly choosing a different coset representative) if and only if $x^{-1}y \in S$. This is equivalent to $(xH)^{-1}(yH) = (x^{-1}y)H = sH$ for some $s \in S$. It follows that $X/\mathcal{P}$ is just $\operatorname{Cay}(G/H, \{sH \mid s \in S\})$, as desired. □

By far the most important example of Cayley graphs is given by circulant graphs, which we will be studying intesively in the following chapters.

Definition 2.29. A *circulant graph* is a Cayley graph of a cyclic group.

Circulants form a wide and diverse class of graphs that includes cycles, Möbius ladders, odd prisms and complete graphs.

2.4.1 Normal Cayley graphs

We now consider a special type of Cayley graphs with a particularly nice and easy to describe automorphism group.

Definition 2.30. Let $X = \operatorname{Cay}(G, S)$ be a Cayley graph of a group G. From Proposition 2.25(1), we know that $\operatorname{Aut}(\operatorname{Cay}(G, S))$ contains the subgroup G_L, which is isomorphic to G. If this subgroup is normal in $\operatorname{Aut}(X)$, then X is called a *normal Cayley graph.*

The following result offers several characterizations of normal Cayley graphs.

Proposition 2.31. *Let $X = \operatorname{Cay}(G, S)$ be a Cayley graph. Then the following are equivalent.*

1. *X is normal,*
2. *$\operatorname{Aut}(X) \cong G_L \rtimes \operatorname{Aut}(G, S)$,*
3. *$\operatorname{Aut}(X)_1 = \operatorname{Aut}(G, S)$,*
4. *$|\operatorname{Aut}(X)| = |G||\operatorname{Aut}(G, S)|$.*

The property of being normal is not preserved by all graph isomorphisms (note that, it is preserved by graph isomorphism coming from isomorphisms of underlying groups). For example, both of the following Cayley graphs

$$X_1 = \mathrm{Cay}(\mathbb{Z}_2^3, \{(1,0,0),(0,1,0),(0,0,1)\})$$

and

$$X_2 = \mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0),(3,0),(0,1)\})$$

are isomorphic to the 3-dimensional cube Q_3.

However, X_1 is a normal Cayley graph, while X_2 is not. Note that they are defined on non-isomorphic groups.

While determining whether a particular Cayley graph is normal is difficult in general, the following result works well for Cayley graphs of abelian groups.

Lemma 2.32 (Baik-Feng-Sim-Xu, [1, Theorem 1.1])**.** *Let* $\mathrm{Cay}(G,S)$ *be a connected Cayley graph on an abelian group* G*. Assume, for all* $s,t,u,v \in S$

$$st = uv \neq 1 \implies \{s,t\} = \{u,v\}.$$

Then the Cayley graph $\mathrm{Cay}(G,S)$ *is normal.*

Lemma 2.33 (Qin-Xia-Zhou [26, Lemma 2.10])**.** *Let* $G = H \times K$ *be a group, where* H *is a subgroup and* K *is a characteristic subgroup of order at least* 5*. Suppose that* $S = T \times (K \setminus \{1\})$ *is inverse-closed, where* $T \subseteq H$*. Then* $\mathrm{Cay}(G,S)$ *is non-normal.*

The following result is a direct corollary of the classification of non-normal Cayley digraphs of order $2p$ given in [5, Theorem 1.6].

Lemma 2.34. *Let* p *be a prime. Let* X *be a non-normal connected, bipartite circulant of order* $2p$ *of valency at most* $p-2$*. Then* X *is isomorphic to* $Y \wr \overline{K_2}$ *for some connected graph* Y*.*

Proof. We inspect the table of 19 examples of non-normal Cayley digraphs given in [5, Theorem 1.6] by Du, Wang and Hu.

- Only the entries in the rows 3-11 are circulants.
- Entries in rows $3, 4$ and 5 are not connected.
- Entries in rows $7, 8$ and 11 are non-bipartite.
- Entries in rows 9 and 10 have the valency at least $p-1$.

This leaves us with row 6, where $X \cong Y \wr \overline{K_2}$ for some digraph $Y \neq pK_1$.

As X is a connected graph, Y must be as well. Note that this makes the remark that $Y \neq pK_1$ superfluous. We have arrived at the desired result. □

Finally, we recall an important classification result for arc-transitive circulants, that we will be using often in later sections.

Theorem 2.35 (Kovács, [15, Theorem 1])**.** *Let $n \geq 1$ be a positive integer. Let X be a connected, arc-transitive circulant graph of order n. Then one of the following holds:*

1. $X = K_n$,
2. *X is normal,*
3. *$X = Y \wr \overline{K_d}$ where $n = md$, $d > 1$ and Y is a connected arc-transitive circulant of order m,*
4. *$X = Y \wr \overline{K_d} - dY$, where $n = md$, $d > 3$, $\gcd(m, d) = 1$ and Y is a connected arc-transitive circulant of order m.*

2.4.2 Paley graphs

Paley graphs are a family of Cayley graphs enjoying several nice properties, such as being self-complementary and strongly regular. They will be the key ingredient of the construction we will discuss in Section 8.3.

We now define Paley graphs and derive some of their many properties.

Definition 2.36. Let $\mathrm{GF}(q)$ be the Galois field of prime power order q. Assume that $q \equiv 1 \pmod 4$. Let S denote the set of non-zero squares of $\mathrm{GF}(q)$, that is, the set of elements $x \in GF(q), x \neq 0$ for which there exists an element $y \in \mathrm{GF}(q)$ such that $x = y^2$.

The *Paley graph of order q* is the graph $\mathrm{P}(q)$ with

- $V(\mathrm{P}(q)) = \mathrm{GF}(q)$,
- $E(\mathrm{P}(q)) = \{\{a, b\} \mid a - b \in S\}$.

Remark 2.37. *Note that the Paley graph $\mathrm{P}(q)$ is just the Cayley graph of the additive group of the field $\mathrm{GF}(q)$ with S, the set of non-zero squares, as its connection set i.e., $\mathrm{P}(q) = \mathrm{Cay}(\mathrm{GF}(q), S)$. The assumption that $q \equiv 1 \pmod 4$ implies that -1 is a square. As the product of non-zero squares is a non-zero square, the fact that -1 is a square ensures that S is closed under additive inverses. Furthermore, note that the inverse of a non-zero square is a non-zero square, so S is also closed under multiplicative inverses (moreover, it is a subgroup of the multiplicative group $\mathrm{GF}(q)^*$).*

We recall the following definitions from graph theory.

Definition 2.38. A graph X is called *self-complementary* if it is isomorphic to its complement $\overline{X}$.

Definition 2.39. A graph X is called *strongly regular with parameters* (v, k, λ, μ) if

1. X has v vertices,
2. X is k-regular,
3. every two adjacent vertices of X have λ neighbours in common,
4. every two non-adjacent vertices of X have μ neighbours in common.

It turns out that Paley graphs enjoy both of these properties.

Proposition 2.40. *Let q be a prime power such that $q \equiv 1 \pmod 4$. The Payley graph* $\mathrm{P}(q)$ *has the following properties.*

1. $\mathrm{P}(q)$ *is vertex-transitive.*
2. $\mathrm{P}(q)$ *is arc-transitive.*
3. $\mathrm{P}(q)$ *is self-complementary.*
4. $\mathrm{P}(q)$ *is strongly regular with parameters* $\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$.

Proof. (1) By Remark 2.37, $\mathrm{P}(q)$ is a Cayley graph, so it is automatically vertex-transitive by Proposition 2.25(2). Given a non-zero element $\gamma \in \mathrm{GF}(q)$, define $\gamma^*(x) = \gamma x$ for $x \in \mathrm{GF}(q)$. Note that γ^* is a group automorphism of the additive group of the field $\mathrm{GF}(q)$.

(2) The conclusion follows by noting that the maps of the form γ^* for $\gamma \in S$ lie in $\mathrm{Aut}(\mathrm{P}(q))_0$.

(3) By Proposition 2.25(6), it holds that $\overline{\mathrm{P}(q)} = \mathrm{Cay}(\mathrm{GF}(q), (\mathrm{GF}(q) \setminus \{0\}) \setminus S)$. If $\gamma \in \mathrm{GF}(q)$ is a non-square, γ^* is an isomorphism of $\mathrm{P}(q)$ and $\overline{\mathrm{P}(q)}$.

(4) The fact that $\mathrm{P}(q)$ is strongly regular follows from the fact that it is arc-transitive and self-complementary. Its parameters can be calculated by counting the number of edges between the sets of non-zero squares and non-squares in $\mathrm{GF}(q)$ and by noting that the inversion map $\iota\colon x \mapsto x^{-1}$ swaps these edges and non-edges. $\square$

2.5 TWIN-FREE GRAPHS

The following property will be of great importance when we will be discussing "trivially unstable" graphs in Section 3.2.

Definition 2.41 (Kotlov-Lovász [14])**.** In a graph X, two distinct vertices x and y are called *twins* if $N_X(x) = N_X(y)$. A graph X is *twin-free* if it does not contain any twins.

Remark 2.42. *Synonymous terms for "twin-free" include "irreducible" [7], "vertex-determining" [26] and "R-thin" [10, p. 91].*

We will now list several results about the property of being twin-free. Their role will be clearer when we move onto more central topics of the thesis, such as canonical double covers and stability of graphs.

Lemma 2.43 (Qin-Xia-Zhou [26, Lemma 2.3])**.** *Let X and Y be graphs. Then $X \times Y$ is twin-free if and only if X and Y are twin-free.*

Proof. The conclusion follows immediately from the following observation coming from the definition of the direct product of graphs (see Definition 2.16(1)).

$$N_{X\times Y}(x, y) = N_X(x) \times N_Y(y), \forall x \in X, \forall y \in Y.$$

□

Lemma 2.44 (Qin-Xia-Zhou [26, Lemma 2.4])**.** *Let X be a graph with at least one edge and $d > 1$ an integer. Then the graph $X \wr \overline{K_d}$ is not twin-free.*

Proof. As X has at least one edge, we can find $x \in V(X)$ so that $N_X(x)$ is non-empty. For $u \in V(\overline{K_d})$, the neighbourhood of (x, u) in $X \wr \overline{K_d}$ is $N_X(x) \times V(\overline{K_d})$. Thus, for distinct vertices u and v of $\overline{K_d}$, (x, u) and (x, v) have the same neighbourhood in $X \wr \overline{K_d}$ and are twins by Definition 2.41. Therefore, $X \wr \overline{K_d}$ is not twin-free. □

Lemma 2.45. *Let $X = \mathrm{Cay}(G, S)$ be a connected Cayley graph of an abelian group G. The following are equivalent.*

1. *X is not twin-free.*
2. *There exists a non-trivial subgroup H of G, such that the connection set S of X is a union of cosets of H in G.*
3. *$X \cong Y \wr \overline{K_m}$ for some $m \geq 2$ and a connected, twin-free graph Y. Note that the vertex sets of copies of $\overline{K_m}$ in X coincide with the sets of twins in X.*

Moreover, in part (3), the copies of $\overline{K_m}$ in X are the cosets of some subgroup H of G of order m, such that the graph Y is isomorphic to a Cayley graph of the group G/H and S is a union of cosets of H.

Proof. (1) $\Leftrightarrow$ (2) If X not twin-free, by vertex-transitivity of X, we can let $h \in G$ be a twin of the identity 1. Then $h \neq 1$ and $S = N_X(1) = N_X(h) = hS$. Let H be the cyclic subgroup of G generated by h. Then it holds that

$$S = \bigcup_{s \in S} Hs.$$

Conversely, if S is a union of cosets of a non-trivial group H, it holds that $N_X(h) = hS = S = N_X(1)$ for all $h \in H$.

(1) $\Leftrightarrow$ (3) If X can be decomposed as $Y \wr \overline{K_m}$ with $m \geq 2$, it is not twin-free by Lemma 2.44. Furthermore, by definition of the wreath product, it is clear that every two vertices of the same copy of $\overline{K_m}$ in X are twins. Vertices of X lying in distinct copies of $\overline{K_m}$ correspond to distinct vertices of Y and are therefore not twins, as Y is assumed to be twin-free.

Assume that X is not twin-free. Let $\sim$ be the relation of "being twins" on X. Then $\sim$ is an equivalence relation and for all $\alpha \in \mathrm{Aut}(X)$, it holds that $x \sim y$ if and only if $\alpha(x) \sim \alpha(y)$ (in the language of [3], this means that $\sim$ is an $\mathrm{Aut}(X)$-congruence). This shows that the partition $\mathcal{P}$ of G into equivalence classes of $\sim$, that is, the sets of twins in X, is a block system for the action of $\mathrm{Aut}(X)$. By Lemma 2.28(3), G has a subgroup H such that the blocks in $\mathcal{P}$ are cosets of H and $Y := X/\mathcal{P}$ is a Cayley graph of the group G/H. Moreover, Y is connected, since X is connected. As its vertices correspond to distinct equivalence classes of twins in X, it follows that Y is twin-free.

As all blocks in $\mathcal{P}$ are conjugate, they are all of the same fixed size m and as X is not twin-free, $m \geq 2$. Recall that twins are non-adjacent vertices. Therefore, elements of $\mathcal{P}$ are independent sets of size m in X. Moreover, if x and y are adjacent, every twin x' of x is adjacent to every twin y' of y. This shows that two vertices of X are adjacent if and only if they lie in distinct equivalence classes of $\sim$, which are adjacent as vertices of Y. Hence, $X \cong Y \wr \overline{K_m}$.

Moreover, note that if we start with the decomposition $Y \wr \overline{K_m}$, by identifying the copies of $\overline{K_m}$ in X with equivalence classes of twins, we can repeat the previous argument and reproduce Y as the Cayley graph of G/H. Furthermore, $S = N_X(1)$ is a union of copies $\overline{K_m}$ i.e., the equivalence class of twins. As they correspond to cosets of H, it follows that S is a union of cosets of H. □

Lemma 2.46. *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a connected circulant graph of order n such that X is not twin-free, and let d be the valency of X.*

1. *There is a connected circulant graph Y and some $m \geq 2$, such that $X \cong Y \wr \overline{K_m}$ and $d = \delta m$, where δ is the valency of Y.*

2. *If d is prime, then $X \cong K_{d,d}$.*

3. *If $d = 4$, then X is isomorphic either to $K_{4,4}$ or to $C_\ell \wr \overline{K_2}$ with $\ell = |V(X)|/2$. Moreover, the unique twin of 0 in the second case is $n/2$.*

Proof. (1) Note that quotients of cyclic groups are cyclic. The desired conclusion follows immediately by applying Lemma 2.45(3).

(2) By (1), we then represent X as $Y \wr \overline{K_m}$, where Y is m-regular and connected, and $m \geq 2$. As $d = \delta m$, and d is prime, it follows that $m = d$ and $\delta = 1$. In particular, $Y = K_2$ and $X = K_2 \wr \overline{K_d} \cong K_{d,d}$. (Conversely, it is clear that $K_{d,d}$ is a connected circulant graph, but is not twin-free.)

(3) By (1), we then represent X as $Y \wr \overline{K_m}$, where Y is m-regular and connected, and $m \geq 2$. As $4 = \delta m$ and $m \geq 2$, it follows that $m \in \{2, 4\}$. If $m = 4$, then $\delta = 1$ and consequently $X \cong K_2 \wr \overline{K_4} \cong K_{4,4}$. If $m = 2$, then Y is connected and 2-regular, so it is isomorphic to the cycle C_ℓ with $\ell = |V(X)|/2$. It follows that $X \cong C_\ell \wr \overline{K_2}$. By Lemma 2.45(3), it is clear that two vertices of X are twins if and only if they lie in the same copy of $\overline{K_2}$, that is, the same coset of some subgroup H of $\mathbb{Z}_n$ of order 2. As H is necessarily $\{0, n/2\}$, the conclusion follows. □

Corollary 2.47. *Let $X = \mathrm{Cay}(\mathbb{Z}_p, S)$ be a circulant of odd prime order p. Then either*

- *$X = \overline{K_p}$, or*

- *X is connected, non-bipartite and twin-free.*

Proof. Assume that $X \neq \overline{K_p}$. In particular, X contains at least one edge. Then $S \neq \emptyset$. Let s be an arbitrary element of S. By definition, S does not contain the identity of the group, so $s \neq 0$. Then s is a generator of $\mathbb{Z}_p$. It follows that S is a generating set of $\mathbb{Z}_p$, so Proposition 2.25(5) implies that X is connected.

Since X is a connected Cayley graph of odd order p, Corollary 2.27 implies that X is non-bipartite.

Finally, assume for contradiction that X is not twin-free. It follows by Lemma 2.46(1), that there exists a connected circulant graph Y and an integer $m \geq 2$ such that $X = Y \wr \overline{K_m}$. Note that

$$p = |V(X)| = |V(Y \wr \overline{K_m})| = |V(Y)||V(\overline{K_m})| = m|V(Y)|.$$

Then $m \geq 2$ is a divisor of a prime p, so it must hold that $m = p$. Consequently, $|V(Y)| = 1$ and $Y = K_1$. It follows that $X \cong \overline{K_p}$, a contradiction. □

2.6 CARTESIAN SKELETON

In this section, we briefly introduce and discuss the properties of the Cartesian skeleton of a graph. The idea behind it, is that it allows one to translate questions about other graph products, in particular the direct product, into questions about the Cartesian product (to recall relevant definitions, see Definition 2.16)).

Lemma 2.51 will be particularly important when we will be proving results about canonical double covers of circulants and more general Cayley graphs of abelian groups of odd order in Chapter 6.

Definition 2.48 ([10, Section 8.3])**.** Let X be a graph. The *Boolean square of* X is the graph $\mathcal{B}(X)$ with

- $V(\mathcal{B}(X)) \coloneqq V(X)$,
- $E(\mathcal{B}(X)) \coloneqq \{\{x,y\} \mid x,y \in V(X), x \neq y, N_X(x) \cap N_X(y) \neq \emptyset\}$.

Definition 2.49 ([10, Section 8.3])**.** Let X be a graph and $\mathcal{B}(X)$ its Boolean square. An edge $\{x,y\} \in E(X)$ is called *dispensable with respect to* X if there exists a vertex $z \in V(X)$ such that

$$N_X(x) \cap N_X(y) \subsetneq N_X(x) \cap N_X(z) \text{ or } N_X(x) \subsetneq N_X(z) \subsetneq N_X(y)$$

and

$$N_X(y) \cap N_X(x) \subsetneq N_X(y) \cap N_X(z) \text{ or } N_X(y) \subsetneq N_X(z) \subsetneq N_X(x).$$

The *Cartesian skeleton of* X is then the graph $\mathcal{S}(X)$ obtained from $\mathcal{B}(X)$ by removing all of the dispensable edges.

Remark 2.50. *The definition of* $\mathcal{B}(X)$ *given above is the same as the one used in [26]. It differs from the one given in [10], where* $\mathcal{B}(X)$ *is required to have a loop at each vertex. However, this modification will not affect the results that follow nor their proofs.*

The definition of $\mathcal{S}(X)$ *matches the ones used in [10] and [26].*

We collect several facts about Cartesian skeletons of graphs that will be used in the proofs of some of the important results we will consider (mostly coming from [26] and [23]). Their proofs can be found in [10]. We give precise statements and references in the following lemma.

Lemma 2.51. *Let* X *and* Y *be connected graphs.*

1. *Every automorphism of* $X \times Y$ *is also an automorphism of* $\mathcal{S}(X \times Y)$ *[10, Proposition 8.11, p. 97].*

2. *If* X *and* Y *are twin-free and have more than one vertex, then* $\mathcal{S}(X \times Y) = \mathcal{S}(X) \square \mathcal{S}(Y)$ *[10, Proposition 8.10, p. 96].*

3. *If* $|V(X)|$ *is relatively prime to* $|V(Y)|$*, then* $\mathrm{Aut}(X \square Y) = \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ *[10, Corollary 6.12, p. 70].*

4. *If X is non-bipartite, then $\mathcal{S}(X)$ is connected [10, Proposition 8.13(i), p. 98].*

5. *If Y is bipartite, then $\mathcal{S}(Y)$ has precisely two connected components, and their vertex sets are the bipartition sets of Y [10, Proposition 8.13(ii), p. 98].*

Lemma 2.52 (Qin-Xia-Zhou [26, Lemma 4.2])**.** *Let X be a graph and α and β two permutations of $V(X)$ such that $\{\alpha(x), \beta(y)\} \in E(X)$ if and only if $\{x, y\} \in E(X)$. Then the following statements hold.*

1. $N_X(\alpha(x)) = \beta(N_X(x))$ *and* $N_X(\beta(x)) = \alpha(N_X(x))$ *for all vertices* $x \in V(X)$.

2. $\alpha, \beta \in \mathrm{Aut}(\mathcal{B}(X))$.

3. $\alpha, \beta \in \mathrm{Aut}(\mathcal{S}(X))$.

3 CANONICAL BIPARTITE DOUBLE COVERS

3.1 BASIC DEFINITIONS AND PROPERTIES

We are now in position to define the main object of our study, the canonical bipartite double cover of a graph.

Definition 3.1. Let X be a graph. Denote by K_2 the complete graph on two vertices. Then $BX \coloneqq X \times K_2$ is called *the canonical bipartite double cover of* X. Note that by the definition of the direct product of graphs, it holds that

- $V(BX) \coloneqq V(X) \times \{0\} \cup V(X) \times \{1\}$,
- $E(BX) \coloneqq \{\{(x, 0), (y, 1)\} \mid x, y \in V(X), \{x, y\} \in E(X)\}$.

We see from Definition 3.1 that, for vertices $(x, i), (y, j) \in V(BX)$, with $x, y \in V(X)$ and $i, j \in \{0, 1\}$, it holds that

$$\{(x, i), (y, j)\} \in E(BX) \text{ if and only if } \{x, y\} \in E(X), i \neq j.$$

From here, as we are only considering finite graphs, it follows that to show that some permutation $\alpha \in \mathrm{Sym}(V(BX))$ is an automorphism of BX, it suffices to check that

$$\{\alpha(x, 0), \alpha(y, 1)\} \in E(BX) \text{ for all } \{x, y\} \in E(X).$$

For simplicity, we will often call BX the "bipartite double cover", or just the "double cover" of a graph X. It will always be clear from context to which object we are referring to.

It is clear from its definition, that the double cover BX is a bipartite graph, with the bipartition consisting of sets $V(X) \times \{0\}$ and $V(X) \times \{1\}$.

Moreover, note that these are two copies of the vertex set $V(X)$ of X, and that every edge $\{x, y\} \in E(X)$ induces two edges in BX, in particular $\{(x, 0), (y, 1)\}$ and $\{(x, 1), (y, 0)\}$.

Finally, BX comes equipped with a natural *covering projection*

$$\pi \colon V(BX) \to V(X)$$
$$(x, i) \mapsto x$$

The following observations are a consequence of the properties of the direct product used to define the bipartite double cover.

Lemma 3.2 ([2, Theorem 3.4]). *Let X be a graph.*

1. *The bipartite double BX of X is connected if and only if X is connected and non-bipartite.*

2. *If X is bipartite with bipartition $V(X) = A \cup B$, then BX consists of two isomorphic copies X_1 and X_2 of X with vertex sets $V(X_1) = A \times \{0\} \cup B \times \{1\}$ and $V(X_2) = B \times \{0\} \cup A \times \{1\}$.*

Another corollary of the fact that BX is defined in terms of the direct product is that it inherits all symmetries of its two factors, X and K_2, as explained in the following result.

Definition 3.3. Let X be a graph and BX its bipartite double cover. Let φ be an automorphism of X. For $x \in V(X)$ and $i \in \{0,1\}$ define $\overline{\varphi}$, called *the lift of* φ, by

$$\begin{aligned}\overline{\varphi}\colon V(BX) &\to V(BX)\\ (x,i) &\mapsto (\varphi(x), i)\end{aligned}$$

We also define the following map.

$$\begin{aligned}\tau\colon V(BX) &\to V(BX)\\ (x,i) &\mapsto (x, i+1)\end{aligned}$$

In the definition of τ, we consider the second coordinate modulo 2 (as $V(BX) = V(X) \times \mathbb{Z}_2$). In particular, $\tau(x,0) = (x,1)$ and $\tau(x,1) = (x,0)$ for all $x \in V(X)$.

The reason $\overline{\varphi}$ is called the lift of φ is because it projects onto φ along the covering projection π, which means it satisfies the equation $\pi\overline{\varphi} = \varphi\pi$. Note that $\tau\overline{\varphi}$ satisfies the same equation and can also be considered a lift of φ. The important distinction is that $\overline{\varphi}$ does not reverse the colour classes of BX.

Note that τ is the lift of the identity automorphism of X that reverses colour classes of BX. The relation between τ and other automorphisms of BX will dictate a lot of its properties as we will establish by the end of this section.

We collect the important observations on BX we made so far in the following lemma for easy reference. We also provide proofs of all statements.

Lemma 3.4. *Let X be a graph and BX its canonical bipartite double cover. Let $\varphi \in \mathrm{Aut}(X)$ be an automorphism of X.*

1. *The lift $\overline{\varphi}\colon (x,i) \mapsto (\varphi(x), i)$ for $x \in V(X), i \in \{0,1\}$ is an automorphism of BX. Moreover, the map $\varphi \mapsto \overline{\varphi}$ is an injective group homomorphism from $\mathrm{Aut}(X)$ into $\mathrm{Aut}(BX)$.*

2. *The map* $\tau\colon (x,i) \mapsto (x,i+1)$ *is an order* 2 *automorphism of* BX*. Moreover,* τ *commutes with all lifts* $\overline{\varphi}$ *for* $\varphi \in \mathrm{Aut}(X)$.

3. *The group* $\mathrm{Aut}(BX)$ *contains a subgroup isomorphic to* $\mathrm{Aut}(X) \times S_2$.

Proof. (1) As φ is invertible and φ^{-1} is also an automorphism of X, it follows by an easy computation that

$$\overline{\varphi}^{-1} = \overline{\varphi^{-1}}.$$

Let $x, y \in V(X)$. Using the fact that φ is an automorphism of X, we obtain that

$$\begin{aligned}\{(x,0),(y,1)\} \in E(BX) &\iff \{x,y\} \in E(X) \iff \{\varphi(x),\varphi(y)\} \in E(X)\\ &\iff \{(\varphi(x),0),(\varphi(y),1)\} \in E(BX)\\ &\iff \{\overline{\varphi}(x,0),\overline{\varphi}(y,1)\} \in E(BX).\end{aligned}$$

It follows that $\overline{\varphi} \in \mathrm{Aut}(BX)$. Consequently, the map $\varphi \mapsto \overline{\varphi}$ is well-defined. To show it is a group homomorphism let $\varphi, \psi \in \mathrm{Aut}(X), x \in V(X)$ and $i \in \{0,1\}$.

$$\overline{\varphi\psi}(x,i) = ((\varphi\psi)(x),i) = (\varphi(\psi(x)),i) = \overline{\varphi}(\psi(x),i) = (\overline{\varphi}\overline{\psi})(x,i)$$

Finally, if $\overline{\varphi} = \mathrm{id}_{BX}$, then $\overline{\varphi}(x,i) = (\varphi(x),i) = (x,i)$ for all $x \in V(X), i \in \{0,1\}$. This proves that $\varphi(x) = x$ for all $x \in V(X)$ i.e., the homomorphism mapping $\varphi \mapsto \overline{\varphi}$ has a trivial kernel and is consequently injective.

(2) We first note that τ is a permutation of $V(BX)$, since it is its own inverse. As it is clearly non-trivial, it is of order 2.

Let $x, y \in V(X)$. Then by the definition of BX, $\{x,y\}$ being an edge of X is equivalent to $\{(x,0),(y,1)\}$ and $\{(x,1),(y,0)\} = \{\tau(x,0),\tau(y,1)\}$ both being edges of BX. This shows that τ is an automorphism of BX.

Finally, let $x \in V(X), i \in \{0,1\}$ and $\varphi \in \mathrm{Aut}(X)$.

$$(\tau\overline{\varphi})(x,i) = \tau(\varphi(x),i) = (\varphi(x),i+1) = \overline{\varphi}(x,i+1) = (\overline{\varphi}\tau)(x,i)$$

This shows that $\tau\overline{\varphi} = \overline{\varphi}\tau$, so τ commutes with $\overline{\varphi}$ for each $\varphi \in \mathrm{Aut}(X)$.

(3) Let $\overline{\mathrm{Aut}(X)}$ denote the image of $\mathrm{Aut}(X)$ under the homomorphism mapping $\varphi \mapsto \overline{\varphi}$ for $\varphi \in \mathrm{Aut}(X)$. Then by (1), $\overline{\mathrm{Aut}(X)}$ is a subgroup of $\mathrm{Aut}(BX)$ and it is isomorphic to $\mathrm{Aut}(X)$. Consider the subgroup $G \coloneqq \langle \overline{\mathrm{Aut}(X)}, \tau \rangle$ generated by τ and elements $\overline{\varphi}$ with $\varphi \in \mathrm{Aut}(X)$.

By (2), it follows that both $\overline{\mathrm{Aut}(X)}$ and $\langle \tau \rangle$ are normal subgroups of G. Moreover, their intersection is trivial as all elements of $\overline{\mathrm{Aut}(X)}$ preserve the colour classes of BX and τ reverses them. It follows that G is isomorphic to the direct product $\overline{\mathrm{Aut}(X)} \times \langle \tau \rangle$. Note that τ is of order 2 by (2).

Putting everything together, we obtain that

$$\mathrm{Aut}(BX) \geq G \cong \overline{\mathrm{Aut}(X)} \times \langle \tau \rangle \cong \mathrm{Aut}(X) \times S_2.$$

This completes the proof. □

Remark 3.5. *In the rest of the thesis, we will treat* $\mathrm{Aut}(X) \times S_2$ *as an actual subgroup of* $\mathrm{Aut}(BX)$*, while we will be define everything in terms of* $G = \langle \overline{\mathrm{Aut}(X)}, \tau \rangle \leq \mathrm{Aut}(BX)$ *defined in the proof of Lemma* 3.4*(*3*), since it satisfies* $G \cong \mathrm{Aut}(X) \times S_2$*.*

Lemma 3.4(3) implies that BX inherits the automorphisms of X via the subgroup $\mathrm{Aut}(X) \times S_2$ of $\mathrm{Aut}(BX)$. A direct corollary of this is that BX also inherits the symmetry properties of X, as explained by the following result.

Corollary 3.6. *Let X be a graph.*

1. *If X is vertex-transitive, then BX is vertex-transitive.*

2. *If X is edge-transitive, then BX is edge-transitive.*

3. *If X is arc-transitive, then BX is arc-transitive.*

As we see from Lemma 3.4, presence of the subgroup $\mathrm{Aut}(X) \times S_2$ in $\mathrm{Aut}(BX)$ does not depend on the structure of the graph X, but is a mere consequence of the definition of BX and properties of the direct product.

This motivates the following definition.

Definition 3.7 ([34])**.** Let X be a graph and BX its bipartite double cover. The automorphisms of BX lying in the subgroup $\mathrm{Aut}(X) \times S_2$ are called *expected automorphisms.* If an automorphism $\alpha \in \mathrm{Aut}(BX)$ is not an element of $\mathrm{Aut}(X) \times S_2$, it is called *unexpected.*

Unexpected automorphisms will be one of our main topics of interest. In particular, we will be interested in distinguishing graphs whose double covers have no unexpected automorphism from the ones that do. With this in mind, we introduce the following terminology.

Definition 3.8 ([21])**.** Let X be a graph. If BX has no unexpected automorphisms i.e., if $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X) \times S_2$, then X is called *stable.* Otherwise, X is *unstable.*

It turns out that there is a very simple way to reformulate the property of being unstable. The following lemma will be one of the main tools in our work with unstable graphs.

Lemma 3.9 ([17, Theorem 3.2], [21, Proposition 4.2])**.** *Let X be a graph. If there exist permutations α and β of $V(X)$, such that $\alpha \neq \beta$ and, for every edge $\{x, y\}$ of X, the vertex $\alpha(x)$ is adjacent to $\beta(y)$, then X is unstable.*

The converse holds if X is connected and non-bipartite.

Proof. ($\Rightarrow$) Define the following map

$$\varphi(x,i) = \begin{cases} \big(\alpha(x),0\big), & \text{if } i=0, \\ \big(\beta(x),1\big), & \text{if } i=1. \end{cases}$$

Then φ is a permutation of $V(BX)$, since α and β are permutations of $V(X)$. Moreover, the condition that α and β satisfy, implies that φ is an automorphism of BX.

Note that φ preserves the colour classes of BX. If φ was an expected automorphism, it would follow that it is actually a lift of some automorphism of $\operatorname{Aut}(X)$. However, this is a contradiction, as the permutations φ induces on the colour classes $V(X)\times\{0\}$ and $V(X)\times\{1\}$ of BX are α and β, respectively, and we have assumed that $\alpha\neq\beta$.

It follows that φ is an unexpected automorphism of BX and consequently, X is unstable.

($\Leftarrow$) Assume that X is connected and non-bipartite graph. If it is also unstable, it has an unexpected automorphism $\varphi\in\operatorname{Aut}BX$, such that $\varphi\notin\operatorname{Aut}X\times S_2$.

Since X is connected and non-bipartite, it follows by Lemma 2.13 that φ either preserves or reverse the colour classes of BX. Therefore, possibly after composing φ with τ, we may assume $\varphi\big(V(X)\times\{i\}\big)=V(X)\times\{i\}$ for $i\in\{0,1\}$. So we may define permutations α and β of $V(X)$ by

$$\varphi(x,0)=\big(\alpha(x),0\big) \quad\text{and}\quad \varphi(y,1)=\big(\beta(y),1\big).$$

Then the condition that φ is an automorphism of BX, implies that $\{\alpha(x),\beta(y)\}\in E(X)$ for all $\{x,y\}\in E(X)$. Finally, if $\alpha=\beta$ then α would be an automorphism of X, implying that $\varphi=\overline{\alpha}\in\operatorname{Aut}(X)\times S_2$, which is a contradiction with φ being an unexpected automorphism of BX. Hence, we conclude that $\alpha\neq\beta$, finishing the proof. □

We consider our first examples of infinite families of stable and unstable graphs. In both cases, the proof will be based on a counting argument. We will generalize the unstable example later in Proposition 3.16 and give a description of an unexpected automorphism.

Example 3.10. Let $n\geq 1$ be an integer. Let C_{2n+1} be the cycle of odd length $2n+1$. Then the automorphism group of C_{2n+1} is the dihedral group D_{4n+2} of order $4n+2$ i.e.,

$$|\operatorname{Aut}(C_{2n+1})| = 4n+2.$$

Odd cycles are connected, non-bipartite and 2-regular graphs. By applying Lemma 3.2(1), we see that BC_{2n+1} is connected and 2-regular. As it is of order $4n+2$, it follows that it is isomorphic to the cycle graph C_{4n+2}. The same conclusion can be derived by applying the definition of the bipartite double cover to C_{2n+1}.

It follows that $\mathrm{Aut}(BX)$ is just D_{8n+4} i.e.,

$$|\mathrm{Aut}(BC_{2n+1})| = 8n + 4.$$

Putting everything together, we get that

$$\frac{|\mathrm{Aut}(BC_{2n+1})|}{|\mathrm{Aut}(C_{2n+1}) \times S_2|} = \frac{8n+4}{2(4n+2)} = 1.$$

In particular, $\mathrm{Aut}(BC_{2n+1}) = \mathrm{Aut}(C_{2n+1}) \times S_2$ and we can conclude that odd cycles are stable.

Example 3.11. Let $n \geq 2$ be an integer. Let C_{2n} be the cycle of even length $2n$. Then we know that the automorphism group of C_{2n} is the dihedral group D_{4n} of order $4n$ i.e.,

$$|\mathrm{Aut}(C_{2n})| = 4n.$$

As C_{2n} is bipartite, Lemma 3.2(2) implies that BC_{2n} consists of two disjoint cycles of length $2n$. It follows that $\mathrm{Aut}(BX)$ is given by the wreath product of groups $\mathrm{Aut}(C_{2n}) \wr S_2$. This follows from the definition of the wreath product of groups [3, p. 46] or by rewriting BX as $\overline{K_2} \wr C_{2n}$ and applying a result of Sabidussi [28].

In particular, it follows that

$$|\mathrm{Aut}(BX)| = |\mathrm{Aut}(C_{2n}) \wr S_2| = |\mathrm{Aut}(C_{2n})|^2 \cdot 2! = (4n)^2 \cdot 2 = 32n^2.$$

Hence, the index $\mathrm{Aut}(C_{2n}) \times S_2$ in $\mathrm{Aut}(BC_{2n})$ is

$$\frac{|\mathrm{Aut}(BC_{2n})|}{|\mathrm{Aut}(C_{2n}) \times S_2|} = \frac{32n^2}{8n} = 4n > 1.$$

It follows that $\mathrm{Aut}(C_{2n}) \times S_2$ is a proper subgroup of $\mathrm{Aut}(BC_{2n})$, so BC_{2n} must have a non-zero number of unexpected automorphisms. In particular, even cycles are unstable.

As we have seen in Example 3.10 and Example 3.11, when checking for existence of unexpected automorphisms of BX, it is natural to consider the index of the subgroup $\mathrm{Aut}(X) \times S_2$ in $\mathrm{Aut}(BX)$. We now give this quantity a name.

Definition 3.12 ([34])**.** Let X be a graph. The *index of instability of* X is the index of the subgroup $\mathrm{Aut}(X) \times S_2$ in $\mathrm{Aut}(BX)$. For finite graphs X, we can define this as the quotient

$$\frac{|\mathrm{Aut}(BX)|}{|\mathrm{Aut}(X) \times S_2|} = \frac{|\mathrm{Aut}(BX)|}{2|\mathrm{Aut}(X)|}.$$

Note that a graph is stable if and only if its index of instability is equal to 1.

In Chapter 6, we will be studying canonical double covers of Cayley graphs. The following simple observations will greatly simplify some of the proofs.

Lemma 3.13. *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G. Let BX be the canonical double cover of X. Then BX is also a Cayley graph. Moreover, it holds that*

$$BX = \mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\}).$$

Proof. As $V(X) = G$, it follows that BX and $\mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\})$ have the same vertex set. Therefore, the claim follows by definitions of adjacency relations of BX and $\mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\})$.

For $x, y \in G$, we have the following equivalence.

$$\begin{aligned} \{(x,0),(y,1)\} \in E(BX) &\iff x^{-1}y \in S \\ &\iff (x,0)^{-1}(y,1) = (x^{-1}y,1) \in S \times \{1\} \\ &\iff \{(x,0),(y,1)\} \in E(\mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\})). \end{aligned}$$

□

Recall from Definition 3.3 that $\tau \colon V(BX) \to V(BX)$ is the automorphism of BX mapping $(x, i) \mapsto (x, i + 1)$ for $\in V(X)$ and $i \in \{0, 1\}$. When X is a Cayley graph, Lemma 3.13 allows us represent τ in another form.

Corollary 3.14. *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G. The automorphism τ equals to the translation automorphism t_L for $t = (1_G, 1)$.*

Proof. As X is a Cayley graph, Lemma 3.13 implies that $BX = \mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\})$. Then as $t \in G \times \mathbb{Z}_2$, we know that $t_L \in (G \times \mathbb{Z}_2)_L \leq \mathrm{Aut}(BX)$.

For $x \in G$ and $i \in \{0, 1\}$, we have that

$$t_L(x, i) = (1_G, 1)(x, i) = (1_G x, 1 + i) = (x, i + 1) = \tau(x, i).$$

This is exactly what we wanted to prove. □

3.2 TRIVIALLY UNSTABLE GRAPHS

In this subsection, we discuss the following three classes of graphs:

- disconnected graphs,
- bipartite graphs with non-trivial automorphism group,
- graphs that are not twin-free.

We will show that elements of each of these three classes are unstable. However, we will treat them as trivial examples of unstable graphs, and later, we will be interested in finding and constructing unstable graphs that are connected, non-bipartite and twin-free.

Proposition 3.15. *Assume that X is not connected. Then X is unstable.*

Proof. As X is disconnected, we can decompose it into two sugraphs X_1 and X_2 with smaller, disjoint vertex sets, where X_1 is a connected component of X. It follows that BX consists of subgraph BX_1 and BX_2. Note that the vertex sets of BX_1 and BX_2 are disjoint and that there are no edges connecting them.

Let τ_j denote the automorphism of BX_j swapping $V(X_j) \times \{0\}$ and $V(X_j) \times \{1\}$ for $j \in \{1, 2\}$. Clearly, τ_j is an automorphism of BX, as $\mathrm{Aut}(BX_j) \leq \mathrm{Aut}(BX)$ for $j \in \{0, 1\}$. Note that τ_j swaps colours on a proper subset $V(X_j) \times \{0, 1\}$ of $V(BX)$ and preserves them on the complement. However, from their definition, it is clear that all expected automorphisms of BX either preserve or reverse the colour classes of BX. From here, $\tau_j \notin \mathrm{Aut}(X) \times S_2$ is an unexpected automorphism of BX. Hence, X is unstable. ☐

We will give two proofs of the following statement. The first one will be a counting argument generalizing Example 3.11 and for the second one, we will construct an explicit unexpected automorphism in order to establish instability.

Proposition 3.16. *Let X be a bipartite graph such that* $\mathrm{Aut}(X) \neq 1$. *Then X is unstable.*

Proof. Let $V(X) = A \cup B$ be a bipartition of X. Note that we can assume that X is connected, otherwise Proposition 3.15 applies. As X is bipartite, Lemma 3.2(2) implies that BX consists of two isomorphic copies of X, denoted X_1 and X_2, with vertex sets $V(X_1) = A \times \{0\} \cup B \times \{1\}$ and $V(X_2) = A \times \{1\} \cup B \times \{0\}$.

We start with the counting argument. As X is connected, X_1 and X_2 are actually connected components of BX. As both are isomorphic to X, we get that $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X) \wr S_2$. In particular, we have that

$$|\mathrm{Aut}(BX)| = |\mathrm{Aut}(X) \wr S_2| = |\mathrm{Aut}(X)|^2 |S_2| = 2|\mathrm{Aut}(X)|^2.$$

By assumption, $\mathrm{Aut}(X)$ is non-trivial, so we know that $|\mathrm{Aut}(X)| > 1$. We can now estimate the index of instability of X.

$$\frac{|\mathrm{Aut}(BX)|}{2|\mathrm{Aut}(X)|} = \frac{2|\mathrm{Aut}(X)|^2}{2|\mathrm{Aut}(X)|} = |\mathrm{Aut}(X)| > 1.$$

Hence, the subgroup $\mathrm{Aut}(X) \times S_2$ of expected automorphisms of BX is a proper subgroup of $\mathrm{Aut}(BX)$, implying that BX has a non-zero number of unexpected automorphisms, proving that X is unstable.

We now present an alternative proof. As $\mathrm{Aut}(X) \neq 1$, we can find a non-trivial automorphism $\alpha \in \mathrm{Aut}(X)$. We can define a map α^* on BX by applying α to X_1 and fixing every vertex of X_2. Note that $E(BX)$ is a disjoint union of $E(X_1)$ and $E(X_2)$.

As both α and the identity permutation are automorphisms of X, it is clear that the map α^* is an automorphism of BX.

Note that the automorphism τ swaps the copies X_1 and X_2 in BX. Hence, $\tau\alpha^*\tau$ acts trivially on X_1 and non-trivially on X_2. In particular, it is not equal to α^*, so α^* and τ do not commute. However, by Lemma 3.4(2), all lifts $\overline{\varphi}$ with $\varphi \in \operatorname{Aut}(X)$ commute with τ, and consequently, so do all expected automorphisms in $\operatorname{Aut}(X) \times S_2$. It follows that α^* must be an unexpected automorphism, so X is unstable. □

Remark 3.17. *Note that even cycles are bipartite graphs with non-trivial automorphism group, so indeed, Example* 3.11 *is just a special case of Proposition* 3.16 *with* $X = C_{2n}, n \geq 2$. *Proposition* 3.16 *implies that all non-trivial path graphs as well as all complete bipartite graphs* $K_{n,n}$ *with* $n \geq 1$ *are unstable.*

Note that the counting argument given in the proof of Proposition 3.16 shows that, when X is connected and bipartite, its index of instability is equal to $|\operatorname{Aut}(X)|$. From here, we obtain the following corollary.

Corollary 3.18. *Let X be a connected bipartite graph. Then X is stable if and only if* $\operatorname{Aut}(X) = 1$.

The last condition we consider is existence of twins (recall Definition 2.41). We start with the following observation.

Lemma 3.19. *Let X be a graph. Let $x, y \in V(X)$ be twins in X. Then the map φ swapping x and y, but fixing all other vertices of X, is an automorphism of X.*

Proof. We observe that φ is its own inverse. Moreover, note that φ

- swaps the edges $\{x, z\}$ and $\{y, z\}$ for $z \in N_X(x) = N_X(y)$,
- fixes all other edges of X.

This shows that φ preserves the set of edges of X. In particular, φ is an automorphism of X. □

Proposition 3.20. *Let X be a graph. If X is not twin-free, then it is unstable.*

Proof. Assume that X is not twin-free. Let $x, y \in V(X)$ be twins. Then $x \neq y$ and $N_X(x) = N_X(y)$. By definition of BX, it holds that

$$N_{BX}(x, 1) = N_X(x) \times \{0\} = N_X(y) \times \{0\} = N_{BX}(y, 1).$$

Hence, $(x, 1)$ and $(y, 1)$ are twins in BX.

Define the following map

$$\alpha\colon V(BX) \to V(BX)$$
$$\alpha(z,i) = \begin{cases} (x,1), & (z,i) = (y,1) \\ (y,1), & (z,i) = (x,1) \\ (z,i), & \text{otherwise} \end{cases}$$

In other words, α swaps the twins $(x,1)$ and $(y,1)$ but fixes all other vertices of BX. By Lemma 3.19, α is an automorphism of BX. However, note that

$$(\alpha\tau)(x,0) = \alpha(x,1) = (y,1)$$
$$(\tau\alpha)(x,0) = \tau(x,0) = (x,1)$$

As $x \neq y$, we conclude that α does not commute with τ. It follows by Lemma 3.4(2) and the same arguments as in the proof of Proposition 3.16, that α is an unexpected automorphism of BX. In particular, X is unstable. □

We have seen in the proof of Proposition 3.20 that if X contains twins, then so does its double cover BX. The converse also turns out to be true. We record this as a corollary.

Corollary 3.21. *Let X be a graph. Then BX is twin-free if and only if X is twin-free.*

Proof. By definition, $BX = X \times K_2$. As K_2 is twin-free, the conclusion follows by Lemma 2.43. □

In conclusion, we have seen three very simple conditions that imply that a graph is unstable. However, as we will see later, there are much more interesting examples of unstable graphs, whose instability stems from deeper structural properties and symmetries that a graph might posses. Motivated by this search for more exotic examples, we introduce the following piece of terminology.

Definition 3.22 ([34, p. 360])**.** Let X be an unstable graph. Then X is *non-trivially unstable* if it is connected, non-bipartite and twin-free. Otherwise, we say that X is *trivially unstable*.

We close the subsection by giving an example of a non-trivially unstable graph, which we will revisit briefly in Chapter 4.

Example 3.23. Let $\mathcal{W}$ be the Bowtie graph shown on the left in Figure 1. From the picture, it is clear that $\mathcal{W}$ is connected, non-bipartite and twin-free.

It is easy to see that $\mathrm{Aut}(\mathcal{W})$ has two orbits on $V(\mathcal{W})$, namely $\{1\}$ and $\{2,3,4,5\}$. Moreover, the only non-trivial element of $\mathrm{Aut}(\mathcal{W})_4$ is the transposition $(2,3)$. Applying Lemma 2.6, we obtain that

$$|\operatorname{Aut}(\mathcal{W})| = |4^{\operatorname{Aut}(\mathcal{W})}||\operatorname{Aut}(\mathcal{W})_4| = 4 \cdot 2 = 8.$$

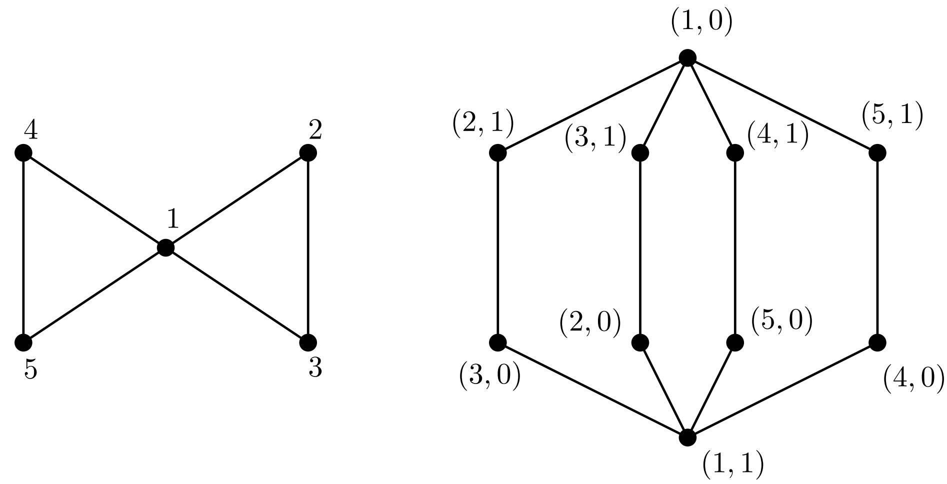

Figure 1: The Bowtie graph $\mathcal{W}$ and its canonical double cover $B\mathcal{W}$

Alternatively, we can note that $\overline{\mathcal{W}} = C_4 \cup K_1$, and from here, it is obvious that $\mathrm{Aut}(\mathcal{W}) \cong \mathrm{Aut}(C_4) \cong D_8$, the dihedral group of order 8.

The canonical double cover $B\mathcal{W}$ is shown on the right in Figure 1. Note that the orbit of $(1,0)$ with respect to $\mathrm{Aut}(B\mathcal{W})$ is $\{(1,0),(1,1)\}$, since $\tau(1,0) = (1,1)$ and these are the only vertices of valency 4 in $B\mathcal{W}$. Moreover, as each of the neighbours of $(1,0)$ has a unique neighbour at distance 2 from $(1,0)$, it is clear that any permutation of neighbours of $(1,0)$ can be extended uniquely to an automorphism of BX. In particular, $\mathrm{Aut}(B\mathcal{W})_{(1,0)}$ is of size $4! = 24$. Applying Lemma 2.6 one more time, we obtain that

$$|\mathrm{Aut}(B\mathcal{W})| = |(1,0)^{\mathrm{Aut}(B\mathcal{W})}||\mathrm{Aut}(B\mathcal{W})_{(1,0)}| = 2 \cdot 24 = 48.$$

Finally, we obtain that the index of instability of $\mathcal{W}$ is 3. It follows that $\mathcal{W}$ is non-trivially unstable. In fact, a simple case-by-case study shows that $\mathcal{W}$ is a non-trivially unstable graph of smallest possible order. The only graphs of order at most 2 are K_1, K_2 and $\overline{K_2}$, none of which are non-trivially unstable. The only non-bipartite graph on three vertices is C_3, which is stable by Example 3.10. Because a non-bipartite graph on four vertices must contain a 3-cycle, up to isomorphism, we have three candidates: a graph obtained by joining a vertex to a triangle, C_4 with a chord and K_4. First and third graph are stable, while the second one is not twin-free. In particular, none of them are non-trivially unstable. A bit more work will show that $\mathcal{W}$ is actually unique up to isomorphism (that is, any non-trivially unstable graph of order 5 is isomorphic to $\mathcal{W}$).

Corollary 3.24. *Up to isomorphism, the Bowtie graph $\mathcal{W}$ is the smallest non-trivially unstable graph.*

4 CONDITIONS IMPLYING STABILITY

We discuss several important conditions that imply that a graph X satisfying them is stable. We will revisit and precisely formulate some of the ideas that were already used when we were studying trivial instability of graphs in Section 3.2. Results of this section also cast some light on the differences between unexpected and expected automorphisms. The automorphism τ of BX, defined in Definition 3.3 and given by $\tau\colon (x,i) \mapsto (x,i+1)$ will play a crucial role. Recall that an element of a group G is *central* if it commutes with all elements of G.

Lemma 4.1. *Let X be a graph and BX its canonical double cover. If X is stable, then τ is central in* $\mathrm{Aut}(BX)$*.*

Proof. By Lemma 3.4(2), τ commutes with all lifts $\overline{\varphi}$ for $\varphi \in \mathrm{Aut}(X)$. As X is stable, the lifts $\overline{\varphi}$ and τ generate $\mathrm{Aut}(BX)$, since $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X) \times S_2$. Consequently, τ commutes with all elements of $\mathrm{Aut}(BX)$ i.e., τ is central in $\mathrm{Aut}(BX)$, as desired. □

Lemma 4.1 shows that if a graph X is stable, then all elements of $\mathrm{Aut}(BX)$ commute with τ. Hence, to show that X is unstable, it suffices to find at least one $\alpha \in \mathrm{Aut}(BX)$ which does not commute with τ (this is exactly what we did in the proofs of Proposition 3.16 and Proposition 3.20).

However, much more is true. When X is connected and non-bipartite, the subgroup $\mathrm{Aut}(X) \times S_2 \leq \mathrm{Aut}(BX)$ consists exactly of the automorphisms of BX that commute with τ. We now prove this.

Proposition 4.2. *Let X be a connected, non-bipartite graph. Let $\alpha \in \mathrm{Aut}(BX)$. Then α commutes with τ if and only if $\alpha \in \mathrm{Aut}(X) \times S_2$.*

Proof. ($\Leftarrow$) If $\alpha \in \mathrm{Aut}(X) \times S_2$, then α is generated by τ and lifts $\overline{\varphi}$ with $\varphi \in \mathrm{Aut}(X)$. As τ commutes with both by Lemma 3.4(2), it follows that α and τ commute.

($\Rightarrow$) Assume that $\alpha \in \mathrm{Aut}(BX)$ commutes with τ. As X is connected and non-bipartite, Lemma 2.13 implies that every automorphism of BX either preserves or reverses the colour classes of BX. Therefore, we have the following cases.

Case 1. *α preserves colour classes of BX*

Let $\varphi \in \mathrm{Sym}(V(X))$ be defined by $\alpha(x,0) = (\varphi(x),0)$ (the fact that φ is a permutation of $V(X)$ follows from the fact that α is a permutation $V(BX) = V(X) \times \{0,1\}$).

Since α commutes with τ, we get that

$$\alpha(x,1) = (\alpha\tau)(x,0) = (\tau\alpha)(x,0) = \tau(\varphi(x),0) = (\varphi(x),1)$$

We conclude that for $x \in V(X), i \in \{0,1\}$, it holds that $\alpha(x,i) = (\varphi(x),i)$.

Let $\{x,y\} \in E(X)$. Then $\{(x,0),(y,1)\} \in E(BX)$ and as α is an automorphism of BX, also $\{(\varphi(x),0),(\varphi(y),1)\} = \{\alpha(x,0),\alpha(x,1)\} \in E(BX)$. It follows that $\{\varphi(x),\varphi(y)\} \in E(X)$, proving that φ is an automorphism of X. Hence, $\alpha = \overline{\varphi} \in \mathrm{Aut}(X) \times S_2$.

Case 2. *α reverses colour classes of BX*

Then $\alpha\tau \in \mathrm{Aut}(BX)$, does not reverse colours and still commutes with τ. By the previous case, $\alpha\tau \in \mathrm{Aut}(X) \times S_2$ and as $\tau \in \mathrm{Aut}(X) \times S_2$, it follows that $\alpha \in \mathrm{Aut}(X) \times S_2$, finishing the proof. □

From Proposition 4.2, it follows that, when X is non-trivially unstable, every automorphism of BX not lying in $\mathrm{Aut}(X) \times S_2$ fails to commute with τ. In particular, for connected and non-bipartite graphs, we obtain the following converse to Lemma 4.1.

Corollary 4.3 (Fernandez-Hujdurović, [7, Lemma 2.2]). *Let X be a connected, non-bipartite graph, and let BX be its canonical double cover. Then X is stable if and only if τ is central in* $\mathrm{Aut}\, BX$.

Proof. ($\Rightarrow$) This direction follows by Lemma 4.1 and does not require any additional assumptions on X.

($\Leftarrow$) Assume that τ is central in $\mathrm{Aut}(BX)$. If X is unstable, BX has unexpected automorphisms. Because X is connected and non-bipartite, Proposition 4.2 implies that these unexpected automorphisms of BX do not commute with τ, contradicting the assumption that τ is central in $\mathrm{Aut}(BX)$. Hence, X must be stable. □

We remark that the advantage of Corollary 4.3 is that it only requires the graph to be connected and non-bipartite in order to be applied. By Proposition 3.15 and Proposition 3.16, being connected and non-bipartite are already prerequisites to being stable anyway. We will use this criteria many times in our proofs. However, its big disadvantage is that it is hard to check for concrete examples, as it requires substantial knowledge about the automorphism group of BX. At the point when we know enough about $\mathrm{Aut}(BX)$ to apply Corollary 4.3, it might be easier to determine whether X is stable or unstable directly, by studying the automorphisms of BX or by simply counting them.

We continue our study of criteria for stability of graphs. The following results make use of the point stabilizers of $\mathrm{Aut}(BX)$ in its natural action on $V(BX)$.

Lemma 4.4. *Let X be a graph. If X is stable, then* $\mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$ *for all $x \in V(X)$.*

Proof. By Lemma 4.1, since X is stable, the automorphism τ is central in $\mathrm{Aut}(BX)$. Let $x \in V(X)$ be arbitrary. Let $\alpha \in \mathrm{Aut}(BX)_{(x,0)}$. Then we have that

$$\alpha(x,1) = (\alpha\tau)(x,0) = (\tau\alpha)(x,0) = \tau(x,0) = (x,1).$$

This shows that $\mathrm{Aut}(BX)_{(x,0)} \subseteq \mathrm{Aut}(BX)_{(x,1)}$. Given $\beta \in \mathrm{Aut}(BX)_{(x,1)}$, we have an analogous computation

$$\beta(x,0) = (\beta\tau)(x,1) = (\tau\beta)(x,1) = \tau(x,1) = (x,0).$$

This shows that $\mathrm{Aut}(BX)_{(x,1)} \subseteq \mathrm{Aut}(BX)_{(x,0)}$, finishing the proof. □

Note that point stabilizers $\mathrm{Aut}(BX)_{(x,0)}$ and $\mathrm{Aut}(BX)_{(x,1)}$ are always conjugate subgroups of $\mathrm{Aut}(BX)$. Indeed, since by Lemma 3.4(2), τ is an order 2 automorphism of BX swapping $(x,0)$ and $(x,1)$, we know that

$$\mathrm{Aut}(BX)_{(x,0)} = \tau\,\mathrm{Aut}(BX)_{(x,1)}\tau \text{ and } \mathrm{Aut}(BX)_{(x,1)} = \tau\,\mathrm{Aut}(BX)_{(x,0)}\tau.$$

This can be used to give an alternative proof of Lemma 4.4, as the action of conjugation by central elements is trivial.

Moreover, if X is a finite graph, establishing only one containment already suffices to conclude that these stabilizers are equal. In proofs, we will often assume that an automorphism $\alpha \in \mathrm{Aut}(BX)$ fixes $(x,0)$ and then derive that it also fixes $(x,1)$.

It turns out that, when X is vertex-transitive, Lemma 4.4 has a converse.

Lemma 4.5 (Fernandez-Hujdurović, [7, Lemma 2.4])**.** *Let X be a connected, non-bipartite, vertex-transitive graph. Then X is stable if and only if* $\mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$ *for some (equivalently, every) vertex $x \in V(X)$.*

Proof. ($\Rightarrow$) This direction follows by Lemma 4.4. Note that it does not require any additional assumptions on X.

($\Leftarrow$) Assume that $\mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$ for some $x \in V(X)$. We claim that the same holds for all $y \in V(X)$. To show this, fix $y \in V(X)$ and, as X is vertex-transitive, let φ be an automorphism of X mapping x to y. Then $\overline{\varphi} \in \mathrm{Aut}(BX)$ and $\overline{\varphi}(x,i) = (\varphi(x),i) = (y,i)$ for $i \in \{0,1\}$. We conclude that

$$\mathrm{Aut}(BX)_{(y,0)} = \overline{\varphi}\,\mathrm{Aut}(BX)_{(x,0)}\overline{\varphi}^{-1} = \overline{\varphi}\,\mathrm{Aut}(BX)_{(x,1)}\overline{\varphi}^{-1} = \mathrm{Aut}(BX)_{(y,1)}.$$

All that is left to prove is that X is stable. By Corollary 4.3, it suffices to prove that τ is central in $\mathrm{Aut}(BX)$. Let $\alpha \in \mathrm{Aut}(BX)$ be arbitrary. As X is connected and non-bipartite, by Lemma 3.2(1), α either preserves or reverses colour classes of BX.

Case 1. *α preserves the colour classes of BX.*

Let $x \in V(X)$ be arbitrary. As α fixes the colour classes, there exists $y \in V(X)$ such that $\alpha(x,0) = (y,0)$. Because X is assumed to be vertex-transitive, we can find $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(y) = x$. Then $\overline{\varphi} \in \mathrm{Aut}(X) \times S_2 \leq \mathrm{Aut}(BX)$ and $\overline{\varphi}(y,i) = (\varphi(y),i) = (x,i)$ for $i \in \{0,1\}$. We calculate:

$$(\overline{\varphi}\alpha)(x,0) = \overline{\varphi}(y,0) = (x,0) \implies \overline{\varphi}\alpha \in \mathrm{Aut}(BX)_{(x,0)}.$$

By assumption, $\mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$, so $(\overline{\varphi}\alpha)(x,1) = (x,1)$. This we can rewrite as

$$\alpha(x,1) = \overline{\varphi}^{-1}(x,1) = \overline{\varphi^{-1}}(x,1) = (\varphi^{-1}(x),1) = (y,1).$$

Finally, we have that

- $(\alpha\tau)(x,0) = \alpha(x,1) = (y,1) = \tau(y,0) = (\tau\alpha)(x,0)$,
- $(\alpha\tau)(x,1) = \alpha(x,0) = (y,0) = \tau(y,1) = (\tau\alpha)(x,1)$.

As $x \in V(X)$ was arbitrary, we conclude that $\alpha\tau = \tau\alpha$. In particular, α commutes with τ.

Case 2. *α reverses colour classes of BX.*

In this case, $\tau\alpha$ preserves colours classes of BX. By the previous case, we know that $\tau\alpha$ commutes with τ. Using the fact that $\tau^2 = 1$ (see Lemma 3.4(2)), we obtain that

$$\tau\alpha = \tau(\tau(\tau\alpha)) = \tau((\tau\alpha)\tau) = \alpha\tau.$$

We conclude that τ commutes with all $\alpha \in \mathrm{Aut}(BX)$. In particular, τ is central in $\mathrm{Aut}(BX)$, as desired. □

Remark 4.6. *Let X be a finite vertex-transitive graph with at least two vertices (so* $\mathrm{Aut}(X)$ *is non-trivial). If we wish to show that X is stable, we need to check that it is connected, non-bipartite and twin-free (otherwise, at least one of the Proposition* 3.15*, Proposition* 3.16 *or Proposition* 3.20 *applies, implying that X is trivially unstable).*

If X satisfies these conditions, then to show it is stable, by Corollary 4.3*, it suffices to show that for some $x \in V(X)$,* $\mathrm{Aut}(BX)_{(x,0)}$ *and* $\mathrm{Aut}(BX)_{(x,1)}$ *are equal. As already discussed, these stabilizers are conjugate and consequently have the same number of elements, so it suffices to show that*

$$\exists x \in V(X) \text{ s.t. } \forall \alpha \in \mathrm{Aut}(BX), \alpha(x,0) = (x,0) \implies \alpha(x,1) = (x,1).$$

In the particular case when X is a Cayley graph $\mathrm{Cay}(G,S)$*, we will usually work with $x = 1_G$.*

Finally, we wish to remark that the assumption that X is vertex-transitive in Lemma 4.5 *is crucial. We recall the Bowtie graph $\mathcal{W}$ from Example* 3.23*. From Figure* 1*, we see that* $(1,0)$ *and* $(1,1)$ *are the only two vertices of valency* 4 *in $B\mathcal{W}$, so an automorphism of $B\mathcal{W}$ fixes* $(1,0)$ *if and only if it fixes* $(1,1)$*. In particular, the condition of Lemma*

4.5 *is satisfied for* $x = 1$. *However, in Example* 3.23, *we have already established that* $\mathcal{W}$ *is non-trivially unstable.*

In Section 8.1, *we will construct examples of non-trivially unstable graphs with trivial automorphism group (we will call such graph asymmetric, see Definition* 8.1*). In particular, in Example* 8.3, *we will discuss the Swift graph* SG, *which is a non-trivially unstable graph such that* $\mathrm{Aut}(SG)$ *is trivial and* $\mathrm{Aut}(B(SG))$ *is semi-regular. Therefore, all stabilizers* $\mathrm{Aut}(B(SG))_{(x,i)}$ *with* $x \in V(SG), i \in \{0,1\}$ *are trivial and consequently equal, so the condition of Lemma* 4.5 *is satisfied by every vertex* $x \in V(SG)$, *but* SG *still fails to be stable.*

We now show that all complete graphs K_n, except K_2, are stable. Recall that $\mathrm{Aut}(K_n) = S_n$ is transitive on $V(K_n)$ making K_n a vertex-transitive graph for all $n \geq 1$.

Example 4.7 (Qin-Xia-Zhou, [26, Example 2.2])**.** The complete graph K_n is unstable if and only if $n = 2$.

Proof. Note that for $x \in V(K_n)$, the vertex $(x, 0)$ of BK_n is adjacent to all vertices in $V(K_n) \times \{1\}$ except $(x, 1)$. It follows that BK_n is actually $K_{n,n} - nK_2$ i.e., the complete bipartite graph $K_{n,n}$ with a perfect matching removed.

We study the following cases:

- If $n = 1$, then $BK_1 = K_2$ and $\mathrm{Aut}(BK_1) = S_2 \cong \mathrm{Aut}(K_1) \times S_2$. It follows that K_1 is stable.

- If $n = 2$, the following calculation shows that K_2 is unstable.

  $$|\mathrm{Aut}(BK_2)| = |\mathrm{Aut}(2K_2)| = |\mathrm{Aut}(\overline{2K_2})| = |\mathrm{Aut}(C_4)| = |D_8| = \\ = 8 > 4 = |S_2 \times S_2| = |\mathrm{Aut}(K_2) \times S_2|.$$

  Alternatively, K_2 is bipartite and $\mathrm{Aut}(K_2) \neq 1$, so K_2 is trivially unstable by Proposition 3.16.

- Assume that $n \geq 3$. Then K_n is connected, non-bipartite and vertex-transitive. Note that for every $x \in V(K_n)$, it holds that:

  1. $d_{BK_n}((x, 0), (y, 1)) = 1$ for all $y \in V(K_n), y \neq x$,
  2. $d_{BK_n}((x, 0), (y, 0)) = 2$ for all $y \in V(K_n), y \neq x$,
  3. $d_{BK_n}((x, 0), (x, 1)) = 3$.

  In particular, $(x, 1)$ is the unique vertex at distance 3 from $(x, 0)$. Consequently, automorphisms of BX fix $(x, 0)$ if and only if they fix $(x, 1)$. Applying Lemma 4.5, we obtain that K_n with $n \geq 3$ is stable, proving the desired.

□

Remark 4.8. *Alternatively, one can note that by Lemma* 2.20*, for* $n \geq 3$ *it holds that*

$$K_n \times K_2 \cong K_2 \times K_n \cong K_2 \wr \overline{K_n} - nK_2 \cong K_2 \wr_d \overline{K_n} \cong K_{n,n} - nK_2$$

Then we have that

$$\mathrm{Aut}(BK_n) = \mathrm{Aut}(K_n \times K_2) \cong \mathrm{Aut}(K_{n,n} - nK_2) = S_n \times S_2 = \mathrm{Aut}(K_n) \times S_2.$$

It turns out that the same idea can be used to establish the stability of a much more interesting class of vertex-transitive graphs.

Definition 4.9 ([8, Chapter 6.10])**.** Let $n \geq 1$ be a positive integer. The *n-Andrásfai graph* is the circulant graph

$$\mathrm{And}(n) := \mathrm{Cay}(\mathbb{Z}_{3n-1}, S) \text{ with } S := \{1 \leq i \leq 3n-2 \mid i \equiv 1 \pmod 3\}.$$

Note that $\mathrm{And}(1) = \mathrm{Cay}(\mathbb{Z}_2, \{1\})$ is isomorphic to K_2. In particular, it is bipartite and trivially unstable. It turns out that this is the only unstable Andrásfai graph.

Proposition 4.10. *Let* $n \geq 2$ *be a positive integer. The n-Andrásfai graph* $\mathrm{And}(n)$ *is stable.*

Proof. Fix an $n \geq 2$ and write $X := \mathrm{And}(n) = \mathrm{Cay}(\mathbb{Z}_{3n-1}, S)$. As already noted, our strategy will be to apply Lemma 4.5. Note that 1 is always contained in S. It follows by Proposition 2.25(5) that X is connected. As $n \geq 2$, 4 is also in S. The vertices $0, 1, 2, 3$ and 4 then lie on a 5-cycle, so X is non-bipartite.

The elements of $\mathbb{Z}_{3n-1}$ represent classes of congruent integers modulo $3n-1$. However, because 3 does not divide the order of the group $3n-1$, different representatives of the same class in $\mathbb{Z}_{3n-1}$ may not be congruent modulo 3 (for instance, when $n = 3$, -2 and 6 are congruent modulo 8 and define the same element of $\mathbb{Z}_8$, but are not congruent modulo 3). This is why given a class $i \in \mathbb{Z}_{3n-1}$, we define $c(i)$ to be its *canonical representative* satisfying

$$0 \leq c(i) \leq 3n-2 \text{ and } c(i) \equiv i \pmod{3n-1}.$$

For $j \in \{0, 1, 2\}$ define

$$N_j := \{i \in \mathbb{Z}_n \setminus \{0\} \mid c(i) \equiv j \pmod 3\}.$$

Then we see that $S = N_1$ and moreover, $\{0\} \cup \bigcup_{j=0}^{2} N_j$ is a partition of the vertex set of X.

Let $x \in N_j$ be arbitrary. Then given $s \in S$, we see that

- $c(x+s) = x+s \equiv j+1 \pmod 3$, or
- $c(x+s) = x+s-(3n-1) = x+s+1-3n \equiv j+2 \pmod 3$.

This shows that neighbours of elements of N_0 are in N_1 and N_2, while neighbours of elements of N_2 are in N_0 and N_1. In particular, N_0 and N_2 are independent sets in X and they induce a bipartite subgraph of X with a bipartition $N_0 \cup N_2$. Applying Lemma 3.2(1), it follows that the double cover of this subgraph is an induced disconnected bipartite subgraph of BX with a bipartition given by $N_0 \times \{j\} \cup N_2 \times \{j\}$ for $j \in \{0,1\}$.

By the same argument as before, the non-zero neighbours of elements of N_1 lie in N_0 and N_2. It follows that all neighbours of elements $N_1 \times \{1\}$ distinct from $(0,0)$ lie in $N_0 \times \{0\} \cup N_2 \times \{0\}$, while all neighbours of elements of $N_1 \times \{0\}$ distinct from $(0,1)$ lie in $N_0 \times \{1\} \cup N_2 \times \{1\}$. Note that $N_1 \times \{1\}$ is exactly the set of neighbours of $(0,0)$, while $N_1 \times \{0\}$ is exactly the set of neighbours of $(0,1)$.

Previous observations let us write down the distance partition of BX with respect to $(0,0)$.

1. $(BX)_1(0,0) = N_1 \times \{1\}$,
2. $(BX)_2(0,0) = N_0 \times \{0\} \cup N_2 \times \{0\}$,
3. $(BX)_3(0,0) = N_0 \times \{1\} \cup N_2 \times \{1\}$,
4. $(BX)_4(0,0) = N_1 \times \{0\}$,
5. $(BX)_5(0,0) = \{(0,1)\}$.

It follows that $(0,1)$ is the unique vertex in BX at distance 5 from $(0,0)$. Hence, an automorphism of BX fixes $(0,0)$ if and only if it fixes $(0,1)$. The conclusion follows by Lemma 4.5. $\square$

The next two stability criteria are specialized to Cayley graphs of abelian groups. The first one illustrates that normality of the double cover, under appropriate assumptions, can force the graph to be stable. The other one shows that if if the connection set of a Cayley graph is sufficiently nice, stability can be deduced by looking at a special type of cycles in the double cover.

Lemma 4.11 (Fernandez-Hujdurović [7, Lemma 2.6]). *Let G be an abelian group of odd order, and let $X = \mathrm{Cay}(G,S)$ be a connected Cayley graph on G. If BX is a normal Cayley graph, then X is stable.*

Proof. Our strategy will be to apply Lemma 4.3. We note that X is assumed to be connected. Because it is a Cayley graph of odd order $|G|$, Corollary 2.27 implies that it is non-bipartite. We now need to establish that the automorphism $\tau(x,i) = (x, i+1)$ is central in $\mathrm{Aut}(BX)$.

As remarked in Corollary 3.14, τ corresponds to the left translation automorphism of BX induced by $t := (1_G, 1)$, that is, $\tau = t_L$. By Lemma 3.13, we know that $BX = \mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\})$. Because BX is normal, Lemma 2.31(2) implies that every automorphism of BX is a product of an element of $(G \times \mathbb{Z}_2)_L$ and some element of $\mathrm{Aut}(G \times \mathbb{Z}_2)$. So to show that $\tau = t_L$ is central, it suffices to show it commutes with all elements of $(G \times \mathbb{Z}_2)_L$ and $\mathrm{Aut}(G \times \mathbb{Z}_2)$.

The first part is obvious as $\tau = t_L \in (G \times \mathbb{Z}_2)_L$ and G is abelian. For the second part, we first note that the element t has order 2 in $G \times \mathbb{Z}_2$. Because $|G \times \mathbb{Z}_2| = 2|G|$ is even but not divisible by 4, it follows that this is the unique element of order 2 in the abelian group $G \times \mathbb{Z}_2$ (indeed, if s and s' were two distinct elements of order 2 in $G \times \mathbb{Z}_2$, then $G \times \mathbb{Z}_2$ would contain a subgroup of order 4, namely $\{1, s, s', ss'\}$, a contradiction). Because group automorphisms preserve the order of an element, this shows that every $\varphi \in \mathrm{Aut}(G \times \mathbb{Z}_2)$ fixes t i.e., $\varphi(t) = t$.

The following calculation shows that τ also commutes with all elements of $\mathrm{Aut}(G \times \mathbb{Z}_2)$. Let $\varphi \in \mathrm{Aut}(G \times \mathbb{Z}_2)$ and $x \in G \times \mathbb{Z}_2$. Then

$$(\varphi\tau)(x) = (\varphi t_L)(x) = \varphi(tx) = \varphi(t)\varphi(x) = t\varphi(x) = t_L(\varphi(x)) = (\tau\varphi)(x).$$

We can now conclude that τ commutes with all elements of $\mathrm{Aut}(BX)$, so τ is central in $\mathrm{Aut}(BX)$, proving the desired. □

Lemma 4.12 (Hujdurović-Mitrović-Morris [12, Lemma 3.5])**.** *Let $X = \mathrm{Cay}(G,S)$ be a connected Cayley graph of an abelian group G, and let $k \in \mathbb{Z}^+$, such that k is odd. Suppose there exists $c \in S$, such that*

1. *$|c| = k$,*
2. *$c^2 \neq st$, for all $s, t \in S \setminus \{c\}$, and*
3. *for all $a \in S$ of order $2k$, there exist $s, t \in S \setminus \{a\}$, such that $a^2 = st$.*

Then X is stable.

Proof. Let us say that a cycle in BX is *exceptional* if, for every pair x_i, x_{i+2} of vertices at distance 2 on the cycle, the unique path of length 2 from x_i to x_{i+2} is x_i, x_{i+1}, x_{i+2}. It is clear that every automorphism of BX must map each exceptional cycle of length k to an exceptional cycle of length k.

Let $c \in S$ be any element satisfying the conditions. Note that X is connected by assumption and that it is non-bipartite, since $c \in S$ is of odd order and generates an odd cycle $(1_G, s, s^2, \ldots, s^{k-1}, s^k = 1_G)$.

Let α be an automorphism of BX fixing $(1_G, 0)$. Note that Lemma 3.13 implies that $BX = \mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\})$. The cycle generated by $(c, 1) \in S \times \{1\}$ is an exceptional cycle of length $2k$ in BX. Furthermore, every exceptional cycle of length $2k$ is of this form. Since $(c, 1)^k = (1_G, 1)$, it follows that $(1_G, 1)$ lies at every exceptional cycle of length $2k$ through $(0, 0)$ and is the unique vertex at distance k from $(1_G, 0)$ in each of these cycle. This implies that α fixes $(1_G, 1)$. Hence, X is stable by Lemma 4.5. $\square$

Next, we discuss a stability criterion, first introduced by Surowski in [30], for graphs whose every edge lies on a triangle. This result has has been subsequently shown not to hold by Lauri, Mizzi and Scapellato in their article [17] on two-fold automorphisms of graphs.

We will first give the original statement of Surowski. Next, we will examine a small counterexample coming from an infinite family constructed by Lauri, Mizzi and Scapellato. Finally, we will point out the mistake in the original proof and fix it by introducing additional assumptions, obtaining a valid stability criterion.

Proposition 4.13 (Surowski (original false statement) [30, Proposition 2.1])**.** *Let X be a connected, vertex-transitive graph of diameter $d \geq 4$. If every edge is contained in a triangle in X, then X is stable.*

Example 4.14 (Lauri-Mizzi-Scapellato [17, Theorem 5.1, Figure 10,p.130])**.** Let $X = C_8 \wr C_8$. Then clearly, X is connected. Its diameter equals to the diameter of C_8, which is 4. Assume for contradiction that X is not twin-free and that $x, y \in V(X)$ are twins. Then x and y are distinct and non-adjacent.

- If x and y lie in the same copy of C_8, then they also need to have the same neighbours inside that copy. This is impossible, since C_8 is twin-free.
- Otherwise, x and y have to correspond to distinct, non-adjacent elements x' and y' of C_8. It follows that x' and y' are twins in C_8, which is again a contradiction with C_8 being twin-free.

Finally, all edges of X lie on a triangle. If an edge has both endpoints in one copy of C_8, we can pick any of the vertices of either of the two adjacent copies to produce a triangle containing the original edge. If an edge has its endpoints in distinct copies of C_8, we can look at one of the endpoints and pick one of its two neighbours in the same copy to produce a triangle containing the original edge. This also shows that X is non-bipartite.

This means that X satisfies all of the conditions of Proposition 4.13. However, by MAGMA calculations, X is unstable with index of instability 16777216.

The original proof Surowski presents uses the assumptions to describe a part of the distance partition of BX with respect to a fixed vertex $(x, 0)$. In particular, Surowski shows that the vertex $(x, 1)$ is at distance 3 from $(x, 0)$. The author then attempts to show that $(x, 1)$ is the unique vertex at distance 3 from $(x, 0)$ in BX with no neighbours lying at distance 4 from $(x, 0)$. This would imply that any automorphism of BX fixing $(x, 0)$ must also fix $(x, 1)$ and X is then stable by Lemma 4.5.

However, this claim is false in general. In fact, if we fix a vertex x of the graph $X = C_8 \wr C_8$ discussed in Example 4.14, then any of the other 5 vertices lying in the same copy of C_8 that are distinct from x and are not adjacent to it, induce vertices of BX at distance 3 from $(x, 0)$, all of whose neighbours are at distance at most 3 from $(x, 0)$. In particular, it is not possible to distinguish $(x, 1)$ from other elements of $(BX)_3(x, 0)$ in the manner proposed by Surowski.

We remedy this situation by adding additional assumptions on X.

Proposition 4.15. *Let X be a non-trivial connected graph. Assume that X satisfies that following conditions.*

1. *Every edge of X lies on a triangle.*
2. *For every $x \in V(X)$, it holds that*
   (a) *If $y \in V(X)$ is at distance 2 from x, then y has a neighbour $z \in V(X)$ at distance 3 from x, and*
   (b) *If $y \in V(X)$ is at distance 3 from x, then it has a neighbour $z \in V(X)$ at distance 4 from x.*

Then X is stable.

Proof. Let $x \in V(X)$ be arbitrary. Note that $X_1(x)$ is non-empty as X is connected and non-trivial.

If $X_2(x)$ is empty, then x is adjacent to all other vertices of X. In particular, if $y \in V(X)$ is distinct from x, every vertex of X is at distance at most 2 from y. However, if $X_2(y)$ is non-empty, we arrive at a contradiction with (2a). It follows that X is complete. As it contains triangles, it is non-bipartite and stable by Example 4.7.

We can now assume that $X_2(x)$ is non-empty for all $x \in V(X)$. Conditions (2a) and (2b) then imply that also $X_3(x)$ and $X_4(x)$ are non-empty for all $x \in V(X)$.

Let $x \in V(X)$ be arbitrary. We analyze the distance partition of BX with respect to the vertex $(x, 0)$. For simplicity, we will write X_i for $X_i(x)$ when we will be performing calculations.

1. $(BX)_1(x, 0) = X_1(x) \times \{1\}$

   This follows by the definition of BX.

2. $(BX)_2(x,0) = X_1(x) \times \{0\} \cup X_2(x) \times \{0\}$

   ($\subseteq$) Clearly, $(BX)_2(x,0) \subseteq V(X) \times \{0\}$. Let $(y,0) \in (BX)_2(x,0)$. Then we have a path $(x,0) \sim (z,1) \sim (y,0)$ in BX for some $z \in V(X)$. By the definition of BX, $x \sim z \sim y$ is a path in X. If $x \sim y$, it follows that $(y,0) \in X_1 \times \{0\}$. Otherwise, $x \not\sim y$ implies that $y \in X_2$ and $(y,0) \in X_2 \times \{0\}$.

   ($\supseteq$) Let $y \in X_1$. As $(BX)_1(x,0) \subseteq V(X) \times \{1\}$, it is clear that $(y,0) \notin (BX)_1(x,0)$. By assumption (1), the edge $\{x,y\}$ is on a triangle in X, so there exists $z \in V(X)$ such that $x \sim z, z \sim y$. It follows that $(x,0) \sim (z,1) \sim (y,0)$ is a path in BX. Consequently, $(y,0) \in (BX)_2(x,0)$

   If $y \in X_2$, we know a common neighbour $z \in V(X)$ of x and y exists, so the same argument applies.

3. $(x,1) \in (BX)_3(x,0)$,
   Clearly, $(x,1) \notin (BX)_1(x,0) \cup (BX)_2(x,0)$, so $d((x,0),(x,1)) \geq 3$. As X is connected and non-trivial, x must have a neighbour. Let $y \in X_1$. By assumption (1), we can find $z \in V(X)$ such that x, y and z form a triangle. Then $(x,0) \sim (y,1) \sim (z,0) \sim (x,1)$ is a path of length 3 in BX. Hence, $(x,1) \in (BX)_3(x,0)$, as desired.

4. $(BX)_3(x,0) \subseteq X_2(x) \times \{1\} \cup X_3(x) \times \{1\} \cup \{(x,1)\}$

   Clearly, $(BX)_3(x,0) \subseteq V(X) \times \{1\}$. If $(w,1) \in (BX)_3(x,0)$, we know that there exist $y, z \in V(X)$ such that $(x,0) \sim (y,1) \sim (z,0) \sim (w,1)$ is a path in BX. Consequently, $x \sim y \sim z \sim w$ in X. If $w = x$, there is nothing to prove, so we may assume $w \neq x$. It follows that $1 \leq d(x,w) \leq 3$. If $d(x,w) = 1$, then $(w,1) \in (BX)_1(x,0)$, a contradiction. Hence, either $d(x,w) = 2$ and $(w,1) \in X_2 \times \{1\}$ or $d(x,w) = 3$ and $(w,1) \in X_3 \times \{1\}$.

Note that $(x,1) \in (BX)_3(x,0)$ has no neighbours in $BX_4(x,0)$. Indeed, since $BX_4(x,0) \subseteq V(X) \times \{0\}$, if $(x,1) \sim (y,0)$, it follows that $x \sim y$ and $y \in X_1$. But then $(y,0) \in X_1 \times \{0\} \subseteq (BX)_2(x,0)$ as previously established, a contradiction.

We now show that $(x,1)$ is the unique element of $(BX)_3(x,0)$ with this property, that is, we will show that every element of $(y,1) \in (BX)_3(x,0) \setminus \{(x,1)\}$ does have a neighbour in $(BX)_4(x,0)$.

If $y \in X_2$, then by assumption (2a), it has a neighbour $z \in V(X)$ such that $z \in X_3$. Then $(z,0) \in V(BX)$ is a neighbour of $(y,1)$ and, by the formulas we derived, it is not contained in $(BX)_i(x,0)$ for $i \in \{1,2,3\}$. As $(y,1) \in (BX)_3(x,0)$, it follows that $(z,0) \in (BX)_4(x,0)$.

If $y \in X_3$, then by assumption (2b), it has a neighbour $z \in V(X)$ such that $z \in X_4$. By the same arguments as before, it follows that $(z,0) \in (BX)_4(x,0)$ is a neighbour of $(y,1)$.

Let $\alpha \in \mathrm{Aut}(BX)$ and $x \in V(X)$. As X is connected and non-bipartite (as it contains triangles), Lemma 2.13 implies that, possibly after composing α with τ, we can assume that $\alpha(V(X) \times \{i\}) = V(X) \times \{i\}$ for $i \in \{0, 1\}$. Let $y \in V(X)$ such that $\alpha(x, 0) = (y, 0)$.

As α is an automorphism, it follows that $\alpha((BX)_i(x, 0)) \subseteq (BX)_i(y, 0)$ for all $i \geq 0$. Then $(x, 1) \in (BX)_3(x, 0)$ will be mapped onto the vertex $\alpha(x, 1) \in (BX)_3(y, 0)$, which will not have any neighbours in $(BX)_4(y, 0)$, as $(x, 1)$ has no neighbours in $(BX)_4(x, 0)$. But the only element with this property in $(BX)_3(y, 0)$ is $(y, 1)$. It follows that $\alpha(x, 1) = (y, 1)$.

This shows that we can find a permutation $\varphi \in \mathrm{Sym}(V(X))$ such that $\alpha(x, i) = (\varphi(x), i)$ for all $x \in V(X), i \in \{0, 1\}$. The fact that α is an automorphism of BX, implies that φ is an automorphism of X. In particular, $\alpha = \overline{\varphi} \in \mathrm{Aut}(X) \times S_2$. It follows that X is stable, as desired. □

Remark 4.16. *As already hinted at in the proof, the updated assumptions rule out the possibility of X being trivially unstable. By assumption X is connected and by* (1)*, X contains triangles and is consequently non-bipartite. If x and y are twins, then y lies in $X_2(x)$. But then the assumption* (2a) *implies that y has a neighbour z at distance* 3 *from x. Clearly, z is a neighbour of y, but not a neighbour of x, contradicting the assumption that x and y are twins. It follows that X must be twin-free to begin with.*

Our updated criterion has the following corollary.

Lemma 4.17. *Let X be a connected distance-transitive graph of diameter $d \geq 4$ that contains a triangle. Then X is stable.*

Proof. Note that X is edge-transitive. In particular, since X contains a triangle, each of its edges lies on a triangle. Moreover, since X has diameter at least 4, we can find two vertices in X and a path of length at least 4 between them. As X is also vertex-transitive and $\mathrm{Aut}(X)_x$ is transitive on the distance sets $X_i(x)$ for all $i \in \{1, \ldots, d\}$, by translating the obtained path, we can show that X satisfies conditions (2a) and (2b). Proposition 4.15 implies that X is stable. □

In the same article, Surowski discusses another stability criterion [30, Proposition 2.2], this time for strongly regular graphs. Before discussing it, we prove the following generalization, of which the original result is just a corollary.

Proposition 4.18. *Let X be a connected, twin-free graph such that every edge of X lies on a triangle. If for every edge $\{x, y\} \in E(X)$ and non-edge $\{z, w\} \notin E(X)$ with $x, y, z, w \in V(X)$, it holds that*

$$|N_X(x) \cap N_X(y)| \neq |N_X(z) \cap N_X(w)|, \tag{4.1}$$

then X is stable.

Proof. We look at the distance partition of BX with respect to $(x,0)$, where $x \in V(X)$ is arbitrary. Note that the analysis of it in the proof of Proposition 4.15 only depended on the assumption that every edge of the graph lies on a triangle. It follows that

1. $(BX)_1(x,0) = X_1(x) \times \{1\}$,

2. $(BX)_2(x,0) = X_1(x) \times \{0\} \cup X_2(x) \times \{0\}$.

Let $\alpha \in \mathrm{Aut}(BX)$ and $x \in V(X)$. As X is connected and non-bipartite (as it contains triangles), it follows by Lemma 2.13 that automorphisms of BX either preserve or reverse the colour classes of BX. Hence, after possibly composing α with τ, we may assume that $\alpha(V(X) \times \{i\}) = V(X) \times \{i\}$. Let $z \in V(X)$ such that $\alpha(x,0) = (z,0)$.

Then as α is an automorphism, $\alpha((BX)_i(x,0)) \subseteq (BX)_i(z,0)$ for all $i \geq 0$. Let $y \in X_1(x)$. Then $(y,0) \in (BX)_2(x,0)$. Denote $\alpha(y,0) = (w,0)$ for $w \in V(X)$. If $\alpha(y,0) \in X_2(z) \times \{0\} \subseteq (BX)_2(z,0)$, then it holds that

$$\begin{aligned}
|N_X(x) \cap N_X(y)| &= |N_{BX}(x,0) \cap N_{BX}(y,0)| = \\
&= |\alpha(N_{BX}(x,0) \cap N_{BX}(y,0))| = \\
&= |\alpha(N_{BX}(x,0)) \cap \alpha(N_{BX}(y,0))| = \\
&= |N_{BX}(\alpha(x,0)) \cap N_{BX}(\alpha(y,0))| = \\
&= |N_{BX}(z,0) \cap N_{BX}(w,0)| = \\
&= |N_X(z) \cap N_X(w)|.
\end{aligned}$$

However, as $(w,0) \in X_2(z) \times \{0\}$, it follows that z and w are not adjacent, so this contradicts Eq. (4.1).

We conclude that $\alpha(X_1(x) \times \{0\}) \subseteq X_1(z) \times \{0\}$. As α is an automorphism, it follows that $(x,0)$ and $(z,0)$ are of the same valency, that is $|X_1(x) \times \{0\}| = |X_1(x)| = |X_1(z)| = |X_1(z) \times \{0\}|$. Hence, we have in fact proven that

$$\alpha(X_1(x) \times \{0\}) = X_1(z) \times \{0\}.$$

Let $y \in V(X)$ be such that $\alpha(x,1) = (y,1)$. Then

$$\begin{aligned}
N_{BX}(y,1) = N_{BX}(\alpha(x,1)) = \alpha(N_{BX}(x,1)) = \alpha(X_1(x) \times \{0\}) = \\
= X_1(z) \times \{0\} = N_{BX}(z,1).
\end{aligned}$$

Hence, $(y,1)$ and $(z,1)$ are twins in BX. However, as X is twin-free, it follows by Corollary 3.21 that BX is twin-free as well. We conclude that $(y,1) = (z,1)$.

This shows that $\alpha(x,i) = (\varphi(x),i), x \in V(X), i \in \{0,1\}$ for some permutation φ of $V(X)$. It is not hard to see that since α is an automorphism of BX, φ is an automorphism of X. In particular, $\alpha = \overline{\varphi} \in \mathrm{Aut}(X) \times S_2$. It follows that X is stable. □

We can now obtain the original criterion as a corollary of the one we just established.

Proposition 4.19 (Surowski [30, Proposition 2.2])**.** *Let X be a strongly regular graph with parameters (v, k, λ, μ). If $k > \mu \neq \lambda \geq 1$, then X is stable.*

Proof. First we note that X must be connected, since vertices in distinct connected components would have no neighbours in common, which contradicts the fact that $\mu > 0$. Next, X is twin-free, as twins are non-adjacent vertices and would have k neighbours in common, which contradicts that assumption that $k > \mu$. Every edge of X lies on a triangle since $\lambda \geq 1$.

Finally, let $x, y, z, w \in V(X)$ be such that $\{x, y\} \in E(X)$ and $\{z, w\} \notin E(X)$. Then

$$|N_X(x) \cap N_X(y)| = \lambda \neq \mu = |N_X(z) \cap N_X(w)|.$$

It follows that X satisfies all conditions of Proposition 4.18. Hence, X is stable, as desired. □

Remark 4.20. *In [30], Surowski constructs an infinite family of strongly regular graphs with $\lambda = \mu$.*

Surowski's original proof distinguishes $(x, 1)$ as the unique element of $(BX)_3(x, 0)$ which has no neighbours in $X_2(x) \times \{0\} \subseteq (BX)_2(x, 0)$. As the diameter of a strongly regular graph is 2, it follows that $(BX)_3(x, 0) \setminus \{(x, 1)\} = X_2(x) \times \{1\}$ and each of its elements has $k - \mu > 0$ neighbours in $X_2(x) \times \{0\}$. What Surowski actually proves is that, $\mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$ for all $x \in V(X)$. However, as X is not necessarily vertex-transitive, the converse of Lemma 4.4 (given in Lemma 4.5) does not necessarily apply (as illustrated by the Bowtie graph in Example 3.23 and the Swift graph in Example 8.3). Hence, while the statement of his result is true, the proof is somewhat incomplete.

A strongly regular graph is said to be a *conference graph*, if its parameters are $\left(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4}\right)$. Note that Proposition 2.40(4) implies that Paley graphs are conference graphs. The following is an immediate corollary of Corollary 4.19.

Corollary 4.21. *Every conference graph is stable. In particular, every Paley graph is stable.*

We now consider the family of *Kneser graphs* $K(n, k)$ and prove, that these graphs are stable under appropriate assumptions on their parameters n and k.

Definition 4.22. Let $n \geq k \geq 1$ be positive integers. The *Kneser graph* is the graph $K(n, k)$ with k-element subsets of the set $[n] := \{1, \ldots, n\}$ as vertices, which are adjacent if and only if they are disjoint as sets.

Corollary 4.23. *Let $n, k \geq 1$ be positive integers such that $n \geq 3k$. Then the Kneser graph $K(n,k)$ is stable.*

Proof. Let $x \in V(K(n,k))$. We claim that

$$[n] \setminus x = \bigcup_{y \in N_{K(n,k)}(x)} y$$

Indeed, if x and y are adjacent, then $x \cap y = \emptyset$, so $y \subset [n] \setminus x$. Conversely, if $a \in [n] \setminus x$, as $|[n] \setminus x| = n - k \geq 2k > k$, we can find a k-subset y disjoint from x and containing a. Then $a \in y \in N_{K(n,k)}(x)$.

Hence, if $N_{K(n,k)}(x) = N_{K(n,k)}(y)$, then the claim implies that $[n] \setminus x = [n] \setminus y$. In particular, $x = y$ and $K(n,k)$ is twin-free.

Let $x, y \in V(K(n,k))$ be distinct vertices and let $r := |x \cap y|$. By Inclusion-exclusion principle we have that

$$|[n] \setminus (x \cup y)| = n - |x \cup y| = n - (|x| + |y| - |x \cap y|) = n - 2k + r. \tag{4.2}$$

Note that common neighbours of x and y are exactly the k-element subsets of $[n] \setminus (x \cup y)$.

- If x and y are adjacent in $K(n,k)$, then they are disjoint and $r = 0$. In particular, by Eq. (4.2), x and y have $\binom{n-2k}{k}$ common neighbours. As $n \geq 3k$, this number is positive and it shows that every edge of $K(n,k)$ lies on a triangle.

- If x and y are not adjacent, then $1 \leq r \leq k-1$. Hence, by Eq. (4.2), x and y have $\binom{n-2k+r}{k}$ common neighbours. Note that this number is also positive. This also shows that $K(n,k)$ is connected.

Moreover, when x and y are not adjacent, as $r \geq 1$, we get that $\binom{n-2k+r}{k} > \binom{n-2k}{k}$, proving that $K(n,k)$ satisfies the condition from Eq. (4.1).

In particular, Proposition 4.18 implies that $K(n,k)$ is stable. $\square$

Note that we have also proven that if $n \geq 3k \geq 1$, it holds that

- $K(n,k)$ is of diameter 2 (so Proposition 4.15 does not apply), and

- When $k \geq 3$, $K(n,k)$ is not strongly regular, as the number of common neighbours of two non-adjacent vertices depends on the size of their intersection as subset of $[n]$ and is consequently not constant (so Proposition 4.19 does not apply).

It is worth noting that the Kneser graph $K(n,2)$ is strongly regular with parameters $\left(\binom{n}{2}, \binom{n-2}{2}, \binom{n-4}{2}, \binom{n-3}{2}\right)$. When $n \geq 6$, Proposition 4.19 applies. Note that $K(2,2) \cong K_1$, $K(3,2) \cong \overline{K_3}$, $K(4,2) \cong 3K_2$ and that $K(5,2)$ is the Petersen graph. Putting these observations together, we obtain the following.

Corollary 4.24. *The Kneser graph $K(n,2)$ is unstable if and only if $n \in \{3,4\}$. Moreover, $K(3,2) \cong \overline{K_3}$ and $K(4,2) \cong 3K_2$ are trivially unstable.*

We are now in position to derive a complete classification of unstable Johnson graphs.

Definition 4.25 ([8, p. 9])**.** Let $n \geq k \geq 1$ be positive integers. The *Johnson graph* is the graph $J(n,k)$ with k-element subsets of $[n] := \{1, \dots, n\}$ as vertices, which are adjacent if and only if the size of their intersection as sets is $k-1$.

The following result lists several well known properties of Johnson graphs, which we will need for their classification.

Lemma 4.26 ([8, Chapter 1.6])**.** *Let $n \geq k \geq 1$ be positive integers. The Johnson graph $J(n,k)$ has the following properties.*

1. *$J(n,k) \cong J(n,n-k)$, where the isomorphism is induced by the complementation map assigning to each k-element subset of $[n]$ its $(n-k)$-element complement.*
2. *$J(n,k)$ is connected.*
3. *$J(n,k)$ is $k(n-k)$-regular.*
4. *$J(n,k)$ is distance-transitive.*
5. *The diameter of $J(n,k)$ is given by $\min(k, n-k)$.*

Lemma 4.27. *Let n and k be positive integers such that $n \geq k+4$ and $k \geq 4$. Then the Johnson graph $J(n,k)$ is stable.*

Proof. It follows from Lemma 4.26 that $J(n,k)$ is a connected, distance-transitive graph with diameter $d = \min(k, n-k)$. Note that $n-k \geq (k+4)-k = 4$ and $k \geq 4$, so $d \geq 4$. Moreover, the following vertices form a triangle

$$\{1, \dots, k-1, k\}, \{1, \dots, k-1, k+1\} \text{ and } \{1, \dots, k-1, k+2\}.$$

Lemma 4.17 implies that $J(n,k)$ is stable. □

We now deal with the remaining non-trivial cases.

Lemma 4.28. *Let $n \geq 2$ be a positive integer. Then $J(n,2)$ is unstable if and only if $n \in \{4,6\}$. Moreover, $J(4,2)$ is not twin-free and is therefore trivially unstable. The graph $J(6,2)$ is non-trivially unstable with index of instabiliy* 28.

Proof. ($\Rightarrow$) Note that $J(2,2) \cong K_1$ and $J(3,2) \cong K_3$ are stable by Example 4.7. Hence, we may assume that $n \geq 4$. We observe that $J(n,2)$ is strongly regular with parameters

$$(v,k,\lambda,\mu) = \left(\frac{n(n-1)}{2}, 2(n-2), n-2, 4\right).$$

To prove this, we could note that $J(n,2)$ is the *triangular graph* $T(n)$ i.e., the line graph of the complete graph K_n, which is known to be strongly regular with above parameters. Alternatively, note that two distinct 2-subsets of $[n]$ are non-adjacent in $J(n,2)$ if and only if they have an empty intersection. This shows us that the complement $\overline{J(n,2)}$ is just the Kneser graph $K(n,2)$. Using the fact that $K(n,2)$ is strongly regular with parameters $\left(\binom{n}{2}, \binom{n-2}{2}, \binom{n-4}{2}, \binom{n-3}{2}\right)$, we again obtain that $J(n,2)$ is strongly regular with above parameters.

Note that, as $n \geq 4$, we have that $k > \lambda > 0$. Next, $k > \mu > 0$, unless $n = 4$, when we get that $k = \mu$. Finally, note that $\lambda \neq \mu$, unless $n = 6$. It follows that for $n \notin \{4,6\}$, $J(n,2)$ is stable by Proposition 4.19.

($\Leftarrow$) As $J(4,2)$ is strongly regular with parameters $(6,4,2,4)$, we conclude that it is not twin-free. Consequently, it is trivially unstable (note $J(4,2)$ is the octahedral graph). Finally, MAGMA calculations show that $J(6,2)$ is non-trivially unstable with index of instability equal to 28. The desired conclusion follows. $\square$

Lemma 4.29. *Let $n \geq 3$. The Johnson graph $J(n,3)$ is unstable if and only if $n = 6$. Moreover, $J(6,3)$ is non-trivially unstable with index of instability 2.*

Proof. ($\Leftarrow$) Calculations in MAGMA show that $J(6,3)$ is non-trivially unstable with index of instability equal to 2.

($\Rightarrow$) Assume that $J(n,3)$ is unstable. Our strategy will be to apply Proposition 4.18 and then handle the remaining cases separately.

Case 1. *$n \geq 7$.*

Note that the vertices $\{1,2,3\}, \{1,2,4\}$ and $\{1,2,5\}$ form a triangle in $J(n,3)$. As $J(n,3)$ is distance-transitive by Lemma 4.26(4), it follows that its every edge lies on a triangle.

Let $A, B \in V(J(n,3))$ be distinct vertices of $J(n,3)$. We calculate the number of common neighbours of A and B.

- If A and B are adjacent, then $A = \{x,y,a\}$ and $B = \{x,y,b\}$ with pair-wise distinct elements $x,y,a,b \in [n]$. Note that common neighbours of A and B are $\{x,a,b\}, \{y,a,b\}$ and vertices of the form $\{x,y,z\}$ with $z \in [n] \setminus \{x,y,a,b\}$. It follows that there is a total of $2 + (n-4) = n-2$ of them.

- If A and B are not adjacent, they are either disjoint as subsets of $[n]$, or they intersect in precisely one element. It is not hard to see that if they are disjoint, they have no common neighbours in $J(n,3)$. If they intersect in exactly one element, then $A = \{a, a', x\}$ and $B = \{b, b', x\}$ with pair-wise distinct elements $a, a', b, b', x \in [n]$. Note that in this case, the common neighbours of A and B are precisely the vertices $\{a, b, x\}, \{a', b, x\}, \{a, b', x\}$ and $\{a', b', x\}$. In total, there is 4 of them.

In conclusion, vertices A and B share $n-2$ neighbours if adjacent and 0 or 4 neighbours if they are not adjacent. As $n \geq 7$, $J(n,3)$ satisfies the condition Eq. (4.1) of Proposition 4.18.

By Lemma 4.26(3), the valency of a vertex of $J(n,3)$ is at least $3 \cdot (7-3) = 12$. In particular, it is twin-free (as non-adjacent vertices share at most 4 neighbours). Finally, by Lemma 4.26(2) $J(n,3)$ is connected.

We conclude by Proposition 4.18 that $J(n,3)$ is stable.

Case 2. $n \in \{3,4,5,6\}$.

Note that $J(3,3) \cong K_1$ and $J(4,3) \cong K_4$ are stable by Example 4.7. Lemma 4.26(1) implies that $J(5,3) \cong J(5,5-3) = J(5,2)$, which is stable by Lemma 4.28.

Hence, the only possibility is that $n = 6$, and indeed, we have already established that $J(6,3)$ is unstable. □

Putting all of the results so far together, we obtain the following result.

Theorem 4.30. *Let $n \geq k \geq 1$ be positive integers. The Johnson graph $J(n,k)$ is unstable if and only if it is one of the following*

1. *the complete graph $J(2,1) = K_2$,*
2. *the octahedral graph $J(4,2)$,*
3. *$J(6,2) \cong J(6,4)$ or*
4. *$J(6,3)$.*

Moreover, $J(2,1)$ is bipartite, while $J(4,2)$ is not twin-free, so both are trivially unstable. Graphs $J(6,2) \cong J(6,4)$ and $J(6,3)$ are non-trivially unstable with indices of instability 28 and 2, respectively.

Proof. ($\Leftarrow$) The graph $J(2,1) \cong K_2$ is bipartite, so trivially unstable. Statements about $J(4,2)$ and $J(6,2)$ have been proven in Lemma 4.28 (note that Lemma 4.26(1) shows that $J(6,2) \cong J(6,6-2) = J(6,4)$). Statements about $J(6,3)$ have been proven in Lemma 4.29.

($\Rightarrow$) Assume that $J(n,k)$ is unstable. We consider the following cases depending on the value of k.

Case 1. $k \in \{1,2,3\}$.

If $k = 1$, then $J(n, 1) \cong K_n$ for all $n \geq 1$. In particular, by Example 4.7, it follows that $n = 2$ and $J(2, 1) \cong K_2$. Note that this graph is bipartite and therefore trivially unstable. Hence, (1) holds.

If $k = 2$, then Lemma 4.28 shows that $n \in \{4, 6\}$, so (2) or (3) applies.

If $k = 3$, then Lemma 4.29 shows $n = 6$, so (4) holds.

Case 2. $k \geq 4$.

From Lemma 4.27 we conclude that $n \in \{k, k+1, k+2, k+3\}$. We consider each case separately and apply Lemma 4.26(1).

- $J(k, k) \cong K_1$ is stable.
- $J(k+1, k) \cong J(k, 1) \cong K_k$. As $k \geq 4$, this graph is stable by Example 4.7.
- $J(k+2, k) \cong J(k+2, 2)$. By Lemma 4.28, this graph is unstable if and only if $k+2 \in \{4, 6\}$, that is, $k \in \{2, 4\}$. As $k \geq 4$, we obtain the graph $J(6, 4) \cong J(6, 2)$, so (3) applies.
- $J(k+3, k) \cong J(k+3, 3)$. As $k \geq 4$, it follows by Lemma 4.29 that this graph is stable.

This finishes the proof. □

We will discuss [12, Lemma 3.6] (in this thesis, it appears as Corollary 7.2) in Chapter 7, as it plays a major role in the classification of non-trivially unstable circulants of low valency that we will discuss. The following result is a generalization of it. We first introduce some additional terminology.

Definition 4.31. Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G. For $g \in G$, we say that an edge $\{x, y\}$ of the complete graph on $G \times \mathbb{Z}_2$ is a *g-edge* if $y \in \{x(g, 1), x(g^{-1}, 1)\}$. Note that $\{x, y\}$ is an edge of BX if and only if it is an s-edge for some $s \in S$.

Lemma 4.32. *Let $X = \mathrm{Cay}(G, S)$ be a connected, non-bipartite Cayley graph of a group G. Let S_0 be a subset of $G \setminus \{1_G\}$ such that $S_0^{-1} = S_0$. If every automorphism of BX maps S_0-edges to S_0-edges, and some (equivalently, every) connected component of* $\mathrm{Cay}(G, S_0)$ *is a stable graph, then X is a stable graph.*

Proof. Let $X_0' \coloneqq \mathrm{Cay}(\langle S_0 \rangle, S_0)$ denote the connected component of $X_0 \coloneqq \mathrm{Cay}(G, S_0)$ containing the identity 1_G. As S_0 is non-empty, X_0' is a non-trivial, vertex-transitive, connected graph. In particular, $\mathrm{Aut}(X_0') \neq 1$ and as X_0' is assumed to be stable, it follows that it must be non-bipartite, as otherwise Lemma 3.16 would apply. Hence, by Lemma 3.2(1), BX_0' is the connected component of BX_0 containing both $(1_G, 0)$ and $(1_G, 1)$.

Suppose that $\alpha \in \mathrm{Aut}(BX)$ fixes the vertex $(1_G, 0)$ of BX. Assumptions imply that α is also an automorphism of BX_0. Moreover, as it fixes $(1_G, 0)$ it reduces to an automorphism of the connected component BX_0'. As X_0' is assumed to be stable, it follows by Lemma 4.4 that α also fixes $(1_G, 1)$. We conclude that X is stable by Lemma 4.5. $\square$

5 UNEXPECTED AUTOMORPHISMS OF UNSTABLE GRAPHS

In his article [34], Wilson studies unstable graphs and the unexpected automorphisms of their canonical double covers. Most of his article is dedicated to formulating different conditions that imply that a graph is unstable. The four instability criteria he derives are among the most general results we have for establishing instability of an arbitrary graph. We will discuss them in detail in Section 5.1.

It turns out that these results are also sufficient to characterize unstable graphs. This is the main result of Wilson's article and we will consider it in Section 5.2 (see Theorem 5.35).

Finally, in the appendix of his article, Wilson considers four families of graphs and applies the four general theorems in an attempt to characterize the unstable members of each family. We will study the conditions he derives for the family of circulant graphs, called Wilson types, in Section 5.3. In Section 5.4 we study their generalizations introduced in [13].

5.1 WILSON'S INSTABILITY CRITERIA

We discuss the four general results of Wilson (see Theorem 5.5, Theorem 5.14, Theorem 5.20 and Theorem 5.28), each of which implies that a graph satisfying certain conditions is unstable.

5.1.1 Sub-components

Let X be a graph and let $\gamma \in \mathrm{Aut}(X)$ be its non-trivial automorphism. Consider the orbits of $\langle\gamma\rangle$ on $V(X)$. We define the following colouring of the edges of the graph X.

- An edge $\{x, y\} \in E(X)$ is *blue with respect to* γ if

  1. x and y lie in distinct orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\langle\gamma\rangle$, and
  2. every two vertices $z \in \mathcal{O}_1$ and $w \in \mathcal{O}_2$ are adjacent in X.

- All other edges of X are *red with respect to* γ.

When the automorphism γ is clear from context, we will often refer to the edges of X as just "blue" or "red" without making a reference to γ.

Observation 5.1. *We make the following observations on the above construction.*

1. *The group $\langle\gamma\rangle$ fixes each of its own orbits set-wise.*
2. *The map γ is an automorphism of X, so all elements of $\langle\gamma\rangle$ preserve both edges and non-edges of X. Furthermore, as $\langle\gamma\rangle$ preserves its own orbits, it also preserves the sets of edges between every two orbits.*
3. *By construction, only the edges connecting vertices in distinct orbits can be blue. It follows that every edge connecting two vertices in the same orbit of $\langle\gamma\rangle$ is automatically red.*
4. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two distinct orbits of $\langle\gamma\rangle$. Let $x \in \mathcal{O}_1, y \in \mathcal{O}_2$ be two adjacent vertices of X.*
   - *If $\{x, y\}$ is blue, from the definition of the colouring, it follows that every edge $\{z, w\} \in E(X)$ with $z \in \mathcal{O}_1, w \in \mathcal{O}_2$ is also blue.*
   - *If $\{x, y\}$ is red, then there exist $z \in \mathcal{O}_1$ and $w \in \mathcal{O}_2$ such that $\{z, w\} \notin E(X)$. In particular, all edges with one endpoint in $\mathcal{O}_1$ and the other in $\mathcal{O}_2$ are red.*
5. *Combining the two previous items, it is clear that γ preserves the set of blue edges. Consequently, it preserves the set of red edges as well.*
6. *Denote by X_b and X_r the subgraphs of X consisting of blue and red edges, respectively. We will call X_b the blue subgraph of X and X_r the red subgraph of X. Note that $V(X_b) = V(X_r) = V(X)$. Previous item implies that γ is an automorphism of both X_b and X_r.*

We will state the following observation as a lemma for easy reference.

Lemma 5.2. *Let X be a graph and $\gamma \in \mathrm{Aut}(X)$ its non-trivial automorphism. Assume that $x \in V(X)$ is fixed by γ, that is, $\gamma(x) = x$. Then every edge of the form $\{x, y\}$ for $y \in V(X)$ is blue with respect to γ.*

Proof. Note that since x is fixed by γ, the orbit of x with respect to $\langle\gamma\rangle$ is just the singleton set $\{x\}$. If $N_X(x) = \emptyset$, that is, if x is not incident with any edge of X, there is nothing to prove.

Assume that $N_X(x) \neq \emptyset$ and let $y \in N_X(x)$ be arbitrary. Then since $\{x, y\} \in E(X)$, by induction, it follows that $\{x, \gamma^k(y)\} = \{\gamma^k(x), \gamma^k(y)\} \in E(X)$ for all $k \in \mathbb{Z}$. In particular, x is adjacent to every element of the orbit of y under the action of $\langle\gamma\rangle$ and all edges of the form $\{x, \gamma^k(y)\}$, including $\{x, y\}$, are blue. $\square$

We now define the key concept for Wilson's first instability criteria.

Definition 5.3. Let X be a graph and γ its non-trivial automorphism. Colour the edges of X as previously described. A non-trivial connected component H of the red subgraph X_r is called a γ *sub-component of* X if it is bipartite, with bipartition $V_1 \cup V_2$, and

1. $\gamma(H) \neq H$, or
2. $\gamma(H) = H$ and γ preserves the sets V_1 and V_2.

We will show that existence of a γ sub-component implies instability. For this, we need X to have at least some red edges. As our overall goal is to explain non-trivial instability, it turns out we can always assume red edges exist as the following lemma explains.

Lemma 5.4. *Let X be a graph and $\gamma \in \mathrm{Aut}(X)$ a non-trivial automorphism of X. If all edges of X are blue with respect to γ, then X is trivially unstable.*

Proof. Let X and γ be as above and assume all edges of X are blue with respect to γ. Note that we can assume that X is connected, as otherwise it is trivially unstable.

Recall that by Observation 5.1(3), any edge between two vertices lying in the same orbit of $\langle\gamma\rangle$ is automatically red. As we have assumed that all edges are blue, it follows that each orbit of $\langle\gamma\rangle$ is an independent set in X. This shows that $\langle\gamma\rangle$ must have at least two orbits on $V(X)$, as otherwise X would be a non-trivial empty graph, which is in contradiction with X being connected.

As γ is non-trivial, it must also have a non-trivial orbit, that is, an orbit $\mathcal{O}$ of size at least 2. Let $x, y \in \mathcal{O}$ be distinct vertices. Note that x and y are not adjacent.

Because X is connected, we know that $N_X(x)$ is non-empty. Let $z \in N_X(x)$ be arbitrary. Because $\mathcal{O}$ is an independent set, z is an element of an orbit of $\langle\gamma\rangle$ distinct from $\mathcal{O}$. As the edge $\{x, z\}$ is blue, it follows that z is adjacent with every element of $\mathcal{O}$, in particular $z \in N_X(y)$. By symmetry, it follows that $N_X(x) = N_X(y)$. We conclude that X is not twin-free. Consequently, X is trivially unstable. □

Theorem 5.5 (Wilson [34, Theorem 1])**.** *Let X be a graph. If there exists a non-trivial automorphism $\gamma \in \mathrm{Aut}(X)$ such that X contains a γ sub-component, then X is unstable.*

Proof. Let X be a graph, $\gamma \in \mathrm{Aut}(X)$ its non-trivial automorphism and H a γ sub-component of X. Then by Definition 5.3, H is a connected component of the red subgraph X_r of X. Moreover, by Observation 5.1(6) γ is an automorphism of X_r. It follows that γ either fixes H, or H and $\gamma(H)$ are two distinct, vertex-disjoint connected components of X_r.

We study the following cases.

Case 1. $\gamma(H) \neq H$

Let $Y := H \cup \gamma(H)$. Then Y is a subgraph of X_r and H and $\gamma(H)$ are exactly its connected components. Note that BY is a subgraph of BX. Moreover, $BY = BH \cup B\gamma(H)$ and as H is bipartite and $\gamma(H) \cong H$ (with bipartition $\gamma(V_1) \cup \gamma(V_2)$), BY consists of 4 connected components each of which is isomorphic to H (see Lemma 3.2(2)).

Recall that $V(H) = V_1 \cup V_2$. We enumerate the connected components of BY by H_1, H_2, H_3 and H_4 so that

1. $V(H_1) = V_1 \times \{0\} \cup V_2 \times \{1\}$,
2. $V(H_2) = V_1 \times \{1\} \cup V_2 \times \{0\}$,
3. $V(H_3) = \gamma(V_1) \times \{0\} \cup \gamma(V_2) \times \{1\}$,
4. $V(H_4) = \gamma(V_1) \times \{1\} \cup \gamma(V_2) \times \{0\}$.

We define the following map.

$$\gamma^* \colon V(BX) \to V(BX)$$
$$\gamma^*(x,i) = \begin{cases} (\gamma(x), i), & (x,i) \in V(H_1) \\ (\gamma^{-1}(x), i), & (x,i) \in V(H_3) \\ (x,i), & \text{otherwise} \end{cases}$$

Note that γ^* swaps the vertex sets of H_1 and H_3, but leaves all of the other vertices fixed. From here, it follows that γ^* is a permutation of $V(BX)$. We check that it is also an automorphism of BX. Let $\{(x,0),(y,1)\} \in E(BX)$. Then $\{x,y\} \in E(X)$.

Subcase 1.1. *The edge $\{x,y\}$ is blue.*

Then by definition of the colouring, all elements of the orbit of x under $\langle\gamma\rangle$ are adjacent to all elements of the orbit of y under $\langle\gamma\rangle$. In particular, for all $z \in \{x, \gamma^{\pm 1}(x)\}$ and $w \in \{y, \gamma^{\pm 1}(y)\}$, $\{z,w\}$ is an edge of X. Therefore, whether $(x,0)$ and $(y,1)$ lie in any of the components $H_i, i \in \{1,2,3,4\}$ is irrelevant and it holds that $\{\gamma^*(x,0), \gamma^*(y,1)\} \in E(BX)$.

Subcase 1.2. *The edge $\{x,y\}$ is red.*

In this case, x and y are adjacent in X_r and consequently, they lie in the same connected component of X_r. As H and $\gamma(H)$ are distinct connected components of X_r, it follows that x and y are either both in $V(H)$, both in $V(\gamma(H))$ or $x, y \notin V(H) \cup V(\gamma(H))$.

Subsubcase 1.2.1. $x, y \in V(H) = V_1 \cup V_2$

As H is bipartite, x and y lie in opposite colour classes of H.

If $x \in V_1, y \in V_2$, then $(x,0),(y,1) \in V(H_1)$. In particular, since γ is an automorphism of X, we get that $\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(\gamma(x),0),(\gamma(y),1)\} \in E(BX)$.

If $x \in V_2, y \in V_1$, then $(x,0),(y,1) \in V(H_2)$, so γ^* fixes both of them. In particular, $\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(x,0),(y,1)\} \in E(BX)$.

Subsubcase 1.2.2. $x, y \in V(\gamma(H)) = \gamma(V_1) \cup \gamma(V_2)$

As $V(\gamma(H)) = \gamma(V_1) \cup \gamma(V_2)$ is a bipartition of $\gamma(H)$ and x, y are adjacent in X_r, they again lie in opposite colour classes.

If $x \in \gamma(V_1), y \in \gamma(V_2)$, then $(x,0),(y,1) \in V(H_3)$. In particular, since γ^{-1} is also an automorphism of X, we get that $\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(\gamma^{-1}(x),0),\, (\gamma^{-1}(y),1)\} \in E(BX)$.

If $x \in \gamma(V_2), y \in \gamma(V_1)$, then $(x,0),(y,1) \in V(H_4)$ and both are fixed by γ^*. It then follows that $\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(x,0),(y,1)\} \in E(BX)$.

Subsubcase 1.2.3. $x, y \notin V(H) \cup V(\gamma(H))$

By definition, γ^* fixes both $(x,0)$ and $(y,1)$. We obtain that $\{\gamma^*(x,0),\, \gamma^*(y,1)\} = \{(x,0),(y,1)\} \in E(BX)$.

This shows that γ^* is an automorphism of BX. Finally, assume for contradiction that X is stable. If $x \in V_2$, then $(x,0) \in V(H_2)$ and $(x,1) \in V(H_1)$. Then $\gamma^*(x,0) = (x,0)$ and $\gamma^* \in \mathrm{Aut}(BX)_{(x,0)}$. As X is stable, Lemma 4.4 implies that $\gamma^* \in \mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$. Hence, it follows that $\gamma^*(x,1) = (x,1)$. However, as $(x,1) \in V(H_1)$, by definition of γ^*, we have that $\gamma^*(x,1) = (\gamma(x),0)$. We conclude that $\gamma(x) = x$. Since H is a connected component of X_r and $\gamma \in \mathrm{Aut}(X_r)$, the fact that γ fixes a vertex of H implies that it must fix H set-wise i.e., $\gamma(H) = H$, a contradiction.

Case 2. $\gamma(H) = H$ *and* $\gamma(V_i) = V_i$ *for* $i \in \{1,2\}$

Then BH is a subgraph of BX consisting of two connected components, each of which is isomorphic to H (again, see Lemma 3.2(2) for details). Denote them by H_1 and H_2 so that

1. $V(H_1) = V_1 \times \{0\} \cup V_2 \times \{1\}$,

2. $V(H_2) = V_1 \times \{1\} \cup V_2 \times \{0\}$.

Define the following map.

$$\gamma^* \colon V(BX) \to V(BX)$$

$$\gamma^*(x,i) = \begin{cases} (\gamma(x), i), & (x,i) \in V(H_1) \\ (x,i), & \text{otherwise} \end{cases}$$

Note that γ^* fixes all vertices of BX, except possibly the vertices of H_1, where it preserves the bipartition into $V_1 \times \{0\} \cup V_2 \times \{1\}$ and induces the permutation γ on each colour class. From here, it is clear that γ^* is a permutation of $V(BX)$.

We now show it is also an automorphism of BX. Let $\{(x,0),(y,1)\} \in E(BX)$. Then $\{x,y\} \in E(X)$.

Subcase 2.1. *The edge $\{x,y\}$ is blue.*

Similarly as in the proof of Theorem 5.5, we conclude that all $z \in \{x,\gamma(x)\}$ and $w \in \{y,\gamma(y)\}$ are adjacent in X, proving that $\{\gamma^*(x,0),\gamma^*(y,1)\} \in E(BX)$.

Subcase 2.2. *The edge $\{x,y\}$ is red.*

As before, it follows that x and y are adjacent in X_r and as H is a connected component of X_r, either both x and y are in H or neither is.

Subsubcase 2.2.1. $x,y \in V(H) = V_1 \cup V_2$

As H is bipartite and x and y are adjacent, they have to lie in opposite colour classes of H.

If $x \in V_1, y \in V_2$, then $(x,0),(y,1) \in V(H_1)$. Because γ is an automorphism of X, we get that $\{\gamma^*(x,0),\gamma^*(y,1)\} = \{(\gamma(x),0),(\gamma(y),1)\} \in E(BX)$.

If $x \in V_2, y \in V_1$, then $(x,0),(y,1) \in V(H_2)$ and both are fixed by γ^*, so we have that $\{\gamma^*(x,0),\gamma^*(y,1)\} = \{(x,0),(y,1)\} \in E(BX)$.

Subsubcase 2.2.2. $x,y \notin V(H) = V_1 \cup V_2$

In this case, $(x,0),(y,1) \notin V(H_1)$, so by definition, they are both fixed by γ^*, again resulting in the same conclusion that $\{\gamma^*(x,0),\gamma^*(y,1)\} = \{(x,0),(y,1)\} \in E(BX)$.

This shows that γ^* is an automorphism of BX.

Assume for contradiction that X is stable. Let $x \in V_2$. Then $(x,0) \in V(H_2)$ and by definition $\gamma^*(x,0) = (x,0)$. As X is stable, Lemma 4.4 implies that $\gamma^* \in \mathrm{Aut}(BX)_{(x,0)} = \mathrm{Aut}(BX)_{(x,1)}$ and $\gamma^*(x,1) = (x,1)$. However, $(x,1) \in V(H_1)$, so by its definition $\gamma^*(x,1) = (\gamma(x),1)$. It follows that $\gamma(x) = x$. Lemma 5.2 implies that all edges incident with x are blue. In particular, x is an isolated vertex in X_r and its own connected component i.e., $H = \{x\}$. This is contradiction as, by definition, H is a non-trivial connected component of X_r. □

Theorem 5.5 allows us to define the following construction.

Proposition 5.6. *Let X be a graph. Let A and B be subsets of $V(X)$. Let ΓX denote the graph with*

- $V(\Gamma X) \coloneqq V(X) \cup \{a_1,a_2,b_1,b_2\}$, *where a_1,a_2,b_1 and b_2 are four distinct vertices with property that $a_1,a_2,b_1,b_2 \notin V(X)$, and*
- $E(\Gamma X) \coloneqq E(X) \cup \{\{a_1,b_1\},\{a_2,b_2\}\} \cup \{\{a_1,a\},\{a_2,a\} \mid a \in A\} \cup \{\{b_1,b\},\{b_2,b\} \mid b \in B\}$.

In particular, ΓX is obtained from X by adding two new edges $\{a_1,b_1\}$ and $\{a_2,b_2\}$ to X, and then joining a_1 and a_2 with every vertex in A and b_1 and b_2 with every vertex in B.

The obtained graph ΓX is unstable.

Proof. Define $\gamma := (a_1, a_2)(b_1, b_2)$. Then γ is a permutation of $V(\Gamma X)$. Moreover, note that

- γ fixes all edges $\{x, y\} \in E(\Gamma X)$ with $x, y \in V(X)$,
- if $A \neq \emptyset$, γ swaps edges $\{a_1, a\}$ and $\{a_2, a\}$ for all $a \in A$,
- if $B \neq \emptyset$, γ swaps edges $\{b_1, b\}$ and $\{b_2, b\}$ for all $b \in B$,
- γ swaps the edges $\{a_1, b_1\}$ and $\{a_2, b_2\}$.

It follows that γ is an automorphism of ΓX. Moreover, the orbits of γ are precisely $\{a_1, a_2\}, \{b_1, b_2\}, \{x\}$ with $x \in V(X)$. It follows that all edges of ΓX are blue with respect to γ except $\{a_1, b_1\}$ and $\{a_2, b_2\}$ which are red (for example, we can apply Lemma 5.4 and note that $\{a_1, b_2\}, \{a_2, b_1\} \notin E(X)$). As γ swaps these edges, it follows that they are both γ sub-components of ΓX. In particular, ΓX is unstable by Theorem 5.5. □

Remark 5.7. *Note that if a graph X is connected and twin-free and at least one of the sets A, B contains a pair of adjacent vertices, the graph ΓX defined in Proposition* 5.6 *is connected, non-bipartite and twin-free. In particular, it is non-trivially unstable.*

Note that the Bowtie graph $\mathcal{W}$ defined in Example 3.23 *is ΓK_1 with both A, B non-empty.*

The following condition we derive, given in Theorem 5.10, is a special instance of the idea used to formulate Theorem 5.5.

Definition 5.8. Let X be a graph. An automorphism $\gamma \in \mathrm{Aut}(BX)$ is called a *half-action* if, for all $x \in V(X)$, γ fixes exactly one of the vertices $(x, 0)$ and $(x, 1)$.

Remark 5.9. *Note that if there exist an automorphism $\gamma \in \mathrm{Aut}(BX)$ and a vertex $x \in V(X)$ such that γ fixes exactly one of the vertices $(x, 0)$ and $(x, 1)$, then X is unstable by Lemma* 4.4.

In particular, if BX has a half-action, then X is unstable.

Theorem 5.10 (Wilson [34, p. 361])**.** *Let X be a graph and $\gamma \in \mathrm{Aut}(X)$ its non-trivial automorphism. Assume that there exists a partition $\{A, B\}$ of $V(X)$ such that*

1. *γ preserves this partition, that is, $\gamma(A) = A$ and $\gamma(B) = B$,*
2. *for every edge $\{x, y\} \in E(X)$ with $x, y \in A$ or $x, y \in B$, it holds that $\{x, \gamma(y)\} \in E(X)$ (equivalently, $\{\gamma(x), y\} \in E(X)$).*

Then X is unstable.

Proof. Define the following map.

$$\gamma^*\colon V(BX) \to V(BX)$$
$$\gamma^*(x,i) = \begin{cases} (x,0), & x \in A, i = 0 \\ (\gamma(x),1), & x \in A, i = 1 \\ (\gamma(x),0), & x \in B, i = 0 \\ (x,1), & x \in B, i = 1 \end{cases}$$

Notice that in the definition of γ^*, we have partitioned $V(BX)$ into $A \times \{0,1\} \cup B \times \{0,1\}$. Since $\gamma(A) = A, \gamma(B) = B$, γ^* preserves each of these parts and induces either the identity permutation or γ on them. As both of these are permutations of A and B, it follows that γ^* is a permutation of $V(BX)$. Let $\{(x,0),(y,1)\} \in E(BX)$. Then $\{x,y\} \in E(X)$.

Case 1. $x, y \in A$

The condition (2) implies that $\{x, \gamma(y)\} \in E(X)$. It follows that
$\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(x,0),(\gamma(y),1)\} \in E(BX)$.

Case 2. $x \in A, y \in B$

In this case, both vertices are fixed by γ^* and $\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(x,0),(y,1)\} \in E(BX)$.

Case 3. $x \in B, y \in A$

As γ is an automorphism of X, we know that $\{\gamma(x), \gamma(y)\} \in E(X)$. We obtain that $\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(\gamma(x),0),(\gamma(y),1)\} \in E(BX)$.

Case 4. $x, y \in B$

The condition (2) implies that $\{\gamma(x), y\} \in E(X)$. It follows that
$\{\gamma^*(x,0), \gamma^*(y,1)\} = \{(\gamma(x),0),(y,1)\} \in E(BX)$.

This shows that γ^* is an automorphism of BX. As γ is non-trivial, we can find $x \in V(X)$ such that $\gamma(x) \neq x$.

If $x \in A$, then $\gamma^*(x,0) = (x,0)$, so $\gamma^* \in \mathrm{Aut}(BX)_{(x,0)}$, but $\gamma^*(x,1) = (\gamma(x),1) \neq (x,1)$, and $\gamma^* \notin \mathrm{Aut}(BX)_{(x,1)}$.

If $x \in B$, then $\gamma^*(x,1) = (x,1)$, so $\gamma^* \in \mathrm{Aut}(BX)_{(x,1)}$, but $\gamma^*(x,0) = (\gamma(x),0) \neq (x,0)$, and $\gamma^* \notin \mathrm{Aut}(BX)_{(x,0)}$.

In either case, we conclude that X is unstable, as otherwise we would obtain a contradiction with Lemma 4.4. □

Remark 5.11. *As we want to establish that X is unstable, by Lemma* 5.4*, we are free to assume that X contains edges that are red with respect to γ. In particular, the red subgraph X_r has at least one non-trivial connected component.*

Note that the condition (2) *implies that all edges of X with both endpoints in A or B are blue. In particular, all of red edges of X have one endpoint in A and one endpoint* *in B. This implies that X_r is bipartite and so are all of its connected components. We conclude that any non-trivial connected component H of X_r is a γ sub-component of X - either $\gamma(H) \neq H$, and Definition* 5.3(1) *applies, or $\gamma(H) = H$ and then Definition* 5.3(2) *applies, because bipartition sets of H are subsets of A and B, which γ preserves by assumption* (1)*.*

Remark 5.12. *In [34], Wilson claims that the automorphism γ^* of BX constructed in the proof of Theorem* 5.10 *is a half-action. This is not necessarily true. In fact, γ^* is a half-action if and only if γ has no fixed points. But as shown in the proof, the instability of X follows from a much weaker assumption that γ is non-trivial.*

5.1.2 Anti-symmetry

Definition 5.13. Let X be a graph. A permutation $\alpha \in \mathrm{Sym}(V(X))$ is an *anti-symmetry* of X if there exists an automorphism $\gamma \in \mathrm{Aut}(X)$ of X of order 2 such that

1. α commutes with γ, and
2. $\{\alpha(x), (\alpha\gamma)(y)\} \in E(X)$ for all edges $\{x, y\} \in E(X)$.

Theorem 5.14 (Wilson [34, Theorem 2])**.** *Let X be a graph. If X has an anti-symmetry, then X is unstable.*

Proof. Let $\gamma \in \mathrm{Aut}(X)$ be an automorphism of X of order 2 for which there exists an anti-symmetry $\alpha \in \mathrm{Sym}(V(X))$. Define the following map

$$\alpha^*\colon V(BX) \to V(BX)$$
$$\alpha^*(x,i) = \begin{cases} (\alpha(x), 0), & i = 0 \\ ((\alpha\gamma)(x), 1), & i = 1 \end{cases} \tag{5.1}$$

We check that α^* is an automorphism. Because both α and $\alpha\gamma$ are permutations of $V(X)$, it is clear that α^* is a permutation.

Let $\{(x, 0), (y, 1)\} \in E(BX)$ be arbtrary. Then $\{x, y\} \in E(X)$. Since α is an anti-symmetry with respect to γ, it follows that $\{\alpha(x), (\alpha\gamma)(y)\} \in E(X)$. Consequently, $\alpha^*(\{(x, 0), (y, 1)\}) = \{(\alpha(x), 0), ((\alpha\gamma)(y), 1)\} \in E(BX)$.

Finally, suppose for contradiction that X is stable. Then by Lemma 4.1, the automorphism τ is central in $\mathrm{Aut}(BX)$, so in particular, $\alpha^*\tau = \tau\alpha^*$. Let $x \in V(X)$ be arbitrary. Then we obtain that

$$(\alpha^*\tau)(x, 0) = \alpha^*(x, 1) = ((\alpha\gamma)(x), 1),$$
$$(\tau\alpha^*)(x, 0) = \tau(\alpha(x), 0) = (\alpha(x), 1).$$

We conclude that $\alpha(\gamma(x)) = \alpha(x)$ for all $x \in V(X)$. As α is a injective, it follows that $\gamma(x) = x$ for all $x \in V(X)$. Hence, γ is trivial, which is a contradiction, as it was assumed to be of order 2. □

Remark 5.15. *We can give an alternative proof of Theorem* 5.14 *by using Lemma* 3.9. *In particular, we can set* $\beta(x) := \alpha(\gamma(x))$. *Then* β *is a permutation of* $V(X)$. *Moreover, using the condition* (2) *and the fact that* X *is finite, we can conclude that* $\{x, y\} \in E(X)$ *if and only if* $\{\alpha(x), \beta(y)\} = \{\alpha(x), (\alpha\gamma)(y)\} \in E(X)$. *Finally, note* $\alpha \neq \beta$ *because* γ *is assumed to be non-trivial.*

Remark 5.16. *Note that the assumptions that* γ *is an automorphism and that* α *and* γ *commute are unnecessary and have not been used in the proof of Theorem* 5.14, *nor in the proof given in the Remark* 5.15. *As a matter of a fact, we only needed the facts that* γ *is a non-trivial permutation of* $V(X)$ *and that the condition* (2) *holds.*

The extra conditions arise from the context of Wilson's original construction. He first observes the quotient graph X/γ *and constructs the anti-symmetry* α *as a*

covering permutation of an automorphism α_1 *of* X/γ, *that additionally satisfies the condition* (2).

This just means that, if $\mu\colon X \to X/\gamma$ *is the projection map mapping* $x \mapsto \{x, \gamma(x)\}$, *then for all* $x \in V(X)$ *we have that* $\mu(\alpha(x)) = \alpha_1(\mu(x))$. *Note that when* α *is obtained in this manner the condition* (1) *follows from the definition of* α *as a covering permutation.*

In conclusion, in case we can realize X *as a covering graph of* X/γ *for some* $\gamma \in \mathrm{Aut}(X)$ *of order* 2, *the search for anti-symmetry can be reduced to covering permutations of automorphisms of* X/γ *(which is a smaller graph).*

It turns out that Theorem 5.14 can explain instability of all non twin-free graphs.

Corollary 5.17. *Let* X *be a graph and assume it is not twin-free. Then* X *has an anti-symmetry and is unstable.*

Proof. As X is not twin-free, we can find a pair of twins $x, y \in V(X)$. Let γ be a map that swaps x and y and fixes all other vertices of X. Then by Lemma 3.19, γ is an automorphism of X of order 2. It is not hard to check directly that the identity permutation α on $V(X)$ is an anti-symmetry of X with respect to γ. Instability of X then follows by Theorem 5.14. □

Remark 5.18. *That graphs which are not twin-free are unstable, we already knew from Proposition* 3.20. *If we apply the construction from the proof of Theorem* 5.14, *we would obtain an automorphism* α^* *of* BX, *that swaps* $(x, 1)$ *and* $(y, 1)$. *This is exactly the map we constructed in the proof of Proposition* 3.20, *which we used to conclude that* X *is unstable.*

5.1.3 Cross-covers

Definition 5.19. Let X be a graph and $n \geq 3$ a positive integer. Let $s\colon E(X) \to \mathbb{Z}_n$ be a function. The *n-cross-cover of* X is a graph $\mathrm{CC}(n, X, s)$ with

- $V(\mathrm{CC}(n, X, s)) = V(X) \times \mathbb{Z}_n$,
- $E(\mathrm{CC}(n, X, s)) = \{\{(x, i), (y, a - i)\} \mid \{x, y\} \in E(X), s(\{x, y\}) = a, i \in \mathbb{Z}_n\}\}$.

Note that the map $\pi\colon V(\mathrm{CC}(n, X, s)) \to V(X)$ given by $\pi(x, i) = x$ is a covering projection from $\mathrm{CC}(n, X, s)$ onto X.

Theorem 5.20 (Wilson [34, Theorem 3])**.** *Every graph that is a cross-cover of some other graph is unstable.*

Proof. Let X be a graph and $n \geq 3$ a positive integer. Let $Y \coloneqq \mathrm{CC}(n, X, s)$ be an n-cross-cover of X for some assignment $s\colon E(X) \to \mathbb{Z}_n$.

Consider the following function

$$\alpha\colon V(BY) \to V(BY)$$
$$\alpha(x, i, j) = \begin{cases} (x, i+1, 0), & j = 0 \\ (x, i-1, 1), & j = 1 \end{cases}$$

Notice that since the second coordinate is considered modulo n, $\alpha^n = 1$. In particular, α is a permutation of $V(BY)$.

Let $\{(x, i, 0), (y, j, 1)\} \in E(BY)$ be arbitrary. Then $\{(x, i), (y, j)\}$ is an edge of $Y = \mathrm{CC}(n, X, s))$. By definition of a cross-cover, it follows that $\{x, y\} \in E(X)$ and that $j = s(\{x, y\}) - i$, that is, $s(\{x, y\}) = i + j$. Since, $s(\{x, y\}) = i + j = (i+1) + (j-1)$, it follows that $\{(x, i+1), (y, j-1)\} \in E(Y)$ and consequently, $\{(x, i+1, 0), (y, j-1, 0)\} \in E(BY)$. This exactly means that $\{\alpha(x, i, 0), \alpha(y, j, 1)\} \in E(BY)$. It follows that $\alpha \in \mathrm{Aut}(BY)$.

Finally, suppose for contradiction that Y is stable. Then Lemma 4.1 implies that τ is central in $\mathrm{Aut}(BY)$ and in particular, it commutes with α. Then for $x \in V(X)$ and $i \in \mathbb{Z}_n$, we have that

$$(\alpha\tau)(x, i, 0) = \alpha(x, i, 1) = (x, i-1, 1),$$
$$(\tau\alpha)(x, i, 0) = \tau(x, i+1, 0) = (x, i+1, 1).$$

It follows that $i - 1 \equiv i + 1 \pmod{n}$ i.e., $1 \equiv -1 \pmod{n}$. This is clearly a contradiction, as we have assumed that $n \geq 3$. Hence, $Y = \mathrm{CC}(n, X, s)$ is unstable, as desired. ☐

The following construction generalizes cross-covers. Unlike cross-covers, which require 3 elements (namely the graph X, the group $\mathbb{Z}_n$ and the assignment $s\colon E(X) \to \mathbb{Z}_n$), the following idea is based on a single permutation of the vertices of a graph satisfying certain conditions.

Definition 5.21. Let X be a graph. Suppose that there exists a permutation $f \in \mathrm{Sym}(V(X))$ of order at least 3, such that

$$\{x, y\} \in E(X) \implies \{f(x), f^{-1}(y)\} \in E(X), \quad \forall x, y \in V(X).$$

Then X is called *a generalized cross-cover* of the quotient graph X/f.

Corollary 5.22 (Wilson [34, p. 369])**.** *Every graph that is a cross-cover of another graph is also a generalized cross-cover of the same graph.*

Proof. Let X be a graph, $n \geq 3$ a positive integer. Let $Y = \mathrm{CC}(n, X, s)$ be an n-cross-cover of X for some $s\colon E(X) \to \mathbb{Z}_n$. Define $f \in \mathrm{Sym}(V(X) \times \mathbb{Z}_n)$ by setting $f(x, i) = (x, i + 1)$. Then f is a permutation of order $n \geq 3$.

Let $\{(x, i), (y, j)\} \in E(Y)$ for $x, y \in V(X), i, j \in \mathbb{Z}_n$. Then $\{x, y\} \in E(X)$ and $s(\{x, y\}) = i + j$. Note that $f(x, i) = (x, i + 1)$ and $f^{-1}(y, j) = (y, j - 1)$. Because $s(\{x, y\}) = i + j = (i + 1) + (j - 1)$, it follows that $\{f(x, i), f^{-1}(y, j)\} \in E(Y)$ as well. Hence, f satisfies all of the condition listed in Definition 5.21 and Y is a generalized cross-cover of the graph Y/f.

The orbits of $\langle f \rangle$ on $V(Y)$ are the sets $\{(x, i) \mid i \in \mathbb{Z}_n\}$ with $x \in V(X)$. In particular, the quotient graph Y/f is clearly isomorphic to X. Hence, $Y = \mathrm{CC}(n, X, s)$ is a generalized cross-cover of X. □

Theorem 5.23 (Wilson [34, p. 369])**.** *Every graph that is a generalized cross-cover of another graph is unstable.*

Proof. Let X be a generalized cross-cover of X/f with $f \in \mathrm{Sym}(V(X))$.

Define the following map

$$\alpha\colon V(BX) \to V(BX)$$
$$\alpha(x, i) = \begin{cases} (f(x), 0), & i = 0 \\ (f^{-1}(x), 1), & i = 1 \end{cases}$$

Note that α has an obvious inverse, namely the map $\beta\colon V(BX) \to V(BX)$ mapping $(x, 0) \mapsto (f^{-1}(x), 0)$ and $(x, 1) \mapsto (f(x), 1)$ for $x \in V(X)$.

Let $\{(x, 0), (y, 1)\} \in E(BX)$. Then $\{x, y\} \in E(X)$. By assumptions on f, it follows that $\{f(x), f^{-1}(y)\} \in E(X)$. In particular, $\{\alpha(x, 0), \alpha(y, 1)\} = \{(f(x), 0), (f^{-1}(y), 1)\} \in E(BX)$, proving that α is an automorphism of BX.

Assume for contradiction that X is stable. Then Lemma 4.1 implies that τ is central in $\mathrm{Aut}(BX)$. In particular, τ and α commute. Hence, for all $x \in V(X)$, the following expressions are equal.

$$(\alpha\tau)(x, 0) = \alpha(x, 1) = (f^{-1}(x), 1)$$
$$(\tau\alpha)(x, 0) = \tau(f(x), 0) = (f(x), 1)$$

Comparing the first coordinates, we obtain that $f(x) = f^{-1}(x)$ for all $x \in V(X)$. In particular, $f = f^{-1}$. However, this implies that f is of order at most 2, a contradiction. It follows that X is unstable, as desired. □

Remark 5.24. *Theorem* 5.20 *can be obtained from Corollary* 5.22 *and Theorem* 5.23*. Moreover, the map* α *defined in the proof of Theorem* 5.20 *is just a particular case of the map with the same name from the proof of Theorem* 5.23*, where* f *is taken to be the permutation of* $V(\mathrm{CC}(n, X, s))$ *described in the proof of Corollary* 5.22*.*

Remark 5.25. *We can also derive Theorem* 5.23 *from Lemma* 3.9*. In particular, we can set* $\alpha = f$ *and* $\beta = f^{-1}$*. Then* α *and* β *are permutations of* $V(X)$ *and as* X *is finite, the condition on* f *given in Definition* 5.21 *implies that* $\{x, y\} \in E(X)$ *if and only if* $\{\alpha(x), \beta(x)\} = \{f(x), f^{-1}(y)\} \in E(X)$*. Finally,* $\alpha \neq \beta$ *as otherwise,* f *would be of order at most* 2*.*

5.1.4 Twist

Let X be a graph and H a group. We will refer to H as the *voltage group.*

Let L, called *a system of labels for* X, be a map that assigns to each vertex $x \in V(X)$ a subgroup $L(x)$ of H.

Let d, called *a system of weights for* X, be a map assigning to each edge $e \in E(X)$ an inverse-closed subset $d(e)$ of H. The value $d(e)$ is called the *weight of an edge* e.

With this data, we define the following crucial concept.

Definition 5.26. Let X be a graph, H a voltage group. Let L and d be systems of labels and weights for X with respect to H, respectively.

Define the *generalized voltage graph of* X, denoted $\mathrm{GV}(X, H, L, d)$, by setting

- $V(\mathrm{GV}(X, H, L, d)) = \{(x, L(x)h) \mid x \in V(X), h \in H\}$,
- $E(\mathrm{GV}(X, H, L, d)) = \{\{(x, L(x)h), (y, L(y)ah)\} \mid \{x, y\} \in E(X),\ a \in d(\{x, y\}),$ $h \in H\}$.

Remark 5.27. *In the original paper [34], Wilson considers directed graphs for the purposes of this construction. In the original construction, there is a requirement that reversed arcs are assigned inverse weights with respect to the group structure of* H*, that is, the following condition holds*

$$d(y, x) = \{h^{-1} \mid h \in d(x, y)\} \text{ for all directed edges } (x, y) \in E(X).$$

Actually, if X *has the property that* $(x, y) \in E(X)$ *implies* $(y, x) \in E(X)$ *for all* $x, y \in V(X)$*, then, due to the original requirement, the directed graph* $\mathrm{GV}(X, H, L, d)$ *will have the same property.*

Indeed, since $a \in d(x,y)$, then $a^{-1} \in d(y,x)$ and by setting $h' \coloneqq ah \in H$, we obtain a directed edge

$$((y, L(y)h'), (x, L(x)a^{-1}h')) = ((y, L(y)ah), (x, L(x)h)).$$

Recall that every graph can be thought of as a directed graph by substituting each of its edges by a pair of opposite arcs. Then the generalized voltage (di)graph can be thought of as an undirected graph by the reverse process of identifying these pairs of arcs with an undirected edge.

To simplify this process, we have changed the requirement so that d must assign only inverse-closed subsets to edges of X, ensuring, by the same proof, that the adjacency relation is symmetric and making $\mathrm{GV}(X,H,L,d)$ *a graph.*

Theorem 5.28 (Wilson [34, Theorem 4]). *Let X be a graph and $n \geq 2$ a positive integer. Let H the group $\mathbb{Z}_2^n$ (written additively). Suppose that there exist $\gamma \in \mathrm{Aut}(X), \varphi \in \mathrm{Aut}(H)$ and systems of labels L and weights d for X with respect to H, such that*

1. *for all $x \in V(X)$, $L(\gamma(x)) = \varphi(L(x))$,*
2. *there exists an element $t \in H$ such that*
   (a) *$t \notin \bigcap_{x \in V(X)} L(x)$, and*
   (b) *for all $x, y \in V(X)$, if $\{x,y\} \in E(X)$, then*
   $$d(\{\gamma(x), \gamma(y)\}) = \varphi(d(\{x,y\})) + t.$$

Then the graph $\mathrm{GV}(X,H,L,d)$ *is unstable.*

Proof. For convenience, write $Y \coloneqq \mathrm{GV}(X,H,L,d)$. Define the following map.

$$\alpha \colon V(BY) \to V(BY)$$
$$\alpha(x, L(x)+h, i) = \begin{cases} (\gamma(x), \varphi(L(x)+h), 0), & i = 0 \\ (\gamma(x), \varphi(L(x)+h)+t, 1), & i = 1 \end{cases}$$

We will first check that α is a well-defined permutation of $V(BY)$ (this is not immediate due to how α acts on the second coordinate). Note that since φ is an automorphism, it holds that $\varphi(L(x)+h) = \varphi(L(x)) + \varphi(h)$. Applying (1), we get that this if further equal to $L(\gamma(x)) + \varphi(h)$. In particular, we have shown that

$$\alpha(x, L(x)+h, 0) = (\gamma(x), L(\gamma(x)) + \varphi(h), 0). \tag{5.2}$$

This is a valid element of the vertex set of BY. By the same argument, we also obtain

$$\alpha(x, L(x)+h, 1) = (\gamma(x), L(\gamma(x)) + \varphi(h) + t, 1). \tag{5.3}$$

This shows that α is well-defined. To prove its a permutation, it suffices to show that its surjective (as it maps between finite sets of the same cardinality). Let $(y, L(y)+g, i)$ be an arbitrary vertex of BY with $y \in V(X), g \in H, i \in \mathbb{Z}_2$. Set $x := \gamma^{-1}(y) \in V(X)$. If $i = 0$, set $h := \varphi^{-1}(g)$. If $i = 1$, set $h := \varphi^{-1}(g - t)$. Then a simple computation shows that $\alpha(x, L(x) + h, i) = (y, L(y) + g, i)$, proving that α is surjective.

Let $\{(x, L(x) + h, 0), (y, L(y) + a + h, 1)\} \in E(BY)$ be arbitrary with $x, y \in V(X), h \in H, a \in d(\{x, y\})$. Note that this implies that $\{x, y\} \in E(X)$.

By Eq. (5.2), α maps the first vertex to $(\gamma(x), L(\gamma(x)) + \varphi(h), 0)$.

By Eq. (5.3), α maps the second vertex to

$$\begin{aligned}(\gamma(y), L(\gamma(x)) + \varphi(a + h) + t, 1) &= (\gamma(y), L(\gamma(x)) + \varphi(a) + \varphi(h) + t, 1) = \\ &= (\gamma(y), L(\gamma(y)) + (\varphi(a) + t) + \varphi(h), 1).\end{aligned}$$

Because $\{x, y\} \in E(X)$ and $\gamma \in \mathrm{Aut}(X)$, it follows that $\{\gamma(x), \gamma(y)\} \in E(X)$. Since $a \in d\{x, y\}$, condition (2b) implies that

$$\varphi(a) + t \in \varphi(d(\{x, y\})) + t = d(\{\gamma(x), \gamma(y)\}).$$

In particular, we have that

$$\{(\gamma(x), L(\gamma(x)) + \varphi(h)), (\gamma(y), L(\gamma(y)) + (\varphi(a) + t) + \varphi(h))\} \in E(Y)$$

and

$$\{(\gamma(x), L(\gamma(x)) + \varphi(h), 0), (\gamma(y), L(\gamma(y)) + (\varphi(a) + t) + \varphi(h), 1)\} \in E(BY),$$

proving that α is an automorphism of BY.

Finally, assume for contradiction that Y is stable. Then Lemma 4.1 implies that τ is central in $\mathrm{Aut}(BY)$. In particular, it must commute with α. For $x \in V(X), h \in H$, by applying Eq. (5.2) and Eq. (5.3) again, we obtain that

$$\begin{aligned}(\alpha\tau)(x, L(x) + h, 0) &= \alpha(x, L(x) + h, 1) = (\gamma(x), L(\gamma(x) + \varphi(h) + t, 1), \\ (\tau\alpha)(x, L(x) + h, 0) &= \tau(\gamma(x), L(\gamma(x)) + \varphi(h), 0) = (\gamma(x), L(\gamma(x)) + \varphi(h), 1).\end{aligned}$$

Comparing the two expressions, in particular their second coordinates, we obtain that

$$\begin{gathered}L(\gamma(x)) + \varphi(h) + t = L(\gamma(x)) + \varphi(h) \\ L(\gamma(x)) + t = L(\gamma(x)) \\ t \in L(\gamma(x))\end{gathered} \tag{5.4}$$

As this holds for all $x \in V(X)$ and γ is a permutation of $V(X)$, we obtain that $t \in \cap_{x \in V(X)} L(x)$. This is a contradiction with (2a). Hence, $Y = \mathrm{GV}(X, H, L, d)$ must be unstable. $\square$

Definition 5.29. Let X be a graph. The generalized voltage graph $\mathrm{GV}(X, \mathbb{Z}_2^n, L, d)$ of X defined in Theorem 5.28 will be called the *twist of* X.

Remark 5.30. *With terminology from Defintion* 5.29*, we can reformulate Theorem* 5.28 *by saying that the twist of every graph is an unstable graph.*

Note that in this case, the requirement that $d(e) \subseteq H$ *is inverse-closed for all* $e \in E(X)$ *is redundant as* H *is an elementary abelian* 2*-group, so every element* $h \in H$ *is of order* 2 *and in particular, its own inverse.*

Moreover, all subgroups are normal, which justifies our computations with cosets in Eq. (5.4)*.*

5.2 CHARACTERIZATION OF NON-TRIVIALLY UNSTABLE GRAPHS

Via a series of lemmas, we will gradually prove that non-trivial instability of any graph can be explained by (at least one of) the criteria discussed in Section 5.1. We will provide a sketch of the proof for each of the auxiliary lemmas and then in Theorem 5.35, we will show that every non-trivially unstable graph has an anti-symmetry, or is a (generalized) cross-cover or a twist of some smaller graph.

Recall from Definition 3.3 that τ is an automorphism of BX given by $\tau\colon (x, i) \mapsto (x, i+1)$ for $x \in V(X), i \in \{0, 1\}$. Our main idea is to look at how τ behaves under conjugation by unexpected automorphisms of BX.

Lemma 5.31 (Wilson [34, Lemma 6.1])**.** *Let* X *be a non-trivially unstable graph. Let* $\alpha \in \mathrm{Aut}(BX)$ *be an unexpected automorphism. Define* $\tau^* := \alpha\tau\alpha^{-1}$ *and* $\gamma := \tau^*\tau$*. If* γ *is of order at least* 3*, then* X *is a generalized cross-cover of some smaller graph.*

Proof. Note that γ does not reverse the colour classes of BX. This allows us to define $f \in \mathrm{Sym}(V(X))$ such that

$$\gamma(x, 0) = (f(x), 0), \forall x \in V(X).$$

As both τ and τ^* are of order 2, a direct computation shows that

$$\gamma(x, 1) = (f^{-1}(x), 1), \forall x \in V(X).$$

The fact that γ is an automorphism of BX of order at least 3 implies that X is a generalized cross cover of X/f. □

Lemma 5.32 (Wilson [34, Lemma 6.2])**.** *Let* X *be a non-trivially unstable graph. Let* $\alpha \in \mathrm{Aut}(BX)$ *be an unexpected automorphism and define* $\tau^* := \alpha\tau\alpha^{-1}$*. Suppose that*

1. τ^* and τ commute, and

2. $\alpha\tau^*\alpha^{-1} = \tau$.

Then X has an anti-symmetry.

Proof. As X is non-trivially unstable, Lemma 2.13 implies that, after possibly composing α with τ, we can assume that α preserves the colour classes of BX. This allows us to define a permutation $\alpha' \in \mathrm{Sym}(V(X))$ by

$$\alpha(x, 0) = (\alpha'(x), 0), \forall x \in V(X).$$

By Proposition 4.2, the condition (1) is equivalent to τ^* being an expected automorphism of BX. It follows that $\gamma \coloneqq \tau^*\tau$ is an expected automorphism of BX of order 2 preserving the colour classes of BX. Hence, we can find an automorphism $\varphi \in \mathrm{Aut}(X)$ of order 2 such that $\gamma = \overline{\varphi}$.

A direct computation using conditions (1) and (2) shows that α' and φ commute and moreover, it holds that

$$\alpha(x, 1) = ((\alpha'\varphi)(x), 1), \forall x \in V(X).$$

In particular, α' is an anti-symmetry of X with respect to φ. □

Lemma 5.33 (Wilson [34, Lemma 6.3])**.** *Let X be a non-trivially unstable graph. Let $\alpha \in \mathrm{Aut}(BX)$ be an unexpected automorphism. Suppose that all conjugates of τ by elements of $\langle\alpha\rangle$ lie in $\mathrm{Aut}(X) \times S_2$ and that there is an even number of them. Then X has an anti-symmetry.*

Proof. Let the number of distinct conjugates of τ by elements of $\langle\alpha\rangle$ be $2k$. It is not hard to see that then conjugating τ by elements of $\langle\alpha^k\rangle$ produces exactly two distinct elements, both of which are expected automorphisms of BX. It follows that we can apply Lemma 5.32 to α^k. □

Lemma 5.34 (Wilson [34, Lemma 6.4])**.** *Let X be a non-trivially unstable graph. Suppose that for every unexpected automorphisms $\alpha \in \mathrm{Aut}(BX)$, all conjugates of τ by elements of $\langle\alpha\rangle$ lie in $\mathrm{Aut}(X) \times S_2$ and that there is an odd number of them (for each α). Then X is a twist of some smaller graph.*

Proof. In order to prove this statement, we will follow the strategy below.

1. We let α denote a colour-preserving unexpected automorphism of BX of minimal possible order. This also allows us to define a permutation $\delta \in \mathrm{Sym}(V(X))$ such that
$$\alpha(x, 0) = (\delta(x), 0), \forall x \in V(X).$$

Note that α^k is an expected automorphism if and only if $\gcd(k, a) > 1$. Here a denotes the order of α. This observation and the assumption that the number of distinct conjugates of τ by powers of α is odd imply that the orders of α and δ are powers of some fixed odd prime p.

2. As all conjugates of τ by elements of $\langle\alpha\rangle$ are expected automorphisms of BX of order 2, it follows by Proposition 4.2, that they generate an elementary abelian 2-group $H^+ \leq \mathrm{Aut}(X) \times S_2$. For our voltage group, we choose the sugroup $H \leq \mathrm{Aut}(X)$ characterized by

$$\varphi \in H \text{ if and only if } \overline{\varphi} \in \overline{H} = H^+ \cap \overline{\mathrm{Aut}(X)}.$$

Note that $H \cong \overline{H}$ is an elementary abelian 2-group (in particular, $H \cong \mathbb{Z}_2^n$ for some $n \in \mathbb{N}$).

3. Note that the quotient graphs $Y' = BX/H^+$ and $Y = X/H$ are isomorphic. In particular, the map $\mu\colon V(Y) \to V(Y')$ given by $x^H \mapsto (x, 0)^{H^+}$ is a well-defined graph isomorphism.

Conjugation by α preserves the generating set of H^+ and consequently, α normalizes H^+. This implies that α has a well-defined action on the orbits of H^+ on $V(BX)$ and therefore, the map

$$(x, i)^{H^+} \mapsto \alpha(x, i)^{H^+}$$

is a well-defined automorphism of Y'. If we let $\pi\colon V(X) \to V(Y)$ be the covering projection of Y by X given by $x \mapsto x^H$, then the identification of Y and Y' via μ shows that the map α^* defined below is an automorphism of Y corresponding to the previously mentioned automorphism of Y'.

$$\begin{aligned} \alpha^*\colon V(Y) &\to V(Y) \\ \pi(x) &\mapsto \pi(\delta(x)) \end{aligned}$$

4. Note that $H^+ = \langle\overline{H}, \tau\rangle$ and that H^+ is exactly the subgroup of colour-preserving elements of H^+. In particular, since α normalizes H^+, it also normalizes $\overline{H}$ and induces a group automorphism of $\overline{H}$. The isomorphism $H \cong \overline{H}$ lets us translate this automorphism to the group automorphism $\varphi\colon H \to H$, where for $h \in H$

$$\varphi(h) \text{ is the unique element of } H \text{ such that } \overline{\varphi(h)} = \alpha\overline{h}\alpha^{-1}.$$

Evaluating the left and right side of this equation at vertices of BX shows that φ is given by

$$\varphi(h) = \delta h \delta^{-1}, \forall h \in H.$$

5. As H is an abelian group, the point stabilizers of every two vertices of X lying in the same orbit of H are equal. This implies that the following is a well-defined system of labels for Y with respect to H

$$L(y) = L(\pi(x)) \coloneqq H_x \leq H, \forall x \in V(X).$$

6. We let $y \in V(Y)$ be arbitrary and let N denote the size of its orbit under the action of $\langle \alpha^* \rangle \leq \mathrm{Aut}(Y)$. Note that $y = \pi(x)$ for some $x \in V(X)$ and as H is an elementary abelian 2-group, size of $\pi(x)$ is a power of 2 (we use Lemma 2.6). As δ normalizes H, it has a well-defined action on each of its orbits, in particular y. A counting argument using Lemma 2.6 will then show that δ^N must have a fixed point on y. We define $r(y) = r(\pi(x))$ to be some fixed point of δ^N on $y = \pi(x)$.

   We extend r to the orbit of y under $\langle \alpha^* \rangle$ by setting

$$r({\alpha^*}^i y) \coloneqq \delta^i(r(y)), \text{ that is, } r(\pi(\delta^i(x))) = \delta^i(r(\pi(x))).$$

   This defines a map $r\colon V(Y) \to V(X)$ satisfying

$$r(\pi(x)) \in \pi(x), \forall x \in V(X).$$

   We think of $r(\pi(x))$ as the canonical representative of the orbit $\pi(x)$ of x under the action of H.

7. We define an auxiliary map f by setting

$$f(x) \coloneqq L(\pi(x))H \text{ with } h \in H \text{ mapping } r(\pi(x)) \text{ to } x.$$

   Note that $f(x)$ is a right coset of $L(\pi(x))$ in H and it is exactly the set of elements in H that map $r(\pi(x))$ to x. The following key properties of f follow by direct computation for all $x \in V(X), h \in H$.

   - $f(\delta(x)) = \varphi(f(x))$,
   - $f((h\delta)(x)) = f(\delta(x))h$,
   - $f(h(x)) = f(x)h$.

8. The following defines a system of weights for Y with respect to H. For an edge $\{y_1, y_2\} \in E(Y)$, we set

$$d(\{y_1, y_2\}) \coloneqq \bigcup_{\substack{\{u,v\} \in E(X) \\ u \in \pi^{-1}(y_1), v \in \pi^{-1}(y_2)}} f(u)f(v).$$

   Notice that this is a union of double cosets of $L(y_1)$ and $L(y_2)$.

9. We are now able to define $X^* = \mathrm{GV}(Y, H, L, d)$. We define the following map.

$$\begin{aligned}\Phi\colon V(X) &\to V(X^*)\\ x &\mapsto (\pi(x), f(x))\end{aligned}$$

A counting argument using Lemma 2.6 shows that Φ is invertible. That Φ is in fact a graph isomorphism of X and X^* can be checked directly.

10. We prove that X^* is a twist of Y. That X^* satisfies condition (1) of Theorem 5.28 follows by definitions of α^*, φ and L, after noting that

$$H_{\delta(x)} = \delta H_x \delta^{-1} = \varphi(H_x).$$

We can define $\tau^* = \alpha\tau\alpha^{-1}$ and $\gamma = \tau\tau^*$. Our assumptions imply that $\gamma \in \overline{H} \leq H^+$, so we can find an element $t \in H$ such that $\gamma = \bar{t}$. Note that

$$t \in \bigcap_{y \in V(Y)} L(y) = \bigcap_{x \in V(X)} L(\pi(x)) = \bigcap_{x \in V(X)} H_x$$

implies that $t = 1$ i.e., $\gamma = 1$. As this is a contradiction with α being unexpected, we conclude that $t \notin \bigcap_{y \in V(Y)} L(y)$ and the condition (2a) is satisfied. Condition (2b) follows by a direct computation, where we use the previously established identities for f. We conclude that $X \cong X^*$ is a twist of Y, finishing the proof.

□

Theorem 5.35 (Wilson [34, Theorem 5])**.** *Let X be a non-trivially unstable graph. Then*

1. *X has an anti-symmetry (and its instability is explained by Theorem* 5.14*), or*

2. *X is a generalized cross-cover of some smaller graph (and its instability is explained by Theorem* 5.23*), or*

3. *X is a twist of some smaller graph (and its instability is explained by Theorem* 5.28*).*

Proof. We consider the conjugates $\alpha\tau\alpha^{-1}$ of τ by unexpected automorphisms α of BX.

Case 1. *There exists an unexpected automorphism $\alpha \in \mathrm{Aut}(BX)$ such that $\alpha\tau\alpha^{-1}$ is unexpected.*

In this case, we can define $\tau^* \coloneqq \alpha\tau\alpha^{-1}$ and $\gamma \coloneqq \tau^*\tau$. As τ^* is unexpected, it does not commute with τ by Proposition 4.2, so the order of γ is at least 3. Then Lemma 5.31 applies and X is a generalized cross-cover of some smaller graph. In particular, case (2) holds.

Case 2. *All conjugates $\alpha\tau\alpha^{-1}$ with $\alpha \in \mathrm{Aut}(BX)$ are expected.*

We consider the number of distinct conjugates of τ produced by every unexpected $\alpha \in \mathrm{Aut}(BX)$.

Subcase 2.1. *There exists an unexpected automorphism $\alpha \in \mathrm{Aut}(BX)$ such that the number of distinct conjugates of τ by elements of $\langle\alpha\rangle$ is even.*

In this case, Lemma 5.33 applies and we conclude that X has an anti-symmetry. In particular, (1) holds.

Subcase 2.2. *For all unexpected automorphisms $\alpha \in \mathrm{Aut}(BX)$, the number of distinct conjugates of τ by elements of $\langle\alpha\rangle$ is odd.*

In this case, Lemma 5.34 applies and we conclude that X is a twist of some smaller graph. In particular, (3) holds. $\square$

5.3 WILSON TYPES

We now come to one of the most important parts of this thesis. Recall that a circulant graph of order n is a Cayley graph of the cyclic group $\mathbb{Z}_n$.

In his article on unexpected automorphisms of unstable graphs, see [34], Wilson attempts to characterize unstable members of four different families of graphs, namely circulants, generalized Petersen graphs, Rose-Window graphs and toroidal graphs. His strategy is to come up with conditions on the defining parameters for each family, that enable one to apply some of the criteria from Section 5.1 (recall that these include sub-components, half-actions, anti-symmetries, (generalized) cross-covers and twists of graphs).

In this subsection, we will discuss the conditions Wilson defined for the family of circulant graphs. It turns out some of these conditions contained errors, and we discuss their corrected versions introduced in [13] and [26]. In Theorem 5.48, we collect the four updated conditions, each implying that a circulant graph satisfying it is unstable, which we will call *Wilson types.*

In Section 8.4, we will see that there are infinitely many non-trivially unstable circulant graphs, which have no Wilson type (contrary to conjecture of Wilson made in [34]). This is why in Section 5.4, we introduce generalizations of Wilson types coming from [13].

Theorem 5.36 (Wilson [34, Theorem C.1.])**.** *Let n be an even integer. Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant. Suppose there exists an even integer $a \in \mathbb{Z}_n$ such that $a + s \in S$ for all even $s \in S$. Then X is unstable.*

Proof. Define $\gamma := a_L$, that is, $\gamma\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is a map given by $\gamma(x) = a + x$. Then by Proposition 2.25(1), γ is an automorphism of X.

Define A and B to be sets of even and odd sets of integers in $\mathbb{Z}_n$, respectively. Then $\{A, B\}$ is a partition of $\mathbb{Z}_n$ and, since a is assumed to be even, γ preserves it (that is, $\gamma(A) = A, \gamma(B) = B$).

Next, assume that $\{x, y\} \in E(X)$ such that $x, y \in A$ or $x, y \in B$. In either case, x and y have the same parity, so $y - x$ is an even element of S. Then by assumption, it follows that $y-x+a = (a+y)-x \in S$. In particular, $\{x, a+y\} = \{x, \gamma(y)\} \in E(X)$. As γ has no fixed points, it follows by Theorem 5.10 that X is unstable due to existence of a half-action. $\square$

The following equivalent to Theorem 5.36 for odd integers has been shown to be false. We start by stating the original version of the result.

Theorem 5.37 (Wilson - original false statement [34, Theorem C.2.])**.** *Let n be an integer divisible by* 4. *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant. Suppose there exists an odd integer $b \in \mathbb{Z}_n$ such that $s + 2b \in S$ for all odd $s \in S$. Then X is unstable.*

The following example, due to Qin, Xia and Zhou, shows that the statement of Theorem 5.37 is false.

Example 5.38 (Qin-Xia-Zhou [26, p.156])**.** Let $X = \mathrm{Cay}(\mathbb{Z}_{12}, \{3, 4, 8, 9\})$. Take $b = 3$. As $3 + 2 \cdot 3 \equiv 9 \pmod{12}$ and $9 + 2 \cdot 3 \equiv 3 \pmod{12}$ are both in S, X satisfies all assumptions of Theorem 5.37. However, MAGMA calculations show that X is stable.

We also provide an infinite family of counterexamples to the original statement of Theorem 5.37.

Example 5.39. Define $X_n := K_{4n}$ for $n \geq 1$. Then $X_n = \mathrm{Cay}(\mathbb{Z}_{4n}, S)$ with $S = \mathbb{Z}_{4n} \setminus \{0\}$. Note that 4 divides the order of X_n. Moreover, S contains all odd integers in $\mathbb{Z}_{4n}$, so X satisfies the condition of Theorem 5.37 for any odd $b \in \mathbb{Z}_{4n}$. However, by Example 4.7, X_n is stable for all $n \geq 1$.

Qin, Xia and Zhou have proposed an alternative version of Theorem 5.37, which we now state.

Theorem 5.40 (Qin-Xia-Zhou [26, p. 156])**.** *Let n be a positive integer divisible by* 4. *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph. Suppose there exists an odd $b \in \mathbb{Z}_n$ such that*

1. *$s + 2b \in S$ for all odd $s \in S$, and*
2. *for each $s \in S$ such that $s \equiv 0$ or $-b \pmod 4$, it holds that $s + b \in S$.*

Proof. Define the following map:

$$\gamma \colon \mathbb{Z}_n \to \mathbb{Z}_n$$

$$\gamma(x) = \begin{cases} x + b, & x \text{ is even} \\ x - b, & x \text{ is odd} \end{cases}$$

Note that if $y \in \mathbb{Z}_n$ is even, then $x = y + b$ is odd and $\gamma(x) = y$. Similarly, if $y \in \mathbb{Z}_n$ is odd, then $x = y - b$ is even and $\gamma(x) = y$. This shows that γ is surjective. As it is a map between finite sets of same cardinality, it follows that it is invertible.

Let $x, y \in \mathbb{Z}_n$. If x and y are of opposite parity, then without loss of generality, assume that y even and x is odd. Then $\gamma(y) - \gamma(x) = (y + b) - (x - b) = (y - x) + 2b$. Then if $\{x, y\} \in E(X)$, it follows that $y - x \in S$ and is odd. By assumption (1), $y - x + 2b \in S$. In particular, $\gamma(y) - \gamma(x) \in S$, proving that $\{\gamma(x), \gamma(y)\} \in E(X)$.

If x and y are of the same parity, then regardless of whether they are both even or both odd, it holds that then $\gamma(y) - \gamma(x) = y - x$. In particular, $\{x, y\} \in E(X)$ if and only if $\{\gamma(y), \gamma(x)\} \in E(X)$.

It follows that γ is an automorphism of X.

We will construct a partition of $\mathbb{Z}_n$ that, together with γ, satisfies the assumptions of Theorem 5.10. It will then follow that BX has a half-action automorphism proving that X is unstable, as desired.

As b is odd, it follows that $b \equiv 1 \pmod 4$ or $b \equiv 3 \pmod 4$. Note that, $-b \in \mathbb{Z}_n$ is still odd (because n is even) and moreover:

1. If $s \in S$ is odd, then $-s$ is also odd (since n is even) and in S (since S is inverse-closed). It follows by (1) that $-s + 2b \in S$. As S is inverse-closed, this further implies that $s + 2(-b) \in S$.

2. Let $s \in S$.

   (a) If $s \equiv 0 \pmod 4$, then $-s \equiv 0 \pmod 4$ as well. It follows by (1) that $-s + b \in S$. As S is inverse-closed, we get that $s + (-b) \in S$.

   (b) If $s \equiv b \pmod 4$, then $-s \equiv -b \pmod 4$. Then (2) implies that $-s + b \in S$. As S is inverse-closed, this further implies that $s + (-b) \in S$.

It follows that b and $-b$ both satisfy conditions (1) and (2) and can be interchanged for the purposes of the proof. Hence, we can assume without loss of generality that $b \equiv 1 \pmod 4$. Define

$$A \coloneqq \{x \in \mathbb{Z}_n \mid x \equiv 0 \text{ or } 1 \pmod 4\} \text{ and } B \coloneqq \{x \in \mathbb{Z}_n \mid x \equiv 2 \text{ or } 3 \pmod 4\}.$$

Then $\{A, B\}$ is a partition of $\mathbb{Z}_n$. Moreover, it is easily checked that $\gamma(A) = A$ and $\gamma(B) = B$. Let $\{x, y\} \in E(X)$. Then $y - x \in S$ and as S is inverse-closed, also $x - y \in S$. We again consider cases.

Case 1. $x, y \in A$

Depending on their residues modulo 4, we have the following.

Subcase 1.1. $x \equiv y \equiv 0 \pmod 4$

Note that in this case, both x and y are even and $y - x \equiv 0 \pmod 4$. Then (2) implies that $(y-x)+b \in S$. Since $\gamma(y) - x = (y+b) - x = (x-y)+b \in S$, we get that $\{x, \gamma(y)\} \in E(X)$.

Subcase 1.2. $x \equiv y \equiv 1 \pmod 4$

It follows that x and y are both odd and $y - x \equiv 0 \pmod 4$. It follows by (2) that $(y-x)+b \in S$. We obtain that $\gamma(y) - x = (y+b) - x = (y-x)+b \in S$ and $\{x, \gamma(y)\} \in E(X)$.

Subcase 1.3. *x and y are not congruent modulo* 4.

In this case, we can assume, without loss of generality, that $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 4$. Then x is odd and y is even and $y - x \equiv -1 \equiv -b \pmod 4$, so (2) implies that $(y-x)+b \in S$. We have that $\gamma(y) - x = (y+b) - x = (y-x)+b \in S$ and $\{x, \gamma(y)\} \in E(X)$.

Case 2. $x, y \in B$

We have the following cases.

Subcase 2.1. $x \equiv y \equiv 2 \pmod 4$

It follows that they are both even and that $y - x \equiv 0 \pmod 4$. We obtain that $(y-x)+b \in S$. Hence, $\gamma(y) - x = (y+b) - x = (y-x)+b \in S$ and $\{x, \gamma(y)\} \in E(X)$.

Subcase 2.2. $x \equiv y \equiv 3 \pmod 4$

It follows that both x and y are odd and that $x - y \equiv 0 \pmod 4$. Hence, by (2), we have that $(x-y)+b \in S$ and S is inverse-closed, also $(y-x)-b \in S$. Finally, $\gamma(y) - x = (y-b) - x = (y-x)-b \in S$ and $\{x, \gamma(y)\} \in E(X)$.

Subcase 2.3. *x and y are not congruent modulo* 4.

Without loss of generality, we may assume that $x \equiv 3 \pmod 4$ and $y \equiv 2 \pmod 4$. Then x is odd and y is even, while $y - x \equiv -1 \equiv -b \pmod 4$. It follows by (2) that $(y-x)+b \in S$. We get that $\gamma(y) - x = (y+b) - x = (y-x)+b \in S$ and $\{x, \gamma(y)\} \in E(X)$.

The desired conclusion now follows by Theorem 5.10. □

Remark 5.41. *We proved that Example* 5.38 *and Example* 5.39 *pose counterexamples to Theorem* 5.37. *We now check that none of the graphs described in these examples satisfy all conditions of Theorem* 5.40*, in particular, the condition* (2).

If $X = \mathrm{Cay}(\mathbb{Z}_{12}, \{3,4,8,9\})$*, then the odd integer* b *satisfying* (1) *must lie in* $\{3, 9\}$. *If* $b = 3$*, then* $8 \equiv 0 \pmod 4$*, but* $8 + 3 \equiv_{12} 11 \notin S$*. If* $b = 9$*, then* $8 \equiv 0 \pmod 4$*, but* $8 + 9 \equiv_{12} 17 \equiv_{12} 5 \notin S$*. Hence, no such* b *can exist.*

Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ *with* $S = \mathbb{Z}_n \setminus \{0\}$*. Suppose that* $b \in \mathbb{Z}_n$ *is odd and satisfies the condition* (1)*. Then* $b \neq 0$ *and* $b, -b \in S$*. Then* $-b \equiv -b \pmod 4$ *and* (2) *implies that* $(-b) + b = 0 \in S$*, a contradiction.*

The next criterion also turned out to be false. We proceed in the same manner as last time. We will first state the original version of the result, provide a counterexample to it and afterwards introduce the updated version of the result.

Theorem 5.42 (Wilson - original false statement [34, Theorem C.3.]). *Let n be an even positive integer. Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant. Let $e \in \mathbb{Z}_n$ such that $e > 1$. Let $B = \{j \in S \mid j + ke \in S, \forall k \in \mathbb{Z}\}$ and let $R = S \setminus B$. Let D be the greatest common divisor of elements in R. Suppose that $D > 1$ and that j/D is odd for all $j \in R$. Then X is unstable.*

The following example shows that Theorem 5.42 is false.

Example 5.43 (Hujdurović-Mitrović-Morris [13, Remark 3.14]). Let n be an even integer. Define $X_n \coloneqq K_n$. Then Example 4.7 shows that X_n is stable for all $n \geq 3$. Note that $X_n = \mathrm{Cay}(\mathbb{Z}_n, S)$ with $S = \mathbb{Z}_n \setminus \{0\}$.

Let $e \coloneqq n/2$. Then for $j \in S$, we have that:

$$ke \equiv_n \begin{cases} 0, & k \text{ even} \\ n/2, & k \text{ odd} \end{cases} \implies j + ke \equiv_n \begin{cases} j, & k \text{ even} \\ j + n/2, & k \text{ odd} \end{cases}$$

It follows that $B = \mathbb{Z}_n \setminus \{0, n/2\}$ and consequently, $R = \{n/2\}$. Then R is non-empty and $D = n/2 > 1$. As R has only one element, the last condition is also satisfied. It follows that X_n satisfies all of the conditions of Theorem 5.42, but is stable.

We now state and prove the corrected version of the result.

Theorem 5.44 (Hujdurović-Mitrović-Morris [13, Remark 3.14]). *Let n be an even integer. Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant. Suppose that there is a subgroup H of $\mathbb{Z}_n$, such that the set*

$$R = \{\, s \in S \mid s + H \not\subseteq S \,\},$$

is non-empty and has the property that if we let $d = \gcd\big(R \cup \{n\}\big)$, then n/d is even, r/d is odd for every $r \in R$, and either $H \not\subseteq d\mathbb{Z}_n$ or $H \subseteq 2d\mathbb{Z}_n$.

Then X is unstable.

Proof. As R is non-empty, H is non-trivial. Let $h \neq 0$ be a generator of H. Define $\gamma \coloneqq h_L$, that is, $\gamma \colon \mathbb{Z}_n \to \mathbb{Z}_n$ is given by $\gamma(x) = x + h$. Then γ is a non-trivial automorphism of X (see Proposition 2.25(1)). Our goal is to show that X has a γ sub-component. The conclusion that X is unstable then follows by Theorem 5.5.

An edge $\{x, x+s\} \in E(X)$ with $x \in \mathbb{Z}_n, s \in S$ is blue with respect to γ if and only if $\{x + kh, (x+s) + \ell h\} \in E(X)$ for all $k, \ell \in \mathbb{Z}$, which means that $s + (\ell - k)h \in S$. This is equivalent to $s + H \subseteq S$, that is, $s \notin R$. In particular, an edge of X is red with respect to γ if and only if it is induced by $s \in R \subseteq S$. This shows that the red subgraph X_r is exactly $\mathrm{Cay}(\mathbb{Z}_n, R)$. As R is assumed to be non-empty and X_r is vertex-transitive, it follows that connected components of X_r are all non-trivial.

Let X_0 be the connected component of X_r containing 0. Note that then $X_0 = \mathrm{Cay}(\langle d \rangle, R)$. Conditions that n/d is even and r/d odd for all $r \in R$ imply that X_0 is a connected, bipartite graph of even order n/d with the bipartition given by the two cosets of $\langle 2d \rangle$ in $\langle d \rangle$. Note that $\langle d \rangle = d\mathbb{Z}_n$ and $\langle 2d \rangle = 2d\mathbb{Z}_n$.

Case 1. $H \not\subseteq d\mathbb{Z}_n$.

As $V(X_0) = d\mathbb{Z}_n$, this implies that γ does not preserve the connected component X_0. In particular, X_0 satisfies the condition (1) of Definition 5.3 and is a γ sub-component of X.

Case 2. $H \subseteq d\mathbb{Z}_n$.

It follows that γ preserves the connected component X_0 and that $H \subseteq 2d\mathbb{Z}_n$. Hence, γ preserves $2d\mathbb{Z}_n$ and its unique coset in $d\mathbb{Z}_n$. In conclusion, γ preserves the bipartition sets of X_0 and X_0 is a γ sub-component of X fulfilling the condition (2) of Definition 5.3. □

Remark 5.45 ([13, Remark 3.14])**.** *The reformulated definition of the set R given in Theorem 5.44 is equivalent to that given in Theorem 5.42, but is more compact, as working with generators of the same subgroup makes no difference. Moreover, the condition that $d > 1$ is unnecessary, as Theorem 5.5 does not require the red subgraph X_r to be disconnected. As we have seen in the proof of Theorem 5.44, the condition that n/d is even is necessary for X_r to have bipartite components and conditions $H \not\subseteq d\mathbb{Z}_n$ and $H \subseteq 2d\mathbb{Z}_n$ correspond to conditions (1) and (2) of Definition 5.3, respectively.*

Remark 5.46. *We saw that Example 5.43 is a counterexample for Theorem 5.42. Let us check that the graphs described there do not satisfy all of the conditions of Theorem 5.44.*

Let n be even and $X_n = K_n = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n \setminus \{0\})$. Suppose that H is a subgroup of $\mathbb{Z}_n$ and R is the subset defined in Theorem 5.44. We show that $R = H \setminus \{0\}$.

If $s \in R$, then $s \neq 0$ and $s + H \not\subseteq S = \mathbb{Z}_n \setminus \{0\}$, so $0 \in s + H$. It follows that $-s \in H$ and as H is a subgroup, $s \in H \setminus \{0\}$ as well. Conversely, if $s \in H \setminus \{0\}$, then $s \neq 0$, so $s \in S$ and $-s \in H$. It follows that $0 = s + (-s) \in s + H \not\subseteq S$, proving that $s \in R$.

As $\mathbb{Z}_n$ is cyclic, so is H. Let h be a generator of H. Then $d = \gcd(R \cup \{n\}) = \gcd(H \setminus \{0\} \cup \{n\}) = \gcd(\{hk, n \mid k \in \mathbb{Z}, k \neq 0\}) = \gcd(h, n)$. Note that then H must be of even order $|H| = \frac{n}{d}$. In particular, H contains $n/2$. It follows that $n/2 \in R$, so $\frac{n}{2d}$ is an odd integer. Hence, $2d$ is a divisor of n.

Note that $d\mathbb{Z}_n$ is a subgroup of $\mathbb{Z}_n$ of order $\frac{n}{\gcd(n,d)} = \frac{n}{d} = |H|$. As subgroups of $\mathbb{Z}_n$ are characterized by their order, it follows that $H = d\mathbb{Z}_n$ (in particular, $H \subseteq d\mathbb{Z}_n$). The subgroup $2d\mathbb{Z}_n$ is of order $\frac{n}{\gcd(n,2d)} = \frac{n}{2d} = \frac{1}{2}|H|$. It follows that $H \not\subseteq 2d\mathbb{Z}_n$.

In conclusion, H fails the newly introduced condition of Theorem 5.44.

The following is the final instability criterion for circulants that Wilson introduced.

Theorem 5.47 (Wilson [34, Theorem C.4.]). *Let $n = 2k$ be an even positive integer. Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant. Let $g \in \mathbb{Z}_n^\times$. If $gs + k \in S$ for all $s \in S$, then X is unstable.*

Proof. Define $\gamma := k_L$, that is, $\gamma\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is given by $\gamma(x) = k + x$. Then by Proposition 2.25(1), γ is an automorphism of X. Moreover, γ is non-trivial (for example, $\gamma(0) = k \neq 0$) and of order 2 (as $n = 2k \equiv 0 \pmod{n}$).

We will show that X has an anti-symmetry with respect to γ. The conclusion then follows by Theorem 5.14.

Since $g \in \mathbb{Z}_n^\times$, the map $\alpha(j) = gj$ for $j \in \mathbb{Z}_n$ is actually a group automorphism of $\mathbb{Z}_n$. In particular, α is a permutation of $\mathbb{Z}_n$. Because $k \in \mathbb{Z}_n$ is the unique element of order 2, every automorphism of $\mathbb{Z}_n$ fixes it. In particular, $\alpha(k) = gk = k$.

Let $x \in \mathbb{Z}_n$. The following calculation shows that α and γ commute proving that α satisfies condition (1) in Definition 5.13.

$$(\alpha\gamma)(x) = \alpha(k + x) = g(k + x) = gk + gx = k + \alpha(x) = (\gamma\alpha)(x)$$

Next, let $x, y \in V(X)$. Note that

$$(\alpha\gamma)(y) - \alpha(x) = g(k + y) - gx = g(y - x) + gk = g(y - x) + k. \tag{5.5}$$

If $\{x, y\} \in E(X)$, then $y - x \in S$. By assumption $g(y - x) + k \in S$. In particular, due to Eq. (5.5), it follows that $\{\alpha(x), (\alpha\gamma)(y)\} \in E(X)$. This shows that α satisfies the condition (2) in Definition 5.13.

It follows that α is an anti-symmetry of X (with respect to γ), as desired. □

We now collect all of the previously proven results and introduce new notation that will be used for the rest of the thesis.

Theorem 5.48 (Wilson [34, Appendix A.1] (and [26, p. 156])). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph, such that n is even. Let $S_e = S \cap 2\mathbb{Z}_n$ and $S_o = S \setminus S_e$. If any of the following conditions is true, then X is unstable.*

1. *There is a non-zero element h of $2\mathbb{Z}_n$, such that $h + S_e = S_e$.*

2. *n is divisible by 4, and there exists $h \in 1 + 2\mathbb{Z}_n$, such that*

   (a) *$2h + S_o = S_o$, and*

   (b) *for each $s \in S$, such that $s \equiv 0$ or $-h \pmod 4$, we have $s + h \in S$.*

3. *There is a subgroup H of $\mathbb{Z}_n$, such that the set*

$$R = \{\, s \in S \mid s + H \not\subseteq S \,\},$$

is non-empty and has the property that if we let $d = \gcd\bigl(R \cup \{n\}\bigr)$, then n/d is even, r/d is odd for every $r \in R$, and either $H \not\subseteq d\mathbb{Z}_n$ or $H \subseteq 2d\mathbb{Z}_n$.

4. *There exists $m \in \mathbb{Z}_n^{\times}$, such that $(n/2) + mS = S$.*

Definition 5.49 ([13, Definition 1.6], [12, Definition 1.6])**.** We say that X has *Wilson type* (C.1), (C.2′), (C.3′), or (C.4), respectively, if it satisfies the corresponding condition of Theorem 5.48.

Remark 5.50 ([12, Remark 1.7])**.** *Wilson type of a graph needs not be unique, that is, a graph may satisfy more than one condition from Theorem* 5.48*. For example, for every odd integer k with* $\gcd(k, 3) = 1$*, the graph*

$$\mathrm{Cay}(\mathbb{Z}_{8k}, \{\pm 2k, \pm 3k\})$$

has Wilson type (C.1) *(with $h = 4k$) as well as Wilson types* (C.3′) *(with $H = \{0, 4k\}, R = \{\pm 3k\}$ and $d = k$) and* (C.4) *(with $g = 3$).*

Example 5.51. While each of the Wilson types implies that the circulant graph satisfying it is unstable, the converse is not true. The following example has already been observed in [26, p. 156].

$$\mathrm{Cay}(\mathbb{Z}_{24}, \{\pm 2, \pm 3, \pm 8, \pm 9, \pm 10\})$$

The above graph is non-trivially unstable but has no Wilson type. Additional five examples of order 24 have been found by a computer search in [13, Observation 6.1].

In Example 8.41, we will construct an infinite family of non-trivially unstable circulants with no Wilson type.

5.4 GENERALIZATIONS OF WILSON TYPES

As we have already discussed in Section 5.3, Wilson types cannot explain instability of all non-trivially unstable circulants. This is why further generalizations are necessary.

We present three results first proven in [13].

- Theorem 5.52, which generalizes Wilson types (C.1),(C.2′) and (C.3′),
- Proposition 5.56, which generalized Wilson type (C.4), and
- Proposition 5.58.

The proofs of all of these results will be based on Lemma 3.9.

Theorem 5.52 (Hujdurović-Mitrović-Morris [13, Theorem 3.2]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph. Choose non-trivial subgroups H and K of $\mathbb{Z}_n$, such that $|K|$ is even, and let $K_o = K \setminus 2K$. If either*

1. $S + H \subseteq S \cup (K_o + H)$ *and* $H \cap K_o = \emptyset$, *or*

2. $(S \setminus K_o) + H \subseteq S \cup K_o$ *and either* $|H| \neq 2$ *or* $|K|$ *is divisible by* 4,

then X is unstable.

Proof. Let h be a generator of H. We will define permutations α and β of $\mathbb{Z}_n$, such that Lemma 3.9 applies.

(1) Define

$$\alpha(x) = \begin{cases} x + h & \text{if } x \in 2K + H; \\ x & \text{otherwise;} \end{cases} \qquad \beta(x) = \begin{cases} x + h & \text{if } x \in K_o + H; \\ x & \text{otherwise.} \end{cases}$$

Note that $0 \notin K_o + H$ (because $H \cap K_o = \emptyset$), so $\beta(0) = 0$. Since $\alpha(0) = h$, this implies $\alpha \neq \beta$.

Given an edge $\{x, y\}$ of X, we wish to show that $\alpha(x)$ is adjacent to $\beta(y)$. We may assume that either x is moved by α or y is moved by β. In fact, we may assume that exactly one of the vertices is moved, for otherwise,

$$\beta(y) - \alpha(x) = (y + h) - (x + h) = y - x \in S.$$

This means we may assume that either $x \in 2K + H$ or $y \in K_o + H$, but not both. Letting $s = y - x \in S$, this implies $s \notin K_o + H$.

Also, we have

$$\beta(y) - \alpha(x) \in (y + H) - (x + H) = (y - x) + H = s + H,$$

so we may write $\beta(y) - \alpha(x) = s + h'$, for some $h' \in H$. By the first assumption of (1), we know $s + h' \in S \cup (K_o + H)$. Since $s \notin K_o + H$, this implies $s + h' \in S$, so $\alpha(x)$ is adjacent to $\beta(y)$.

(2) If $h \in 2K$, then $K_o + H = K_o$, so

$$(S \cap K_o) + H \subseteq K_o + H = K_o.$$

Since, by the first assumption of (2), we also have $(S \setminus K_o) + H \subseteq S \cup K_o$, this implies that (1) applies. Therefore, we may assume

$$h \notin 2K.$$

Define

$$\alpha(x) = \begin{cases} x+h & \text{if } x \in 2K; \\ x-h & \text{if } x \in 2K+h; \\ x & \text{otherwise} \end{cases} \qquad \beta(x) = \begin{cases} x+h & \text{if } x \in K_o; \\ x-h & \text{if } x \in K_o+h; \\ x & \text{otherwise.} \end{cases}$$

We claim that $\alpha \neq \beta$. Note that $\alpha(0) = h$. Therefore, if $\alpha = \beta$, then we must have $\beta(0) = h$. Since $0 \notin K_o$, this implies that $0 \in K_o + h$ (which means $h \in K_o$) and $-h = h$ (which means $|h| = 2$). Since $|h| = 2$ and $h \in K_o$, we see that $|H| = 2$ and that $|K|$ is not divisible by 4. This contradicts the second half of assumption (2), so the proof of the claim is complete.

Given an edge $\{x, y\}$ of X, we wish to show that $\alpha(x)$ is adjacent to $\beta(y)$. That is, we wish to show $\beta(y) - \alpha(x) \in S$. We have $y = x + s$ for some $s \in S$. We may assume

$$\beta(y) - \alpha(x) \neq y - x.$$

In particular, we cannot have both $\alpha(x) = x$ and $\beta(y) = y$. Therefore,

$$\text{either } \ x \in 2K \cup (2K+h) \ \text{ or } \ y \in K_o \cup (K_o+h).$$

Case 1. *Assume* $x \in 2K \cup (2K+h)$ ***and*** $y \in K_o \cup (K_o+h)$. We consider two different possibilities, but both of the arguments are very similar.

Subcase 1.1. *Assume* $x \in 2K$. Then $\alpha(x) = x + h$. Since $\beta(y) - \alpha(x) \neq y - x$, this implies $\beta(y) \neq y + h$, so $y \notin K_o$. By the assumption of Case 1, this implies $y \in K_o + h$, so $\beta(y) = y - h$. Hence, $\beta(y) - \alpha(x) = s - 2h$.

We have $x \in 2K$ and $y \in K_o + h$, so $s = y - x \in K_o + h$, which means $s - h \in K_o$. Since $h \notin 2K$, this implies $s \notin K_o$ and $s - 2h \notin K_o$. Since $s \notin K_o$, the first assumption of (2) tells us $s + H \subseteq S \cup K_o$. Since $s - 2h \notin K_o$, this implies $s - 2h \in S$. So $\alpha(x)$ is adjacent to $\beta(y)$.

Subcase 1.2. *Assume* $x \in 2K + h$. We have $\alpha(x) = x - h$. Since $\beta(y) - \alpha(x) \neq y - x$, this implies $\beta(y) \neq y - h$, so $y \notin K_o + h$. By the assumption of Case 1, this implies $y \in K_o$, so $\beta(y) = y + h$. Hence, $\beta(y) - \alpha(x) = s + 2h$.

We have $x \in 2K + h$ and $y \in K_o$, so $s = y - x \in K_o - h$, which means $s + h \in K_o$. Since $h \notin 2K$, this implies $s \notin K_o$ and $s + 2h \notin K_o$. Since $s \notin K_o$, the first assumption of (2) tells us $s + H \subseteq S \cup K_o$. Since $s + 2h \notin K_o$, this implies $s + 2h \in S$. So $\alpha(x)$ is adjacent to $\beta(y)$.

Case 2. *Assume Case* 1 *does not apply.* As in Case 1, we consider two different possibilities, but both of the arguments are very similar.

Subcase 2.1. *Assume* $x \in 2K \cup (2K + h)$. Choose $\delta \in \{0, 1\}$, such that $x \in 2K + \delta h$. We have $\alpha(x) = x + \epsilon h$, where $\epsilon = 1 - 2\delta$, and we also have $y \notin K_o \cup (K_o + h)$, since

Case 1 does not apply, so $\beta(y) = y$. Since $x \in 2K + \delta h$, but $x + s = y \notin K_o + \delta h$, we have $s \notin K_o$. So the first assumption of (2) tells us $s + H \subseteq S \cup K_o$, so $s - \epsilon h \in S \cup K_o$. Since $\beta(y) - \alpha(x) = s - \epsilon h$, then we may assume $s - \epsilon h \in K_o$ (otherwise, $\alpha(x)$ is adjacent to $\beta(y)$, as desired), so $s \in K_o + \epsilon h$. Then

$$y = x + s \in (2K + \delta h) + (K_o + \epsilon h) = K_o + (\delta + \epsilon)h = K_o + (1 - \delta)h.$$

Since $1 - \delta \in \{0, 1\}$, but $y \notin K_o \cup (K_o + h)$, this is a contradiction.

Subcase 2.2. *Assume* $y \in K_o \cup (K_o + h)$. Choose $\delta \in \{0, 1\}$, such that $y \in K_o + \delta h$. We have $\beta(y) = y + \epsilon h$, where $\epsilon = 1 - 2\delta$, and we also have $x \notin 2K \cup (2K + h)$, since Case 1 does not apply, so $\alpha(x) = x$. Since $y \in K_o + \delta h$, but $y - s = x \notin 2K + \delta h$, we have $s \notin K_o$. So the first assumption of (2) tells us $s + H \subseteq S \cup K_o$, so $s + \epsilon h \in S \cup K_o$. Since $\beta(y) - \alpha(x) = s + \epsilon h$, then we may assume $s + \epsilon h \in K_o$ (otherwise, $\alpha(x)$ is adjacent to $\beta(y)$, as desired), so $s \in K_o - \epsilon h$. Then

$$x = y - s \in (K_o + \delta h) - (K_o - \epsilon h) = 2K + (\delta + \epsilon)h = 2K + (1 - \delta)h.$$

Since $1 - \delta \in \{0, 1\}$, but $x \notin 2K \cup (2K + h)$, this is a contradiction. □

Wilson types (C.1), (C.2′), and (C.3′) are special cases of Theorem 5.52(2):

Proposition 5.53 (Hujdurović-Mitrović-Morris [13, Proposition 3.4]). *If* $\mathrm{Cay}(\mathbb{Z}_n, S)$ *has Wilson type* (C.1), (C.2′), *or* (C.3′), *then there are non-trivial subgroups H and K of $\mathbb{Z}_n$ that satisfy the conditions given in part* (2) *of Theorem* 5.52 (*and $|K|$ is even*).

Proof. (C.1) Let $K = \mathbb{Z}_n$ and $H = \langle h \rangle$. Then

$$(S \setminus K_o) + H = S_e + \langle h \rangle = S_e \subseteq S,$$

so the first condition of Theorem 5.52(2) is satisfied. Also, since $h \in 2\mathbb{Z}_n = 2K$, it must be true that either $|H| \neq 2$ or $|K|$ is divisible by 4.

(C.2′) Let $H = \langle h \rangle$ and $K = 2\mathbb{Z}_n$. (Note that $|H| > 2$, since h is odd and n is divisible by 4.) Then

$$K_o = \{\, x \in \mathbb{Z}_n \mid x \equiv 2 \pmod 4 \,\}.$$

We will show that $(S \setminus K_o) + H \subseteq S \cup K_o$.

We may assume $h \equiv 1 \pmod 4$, by applying the graph automorphism $x \mapsto -x$ if necessary. Fix some $s \in S \setminus K_o$.

Suppose, first, that $s \not\equiv 0 \pmod 4$ (and recall that $s \notin K_o$, so $s \not\equiv 2 \pmod 4$), so s is odd. This means $s \in S_o$, so we see from C.2′(a) that $s + 2kh \in S$ for all $k \in \mathbb{Z}$. If $s + 2kh \equiv 3 \pmod 4$, then $s + (2k + 1)h \in S$ (by C.2′(b)). If $s + 2kh \equiv 1 \pmod 4$, then $s + (2k + 1)h \in K_o$. Thus, we have $s + H \subseteq S \cup K_o$.

Now, suppose $s \equiv 0 \pmod 4$. Then $s + h \in S$ (by C.2′(b)). Now, since $s + h \not\equiv 0 \pmod 4$, the previous case tells us that $s + h + H \subseteq S \cup K_o$. Since $h + H = H$, this means $s + H \subseteq S \cup K_o$.

(C.3′) Let $K = \langle R \rangle = \langle d \rangle$. Since n/d is even, we know that K has even order. Then, since r/d is odd for every $r \in R$, we see that $R \subseteq K_o$. By the definition of R, this means $(S \setminus K_o) + H \subseteq S$, so the first condition of Theorem 5.52(2) is satisfied.

Also, since either $H \nsubseteq d\mathbb{Z}_n$ or $H \subseteq 2d\mathbb{Z}_n$, we know that either $H \nsubseteq K$ or $H \subseteq 2K$. If $H \nsubseteq K$ and $|H| = 2$, then it is clear that $H \cap K = \{0\} \subseteq 2K$. Thus, in both cases, we have $H \cap K \subseteq 2K$, which easily implies that either $|H| \neq 2$ or $|K|$ is divisible by 4. □

There is a strong converse to Proposition 5.53 when n is not divisible by 4:

Proposition 5.54 (Hujdurović-Mitrović-Morris [13, Proposition 3.5])**.** *Let X be a circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$*. If X, H and K satisfy the conditions of Theorem* 5.52(2)*, and n is not divisible by* 4*, then X has Wilson type* (C.1).

Proof. Since n is not divisible by 4, it is not possible for $|K|$ to be divisible by 4, so the second half of Theorem 5.52(2) tells us that $|H| > 2$. This implies that $H_e \coloneqq H \cap 2\mathbb{Z}_n$ is non-trivial. Also, since n is not divisible by 4, we know that $K_o \cap 2\mathbb{Z}_n = \emptyset$, so

$$S_e + H_e = (S \cap 2\mathbb{Z}_n) + H_e \subseteq (S \setminus K_o) + H \subseteq S \cup K_o \subseteq S \cup (\mathbb{Z}_n \setminus 2\mathbb{Z}_n).$$

Since $H_e \subseteq 2\mathbb{Z}_n$, we also know that $S_e + H_e \subseteq 2\mathbb{Z}_n$. Therefore, we conclude that $S_e + H_e \subseteq S_e$, so X has Wilson type (C.1). □

Remark 5.55 (Hujdurović-Mitrović-Morris [13, Remark 3.6])**.** *By combining Proposition* 5.53 *and Proposition* 5.54*, we see that if X has a Wilson type, and n is not divisible by* 4*, then X must have Wilson type* (C.1) *or* (C.4).

Proposition 5.56. *Assume $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ is a circulant graph of even order. If $X \cong \mathrm{Cay}\big(\mathbb{Z}_n, S + (n/2)\big)$, then X is unstable.*

Proof. Let α be an isomorphism from $\mathrm{Cay}(\mathbb{Z}_n, S)$ to $\mathrm{Cay}(\mathbb{Z}_n, S+(n/2))$. Define $\beta(x) \coloneqq \alpha(x) + (n/2)$. Then α and β are distinct permutations of $\mathbb{Z}_n$.

Let $\{x, y\} \in E(X)$. From the isomorphism property of α, it follows that we can find an $s \in S$ such that $\alpha(y) - \alpha(x) = s + (n/2)$. We obtain that

$$\beta(y) - \alpha(x) = \alpha(y) + (n/2) - \alpha(x) = (\alpha(y) - \alpha(x)) + (n/2) = s \in S.$$

In particular, $\{\alpha(x), \beta(y)\} \in E(X)$. Lemma 3.9 implies that X is unstable. □

Remark 5.57 (Hujdurović-Mitrović-Morris [13, Proposition 3.8]).

Proposition 5.56 *is a generalization of Wilson type* (C.4). *If* X *has Wilson type* (C.4), *we can find* $m \in \mathbb{Z}_n^\times$ *inducing a group automorphism of* $\mathbb{Z}_n$ *such that* $(n/2) + mS = S$. *Then* $mS = S + (n/2)$ *and* $\mathrm{Cay}(\mathbb{Z}_n, S) \cong \mathrm{Cay}(\mathbb{Z}_n, mS)$. *We conclude that* $\mathrm{Cay}(\mathbb{Z}_n, S) \cong \mathrm{Cay}(\mathbb{Z}_n, S + (n/2))$, *so Proposition* 5.56 *applies.*

Proposition 5.58 (Hujdurović-Mitrović-Morris [13, Proposition 3.12]). *Assume* $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ *is a circulant graph of even order. If there exist permutations* α *and* β *of* $2\mathbb{Z}_n$, *and a subgroup* H *of* $2\mathbb{Z}_n$, *such that:*

1. $\alpha \neq \beta$,
2. *if the vertices* $x, y \in 2\mathbb{Z}_n$ *are adjacent, then the vertices* $\alpha(x)$ *and* $\beta(y)$ *are also adjacent,*
3. $s + H \subseteq S$, *for all odd* $s \in S$, *and*
4. $\alpha(x) - x \in H$ *and* $\beta(x) - x \in H$, *for all* $x \in 2\mathbb{Z}_n$,

then X *is unstable.*

Proof. We define the following permutations of $\mathbb{Z}_n$.

$$\alpha^*(x) := \begin{cases} x, & x \text{ is odd} \\ \alpha(x), & x \text{ is even} \end{cases} \quad \text{and } \beta^*(x) := \begin{cases} x, & x \text{ is odd} \\ \beta(x), & x \text{ is even} \end{cases}$$

As $\alpha \neq \beta$, we conclude that $\alpha^* \neq \beta^*$. Let $\{x, y\} \in E(X)$. Then we can find an $s \in S$ such that $y = x + s$. We consider the following cases.

Case 1. *x and y are both odd.*

Then $\{\alpha^*(x), \beta^*(y)\} = \{x, y\} \in E(X)$.

Case 2. *Exactly one among x and y is odd.*

We will assume that x is odd and y is even. By (4), $\beta(y) - y \in H$. Then, as $s = y - x$ is odd, it follows by (3) that $(\beta(y) - y) + s \in S$. We obtain that $\beta^*(y) - \alpha^*(x) = \beta(y) - x = (\beta(y) - y) + s \in S$. In particular, $\{\alpha^*(x), \beta^*(y)\} \in E(X)$.

An analogous proof applies to the case when x is even and y is odd.

Case 3. *x and y are both even.*

It this case, it follows by (2) that $\{\alpha^*(x), \beta^*(y)\} = \{\alpha(x), \beta(y)\} \in E(X)$.

Lemma 3.9 now implies that X is unstable. $\square$

6 STABILITY OF CAYLEY GRAPHS OF ABELIAN GROUPS

In this section, we will be studying stability of Cayley graphs of abelian groups, with a particular emphasis on circulant graphs. We will tend to consider and prove results in their chronological order of appearance in the literature, from the least general to the most general. This will show the development of the stability theory of Cayley graphs and the gradual strengthening of the results and techniques applied.

For the case of abelian groups of odd order, we will proceed according to the following list of results.

- In Section 6.1, we will show that there are no non-trivially unstable circulants of prime order. This result has first been proven by Qin, Xia and Zhou in [26]. In this thesis, it appears as Theorem 6.1.

- In Section 6.2, we will generalize the previous result to circulants of an arbitrary odd order. This is a result of Fernandez and Hujdurović from [7]. We have it as Theorem 6.8.

- In Section 6.3, we will prove that there are no non-trivially unstable Cayley graphs on abelian groups of odd order. This is Theorem 6.11, which has been first proven by D.W. Morris in [23].

Having resolved the issue of stability of circulant graphs of odd order, we move onto circulants of even order. It turns out that, in the even case, the behaviour of these graphs is a lot more exotic and difficult to describe. For example, in Section 7, we will see that even when the valency is assumed to be low, there are infinitely many examples of non-trivially unstable circulant graphs. This is why in Section 6.4, we consider the easiest even case, namely unstable circulants of order twice a prime. We will derive a complete classification of these graphs, which is one of the main results of [13] by Hujdurović, Mitrović and Morris.

Finally, in Section 6.5, we will also consider another result of Qin, Xia and Zhou from [26], proving that there are no non-trivially unstable arc-transitive circulants. Note that this is the only result from this section that does not make any assumptions on the order of the graphs it considers.

6.1 CIRCULANTS OF PRIME ORDER

We begin with the first non-trivial result on stability of circulant graphs. Note that the only simple graphs of order 2 are K_2 and its complement $\overline{K_2}$, both of which are trivially unstable. Therefore, in the following result, we only consider odd primes.

Theorem 6.1 (Qin-Xia-Zhou [26, Theorem 1.6])**.** *A circulant of odd prime order p is either stable or isomorphic to $\overline{K_p}$. In particular, there is no non-trivially unstable circulant of odd prime order.*

Proof. Let $X = \mathrm{Cay}(\mathbb{Z}_p, S)$ be a circulant of odd prime order p. Assume that X is not isomorphic to $\overline{K_p}$. It follows by Corollary 2.47, that X is connected, non-bipartite and twin-free.

By Lemma 3.13, we have that

$$BX = B\,\mathrm{Cay}(\mathbb{Z}_p, S) = \mathrm{Cay}(\mathbb{Z}_p \times \mathbb{Z}_2, S \times \{1\}).$$

Because p is odd, it holds that $\mathbb{Z}_p \times \mathbb{Z}_2 \cong \mathbb{Z}_{2p}$, and BX is isomorphic to a circulant of order $2p$.

Case 1. *BX is a normal Cayley graph.*

As X is connected, Lemma 4.11 applies proving that X is stable.

Case 2. *BX is non-normal.*

Note that S is non-empty, as X is assumed not to be isomorphic with $\overline{K_p}$. If $S = \mathbb{Z}_p \setminus \{0\}$, then $X \cong K_p$ and it is stable by Example 4.7. Hence, we can assume $S \neq \mathbb{Z}_p \setminus \{0\}$. Then as p is odd, $\mathbb{Z}_p$ has no self-inverse elements and $|S|$ is an even integer between 1 and $p-2$. As X is connected and non-bipartite, Lemma 3.2(1) shows that BX is connected.

It follows that BX is a non-normal, connected, bipartite circulant of order $2p$ and valency at most $p-2$. Then Lemma 2.34 implies that $BX = Y \wr \overline{K_2}$ for some connected graph Y. From Lemma 2.44, we conclude that BX is not twin-free. However, Corollary 3.21 then implies that X is also not twin-free, a contradiction. □

Remark 6.2. *The original version of this result, given in [26], claims that every circulant of odd prime order is stable. We have chosen to reformulate this result, because by our definition of a Cayley graph (see Definition* 2.22*), it could happen that S is the empty set, so $\overline{K_p}$ is a circulant. However, $\overline{K_p}$ is disconnected and therefore trivially unstable by Proposition* 3.15*.*

6.2 CIRCULANTS OF ODD ORDER

We start by considering Corollary 4.3 and Lemma 4.5 in the context of circulant graphs of odd order, in order to obtain simpler formulations of these results, which will greatly simplify the arguments that follow.

Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph of odd order n. By Lemma 3.13, we have that

$$BX = B\,\mathrm{Cay}(\mathbb{Z}_n, S) = \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S \times \{1\}).$$

Since n is assumed to be odd, the map $\psi(k, i) = 2k + ni$ for $(k, i) \in \mathbb{Z}_n \times \mathbb{Z}_2$, is a group isomorphism between $\mathbb{Z}_n \times \mathbb{Z}_2$ and the cyclic group $\mathbb{Z}_{2n}$. This allows us to think of BX as the circulant graph $\mathrm{Cay}(\mathbb{Z}_{2n}, \psi(S \times \{1\}))$. We conclude that

$$BX \cong \mathrm{Cay}(\mathbb{Z}_{2n}, 2S + n).$$

As X is a Cayley graph, Corollary 3.14 shows that the automorphism τ equals to the translation automorphism t_L induced by $t = (0, 1)$ (since $G = \mathbb{Z}_n$ in this case). Under the above identification, this automorphism corresponds to

$$\psi(t)_L = (2 \cdot 0 + n \cdot 1)_L = n_L.$$

We can now reformulate Corollary 4.3.

Lemma 6.3 (Fernandez-Hujdurović [7, Lemma 2.3]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a connected circulant of odd order n. Then X is stable if and only if the permutation n_L is central in the automorphism group of $BX = \mathrm{Cay}(\mathbb{Z}_{2n}, n + 2S)$.*

Note that the group isomorphism ψ identifies vertices $(0, 0)$ and $(0, 1)$ of BX with 0 and n in $\mathbb{Z}_{2n}$, respectively. Using this conclusion and letting $x = 0$ in Corollary 4.5, lets us reformulate it as follows.

Lemma 6.4 (Fernandez-Hujdurović [7, Lemma 2.5]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a connected circulant of odd order n and let $BX = \mathrm{Cay}(\mathbb{Z}_{2n}, n + 2S)$ be its canonical double cover represented as a Cayley graph on $\mathbb{Z}_{2n}$. Then X is stable if and only if $\mathrm{Aut}(BX)_0 = \mathrm{Aut}(BX)_n$.*

We refer the reader to the original papers for further details on the following two results.

Lemma 6.5 ([7, Lemma 2.12]). *Let $X = \mathrm{Cay}(G, S)$, $A = \mathrm{Aut}(X)$ and let $K \subseteq S$ such that $\varphi(K) = K$ for every $\varphi \in A_1$. Then $\varphi(\langle K \rangle) = \langle K \rangle$ for every $\varphi \in A_1$. Moreover, if K is inverse-closed, then φ induces an automorphism of $\mathrm{Cay}(\langle K \rangle, K)$.*

Lemma 6.6 (Fernandez-Hujdurović [7, Lemma 3.1]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a nontrivially unstable circulant of odd order n. Then BX is not arc-transitive.*

Lemma 6.7 (Fernandez-Hujdurović [7, Lemma 3.2]). *Let n be an odd positive integer, and let X be a connected bipartite arc-transitive circulant of order $2n$ and even valency. Then one of the following holds.*

1. $\mathrm{Aut}(X)_0 = \mathrm{Aut}(X)_n$, *or*

2. $X \cong Y \wr \overline{K_d}$ *where* Y *is a twin-free arc-transitive circulant of even order* $2m$ *and* $\mathrm{Aut}(Y)_0 = \mathrm{Aut}(Y)_m$.

Proof. As X is a circulant of order $2n$ and even valency, we know that $X \cong \mathrm{Cay}(\mathbb{Z}_{2n}, S)$, where S is inverse-closed and does not contain 0 nor n. Then [25, Proposition 2.4] implies that X is isomorphic to the canonical double cover BZ of a graph Z, obtained as the quotient graph X/n_L. Because it is a quotient of a connected circulant X, Z is a connected circulant itself. Moreover, as Z is of odd order n, by Corollary 2.27, it is also non-bipartite.

Case 1. *Z is stable.*

Then the conclusion of Lemma 6.4 exactly translates to the case (1).

Case 2. *Z is unstable.*

As Z is a circulant of odd order n with an arc-transitive double cover X, Lemma 6.6 implies that Z cannot be non-trivially unstable. It follows that Z is trivially unstable. As we have already established that it is connected and non-bipartite, it follows that Z is not twin-free.

By Lemma 2.46(1), there exists a connected twin-free circulant W of order m and an integer $d \geq 2$ such that $Z \cong W \wr \overline{K_d}$. Note that since $n = |V(Z)| = md$ and n is odd, m and d are both odd.

We then have

$$X \cong BZ = Z \times K_2 \cong (W \wr \overline{K_d}) \times K_2 \cong (W \times K_2) \wr \overline{K_d}.$$

Let $Y \coloneqq BW = W \times K_2$. Then $X \cong Y \wr \overline{K_d}$ and Y is of even order $2m$. Moreover, as W is twin-free, it follows by Corollary 3.21 that Y is twin-free. Because X is an arc-transitive circulant, so is Y (see [20, Remark 1.2]).

As W is a circulant of odd order m with an arc-transitive double cover Y, Lemma 6.6 implies that W is not non-trivially unstable. As it is also connected, non-bipartite and twin-free, it is not trivially unstable either. It follows that W is stable. Then Lemma 6.4 implies that $\mathrm{Aut}(Y)_0 = \mathrm{Aut}(Y)_m$, finishing the proof. □

We have arrived to the main result of the subsection.

Theorem 6.8 (Fernandez-Hujdurović, [7, Theorem 1.2]). *Every connected, twin-free circulant of odd order is stable.*

Proof. Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a connected, twin-free circulant of odd order n. By Corollary 2.27, X is non-bipartite. In particular, X is not trivially unstable.

As discussed before, we can identify BX with the circulant $\mathrm{Cay}(\mathbb{Z}_{2n}, S')$ with $S' = n + 2S$. Denote by $A \coloneqq \mathrm{Aut}(BX)$. We consider the following cases.

Case 1. *A_0 is transitive on S'.*

It follows by Lemma 2.11 that BX is arc-transitive. Lemma 6.6 then implies that X is not non-trivially unstable, so it must be stable.

Case 2. *A_0 is not transitive on S'.*

Let the orbits of A_0 on S' be $S_1, \ldots, S_k$. Note that S' is inverse-closed and does not contain 0. As $(0,1) \in \mathbb{Z}_n \times \mathbb{Z}_2$ is identified with $n \in \mathbb{Z}_{2n}$, it also follows that S' does not contain n, since S does not contain 0.

Because S' is the union of $S_1, \ldots, S_k$, it follows that none of the sets S_i contain 0 nor n. Observe that each S_i is inverse-closed, since the inversion map $x \mapsto -x$ on $\mathbb{Z}_{2n}$ lies in A_0. This shows that $|S_i|$ is even for all $i \in \{1, \ldots, k\}$. As all elements in S' are odd, they are of even order in $\mathbb{Z}_{2n}$, so $|\langle S_i \rangle|$ is even and $\langle S_i \rangle$ contains n (the unique element of order 2 in $\mathbb{Z}_{2n}$). We define $X_i \coloneqq \mathrm{Cay}(\langle S_i \rangle, S_i)$. Then each X_i is a connected circulant graph of even valency and even order not divisible by 4. Each X_i is a subgraph of BX, so they are all bipartite.

Because each S_i is an orbit of A_0, it is invariant under all elements of A_0. It follows by Lemma 6.5 that every element of A_0 induces an automorphism of X_i fixing 0, that is, $A_0 \leq \mathrm{Aut}(X_i)_0 \leq \mathrm{Aut}(X_i)$ for all $i \in \{1, \ldots, k\}$. This also shows that $\mathrm{Aut}(X_i)_0$ is transitive on S_i, so by Lemma 2.11, we conclude that each X_i is arc-transitive.

We can now apply Lemma 6.7 to each X_i.

Subcase 2.1. *There exists an $i \in \{1, \ldots, k\}$ such that* $\mathrm{Aut}(X_i)_0 = \mathrm{Aut}(X_i)_n$.

As $A_0 \leq \mathrm{Aut}(X_i)_0 = \mathrm{Aut}(X_i)_n$, it follows that if $\alpha \in \mathrm{Aut}(BX)$ fixes 0, then it must also fix n. We conclude that $A_0 = A_n$. Hence, X is stable by Lemma 6.4.

Subcase 2.2. *For every $i \in \{1, \ldots, k\}$, it holds that* $\mathrm{Aut}(X_i)_0 \neq \mathrm{Aut}(X_i)_n$.

In this case, each X_i must satisfy the condition (2) of Lemma 6.7. In particular, for each $i \in \{1, \ldots, k\}$ we can find a twin-free, arc-transitive circulant Y_i of even order satisfying the condition (1) of Lemma 6.7 such that $X_i = Y_i \wr \overline{K_{d_i}}$.

Because Y_i is twin-free, we can apply Lemma 2.45 to conclude that there exists a subgroup H_i of $\langle S_i \rangle$ of order d_i, such that

- the copies of $\overline{K_{d_i}}$ in X_i are exactly the cosets of H_i,
- Y_i is a quotient graph of X and it is isomorphic to a Cayley graph on the quotient group $\langle S_i \rangle / H_i$, and
- S_i is a union of cosets of H_i.

Observe that the unique element of order 2 in the quotient group $\langle S_i \rangle / H_i$ is $m + H_i$. Then since Y_i satisfies Lemma 6.7(1), it follows that $\mathrm{Aut}(Y_i)_{H_i} = \mathrm{Aut}(Y_i)_{m+H_i}$. Let $\alpha \in \mathrm{Aut}(X_i)$. If α fixes 0, it must also fix the coset H_i. Therefore, α induces an automorphism of Y_i that fixes the vertex H_i. It follows that α also fixes the coset $m + H_i$.

Define $d := \gcd(d_1, \dots, d_k)$. Then $H = \bigcap_{i=1}^k H_i$ is of order d.

Subsubcase 2.2.1. $d > 1$

As S_i is a union of cosets of H_i, it is also a union of cosets of H. It follows that $S' = \bigcup_{i=1}^k S_i$ is a union of cosets of a non-trivial group H. In particular, by Lemma 2.45(2), BX is not twin-free. By Corollary 3.21, it follows that X is not twin-free, a contradiction.

Subsubcase 2.2.2. $d = 1$

It follows that $H = \{0\}$. Since every element of A_0 fixes $m + H_i$, it must also fix their intersection

$$\bigcap_{i=1}^k (m + H_i) = m + \bigcap_{i=1}^k H_i = \{m\}.$$

It follows by Lemma 6.4 that X is stable. This finishes the proof. □

6.3 CAYLEY GRAPHS OF ABELIAN GROUPS OF ODD ORDER

The following elementary lemma on isomorphisms between Cayley graphs of abelian groups will be the main ingredient of the proof of one of the main results of this subsections, namely Theorem 6.11. It also has some interesting corollaries that we will consider in Chapter 7.

Lemma 6.9 (Morris, [23, Lemma 2.2], [13, Lemma 4.2])**.** *Let $m \in \mathbb{Z}^+$, and let $X_1 = \mathrm{Cay}(G_1, S_1)$ and $X_2 = \mathrm{Cay}(G_2, S_2)$ be Cayley graphs, such that*

1. *G_1 and G_2 are abelian groups (written additively), and*
2. *for $j \in \{1, 2\}$, we have $ms \neq mt$ for all $s, t \in S_j$, such that $s \neq t$.*

If φ is any isomorphism from X_1 to X_2, then φ is also an isomorphism of graphs $\mathrm{Cay}(G_1, mS_1)$ and $\mathrm{Cay}(G_2, mS_2)$, where $mS_j = \{\, ms \mid s \in S_j \,\}$.

Proof. Write $m = p_1 p_2 \dots p_r$ as a product of primes. Define $m_i := p_1 \dots p_i$ with $0 \leq i \leq r$. We will prove by induction on i that φ is an isomorphism from $\mathrm{Cay}(G_1, m_i S_1)$ to $\mathrm{Cay}(G_2, m_i S_2)$.

As $m_0 = 1$ and $m_0 S_j = S_j$ for $j \in \{1, 2\}$, the claim holds by assumption.

Given $x, y \in G_j$, let $\mathcal{W}_i(x, y)$ be the set of all walks of length p_i from x to y in $\mathrm{Cay}(G_j, m_{i-1} S_j)$. We will define $\#_i(x, y)$ to be the number of such walks, that is, $\#_i(x, y) = |\mathcal{W}_i(x, y)|$.

As every edge of a Cayley graph is induced by an element of its connection set, it follows that the elements of $\mathcal{W}_i(x,y)$ can be identified with p_i-tuples $(s_1,\ldots,s_{p_i})$ of elements $s_i \in m_{i-1}S_j$ satisfying $s_1+\ldots+s_{p_i} = y-x$. However, since G_j is abelian, cyclic rotations of the entries of such a p_i-tuple do not affect the total sum. In particular, any cyclic rotation of $(s_1,\ldots,s_{p_i})$ also corresponds to a walk of length p_i from x to y in $\mathrm{Cay}(G_j, m_{i-1}S_j)$.

It follows that the cyclic group $\mathbb{Z}_p \cong \langle(1,2,\ldots,p_i)\rangle \in S_{p_i}$ has an action on $\mathcal{W}_i(x,y)$, understood as the set of p_i-tuples, by permuting the indices of the entries of p_i-tuples.

Given $t \in \mathcal{W}_i(x,y)$, it follows by Lemma 2.6 that $p = |\mathbb{Z}_p| = |t^{\mathbb{Z}_p}||(\mathbb{Z}_p)_t|$.

In particular, the orbit of t has size p, unless t is fixed by every cyclic permutation in $\mathbb{Z}_p$. In this case, $t = (s,\ldots,s)$ for some $s \in m_{i-1}S_j$ and $y = x + p_i s$. Note that since $s \in m_{i-1}S_j$, then $p_i s \in p_i m_{i-1}S_j = m_i S_j$ and x and y are adjacent in $\mathrm{Cay}(G_j, m_i S_j)$. Moreover, note that if $s' \in m_i S_j$ had the same property, it would follow that $p_i s = p_i s' = y - x$ and consequently, $ms = ms'$. This contradicts our second assumption, unless $s = s'$.

Hence, we have proven that $\mathcal{W}_i(x,y)$ contains an element whose orbit is of size 1 if and only if $\{x,y\} \in E(\mathrm{Cay}(G_j, m_i S_j))$. Moreover, if such an element exists, it is unique. It follows that

$$\#_i(x,y) \not\equiv 0 \pmod{p_i} \iff \{x,y\} \in E(\mathrm{Cay}(G_j, m_i S_j)).$$

By the inductive hypothesis, it holds that φ is an isomorphism of $\mathrm{Cay}(G_1, m_{i-1}S_1)$ and $\mathrm{Cay}(G_2, m_{i-1}S_2)$. Then φ maps the elements of $\mathcal{W}_i(x,y)$ onto the elements of $\mathcal{W}_i(\varphi(x),\varphi(y))$. As φ is bijective, we get that it also preserves the value of the function $\#_i(\cdot,\cdot)$ i.e., we have that

$$\#_i(\varphi(x),\varphi(y)) = \#_i(x,y), \forall x,y \in G_1.$$

Putting everything together, we obtain that

$$\begin{aligned}
\{\varphi(x),\varphi(y)\} \in E(\mathrm{Cay}(G_2, m_i S_2)) &\iff \#_i(\varphi(x),\varphi(y)) \not\equiv 0 \pmod{p_i} \\
&\iff \#_i(x,y) \not\equiv 0 \pmod{p_i} \\
&\iff \{x,y\} \in E(\mathrm{Cay}(G_1, m_i S_1)).
\end{aligned}$$

Hence, φ is an isomorphism between $\mathrm{Cay}(G_1, m_i S_1)$ and $\mathrm{Cay}(G_2, m_i S_2)$, proving the desired. □

Lemma 6.10 (Hujdurović-Mitrović-Morris [13, Corrolary 4.3]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph of even order, let φ be an automorphism of BX, and let*

$$S' = \{\, s \in S \mid s + (n/2) \notin S \,\}.$$

Then φ is an automorphism of $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, 2S' \times \{0\})$.

Proof. We apply a similar idea to that used in the proof of Lemma 6.9. Note that by Lemma 3.13, we know that

$$BX = B\,\mathrm{Cay}(\mathbb{Z}_n, S) = \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S \times \{1\}).$$

For $x, y \in V(BX)$, let $\#(x, y)$ be the number of walks of length 2 from x to y in BX. These walks are in one-to-one correspondence with the elements of

$$\mathcal{W}_{x,y} \coloneqq \{\, (s_1, s_2) \in S \times S \mid (s_1, 1) + (s_2, 1) = y - x \,\},$$

so

$$\#(x, y) = |\mathcal{W}_{x,y}|.$$

Since $\mathbb{Z}_n \times \mathbb{Z}_2$ is abelian, the map $\pi\colon (s_1, s_2) \mapsto (s_2, s_1)$ is a permutation of $\mathcal{W}$. Let

$$\mathcal{F}_{x,y} \coloneqq \{\, (s, s) \mid s \in S,\ 2(s, 1) = y - x \,\}$$

be the set of fixed points of π in its action on $\mathcal{W}_{x,y}$.

Since the cardinality of every orbit of π is either 1 or 2, we see that

$$|\mathcal{W}_{x,y}| \equiv |\mathcal{F}_{x,y}| \pmod 2.$$

We claim that $|\mathcal{F}_{x,y}|$ is either 0, 1, or 2. (So $|\mathcal{F}_{x,y}|$ is odd if and only if $|\mathcal{F}_{x,y}| = 1$.) To see this, suppose (s, s) and (t, t) are two different elements of $\mathcal{F}_{x,y}$. This means that $s \neq t$ and $2(s, 1) = y - x = 2(t, 1)$, so $2s = 2t$. Then $t - s$ must be an element of order 2, so $t = s + n/2$ (because $n/2$ is the only element of order 2 in $\mathbb{Z}_n$). This completes the proof of the claim. Furthermore, the argument establishes that

$$|\mathcal{F}_{x,y}| = 1 \iff y - x \in 2(S' \times \{1\}).$$

So

$$|\mathcal{F}_{x,y}| \text{ is odd} \quad \iff \quad y - x \in 2(S' \times \{1\}).$$

Combining the above facts implies that

$$\#(x, y) \text{ is odd} \quad \iff \quad x - y \in 2(S' \times \{1\}).$$

Since every automorphism of BX must preserve the value of $\#(x, y)$, and $2(S' \times \{1\}) = 2S' \times \{0\}$, this implies that every automorphism of BX is an automorphism of $\mathrm{Cay}\big(\mathbb{Z}_n \times \mathbb{Z}_2,\ 2S' \times \{0\}\big)$. □

Theorem 6.11 (Morris, [23, Theorem 1.1])**.** *Every connected, twin-free Cayley graph of an abelian group of odd order is stable.*

Proof. Let G be an abelian group of odd order. Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph of the group G. Assume that X is connected and twin-free.

As X is of odd order $|G|$, it follows by Corollary 2.27 that it is non-bipartite.

By Lemma 3.13, we know that

$$BX = \mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{1\}).$$

As X is connected and non-bipartite, Lemma 3.2(1) implies that BX is connected. Then Lemma 2.13 shows that automorphisms of BX either preserves or reverses the colour classes $G \times \{i\}, i \in \{0,1\}$ of BX. Hence, if we let $\varphi \in \mathrm{Aut}(BX)$, after potentially multiplying φ by τ from $\mathrm{Aut}(X) \times S_2$, we may assume that $\varphi(x,i) \in G \times \{i\}$ for all $x \in G, i \in \{0,1\}$.

Let now $m := |G| + 1$. Then m is even and $m \equiv 1 \pmod{|G|}$. It follows that $m(s,1) = (s,0)$, so in particular $m(s,1) \neq m(t,1)$ for $s,t \in S, s \neq t$. We can now apply Lemma 6.9 with $G_1 = G_2 = G$ and $S_1 = S_2 = S$, to conclude that

$$\varphi \in \mathrm{Aut}(\mathrm{Cay}(G \times \mathbb{Z}_2, m(S \times \{1\}))) = \mathrm{Aut}(\mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{0\})).$$

Note that the graph $\mathrm{Cay}(G \times \mathbb{Z}_2, S \times \{0\})$ consists of two disjoint copies of X with vertex set $G \times \{0\}$ and $G \times \{1\}$. As $\varphi(G \times \{i\}) = G \times \{i\}$, we conclude that φ restricts to automorphisms of these copies of X. By multiplying φ by the inverse of the lift of the automorphism induced on the copy of X with the vertex set $G \times \{0\}$, we can assume $\varphi(v) = v$ for all $v \in G \times \{0\}$.

Let $x \in G$ be arbitrary and let $y \in G$ be such that $\varphi(x,1) = (y,1)$. We have the following.

$$\begin{aligned}
N_X(x) \times \{0\} &= \varphi(N_X(x) \times \{0\}) && (\varphi(v) = v \text{ for all } v \in G \times \{0\}) \\
&= \varphi(N_{BX}(x,1)) && (\text{definition of } BX) \\
&= N_{BX}(\varphi(x,1)) && (\varphi \in \mathrm{Aut}(BX)) \\
&= N_{BX}(y,1) && (\varphi(x,1) = (y,1)) \\
&= N_X(y) \times \{0\} && (\text{definition of } BX)
\end{aligned}$$

From here, we obtain that $N_X(x) = N_X(y)$. However, X has been assumed to be twin-free, meaning that the above is possible if and only if $x = y$. We conclude that $\varphi(x,1) = (y,1) = (x,1)$ for all $x \in V(X)$ and φ is the identity on $V(BX)$.

Hence, after multiplying φ by elements of $\mathrm{Aut}(X) \times S_2$, we obtained the identity automorphism of BX. It follows that $\varphi \in \mathrm{Aut}(X) \times S_2$ to begin with. In conclusion, $\mathrm{Aut}(BX) = \mathrm{Aut}(X) \times S_2$, proving that X is stable. □

Remark 6.12 ([23, Example 1.3])**.** *Theorem* 6.11 *does not generalize to non-abelian groups of odd order as the following example shows.*

Let G denote the non-abelian group of order 21 with presentation

$$G = \langle a, x \mid a^3 = x^7 = 1, a^{-1}xa = x^2 \rangle.$$

Let $X \coloneqq \mathrm{Cay}(G, S)$ be the Cayley graph of G with $S \coloneqq \{a^{\pm 1}, x^{\pm 1}, (ax)^{\pm 1}\}$. As S contains generators a and x, it generates G, so X is connected by Proposition 2.25(5). As it is of odd order, X is non-bipartite by Corollary 2.27. Finally, MAGMA calculations show that X is twin-free, $|\mathrm{Aut}(X)| = 42$ and $|\mathrm{Aut}(BX)| = 252$.

It follows that X is non-trivially unstable.

Theorem 6.11 can be used to derive the automorphism group of the double cover of an arbitrary Cayley graph of an abelian group of odd order. To achieve this we need the already mentioned *wreath product* of groups, denoted by $H \wr K$ for two groups H, K (see [3, p. 46] for more details).

Proposition 6.13 ([23, Remark 1.4], [7, Remark 1.3])**.** *Let X be a Cayley graph on an abelian group of odd order. Then there exist a connected, twin-free Cayley graph Y of an abelian group of odd order and integers $c, d \geq 1$ that satisfy the following conditions:*

1. *X is connected if and only if $c = 1$,*
2. *X is twin-free if and only if $d = 1$,*
3. *$X \cong \overline{K_c} \wr (Y \wr \overline{K_d})$,*
4. *$BX \cong \overline{K_c} \wr (BY \wr \overline{K_d})$.*

Moreover, it holds that

$$\mathrm{Aut}(BX) \cong S_c \wr ((\mathrm{Aut}(Y) \times S_2) \wr S_d).$$

Proof. Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on an abelian group G of odd order. Let X have $c \geq 1$ connected components. By definition, X is connected if and only if $c = 1$, so (1) is satisfied. As X is vertex-transitive by Proposition 2.25(2), all of its connected components are isomorphic. Let X_0 be the connected component of X containing the identity of G. Then $X \cong cX_0 \cong \overline{K_c} \wr X_0$. Moreover, X_0 is the graph $\mathrm{Cay}(H, S)$ with $H \coloneqq \langle S \rangle$. Note that the group H is abelian and of odd order.

Using the proof of Lemma 2.45, we conclude that there exists a subgroup N of H, such that we can decompose X_0 as $Y \wr \overline{K_d}$, where Y is a connected, twin-free Cayley graph on the quotient group H/N and d is the size of each equivalence class of twins of X_0. Note that then $d = 1$ if and only if X_0 is twin-free (when this decomposition is trivial), which is equivalent to X being twin-free, so the condition (2) is satisfied.

Since H is an abelian group of odd order, the group H/N is also an abelian group of odd order.

Hence, we have the following decomposition of X.

$$X \cong \overline{K_c} \wr (Y \wr \overline{K_d}).$$

Here, Y is a connected, twin-free Cayley graph on an abelian group of odd order. Hence, (3) holds.

From here, we obtain (4) by a direct computation.

$$BX = X \times K_2 \cong \overline{K_c} \wr ((Y \times K_2) \wr \overline{K_d}) = \overline{K_c} \wr (BY \wr \overline{K_d}).$$

Note that Y is stable by Theorem 6.11, so $\operatorname{Aut}(BY) \cong \operatorname{Aut}(Y) \times S_2$. By a well-known theorem of Sabidussi [28] on the automorphism group of the wreath product of graphs, we obtain that

$$\operatorname{Aut} BX \cong S_c \wr ((\operatorname{Aut} BY) \wr S_d) \cong S_c \wr ((\operatorname{Aut}(Y) \times S_2) \wr S_d).$$

This finishes the proof. □

The following well known theorem describes the automorphism group of a direct product of two non-bipartite graphs satisfying mild regularity conditions.

Theorem 6.14 (Dörfler [10, Theorem 8.18, p. 103])**.** *Let X and Y be connected, non-bipartite, twin-free graphs of relatively prime orders. Then*

$$\operatorname{Aut}(X \times Y) = \operatorname{Aut} X \times \operatorname{Aut} Y.$$

The case when at least one of the factors is bipartite is much more complicated. However, Theorem 6.11 can be used to calculate the automorphism group of this type of a direct product in the case of Cayley graphs.

Using the facts on Cartesian skeletons listed in Lemma 2.51, we first prove a general lemma about automorphism groups of direct products of graphs, where exactly one of the factors is bipartite.

Proposition 6.15 (Morris [23, Proposition 5.6])**.** *Let X and Y be twin-free, connected graphs that have at least one edge, such that:*

1. *X is not bipartite,*
2. *Y is bipartite, with bipartition $V(Y) = Y_0 \cup Y_1$, such that*
   - *(a) $|Y_0|$ and $|Y_1|$ are relatively prime to $|V(X)|$, and*
   - *(b) either Y has an automorphism that interchanges Y_0 and Y_1, or $|Y_0| \neq |Y_1|$, and*
3. $\operatorname{Aut} BX = \operatorname{Aut} X \times S_2$ *(that is, X is stable).*

Then $\mathrm{Aut}(X \times Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$.

Proof. Let $\varphi \in \mathrm{Aut}(X \times Y)$. As $\mathrm{Aut}(X) \times \mathrm{Aut}(Y) \leq \mathrm{Aut}(X \times Y)$, it suffices to show that $\varphi \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$.

As both X and Y are connected, we can apply statements from Lemma 2.51. As Y is bipartite, Lemma 2.51(5) implies that $\mathcal{S}(Y)$ has two connected components C_0 and C_1 such that $V(C_i) = Y_i$ for $i \in \{0, 1\}$.

By Lemma 2.51(1), it follows that $\varphi \in \mathrm{Aut}(\mathcal{S}(X \times Y))$. Lemma 2.51(2) implies that $\mathcal{S}(X \times Y) = \mathcal{S}(X) \square \mathcal{S}(Y)$. Putting everything together, we conclude that

$$\varphi \in \mathrm{Aut}(\mathcal{S}(X \times Y)) = \mathrm{Aut}(\mathcal{S}(X) \square \mathcal{S}(Y)) = \mathrm{Aut}(\mathcal{S}(X) \square (C_0 \cup C_1)).$$

Note that $\mathcal{S}(X)$ is connected by Lemma 2.51(4) as X is non-bipartite. It follows that connected components of $\mathcal{S}(X) \square \mathcal{S}(Y)) = \mathcal{S}(X) \square (C_0 \cup C_1)$ are exactly $\mathcal{S}(X) \square C_i$ for $i \in \{0, 1\}$.

If $|V(C_0)| = |Y_0| \neq |Y_1| = |V(C_1)|$, then components $\mathcal{S}(X) \square C_0$ and $\mathcal{S}(X) \square C_1$ are of different orders. If $|Y_0| = |Y_1|$, then by (2b), Y has an automorphism that interchanges C_0 and C_1. This induces an automorphism of $X \times Y$ that swaps $\mathcal{S}(X) \square C_0$ and $\mathcal{S}(X) \square C_1$.

In either case, we can assume that φ fixes $\mathcal{S}(X) \square C_i$ set-wise for $i \in \{0, 1\}$. Hence, φ restricts to an automorphism φ_i of $\mathcal{S}(X) \square C_i$ (for $i \in \{0, 1\}$).

Assumption (2a) and the fact that $|V(\mathcal{S}(X))| = |V(X)|$ allow us to apply Lemma 2.51(3) to conclude that for $i \in \{0, 1\}$

$$\mathrm{Aut}(\mathcal{S}(X) \square C_i) = \mathrm{Aut}(\mathcal{S}(X)) \times \mathrm{Aut}(C_i).$$

From here, we obtain that for $i \in \{0, 1\}$ there exist $\alpha_i \in \mathrm{Sym}(V(X))$ and $\beta_i \in \mathrm{Sym}(Y_i)$ such that

$$\varphi(x, y) = \varphi_i(x, y) = (\alpha_i(x), \beta_i(y)), x \in V(X), y \in Y_i.$$

By assumption, Y has at least one edge so there exist $y_i \in Y_i$ for $i \in \{0, 1\}$ such that $\{y_0, y_1\} \in E(Y)$.

Let $x_0, x_1 \in V(X)$. Then

$$\begin{aligned} \{x_0, x_1\} \in E(X) &\implies \{(x_0, y_0), (x_1, y_1)\} \in E(X \times Y) \\ &\implies \{\varphi(x_0, y_0), \varphi(x_1, y_1)\} \in E(X \times Y) \\ &\implies \{(\alpha_0(x_0), \beta_0(y_0)), (\alpha_1(x_1), \beta_1(y_1))\} \in E(X \times Y) \\ &\implies \{\alpha_0(x_0), \alpha_1(x_1)\} \in E(X). \end{aligned} \tag{6.1}$$

Conversely, assume that $\{\alpha_0(x_0), \alpha_1(x_1)\} \in E(X)$. Let n be the least common multiple of orders of α_0 and α_1 as permutations of $V(X)$. Applying Eq. (6.1) $n - 1$ times to the edge $\{\alpha_0(x_0), \alpha_1(x_1)\} \in E(X)$, we obtain that

$$\{x_0, x_1\} = \{\alpha_0^n(x_0), \alpha_1^n(x_1)\} \in E(X).$$

In particular, we have shown that

$$\{x_0, x_1\} \in E(X) \iff \{\alpha_0(x_0), \alpha_1(x_1)\} \in E(X).$$

As X is connected, non-bipartite and stable (by assumption (3)), it follows from Lemma 3.9 that $\alpha_0 = \alpha_1 = \alpha$. Note that then $\alpha \in \mathrm{Aut}(X)$.

As X also contains an edge, we can apply an analogous argument to Y. It follows that $\beta_1 = \beta_2 = \beta \in \mathrm{Aut}(Y)$. In particular, $\varphi = (\alpha, \beta) \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$, as desired. □

We can now derive the following result for the case when one of the factors of the direct product is a Cayley graph of a finite abelian group of odd order.

Corollary 6.16 (Morris [23, Corollary 1.5])**.** *Let X be a twin-free, connected Cayley graph on a finite abelian group of odd order, and let Y be any twin-free, connected graph, such that either:*

1. *Y is not bipartite, and $|V(Y)|$ is relatively prime to $|V(X)|$, or*

2. *Y is bipartite, with bipartition $V(Y) = Y_0 \cup Y_1$, such that*

   (a) *$|Y_0|$ and $|Y_1|$ are relatively prime to $|V(X)|$, and*

   (b) *either $|Y_0| = |Y_1|$, or Y has an automorphism that interchanges Y_0 and Y_1.*

Also assume that neither X nor Y is the one-vertex trivial graph. Then

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y.$$

Proof. Note that since neither X nor Y are trivial, but both are connected, each of them contains at least one edge. Moreover, Corollary 2.27 implies that X is non-bipartite.

(1) In this case, the conclusion follows by Theorem 6.14.

(2) Theorem 6.11 implies that X is stable. The conclusion follows by Proposition 6.15. □

Corollary 6.16 has an even nicer form when both factors are Cayley graphs of abelian groups.

Corollary 6.17 (Morris [23, Corollary 1.6])**.** *Let X and Y be twin-free, connected Cayley graphs on abelian groups, such that $|V(X)|$ is relatively prime to $|V(Y)|$. Also assume that neither X nor Y is the one-vertex trivial graph. Then*

$$\mathrm{Aut}(X \times Y) = \mathrm{Aut}\, X \times \mathrm{Aut}\, Y.$$

Proof. As $|V(X)|$ and $|V(Y)|$ are relatively prime, at least one of them is odd. Therefore, after possibly interchaning X and Y, we can assume that $|V(X)|$ is odd.

If Y is non-bipartite, we can apply Corollary 6.16(1).

We can now assume that Y is bipartite. As it is connected by assumption and vertex-transitive (by Proposition 2.25(2)), Lemma 2.13 implies that Y_0 and Y_1 are conjugate blocks for the action of $\mathrm{Aut}(Y)$. In particular, we have that

- $|Y_0| = |Y_1| = |V(Y)|/2$, so the condition (2a) of Corollary 6.16 is satisfied, since $|V(X)|$ and $|V(Y)|$ are assumed to be relatively prime.
- Y has an automorphism interchaning Y_0 and Y_1, so the condition (2b) of Corollary 6.16 is satisfied as well.

The desired conclusion now follows by Corollary 6.16(2). □

6.4 CIRCULANTS OF ORDER TWICE A PRIME

Theorem 6.18 (Hujdurović-Mitrović-Morris [13, Theorem 5.1]). *If p is a prime number, then every non-trivially unstable circulant graph of order $2p$ has Wilson type* (C.4).

Proof. Let $X = \mathrm{Cay}(\mathbb{Z}_{2p}, S)$ be a non-trivially unstable, circulant graph of order $n = 2p$. Note that X is connected, non-bipartite and twin-free.

If $p = 2$, X is of order 4. As it is non-bipartite, it must contain a 3-cycle. Because it is also connected and regular, it must be isomorphic to K_4 and consequently, it is stable by Example 4.7. As this is a contradiction, we conclude that there are no non-trivially unstable circulants of order 4, so p is odd and $n = 2p$ is square-free.

Let
$$S' := S \setminus (S + p) = \{s \in S | s + p \notin S\}.$$
Since X is twin-free, we know that $S + p \neq S$, which means that $2S'$ is non-empty.

Case 1. $2S' \neq \{0\}$.

Since $2\mathbb{Z}_{2p}$ has order p, which is prime, every non-zero element is a generator. It follows that $2S'$ generates $2\mathbb{Z}_{2p}$. We also know from Lemma 6.10 that every automorphism of BX is an automorphism of
$$\mathrm{Cay}\big(\mathbb{Z}_{2p} \times \mathbb{Z}_2, 2S' \times \{0\}\big).$$

As $\langle 2S' \times \{0\}\rangle = 2\mathbb{Z}_{2p} \times \{0\}$ is a connected component of this graph, we conclude that it is a block for the action of $\mathrm{Aut}\, BX$. Therefore, [13, Corollary 5.6(3)] applies and X has Wilson type (C.1) or (C.4).

Assume that X has Wilson type (C.1). Then we can find a non-zero $a \in 2\mathbb{Z}_{2p}$ such that $a + S_e = S_e$. As $2\mathbb{Z}_{2p}$ is of order p, the element a generates it. Note that S_e is non-empty (otherwise, X would be bipartite), so it follows that we can find an $s \in S_e$ such that $0 = ka + s \in S_e \subseteq S$, a contradiction.

Therefore, X must have Wilson type (C.4), which is exactly what we needed to prove.

Case 2. $2S' = \{0\}$.

This means that $S' = \{p\}$, so $p \in S$. Since X is unstable, we may let α be an automorphism of BX, such that $\alpha(0,1) = (t,1)$ with $t \neq 0$. From [13, Corollary 4.7], we obtain that for $x \in \mathbb{Z}_n$, $|x| = |t|$ implies that $x \notin S$. Since we already know that $p \in S$, it follows that $|t| \neq 2$. So $|t|$ is either p or $2p$. Therefore, either S does not contain any element of order p, or S does not contain any element of order $2p$.

However, since $2S' = \{0\}$, we also know that $s + p \in S$ for all $s \in S \setminus \{p\}$. Also note that

$$|s| = p \iff |s+p| = 2p.$$

This implies that S contains an element of order p if and only if it contains an element of order $2p$.

The only possibility is that $S = \{p\}$. This contradicts the fact that the non-trivially unstable graph X must be connected. □

We can now characterize non-trivially unstable circulants of order $2p$.

Corollary 6.19 (Hujdurović-Mitrović-Morris [13, Corollary 5.7])**.** *Let $X = \mathrm{Cay}(\mathbb{Z}_{2p}, S)$ be a circulant graph of order $2p$, where p is an odd prime, and let $S_e = S \cap 2\mathbb{Z}_{2p}$. The graph X is unstable if and only if either it is trivially unstable, or there exists $m \in \mathbb{Z}_{2p}^{\times}$, such that $m^2 S_e = S_e$, $mS_e \neq S_e$, and $S = S_e \cup \big((n/2) + mS_e\big)$.*

Proof. ($\Leftarrow$) Assume that X is not trivially unstable. Let $m \in \mathbb{Z}_{2p}^{\times}$ an element satisfying above conditions. Note that $m(n/2) = (n/2)$ as $(n/2)$ is the unique element of order 2 in $\mathbb{Z}_{2p}$.

$$\begin{aligned}(n/2) + mS &= (n/2) + m\big(S_e \cup \big((n/2) + mS_e\big)\big)\\ &= (n/2) + \big(mS_e \cup \big((n/2) + m^2 S_e\big)\big)\\ &= \big((n/2) + mS_e\big)\big) \cup S_e = S\end{aligned}$$

It follows that X has Wilson type (C.4).

($\Rightarrow$) Assume X is non-trivially unstable. We conclude from Theorem 6.18 that X has Wilson type (C.4), so there is some $m \in \mathbb{Z}_{2p}$, such that $S = mS + p$. As both m and p are odd, this implies that $mS_e + p = S \setminus S_e = S_o$ and $mS_o + p = S_e$. This shows that

$$S = S_e \cup S_o = S_e \cup (p + mS_e).$$

Moreover, as $mp = p$, we obtain that

$$m^2 S_e = m(S_o + p) = mS_o + p = S_e.$$

If $mS_e = S_e$, then $S_o = S_e + p$, so $S = S + p$, which contradicts the fact that X is non-trivially unstable and, in particular, twin-free. Hence, $mS_e \neq S_e$, finishing the proof. □

Corollary 6.20 (Hujdurović-Mitrović-Morris [13, Corollary 5.8]). *For $n \in \mathbb{Z}^+$, there does **not** exist a non-trivially unstable circulant graph of order n if and only if either n is odd, or $n < 8$, or $n = 2p$, for some prime number $p \equiv 3 \pmod 4$.*

Proof. ($\Rightarrow$) If $n/2$ is not prime, then $2\mathbb{Z}_n \cong \mathbb{Z}_{n/2}$ has a non-trivial, proper subgroup A. Choose some $b \in 2\mathbb{Z}_n \setminus A$, and let $S = \{\pm 1\} \cup (\pm b + A)$, so $S_e := S \cap 2\mathbb{Z}_n = \pm b + A$. Then $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ has Wilson type (C.1), so it is unstable.

If $n/2$ is prime, and $n/2 \not\equiv 3 \pmod 4$ (and $n \geq 8$), then $n/2 \equiv 1 \pmod 4$, so there exists $m \in \mathbb{Z}_n^\times$, such that $m^2 = -1$. Let $S = \{\pm 1, n/2 \pm m\}$, so $S_e := S \cap 2\mathbb{Z}_n = \{\pm m + n/2\}$ and $S = mS + n/2$. Then $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ has Wilson type (C.4), so it is unstable.

In either case, X is also connected (because $1 \in S$) and non-bipartite (because $S_e \neq \emptyset$). Hence, if X is not non-trivially unstable, then it must not be twin-free, so there is a nonzero $h \in \mathbb{Z}_n$, such that $h + S = S$. Note that in both cases, $S_o = \{\pm 1\}$ and since $n > 4$ and h is nonzero, it cannot happen that $\{\pm 1\} + h = \{\pm 1\}$. It follows that $S_o + h = S_e$ and $S_e + h = S_o$ (and h is odd).

Since $S_o + 2h = S_o$ (and $S_0 = \{\pm 1\}$), we must have $2h = 0$, which means $h = n/2$, so $S_e = S_o + n/2 = \{n/2 \pm 1\}$.

If $n/2$ is not prime, then, since $\{n/2 \pm 1\} = S_e = \pm b + A$, we must have $\langle 2 \rangle \subseteq A$. Since $n > 4$, this implies $|b + A| = |A| \geq n/2 > 2$, which is a contradiction.

If $n/2$ is prime, we must have $m = \pm 1$ (since $S_e = \{n/2 \pm m\}$, which contradicts the fact that $m^2 = -1$.

($\Leftarrow$) We prove the contrapositive: supposing there does exist a non-trivially unstable circulant graph of order n, we will show that n is odd, that $n \geq 8$, and that $n/2$ is *not* a prime number that is congruent to $3 \pmod 4$.

The fact that n is odd is immediate from Theorem 6.8. Also, it is easy to see, by inspection, that there are no non-trivially unstable circulant graphs of order 2 or 4; so $n \geq 6$.

Now suppose $X = \mathrm{Cay}(\mathbb{Z}_{2p}, S)$ is a non-trivially unstable circulant graph of order $2p$, where p is prime, and $p \equiv 3 \pmod 4$. (This includes the case where $n = 6$.) We will show that this leads to a contradiction. By Theorem 6.18, we know that X has Wilson type (C.4), so there is some $m \in \mathbb{Z}_{2p}^\times$, such that $S = mS + n/2$. Write $m = m_o m_2$, where m_o has odd order (as an element of the group $\mathbb{Z}_{2p}^\times$), and the order of

m_2 is a power of 2. Since $\mathbb{Z}_{2p}^{\times}$ is cyclic of order $p-1 \equiv 2 \pmod 4$, there are no elements of order 4 in $\mathbb{Z}_{2p}^{\times}$, so $m_2 \in \{\pm 1\}$. Since $S = -S$, this implies $S = m_2 S$, so we conclude that $S = m_o S + n/2$. After repeatedly multiplying both sides of this equation by m_o, we see that $S = m_o^k S + n/2$ for any odd number k, including $k = |m_o|$. Hence, we have $S = S + n/2$. This contradicts the fact that X is twin-free. □

6.5 ARC-TRANSITIVE CIRCULANTS

We start with the following lemma that will be needed for the proof of the main result of this subsection (see Theorem 6.22), but it is an interesting result on its own.

Lemma 6.21 (Qin-Xia-Zhou [26, Lemma 4.3])**.** *Let X be a graph of order m. Let $d > 2$ such that* $\gcd(m, d) = 1$*. If $X \times K_d$ is non-trivially unstable, then X is non-trivially unstable.*

Proof. Let $Y \coloneqq X \times K_d$. As Y is non-trivially unstable, it follows that it is connected, non-bipartite and twin-free. As Y is connected, X must be connected. As Y is non-bipartite, it contains an odd cycle. The definition of the direct product (see Definition 2.16(1)) then implies that this produces an odd cycle in X, so X is non-bipartite. As both Y and K_d are twin-free, Lemma 2.43 implies that X is also twin-free.

As Y is connected, non-bipartite and unstable, Lemma 3.9 implies that there exist two distinct permutations $\alpha, \beta \in \mathrm{Sym}(V(Y))$ such that $\{y_1, y_2\} \in E(Y)$ if and only if $\{\alpha(y_1), \beta(y_2)\} \in E(Y)$. It follows by Lemma 2.52(3) that $\alpha, \beta \in \mathrm{Aut}(\mathcal{S}(Y))$. Note that as $d > 2$, it follows that $\mathcal{B}(K_d) = K_d$, because every two distinct vertices have a common neighbour. Moreover, no edge of K_d is dispensable and $\mathcal{S}(K_d) = K_d$. Applying Lemma 2.51(2) we conclude that

$$\mathcal{S}(Y) = \mathcal{S}(X \times K_d) \cong \mathcal{S}(X) \square \mathcal{S}(K_d) = \mathcal{S}(X) \square K_d.$$

As X is connected and non-bipartite, Lemma 2.51(4) implies that $\mathcal{S}(X)$ is connected. Note that

$$\gcd(|V(\mathcal{S}(X))|, d) = \gcd(|V(X)|, d) = \gcd(m, d) = 1.$$

We can then apply Lemma 2.51(3) to obtain that

$$\mathrm{Aut}(\mathcal{S}(Y)) = \mathrm{Aut}(\mathcal{S}(X) \square K_d) \cong \mathrm{Aut}(\mathcal{S}(X)) \times \mathrm{Aut}(K_d).$$

Consequently, since $\alpha, \beta \in \mathrm{Aut}(\mathcal{S}(Y))$, we can find $\alpha_1, \beta_1 \in \mathrm{Sym}(V(X))$ and $\alpha_2, \beta_2 \in \mathrm{Sym}(V(K_d))$ such that $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$.

Let $x, x' \in V(X)$. As $d \geq 3$, we can find distinct $i, j \in V(K_d)$. Then $\{i, j\} \in E(K_d)$. We have the following.

$$\begin{aligned}
\{x, x'\} \in E(X) &\implies \{(x, i), (x', j)\} \in E(X \times K_d) = E(Y) \\
&\implies \{\alpha(x, i), \beta(x', j)\} \in E(Y) \\
&\implies \{(\alpha_1(x), \alpha_2(i)), (\beta_1(x'), \beta_2(j)\} \in E(Y) \\
&\implies \{\alpha_1(x), \beta_1(x')\} \in E(X).
\end{aligned}$$

As $d \geq 3$, we can always find $i, j \in V(K_d)$ such that $\alpha_2(i) \neq \beta_2(j)$. Then $\{\alpha_2(i), \beta_2(j)\} \in E(K_d)$ and we have the following

$$\begin{aligned}
\{\alpha_1(x), \beta_1(x')\} \in E(X) &\implies \{(\alpha_1(x), \alpha_2(i)), (\beta_1(x'), \beta_2(j))\} \in E(Y) \\
&\implies \{\alpha(x, i), \beta(x', j)\} \in E(Y) \\
&\implies \{(x, i), (x', j)\} \in E(Y) \\
&\implies \{x, x'\} \in E(X).
\end{aligned}$$

Therefore, we have proven that

$$\{x, x'\} \in E(X) \iff \{\alpha_1(x), \beta_1(x')\} \in E(X). \tag{6.2}$$

Note that X is not K_1 (as the assumptions would then imply that $X \times K_d \cong K_d$ is unstable with $d \geq 3$, a contradiction with Example 4.7). Therefore, X has at least two vertices and as it is connected, it must contain an edge. Fixing $\{x, x'\} \in E(X)$, lets us repeat the previous argument for $i, j \in V(K_d)$. We conclude that

$$\{i, j\} \in E(K_d) \iff \{\alpha_2(i), \beta_2(j)\} \in E(K_d). \tag{6.3}$$

If $\alpha_2 \neq \beta_2$, then Eq. (6.3) and Lemma 3.9 would imply that K_d is unstable. As $d \geq 3$, this is a contradiction with Example 4.7. It follows that $\alpha_2 = \beta_2$. As $\alpha \neq \beta$ by assumption, it must hold that $\alpha_1 \neq \beta_1$. Then Eq. (6.2) and Lemma 3.9 imply that X is unstable.

As we have already shown that X is connected, non-bipartite and twin-free, we conclude that it is non-trivially unstable, as desired. □

We are now ready to prove the main result of this subsection. The main ingredient of this proof is the Theorem 2.35 that classifies arc-transitive circulants.

Theorem 6.22 (Qin-Xia-Zhou [26, Theorem 1.6])**.** *There is no arc-transitive non-trivially unstable circulant. In other words, a connected arc-transitive circulant is stable if and only if it is non-bipartite and twin-free.*

Proof. Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a connected circulant graph.

($\Rightarrow$) Assume that X is stable. Then it must be non-bipartite by Proposition 3.16 and twin-free by Proposition 3.20.

($\Leftarrow$) Suppose that X is arc-transitive and non-trivially unstable and choose is it so it is a circulant of minimal possible order with these properties. Then it is connected, non-bipartite and twin-free. As it is arc-transitive, Theorem 2.35 implies that we have the following cases.

Case 1. *X is complete.*

By Example 4.7, the only unstable graph among the complete graphs is K_2, hence $X \cong K_2$. This is a contradiction with X being non-bipartite.

Case 2. *$X \cong Y \wr \overline{K_d}$ where $n = md$, $d > 1$ and Y is a connected arc-transitive circulant of order m.*

As $d > 2$, Lemma 2.44 implies that X is not twin-free, a contradiction.

Case 3. *$X \cong Y \wr_d \overline{K_d} \cong Y \wr \overline{K_d} - dY$, where $n = md$, $d > 3$, $\gcd(m, d) = 1$ and Y is a connected arc-transitive circulant of order m.*

By Lemma 2.20, $X \cong Y \wr_d \overline{K_d} \cong Y \times K_d$. Then as X is non-trivially unstable and $d \geq 4$, Lemma 6.21 implies that Y is also non-trivially unstable. However, then Y is a non-trivially unstable, arc-transitive circulant of order strictly smaller than X, which is a contradiction with minimality of X.

Case 4. *X is a normal Cayley graph.*

As X is normal, Proposition 2.31(3) implies that $\mathrm{Aut}(X)_0 = \mathrm{Aut}(\mathbb{Z}_n, S)$. Moreover, as X is also arc-transitive, it follows by Lemma 2.11 that this group is transitive on S.

Subcase 4.1. *n is even.*

Every automorphism of $\mathbb{Z}_n$ is given by multiplication by an integer coprime to n. As n is even, all of them are odd. If S contained only even integers, $\langle S \rangle \leq 2\mathbb{Z}_n$ would be a proper subgroup of $\mathbb{Z}_n$. By Proposition 2.25(5), this is a contradiction with X being connected. Hence, S contains an odd integer. But then all elements of S are odd by transitivity of $\mathrm{Aut}(\mathbb{Z}_n, S)$ on S. It follows that X is bipartite (with bipartition sets $2\mathbb{Z}_n$ and $1 + 2\mathbb{Z}_n$), a contradiction.

Subcase 4.2. *n is odd.*

As X is arc-transitive, it follows by Corollary 3.6(3) that BX is also arc-transitive. As X is a circulant, by applying Lemma 3.2(1) we obtain that

$$BX = B\,\mathrm{Cay}(\mathbb{Z}_n, S) = \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S \times \{1\}).$$

Moreover, as n is odd, there is a group isomorphism $\mathbb{Z}_n \times \mathbb{Z}_2 \cong \mathbb{Z}_{2n}$. From here, we conclude that BX is isomorphic to a circulant of order $2n$.

In conclusion, BX is an arc-transitive circulant, so Theorem 2.35 applies.

Subsubcase 4.2.1. *BX is complete.*

As BX is bipartite and of even order, it follows that $BX \cong K_2$. But then $2n = |V(BX)| = 2$ and therefore, $n = 1$, so $X \cong K_1$. This is a contradiction with X being unstable (see Example 4.7).

Subsubcase 4.2.2. *BX is normal.*

In this case, Lemma 4.11 implies that X is stable, a contradiction.

Subsubcase 4.2.3. *$BX \cong Y \wr \overline{K_d}$ with $2n = md, d > 1$ and Y is a connected arc-transitive circulant of order m.*

It follows by Lemma 2.44 that BX is not twin-free. However, then Corollary 3.21 implies that X is not twin-free, a contradiction.

Subsubcase 4.2.4. *$BX \cong Y \wr \overline{K_d} - dY \cong Y \wr_d \overline{K_d}$, where $2n = md, d > 3, \gcd(m, d) = 1$ and Y is a connected arc-transitive circulant of order m.*

By Lemma 2.20, we know that $BX \cong Y \times K_d$. As BX is bipartite and K_d is not (note that $d > 3$), it follows that Y has to be bipartite. As Y is a circulant (so in particular, a Cayley graph), Corollary 2.27 implies that $m = |V(Y)|$ is even.

Write $Y = \mathrm{Cay}(\mathbb{Z}_m, S_1)$ and $K_d = \mathrm{Cay}(\mathbb{Z}_d, S_2)$ with $S_2 = \mathbb{Z}_d \setminus \{0\}$. Generalizing Lemma 3.13, we get that

$$BX = Y \times K_d = \mathrm{Cay}(\mathbb{Z}_m, S_1) \times \mathrm{Cay}(\mathbb{Z}_d, S_2) = \mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_d, S_1 \times S_2). \tag{6.4}$$

As n is odd, it is easy to check that the map $(x, y) \mapsto (1-n)x + ny$ is a group isomorphism between $\mathbb{Z}_n \times \mathbb{Z}_2$ and $\mathbb{Z}_{2n}$. Moreover, as $2n = md$ and $\gcd(m, d) = 1$, the map $t \mapsto (t \pmod m, t \pmod d)$ is a group isomorphism between $\mathbb{Z}_{2n}$ and $\mathbb{Z}_m \times \mathbb{Z}_d$. By composing these two maps, we obtain the following group isomorphism.

$$\varphi \colon \mathbb{Z}_n \times \mathbb{Z}_2 \to \mathbb{Z}_m \times \mathbb{Z}_d$$

$$(x, y) \mapsto ((1-n)x + ny \pmod m, x \pmod d)$$

We have used the fact that $d = \frac{2n}{m}$ divides n to simplify the second coordinate, which follows from the fact that m is even.

The group isomorphism φ induces a graph isomorphism.

$$BX = \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S \times \{1\}) \cong \mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_d, \varphi(S \times \{1\})). \tag{6.5}$$

It then follow from Eq. (6.4) and Eq. (6.4) that

$$BX \cong \mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_d, S_1 \times S_2) \cong \mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_d, \varphi(S \times \{1\})).$$

As BX is an arc-transitive circulant, it has the *Cayley Isomorphism property* by a result from [19, Defn. 3.1]. This means that we can find a group automorphism $\sigma = (\sigma_1, \sigma_2) \in \mathrm{Aut}(\mathbb{Z}_m \times \mathbb{Z}_d) \cong \mathrm{Aut}(\mathbb{Z}_m) \times \mathrm{Aut}(\mathbb{Z}_d)$ such that

$$\varphi(S \times \{1\}) = \sigma(S_1 \times S_2) = \sigma_1(S_1) \times \sigma_2(S_2) = \sigma_1(S_1) \times S_2. \tag{6.6}$$

In the last step, we have used the fact that every automorphism of $\mathbb{Z}_d$ fixes 0 and consequently preserves $S_2 = \mathbb{Z}_d \setminus \{0\}$.

Define

$$T \coloneqq \{t \pmod{m/2} \mid t \in \sigma_1(S_1)\} \subseteq \mathbb{Z}_{m/2}.$$

As m is even, we can write $n = \frac{md}{2} = \frac{m}{2}d$. Moreover, $\gcd(m, d) = 1$ implies that $\gcd(m/2, d) = 1$, so we have another group isomorphism

$$\psi\colon \mathbb{Z}_n \to \mathbb{Z}_{m/2} \times \mathbb{Z}_d$$

$$x \mapsto (x \pmod{m/2}, x \pmod{d})$$

Let $s \in S$. Then Eq. (6.6) implies that

$$((1-n)s + n \pmod{m}, s \pmod{d}) = \varphi(s, 1) \in \varphi(S, 1) = \sigma_1(S_1) \times S_2.$$

We conclude that $(1-n)s+n \in \sigma_1(S_1)$. Because $m/2$ divides n, the first coordinate of $\psi(s)$, which is $s \pmod{m/2}$, can be rewritten as $(1-n)s+n \pmod{n/2}$. We conclude that $(1-n)s + n \in T$ and obtain that

$$\psi(s) = ((1-n)s + n \pmod{m/2}, s \pmod{d}) \in T \times S_2.$$

It follows that $\psi(S) \subseteq T \times S_2$. As φ and ψ are isomorphisms and $T \subseteq \sigma_1(S_1)$, we also conclude that

$$|\psi(S)| = |S| = |S \times \{1\}| = |\varphi(S \times \{1\})| = |\sigma(S_1) \times S_2| = |\sigma_1(S_1)||S_2| \geq |T||S_2|.$$

In particular, $\psi(S) = T \times S_2$. As S is inverse-closed and ψ is group isomorphism, this shows that $T \times S_2 = T \times (\mathbb{Z}_d \setminus \{0\})$ is inverse-closed. Finally, note that as m is even and by assumptions $d > 3$ and $\gcd(m, d) = 1$, it follows that $d \geq 5$.

We can now apply Lemma 2.33 with $H = \mathbb{Z}_{m/2}, K = \mathbb{Z}_d$ and $T \subset H$ to conclude that $\mathrm{Cay}(\mathbb{Z}_{m/2} \times \mathbb{Z}_d, T \times (\mathbb{Z}_d \setminus \{0\}) = \mathrm{Cay}(\mathbb{Z}_{m/2} \times \mathbb{Z}_d, T \times S_2)$ is non-normal. However, as $\psi(S) = T \times S_2$, this graph is isomorphic to X via the graph isomorphism induced by ψ. We conclude that X is non-normal, a contradiction. □

Remark 6.23 (Qin-Xia-Zhou [26, Remark 4.4])**.** *Theorem 6.22 fails for Cayley graphs of general abelian groups. Calculations in MAGMA prove that the following graph is arc-transitive and non-trivially unstable.*

$$\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_4, \{\pm(1,3), \pm(0,1), (0,2), (2,2)\}).$$

7 UNSTABLE CIRCULANTS OF LOW VALENCY

In [12], Hujdurović, Mitrović and Morris showed that every non-trivially unstable circulant graph of valency at most 7 has a Wilson type. Moreover, for each valency, an explicit list of non-trivially unstable circulants, along with their Wilson types, has been derived. These classifications are discussed in Section 7.2.

The main corollary of these classifications is the following result.

Theorem 7.1 (Hujdurović-Mitrović-Morris [12, Theorem 1.9]). *Every non-trivially unstable circulant graph of valency at most* 7 *has Wilson type* (C.1), (C.2′), (C.3′), *or* (C.4).

7.1 MAIN IDEAS AND METHODS

We will start by stating some technical lemmas that were intensively used in the proof of the classifications.

Recall the definition of an s-edge (see Definition 4.31). The first result we will discuss is a particular case of Lemma 4.32, when the graph X is a circulant (so $G = \mathbb{Z}_n$ for some $n \in \mathbb{N}$) and $S_0 \subseteq S$.

Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph. The idea is that we look at proper, non-empty inverse-closed subsets of the connection set S and the subgraphs of X that these sets induce. Suppose we are able to find a subset $S_0 \subseteq S$, such that every automorphism $\alpha \in \mathrm{Aut}(BX)$ maps S_0-edges to S_0-edges. Note that then its complement in S i.e., the set $S_1 \coloneqq S \setminus S_0$ has the same property. This allows us to define two subgraphs of X

$$X_0 = \mathrm{Cay}(\mathbb{Z}_n, S_0) \text{ and } X_1 = \mathrm{Cay}(\mathbb{Z}_n, S_1).$$

Hence, we have split X into X_1 and X_2. Note that X_1 and X_2 are circulants and that $E(X)$ is a disjoint union of $E(X_0)$ and $E(X_1)$. Similarly, BX_1 and BX_2 are subgraphs of BX and $E(BX)$ is a disjoint union of $E(BX_1)$ and $E(BX_2)$.

The crucial property of X_0 and X_1 is that for $i \in \{0, 1\}$, it holds that

$$\alpha \in \mathrm{Aut}(BX) \implies \alpha \in \mathrm{Aut}(BX_i), \forall \alpha \in \mathrm{Aut}(BX).$$

Therefore, by studying properties of BX_i and in particular, stability of X_i and its connected components, we are able to derive conclusions about stability of X, as the following results will show. Two particularly nice cases are when X is of odd valency and S_0 can be taken to be $\{n/2\}$ and when X_0 or X_1 turn out to have a connected component that is a stable graph (see Corollary 7.2 for details).

One of the advantages of this approach is that the obtained circulant graphs X_1 and X_2 are of lower valency than X and have fewer edges. Often, they are easier to study. Moreover, results on circulants of valency lower than X become available.

Corollary 7.2 (Hujdurović-Mitrović-Morris [12, Lemma 3.4]). *Let n be a positive integer and $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ a connected, non-bipartite circulant graph of order n. Let S_0 be a non-empty subset of $\mathbb{Z}_n \setminus \{0\}$ such that $S_0 = -S_0$. If every automorphism of BX maps S_0-edges to S_0-edges, and some (or, equivalently, every) connected component of $\mathrm{Cay}(\mathbb{Z}_n, S_0)$ is a stable graph, then X is stable.*

The two following results are an extension of this idea.

Lemma 7.3 (Hujdurović-Mitrović-Morris [12, Lemma 3.6]). *Let $X := \mathrm{Cay}\ (\mathbb{Z}_n, S)$ be a circulant graph of even order and odd valency. Let $S_0 \subseteq S$ be non-empty such that $n/2 \in \langle S_0 \rangle$. Assume that the set of S_0-edges is invariant under the elements of $\mathrm{Aut}\, BX$ (and $S_0 = -S_0$). If some (equivalently every) connected component X_0' of $\mathrm{Cay}(\mathbb{Z}_n, S_0)$ is non bipartite and has the property that BX_0' is normal, then X is stable.*

Proof. We can take X_0' to be the connected component of $\mathrm{Cay}(\mathbb{Z}_n, S_0)$ containing 0. Then $X_0' = \mathrm{Cay}(\langle S_0 \rangle, S_0)$ and all other connected components of $\mathrm{Cay}(\mathbb{Z}_n, S_0)$ are isomorphic to X_0'. Because X is of odd valency, we know $n/2 \in S$. By assumption $n/2 \in \langle S_0 \rangle$, so $n/2$ is a vertex of X_0'. As X_0' is connected and assumed to be non-bipartite, BX_0' is connected.

Let $\alpha \in \mathrm{Aut}(BX)_{(0,0)}$. Then by our assumptions, $\alpha \in \mathrm{Aut}(BX_0)$. Because α fixes $(0, 0)$, it also fixes the connected component of BX_0 containing it, which is BX_0'. As BX_0' is normal, by Proposition 2.31(3), the restriction of α onto BX_0' is a group automorphism of $\langle S_0 \rangle \times \mathbb{Z}_2$.

Note that as $(0, 1), (n/2, 0), (n/2, 1)$ are the only elements of order 2 in $\langle S_0 \rangle \times \mathbb{Z}_2$, it follows that α must permute them among themselves. As α fixes the colours of BX_0', it follows that α fixes $(n/2, 0)$ because this is the unique element of order 2 in the set $\langle S_0 \rangle \times 0$. By definition, $0 \notin S$ and the only element of order 2 in the connection set of BX, which is $S \times \{1\}$, is $(n/2, 1)$. Since α fixes $N_{BX}(0, 0) = S \times \{1\}$ set-wise, it must hold that α fixes $(n/2, 1)$ and consequently, it also fixes $(0, 1)$. It follows by Lemma 4.5 that X is stable. □

Corollary 7.4 (Hujdurović-Mitrović-Morris [12, Corollary 4.5]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a circulant graph of even order and odd valency, and let $S_0 \subseteq S$ with $|S_0| = 4$. Let $X_0 := \mathrm{Cay}(\mathbb{Z}_n, S_0)$ and let X_0' be a connected component of X_0. Assume that the set of S_0-edges is invariant under $\mathrm{Aut}(BX)$. If either*

1. *$|V(X_0')|$ is odd, or*

2. *X_0 is twin-free and non-bipartite,*

then X is stable.

Proof. Let us first assume that $|\langle S_0\rangle| = |V(X_0')|$ is odd. Note that then X_0' is twin-free, as otherwise, because it is 4-valent, Lemma 2.46(3) would imply that X_0' is isomorphic to $K_{4,4}$ or $C_\ell \wr \overline{K_2}$, both of which are of even order. It follows that X_0' is a connected, twin-free, circulant graph of odd order, so, by Theorem 6.8, it must be stable. It follows by Corollary 7.2 that X is stable.

We can now assume that $|V(X_0')|$ is even. In this case, assumption (2) must hold, so X_0 is twin-free and non-bipartite. As all of its connected components are isomorphic to X_0', it follows that X_0' is twin-free and non-bipartite. In particular, X_0' is not trivially unstable. If it is stable, we conclude that X is stable by Corollary 7.2. If it is not stable, it is non-trivially unstable, and as $|\langle S_0\rangle| = |V(X_0')|$ being even implies that $n/2 \in \langle S_0\rangle$, [12, Corollary 4.4] shows that BX_0' is normal. Applying Lemma 7.3, we conclude that X is stable. □

The previously described strategy is not always successful. Even if we are able to find a subset $S_0 \subseteq S$ with the desired property, the resulting subgraphs $X_0 = \mathrm{Cay}(\mathbb{Z}_n, S_0)$ may not provide us with enough useful information. The following result is meant to simplify this situation. It turns out that, under appropriate assumptions, existence of automorphisms of BX not preserving the edge type can still tell us a lot about the connection set of X.

Lemma 7.5 (Hujdurović-Mitrović-Morris [13, Corollary 4.6], [12, Proposition 2.18])**.** *Let α be an automorphism of BX, where $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ is a circulant graph, and let $s, t \in S$. If α maps some s-edge to a t-edge, and either* $\gcd(|s|, |t|) = 1$*, or S contains every element that generates $\langle s\rangle$ (e.g., if $|s| \in \{1, 2, 3, 4, 6\}$), then S contains every element that generates $\langle t\rangle$.*

Proof. Let ℓt be a generator of $\langle t\rangle$, so $\gcd\big(\ell, |t|\big) = 1$. It suffices to find $k \in \mathbb{Z}^+$, such that

$$ks \in S, \;\; k \equiv \ell \pmod{\gcd\big(|s|, |t|\big)}, \;\text{ and }\; \gcd\big(k, |s|\big) = 1,$$

for then [12, Corollary 6.1] tells us that $\ell t \in S$.

If $\gcd(|s|, |t|\big) = 1$, we may let $k = 1$.

Since $\gcd\big(\ell, |t|\big) = 1$, we know that ℓ is relatively prime to $\gcd\big(|s|, |t|\big)$, so there is some $k \in \mathbb{Z}^+$, such that

$$k \equiv \ell \pmod{\gcd\big(|s|, |t|\big)} \;\text{ and }\; \gcd\big(k, |s|\big) = 1.$$

(For example, we could take k to be a large prime.) If S contains every element that generates $\langle s\rangle$, then $ks \in S$. □

If S contains a non-trivial element s of order $2, 3, 4$ or 6 (which implies that s and $-s$ are the only generators of $\langle s\rangle$), then Lemma 7.5 shows that no automorphism of BX can map an s-edge onto a t-edge, if $\langle t\rangle$ with $t \in S$ has more than $|S| - |s|$ generators. This observation has been used several times to find subsets $S_0 \subseteq S$ to which Corollary 7.2 can be applied.

The last result we want to point out has already been introduced as Lemma 2.32 in Section 2. It enables us to derive additional relations satisfied by the elements of the connection set of a circulant graph in the case when the graph is not a normal Cayley graph.

7.2 CLASSIFICATIONS OF NON-TRIVIALLY UNSTABLE CIRCULANT GRAPHS OF VALENCY AT MOST 7

In this section, we discuss the classifications of non-trivially unstable circulants of valency at most 7.

Proposition 7.6 (Hujdurović-Mitrović-Morris [12, Proposition 4.2]). *There are no non-trivially unstable circulant graphs of valency at most* 3.

Proof. Let X be a connected, non-bipartite, twin-free circulant graph of valency d with $d \leq 3$. We consider the following cases.

- If $d = 0$, then $X = K_1$ is stable by Example 4.7.
- If $d = 1$, it follows that $X = K_2$, which is bipartite.
- If $d = 2$, X is connected and 2-regular, so it is isomorphic to a cycle. As it is also non-bipartite, it is of odd order. At this point, both Theorem 6.8 and Example 3.10 can be used to show that X is stable.
- Assume that $d = 3$. Using Theorem 6.8, we can assume that X is of even order n. Write $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ with $S = \{\pm a, n/2\}$. As X is connected, $\langle a, n/2\rangle = \mathbb{Z}_n$. From here, a either generates $\mathbb{Z}_n$ or it generates a proper subgroup $2\mathbb{Z}_n$ of index 2, in which case $n/2$ must be odd.

  By applying appropriate group automorphisms of $\mathbb{Z}_n$, we can conclude that X is isomorphic to $\mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, n/2\})$ with $n/2$ even (otherwise, the graph is bipartite) or $\mathrm{Cay}(\mathbb{Z}_n, \{\pm 2, n/2\})$ with $n/2$ odd (otherwise, the graph is disconnected). In particular, X is isomorphic to a non-bipartite Möbius ladder or an odd prism. In either case, $|\operatorname{Aut}(X)| = n$ and BX is an even prism of order $2n$ with $|\operatorname{Aut}(BX)| = 4n$.

  In either case, we see that the index of instability of X is 1, that is, X is stable.

□

Theorem 7.7 (Hujdurović-Mitrović-Morris [12, Theorem 4.3])**.** *A circulant graph* $X = \mathrm{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\})$ *of valency* 4 *is unstable if and only if either it is trivially unstable, or one of the following conditions is satisfied (perhaps after interchanging* a *and* b*):*

1. $n \equiv 2 \pmod 4$*,* $\gcd(a, n) = 1$*, and* $b = ma + (n/2)$*, for some* $m \in \mathbb{Z}_n^\times$*, such that* $m^2 \equiv \pm 1 \pmod n$*, or*

2. n *is divisible by* 8 *and* $\gcd\big(|a|, |b|\big) = 4$*.*

In both of these cases, X *has Wilson type* (C.4)*.*

Proof. ($\Leftarrow$) A direct computation shows that the condition (1) implies that $mS + (n/2) = S$, so X has Wilson type (C.4).

We can therefore assume that (2) applies. As X is connected, we can assume that a is odd. Then $n/|a|$ is odd. As $\gcd(|a|, |b|) = 4$, we conclude that $|b| = 4n/|a|$. Write $|a| = 2^r\ell$, where ℓ is an odd integer. Since ℓ and $n/|a|$ are odd, we have $\gcd(2^r, \ell) = \gcd(2^r, n/|a|) = 1$. Also note that

$$4 = \gcd\big(|a|, |b|\big) = \gcd\big(2^r\ell, 4n/|a|\big) = 4\gcd\big(2^{r-2}\ell, n/|a|\big).$$

Therefore 2^r, ℓ, and $n/|a|$ are pairwise relatively prime, so we may choose $m \in \mathbb{Z}_n^\times$, such that

$$m \equiv 2^{r-1} + 1 \pmod{2^r}, \quad m \equiv 1 \pmod{\ell}, \quad \text{and} \quad m \equiv -1 \pmod{n/|a|}.$$

It can be checked that $mS + (n/2) = S$, proving that X has Wilson type (C.4).

($\Rightarrow$) Assume that X is non-trivially unstable. Note that n must be even (see Theorem 6.8). Since X is connected and non-bipartite, it follows that exactly one of the elements of $\{a, b\}$ is even.

Let α be an unexpected automorphism of BX that fixes $(0, 0)$. Without loss of generality, after possibly composing α with τ, we may assume that α preserves the colour classes of BX, that is, $\alpha(\mathbb{Z}_n \times \{i\}) = \mathbb{Z}_n \times \{i\}$ for $i \in \{0, 1\}$.

Case 1. *Assume* α *is a group automorphism.* Since $\mathbb{Z}_n \times \{0\}$ is α-invariant, this implies there is some $m \in \mathbb{Z}_n^\times$, such that $\alpha(x, 0) = (mx, 0)$ for all $x \in \mathbb{Z}_n$. Note that α fixes $(n/2, 0)$. Since $\alpha(0, 1)$ is an element of order 2 contained in $\mathbb{Z}_n \times \{1\}$, we must have $\alpha(0, 1) \in \{(0, 1), ((n/2), 1)\}$. If $\alpha(0, 1) = (0, 1)$, then $\alpha(x, i) = (mx, i)$, which contradicts the assumption that $\alpha \notin \mathrm{Aut}\, X \times S_2$. Therefore, we have $\alpha(0, 1) = (n/2, 1)$. So

$$\alpha(x, i) = \big(mx + i(n/2), i\big) \quad \text{for all } (x, i) \in BX.$$

Since $S \times \{1\}$ is α-invariant, this implies that $mS + (n/2) = S$, so X is of Wilson type (C.4).

Subcase 1.1. *Assume that* $(n/2)$ *is odd.* Since X is connected, we may assume, without loss of generality, that a is odd and b is even. Then $ma + (n/2)$ is an even element of S, so we must have $\alpha(a,1) \in \{(\pm b, 1)\}$. Therefore $b = ma + (n/2)$ (perhaps after composing α with the group automorphism $x \mapsto -x$, which replaces m with $-m$).

Now, we have $\alpha(a,1) = (ma + (n/2), 1) = (b,1)$, so $\alpha(\pm a, 1) = (\pm b, 1)$. Since α is a group automorphism that preserves the set $S \times \{1\}$, this implies $\alpha(\pm b, 1) = (\pm a, 1)$, so we may write $\alpha(b,1) = (\epsilon a, 1)$ with $\epsilon \in \{\pm 1\}$. Then we have $m^2(a,1) = \alpha^2(a,1) = \epsilon(a,1)$ and $m^2(b,1) = \alpha^2(b,1) = \epsilon(b,1)$, so $m^2 x = \epsilon x$ for all $x \in \mathbb{Z}_n \times \mathbb{Z}_2$. This implies $m^2 \equiv \epsilon \equiv \pm 1 \pmod{n}$. So X is as described in (1).

Subcase 1.2. *Assume that* $(n/2)$ *is even.* Then $ma + a$ has the same parity as a (and $mb + b$ has the same parity as b), so we must have $\alpha(a,1) \in \{(\pm a, 1)\}$ and $\alpha(b,1) \in \{(\pm b, 1)\}$. There is no harm in assuming $\alpha(a,1) = (a,1)$ (by replacing m with $-m$ if necessary). Then, since α is not the identity map, we must have $\alpha(b,1) = (-b,1)$. Therefore

$$\big(ma + (n/2), 1\big) = \alpha(a,1) = (a,1), \text{ so } (m-1)a = (n/2),$$

and

$$\big(mb + (n/2), 1\big) = \alpha(b,1) = (-b,1), \text{ so } (m+1)b = (n/2).$$

Since $m-1$ and $m+1$ are even (and $(n/2)$ has order 2), this implies that $|a|$ and $|b|$ are divisible by 4.

This also implies that $2(m-1)a = 0$ and $2(m+1)b = 0$, so $|a|$ is a divisor of $2(m-1)$ and $|b|$ is a divisor of $2(m+1)$. Therefore $\gcd\big(|a|, |b|\big)$ is a divisor of $\gcd\big(2(m-1), 2(m+1)\big) \leq 4$. By combining this with the conclusion of the preceding paragraph, we conclude that $\gcd\big(|a|, |b|\big) = 4$. Then, since X is not bipartite, we must have $n \equiv 0 \pmod{8}$. This establishes that conclusion (2) holds.

Case 2. *Assume* $2s \neq 2t$*, for all* $s, t \in S$*, such that* $s \neq t$*.* We may assume α is not a group automorphism. Then by Proposition 2.31(3), BX is not normal. Lemma 2.32 implies there exist $s, t, u, v \in S$ such that $s + t = u + v \neq 0$ and $\{s,t\} \neq \{u,v\}$. From the assumption of this case, we see that this implies $3a = \pm b$ (perhaps after interchanging a with b). This implies that a and b have the same parity, which contradicts the assumption that X is connected and non-bipartite.

Case 3. *The remaining case.*

We have $2s = 2t$, for some $s, t \in S$, such that $s \neq t$. We may assume $s = a$.

Subcase 3.1. *Assume that* $t = -s = -a$.

Then $|a| = 4$. If n is divisible by 8, then $|b|$ must be divisible by 8, so $\gcd\big(|a|, |b|\big) = 4$. It follows that the condition (2) is satisfied. So we may assume $n = 4k$, where k is odd. Since X is non-bipartite, we know that $|b|$ is not divisible by 4, so the fact that $|a| = 4$ implies $|\langle a \rangle \cap \langle b \rangle| \leq 2$. Hence, there is an automorphism of $\mathbb{Z}_n$ that fixes a, but inverts b. Together with the inversion map $\iota\colon x \mapsto -x$, these automorphisms generate a subgroup of order 4 in $\operatorname{Aut}(X)_0$. If follows by Lemma 2.6 that

$$|\operatorname{Aut} X| = |V(X)||\operatorname{Aut}(X)_0| = |\mathbb{Z}_n||\operatorname{Aut}(X)_0| \geq 4n.$$

Also, since k is odd and X is non-bipartite, we must have $kb \neq \pm a$. Since $4kb = nb = 0$, this implies $k(b,1) \in \{(0,1),(2a,1)\}$. Since $(2a,1) \notin \langle (a,1)\rangle$, this implies that $\langle (a,1)\rangle \cap \langle (b,1)\rangle = \{(0,0)\}$, so $BX \cong C_4 \square C_{n/2}$. Therefore, by Lemma 2.51(3), we have

$$\begin{aligned}|\operatorname{Aut} BX| &= |\operatorname{Aut}(C_4 \square C_{n/2})| = |\operatorname{Aut} C_4 \times \operatorname{Aut} C_{n/2}| \\ &= |\operatorname{Aut} C_4| \cdot |\operatorname{Aut} C_{n/2}| = 8 \cdot n = 2 \cdot 4n \leq 2 \cdot |\operatorname{Aut} X|.\end{aligned}$$

This contradicts the assumption that X is unstable.

Subcase 3.2. *Assume that $t \neq -s$.*

Therefore, we may assume $s = a$ and $t = b$, so $2a = 2b$. This means that $a - b$ has order 2, and must therefore be equal to $(n/2)$, so $S + (n/2) = S$. This contradicts the fact that $\operatorname{Cay}(\mathbb{Z}_n, S)$ is twin-free. □

Theorem 7.8 (Hujdurović-Mitrović-Morris [12, Theorem 5.1])**.** *A circulant graph $\operatorname{Cay}(\mathbb{Z}_n, S)$ of valency 5 is unstable if and only if either it is trivially unstable, or it is one of the following:*

1. $\operatorname{Cay}(\mathbb{Z}_{12k}, \{\pm s, \pm 2k, 6k\})$ *with s odd, which has Wilson type* (C.1),
2. $\operatorname{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$, *which has Wilson type* (C.3′).

Proof. ($\Leftarrow$) For any member of the infinite family listed under (1), it holds that $S_e = \{2k, 6k, 10k\}$. Hence, $4k + S_e = S_e$ and the graph has Wilson type (C.1). The graph $\operatorname{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$ listed under (2) has Wilson type (C.3′) with parameters $H = \langle 2\rangle = \{0,2,4,6\}$, $R = \{4\}$, and $d = 4$. (Then $n/d = 2$ is even, $r/d = 1$ for the unique element r of R, and $H = 2\mathbb{Z}_8 \nsubseteq 4\mathbb{Z}_8 = d\mathbb{Z}_8$.)

($\Rightarrow$) Let $X = \operatorname{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, (n/2)\})$ be a non-trivially unstable circulant graph of valency 5. The proof proceeds in two steps. We first assume that $(n/2)$-edges are not invariant under automorphisms of BX and later, we address the case when every automorphism of BX maps $(n/2)$-edges to $(n/2)$-edges.

Assume that $(n/2)$-edges are not invariant. Without loss of generality, we can assume that some automorphism of BX maps an $(n/2)$-edge to an a-edge (perhaps after interchanging a and b).

Then Lemma 7.5 shows that every generator of $\langle a\rangle$ is in $S \setminus \{(n/2)\}$. So the number of generators of $\langle a\rangle$ is ≤ 4 (and $|a| > 2$), so $|a| \in \{3,4,5,6,8,10,12\}$.

Case 1. *Assume $|a| \in \{5,8,10,12\}$.* The four generators of $\langle a\rangle$ are in S, so they must coincide with $\pm a$ and $\pm b$. Therefore, $\langle a, (n/2)\rangle = \langle a, b, (n/2)\rangle = \mathbb{Z}_n$. We have the following cases.

- If $|a| = 5$, then $n = 10$ and $X = \mathrm{Cay}(\mathbb{Z}_{10}, \{\pm 2, \pm 4, 5\})$. This graph is stable.
- If $|a| = 8$, then $n = 8$ and $X \cong \mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$. This graph appears in Theorem 7.8, however, its $(n/2)$-edges are actually invariant, so it is not permissible in this case.
- If $|a| = 10$, then $n = 10$ and $X \cong \mathrm{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 3, 5\})$. This graph is bipartite.
- If $|a| = 12$, then $n = 12$ and $X \cong \mathrm{Cay}(\mathbb{Z}_{12}, \{\pm 1, \pm 5, 6\})$. This graph is stable.

Case 2. *Assume* $|a| \in \{3, 4, 6\}$. Note that then $\pm a$ are the only generators of $\langle a \rangle$. Because all elements of S are pairwise distinct, it follows that $\langle a \rangle \neq \langle b \rangle$. Therefore, $|a| \neq |b|$.

Subcase 2.1. *Assume* $|b| \in \{3, 4, 6\}$. We consider each of the three possibilities for $\{|a|, |b|\} \in \{\{3, 4\}, \{3, 6\}, \{4, 6\}\}$. The only unstable graph we obtain is

$$X = \mathrm{Cay}(\mathbb{Z}_{12}, \{\pm 2, \pm 3, 6\}).$$

This graph appears in part (1) of the statement of Theorem 7.8 with parameters $s = 3$ and $k = 1$.

Subcase 2.2. *Assume* $|b| \notin \{3, 4, 6\}$. From here, we see from Lemma 7.5 that no automorphism of BX can map an $(n/2)$-edge to a b-edge (because S cannot contain more than 2 generators of $\langle b \rangle$, in addition to $\pm a$). Hence, the set of b-edges is invariant under all automorphisms of BX. Now, we see that every automorphism of BX is also an automorphism of the graphs

$$BX_1 \coloneqq \mathrm{Cay}\big(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm a, 1), ((n/2), 1)\}\big) \text{ and } BX_2 \coloneqq \mathrm{Cay}\big(\mathbb{Z}_n \times \mathbb{Z}_2, \{(\pm b, 1)\}\big)$$

which are the canonical double covers of

$$X_1 \coloneqq \mathrm{Cay}\big(\mathbb{Z}_n, \{\pm a, (n/2)\}\big) \quad \text{and} \quad X_2 \coloneqq \mathrm{Cay}\big(\mathbb{Z}_n, \{\pm b\}\big),$$

respectively. Note that BX_1 is arc-transitive (because a-edges and $(n/2)$-edges are in the same orbit of $\mathrm{Aut}\, BX$).

If $|a| = 3$, then every connected component of BX_1 is isomorphic to a 6-prism, which is not arc-transitive. If $|a| = 4$, then every connected component of X_1 is isomorphic to K_4, which is a stable graph by Example 4.7, so it follows from Corollary 7.2 that X is stable, a contradiction. Therefore, we must have $|a| = 6$.

The connected components of X_2 are $|b|$-cycles. If $|b|$ is odd, then these are stable (by Theorem 6.8 or Example 3.10). By another application of Corollary 7.2, it follows that X is stable, a contradiction.

So we can now assume that $|a| = 6$ and $|b|$ is even. Write $n = 6\ell$. From here $n/2 = 3\ell$ and $\{\pm a\} = \{\ell, 5\ell\}$.

Note that if ℓ is odd, then a and b must both be odd (since $|a|$ and $|b|$ are even). Since $n/2 = 3\ell$ is also odd, this means that all elements of S are odd, so X is bipartite, a contradiction.

Therefore, we know that ℓ is even, so we may write $\ell = 2k$. Then $n = 12k$, $n/2 = 6k$ and $\{\pm a\} = \{\pm 2k\}$. In particular, $\pm a$ and $n/2$ are all even. So b must be odd (since X is connected). This means that X appears in part (1) of the statement of the Theorem 7.8 with parameter $s = b$. This finishes the first case.

Assume that $(n/2)$-edges are invariant. This implies that every automorphism of BX is an automorphism of BX_0, where

$$X_0 = \mathrm{Cay}(\mathbb{Z}_n, \{\pm a, \pm b\}).$$

If X_0 is stable, then by Corollary 7.2 it follows that X is stable, a contradiction. So we may assume now that X_0 is unstable.

Case 1. *Assume X_0 is non-trivially unstable.*

In this case, [12, Corollary 4.4] and [12, Corollary 2.11] show that this leads to a contradiction.

Case 2. *Assume X_0 is trivially unstable.*

There are three possibilities to consider:

Subcase 2.1. *Assume X_0 is not connected.* Then a and b generate a proper subgroup of $\mathbb{Z}_n$, while a, b and $n/2$ generate the whole group. From here, $n = 2k$, where k is odd, and $\langle a, b\rangle = 2\mathbb{Z}_n$ has order k. The connected components of X_0 then have order k, and therefore have odd order. By applying [12, Corollary 4.5(1)] we conclude that X is stable, a contradiction.

Subcase 2.2. *Assume X_0 is connected, but is not twin-free.*

By Lemma 2.46(1), we can represent X_0 as a wreath product $Y \wr \overline{K_m}$, where Y is a δ-regular connected graph and $m > 1$ is an integer such that $\delta m = 4$.

Subsubcase 2.2.1. *Assume $m = 4$.* Then $\delta = 1$, so we get that $X_0 = K_2 \wr \overline{K_4} = K_{4,4}$. Hence, X_0 is a connected, 4-valent Cayley graph on $\mathbb{Z}_8$ and its connection set can only contain odd numbers, because it is also bipartite. This uniquely determines X_0 as $\mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3\})$. From here, $X = \mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 3, 4\})$, so X is the graph listed in the part (2) of Theorem 7.8.

Subsubcase 2.2.2. *Assume $m = 2$.* Then $\delta = 2$ and

$$|V(Y)| = |V(X)|/m = 2k/2 = k,$$

so Y is a k-cycle, so $X_0 \cong C_k \wr \overline{K_2}$. Consequently, $X \cong C_k \wr K_2$. It can be checked directly that this graph is only unstable when $k = 4$, i.e., $X \cong C_4 \wr K_2$. It is easy to see that this implies $X = \mathrm{Cay}(\mathbb{Z}_8, \{\pm1, \pm3, 4\})$. Hence, X is the graph appearing under (2).

Subcase 2.3. *Assume X_0 is bipartite, connected and twin-free.*

In this case, if follows that [12, Lemma 2.12] and [12, Lemma 5.3] that $X_0 = \mathrm{Cay}(\mathbb{Z}_{10}, \{\pm1, \pm3\})$, meaning that $X = \mathrm{Cay}(\mathbb{Z}_{10}, \{\pm1, \pm3, 5\})$. But then X is bipartite, a contradiction. □

The following is a corollary of the proof of Theorem 7.8.

Corollary 7.9 ([12, Lemma 5.4]). *Let $X = \mathrm{Cay}(\mathbb{Z}_n, S)$ be a non-trivially unstable circulant graph of valency 5. If every automorphism of BX maps $(n/2)$-edges to $(n/2)$-edges, then $X = \mathrm{Cay}(\mathbb{Z}_8, \{\pm1, \pm3, 4\})$.*

Remark 7.10. *Note that we can obtain the graph $\mathrm{Cay}(\mathbb{Z}_8, \{\pm1, \pm3, 4\})$ listed under (2) of Theorem 7.8 via Proposition 5.6 as $\Gamma(2K_2)$ where $A = B = V(2K_2)$.*

Similar ideas can be applied when studying circulants of valency 6 or 7. However, the number of subcases that emerge is a lot bigger and significant effort is needed to analyze all of them. Often, local analysis of particular examples is necessary in order to determine their stability. We refer the reader to the original article [12] for complete proofs and give only the final results that have been obtained.

Theorem 7.11 (Hujdurović-Mitrović-Morris [12, Corrolary 6.8]). *A circulant graph $X = \mathrm{Cay}(\mathbb{Z}_n, \{\pm a, \pm b, \pm c\})$ of valency 6 is unstable if and only if either it is trivially unstable, or it is one of the following:*

1. $\mathrm{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm b, \pm 2k\})$, *where a and b are odd, which is of Wilson type* (C.1).

2. $\mathrm{Cay}(\mathbb{Z}_{4k}, \{\pm a, \pm b, \pm b + 2k\})$, *where a is odd and b is even, which is of Wilson type* (C.1).

3. $\mathrm{Cay}\big(\mathbb{Z}_{4k}, \big\{\pm a, \pm(a + k), \pm(a - k)\big\}\big)$, *where $a \equiv 0 \pmod 4$ and k is odd, which is of Wilson type* (C.2′).

4. $\mathrm{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm b, \pm b + 4k\})$, *where a is even and $|a|$ is divisible by 4, which is of Wilson type* (C.3′).

5. $\mathrm{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm k, \pm 3k\})$, *where $a \equiv 0 \pmod 4$ and k is odd, which is of Wilson type* (C.3′).

6. $\mathrm{Cay}(\mathbb{Z}_{4k}, \{\pm a, \pm b, \pm mb + 2k\})$, *where*
$$\gcd(m, 4k) = 1, \quad (m-1)a \equiv 2k \pmod{4k}, \quad \textit{and}$$
$$\textit{either } m^2 \equiv 1 \pmod{4k} \;\; \textit{or} \;\; (m^2+1)b \equiv 0 \pmod{4k},$$
which is of Wilson type (C.4).

7. $\mathrm{Cay}(\mathbb{Z}_{8k}, \{\pm a, \pm b, \pm c\})$, *where there exists* $m \in \mathbb{Z}$, *such that*

$$\gcd(m, 8k) = 1, \quad m^2 \equiv 1 \pmod{8k}, \quad \textit{and}$$

$$(m-1)a \equiv (m+1)b \equiv (m+1)c \equiv 4k \pmod{8k},$$

which is of Wilson type (C.4).

Theorem 7.12 (Hujdurović-Mitrović-Morris [12, Theorem 7.1]). *A circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$ *of valency 7 is unstable if and only if either it is trivially unstable, or it is one of the following:*

1. $\mathrm{Cay}(\mathbb{Z}_{6k}, \{\pm 2t, \pm 2(k-t), \pm 2(k+t), 3k\})$, *with* k *odd, which has Wilson type* (C.1).
2. $\mathrm{Cay}(\mathbb{Z}_{12k}, \{\pm 2k, \pm b, \pm c, 6k\})$, *with* b *and* c *odd, which has Wilson type* (C.1).
3. $\mathrm{Cay}(\mathbb{Z}_{20k}, \{\pm t, \pm 2k, \pm 6k, 10k\})$, *with* t *odd, which has Wilson type* (C.1).
4. $\mathrm{Cay}(\mathbb{Z}_{4k}, \{\pm t, \pm(k-t), 2k \pm t, 2k\})$, *with* k *odd and* $t \equiv k \pmod 4$, *which has Wilson type* (C.2′).
5. $\mathrm{Cay}(\mathbb{Z}_{8k}, \{\pm 4t, \pm k, \pm 3k, 4k\})$, *with* k *and* t *odd, which has Wilson type* (C.3′).
6. $\mathrm{Cay}(\mathbb{Z}_{12k}, \{\pm t, \pm(4k-t), \pm(4k+t), 6k\})$, *with* t *odd, which has Wilson type* (C.3′).

8 CONSTRUCTIONS OF UNSTABLE GRAPHS

8.1 ASYMMETRIC NON-TRIVIALLY UNSTABLE GRAPHS

We first recall the following definition from algebraic graph theory.

Definition 8.1. A graph X is called *asymmetric* if its automorphism group is trivial. This means X has no non-trivial automorphisms.

At first, it might seem that asymmetric graphs are rare, especially since we have been focusing on highly symmetric graphs so far. However, the following result illustrates just how wrong this intuition is.

Corollary 8.2 ([8, Corollary 2.3.3])**.** *Almost all graphs are asymmetric.*

More formally, what the above result states is that, the proportion of graphs on n vertices that are asymmetric goes to 1 as n tends to ∞.

In this section, we will construct some examples of non-trivially unstable graphs that are asymmetric. The first one has been constructed by Wilson in [34].

Recall that neither the cross-cover construction (see Definition 5.19) nor Theorem 5.20, which shows that every graph that is a cross-cover of another graph is unstable, assume any symmetries of the underlying graph. This is what enables the following construction.

Example 8.3 (Wilson [34, p. 369])**.** Let X be a graph in Figure 2. The labels of the edges define a map $s\colon E(X) \to \mathbb{Z}_3$.

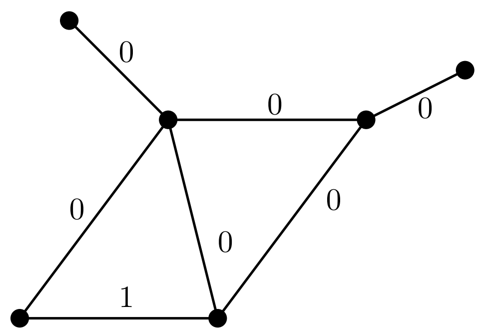

Figure 2: The base graph X with the edge labelling $s\colon E(X) \to \mathbb{Z}_3$

Define the *Swift graph* to be $SG \coloneqq \mathrm{CC}(3, X, s)$, that is, it is the 3-cross-cover of X with respect to s. This graph is shown in Figure 3.

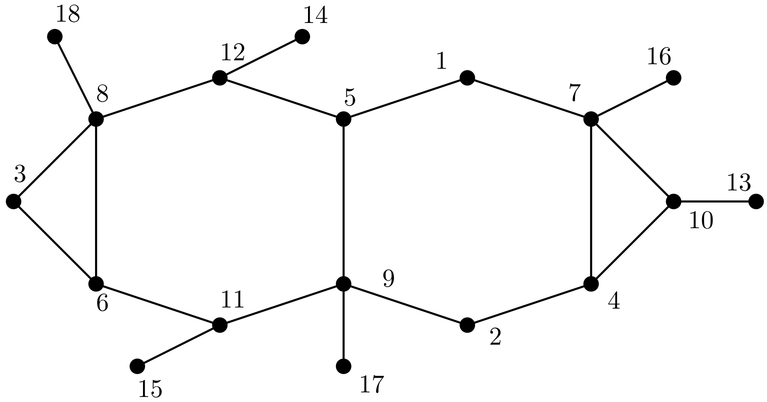

Figure 3: The Swift graph SG

The Swift graph SG has the following properties.

1. SG is connected, non-bipartite and twin-free.

2. SG is non-trivially unstable.

3. SG is asymmetric.

4. $\mathrm{Aut}(B(SG))$ is a group of order 6 acting semi-regularly on $V(B(SG))$.

5. The index of instability of SG is 3.

Proof. (1) It can be seen from Figure 3 that SG is connected and twin-free. It has two obvious odd cycles formed by the vertices $3, 6, 8$ and $4, 7, 10$, so SG is also non-bipartite.

(2) As SG is constructed as a cross-cover of the graph X (see Figure 2), SG is unstable by Theorem 5.20. Since it is also connected, non-bipartite and twin-free, SG is non-trivially unstable.

(3) Let $\gamma \in \mathrm{Aut}(SG)$. We will show that γ fixes all vertices of SG and is consequently trivial. We use the labelling given in Figure 3.

Note that 9 is the unique vertex of valency 4 that does not lie on a 3-cycle. Consequently, it must be fixed by γ. Vertices 17 and 2 are unique neighbours of 9 of valencies 1 and 2, respectively. It follows that both are fixed. The remaining neighbours of 9 are 5 and 11, both of valency 3, but 11 has a neighbour of valency 1, namely vertex 15, while 5 does not. It follows that $5, 11$ and 15 are all fixed. As the only other neighbour of 2, namely the vertex 9, is fixed, vertex 3 must also be fixed. By the same argument, we conclude that 6 is fixed. Neighbours of 6 are 3 and 8 and they are of different valencies, so both are fixed. It follows that 18 is fixed as well. As all other neighbours of 8 are fixed, 12 must be fixed as well and therefore, also 14. By applying the same argument to 5, we get that 1 is fixed. The same argument applied to 1 shows that 7 is fixed. The remaining neighbours of 7 are 10 and 16, but they have different valencies, so both are fixed. Finally, since 10 is fixed, so is 13.

This concludes the argument and shows that γ fixes all vertices of SG. Equivalently, $\gamma = 1$, as desired.

Statements (4) and (5) can be checked by MAGMA. □

The Swift graph serves as a very important counterexample, as it shows that the converse of Lemma 4.4 does not hold for general graphs. Indeed, as $\mathrm{Aut}(B(SG))$ is semi-regular by Example 8.3(4), the condition of Lemma 4.4 is satisfied trivially. However, SG is unstable by Example 8.3(2). In particular, the assumption on vertex-transitivity in Lemma 4.5 cannot just be dropped (it would need to be substituted by additional assumptions).

The Swift graph SG is our first example of a non-trivially unstable asymmetric graph. It is quite a small example, as its order is only 18 (see Figure 3). However, it turns out this is not the smallest asymmetric non-trivially unstable graph.

In [18], Lauri, Mizzi and Scapellato construct an infinite family of asymmetric non-trivially unstable graphs of arbitrary large index of instability using *TF-automorphisms* (here "TF" stands for *two-fold*). They also proved the following lower bound on the order of an unstable asymmetric graph, which is achieved by the smallest member of their infinite family.

Theorem 8.4 (Lauri-Mizzi-Scapellato [18, Theorem 2.4])**.** *Every asymmetric unstable graph has at least* 12 *vertices.*

We first describe the minimal example.

Example 8.5 (Lauri-Mizzi-Scapellato [18])**.** Let $\mathcal{U}$ denote the graph depicted in Fig. 4.

The following properties of $\mathcal{U}$ can be checked by MAGMA.

1. $\mathcal{U}$ is non-trivially unstable of order 12.

2. $|\,\mathrm{Aut}(B\mathcal{U})| = 6$ and the index of instability of $\mathcal{U}$ is 3.

3. $\mathcal{U}$ is asymmetric.

By Theorem 8.4, it follows that $\mathcal{U}$ is an asymmetric non-trivially unstable graph of minimal possible order.

In order to explain how the graph $\mathcal{U}$ has been constructed, we briefly introduce the concepts of mixed graphs and TF-automorphisms.

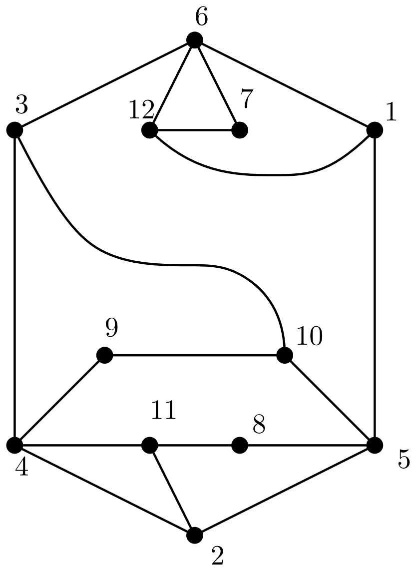

Figure 4: The asymmetric unstable graph $\mathcal{U}$ of order 12

Definition 8.6 ([18, p. 85-86])**.** A *mixed graph* X is a graph with both directed edges, also called *arcs*, and undirected edges, where we identify the undirected edge $\{x, y\}$ with the union of two arcs $\{(x, y), (y, x)\}$. Note that digraphs are mixed graphs and that every graph is just a mixed graphs that contains only undirected edges.

We will denote the set of all edges (both directed and undirected) of a mixed graph X by $A(X)$.

Definition 8.7 ([18, p. 85-86])**.** Let X be a mixed graph. Given two permutations α and β of $V(X)$, we say that (α, β) is a *TF-automorphism of* X if for all $x, y \in V(X)$

$$(x, y) \text{ is an arc of } X \text{ if and only if } (\alpha(x), \beta(y)) \text{ is an arc of } X.$$

If $\alpha \neq \beta$, the TF-automorphism (α, β) is called *non-trivial.*

Definition 8.8 ([18, p. 85-86])**.** Let Y be a mixed graph. The *underlying graph of* Y is the graph X with

- $V(X) \coloneqq V(Y)$, and
- $E(X) \coloneqq \{\{x, y\} \mid (x, y) \in A(Y) \text{ or } (y, x) \in A(Y)\}$.

To understand the construction of Example 8.5, we will need the following facts on TF-automorphisms discussed in [18].

Facts 8.9. *Let Y be a graph, understood as a mixed graph containing only undirected edges. Let γ be a permutation of $V(Y)$.*

1. *Y is unstable if and only if it it has a non-trivial TF-automorphism ([18, Theorem 2.1]).*

2. *If (γ, γ^{-1}) is a TF-automorphism of Y, then (γ, γ^{-1}) is also a TF-automorphism of the underlying graph X of Y ([18, Proposition 2.3]).*

The main idea of the construction is the following. We start with a finite set V, that will become the vertex set of our graph. We pick a permutation $\gamma \in \mathrm{Sym}(V)$ of order at least 3 (so that $\gamma \neq \gamma^{-1}$). Then we let (γ, γ^{-1}) act on the set $V \times V$ in the obvious way.

We pick an ordered pair (u, v) of elements of $u, v \in V$ and consider its orbit $\mathcal{O}_\gamma(u, v)$ under the action of (γ, γ^{-1}). We then consider a digraph Y with the vertex set V and set of arcs equal to the orbit $\mathcal{O}_\gamma(u, v)$. The important thing to note is that, by construction, (γ, γ^{-1}) is a non-trivial TF-automorphism of Y. Then by Facts 8.9(2), (γ, γ^{-1}) is also a non-trivial TF-automorphism of the underlying graph X of Y. Hence, Facts 8.9(1) implies that X is unstable.

We can now keep on choosing ordered pairs of elements of V and adding their orbits under (γ, γ^{-1}) to the arc set of Y. The key observation is that (γ, γ^{-1}) will still be a non-trivial TF-automorphism of the obtained digraph, so the previous argument applies and the underlying graph of the constructed digraph is unstable.

Our goal is to keep on adding arcs, in order to produce a digraph whose underlying graph is not only unstable, but also asymmetric, connected, non-bipartite and twin-free.

The Example 8.5 has been obtained in this manner.

- Let $V \coloneqq \{1, \dots, 12\}$.
- Let $\gamma \coloneqq (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$.
- Let Y be the digraph with $V(Y) \coloneqq V$ and arc set equal to the union of orbits of pairs $(1, 5), (6, 1), (6, 7), (12, 1), (12, 6)$ and $(12, 7)$ under (γ, γ^{-1}).
- The underlying graph X is then exactly the graph $\mathcal{U}$ shown in Figure 4.

The success of the above strategy obviously depends on how we choose the pairs of elements of V whose orbits we are going to include into the arc set of our digraph. A priori, there is no guarantee that by continuously adding arcs we will end up with a graph having all of the desired properties. Although, instability is guaranteed, as we have seen, and connectedness and non-bipartiteness are likely to also be achieved as they require that the graph has more edges anyway.

The authors were able to come up with the following recipe for choosing these pairs, which generalizes the construction of Example 8.5 we just explained.

Example 8.10 (Lauri-Mizzi-Scapellato [18, p.88-91])**.** Let $k \geq 3$ be a positive integer. Define the following permutation of $\{1, \dots, 4k\}$.

$$\gamma = (1, \dots, k)(k+1, k+2, \dots, 2k)(2k+1, \dots, 3k)(3k+1, \dots, 4k)$$

Consider the digraph $\mathcal{G}_k$ with

- $V(\mathcal{G}_k) \coloneqq \{1, \ldots, 4k\}$,
- $A(\mathcal{G}_k)$ consisting of the orbits of pairs $(1, 2k-1), (1, 2k), (1, 4k), (k+1, 3k), (2k, 4k)$ and $(2k+1, 4k)$ under the action of (γ, γ^{-1}).

Let $\mathcal{U}_k$ be the underlying graph of $\mathcal{G}_k$. Then $\mathcal{U}_k$ is a non-trivially unstable and asymmetric graph with a non-trivial TF-automorphism (γ, γ^{-1}).

Remark 8.11. *Every non-trivial TF-automorphism of a graph corresponds to an unexpected automorphism of its double cover (this is how Facts* 8.9(1) *is proved).*

Note that γ from Example 8.10 *is of order k. Then each (γ^j, γ^{-j}) with $1 \leq j \leq k-1$ induces an unexpected automorphism of $B\mathcal{U}_k$, if $\gamma^j \neq \gamma^{-j}$.*

In particular, the index of instability of $\mathcal{U}_k$ is at least k for odd k and at least $k-1$ for even k (we have to exclude the case $j = k/2$).

Remark 8.12. *Note that $\mathcal{U}$ is just $\mathcal{U}_3$. Indeed, by setting $k = 3$ in Example* 8.10 *we obtain the same γ and the same ordered pairs we used to construct $\mathcal{U}$.*

8.2 GENERALIZED CAYLEY GRAPHS

As Lemma 3.13 shows, the double cover BX of a Cayley graph X is also a Cayley graph. However, the converse does not hold (see for example [11, Example 3.4, Example 4.4]). Generalized Cayley graphs have been first introduced in [22] in attempt to characterize graphs whose canonical double cover is a Cayley graph.

Definition 8.13 ([22, p. 281], [11, Definition 2.1])**.** Let G be a group, $S \subseteq G$ a non-empty subset and $\alpha \in \mathrm{Aut}(G)$ an automorphism of the group G such that

1. $\alpha^2 = 1$,
2. for $x \in G$, $\alpha(x^{-1})x \notin S$,
3. for $x, y \in G$, if $\alpha(x^{-1})y \in S$, then $\alpha(y^{-1})x \in S$.

The Generalized Cayley graph $\mathrm{GCay}(G, S, \alpha)$ is the graph given by the following parameters:

- $V(\mathrm{GCay}(G, S, \alpha)) \coloneqq G$,
- $E(\mathrm{GCay}(G, S, \alpha)) \coloneqq \{\{x, y\} \mid x, y \in G, \alpha(x^{-1})y \in S\}$.

Remark 8.14. *Let G, S and α be as in Definition* 8.13 *and write $X := \mathrm{GCay}(G, S, \alpha)$ the corresponding Generalized Cayley graph. The condition* (2) *ensures that X has no loops. The condition* (3) *ensures that the relation of adjacency is symmetric i.e., X is an undirected graph.*

Remark 8.15. *Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph on a group G with a connection set S. Let $\alpha = 1$ be the trivial automorphism of G.*

Then $\alpha^2 = 1$, so (1) *is satisfied. By definition of a Cayley graph, S is does not contain the identity, so* (2) *holds. Finally, if $\alpha(x^{-1})y \in S$, then as S is inverse-closed, it holds that $y^{-1}x = \alpha(y^{-1})x \in S$, proving that* (3) *also holds.*

In conclusion, X is exactly the Generalized Cayley graph $\mathrm{GCay}(G, S, 1)$.

Lemma 8.16 ([22, p. 281])**.** *Let G, S and α be as in Definition* 8.13. *Let $X = \mathrm{GCay}(G, S, \alpha)$. Then the following hold.*

1. *$\alpha(S) = S^{-1}$, and conversely*
2. *if $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$, then $\alpha(S) = S^{-1}$ implies that α satisfies the condition* 8.13(3).
3. *For $x \in V(X)$, $N_X(x) = \alpha(x)S = \{\alpha(x)s \mid s \in S\}$.*

Proof. (1) Let $y \in S$. Then by setting $x = 1$ in Definition 8.13(3), we get that $\alpha(y^{-1})x = \alpha(y^{-1}) \in S$. Using Definition 8.13(1), we get that $y^{-1} \in \alpha(S)$. Hence, $S^{-1} \subseteq \alpha(S)$ and as α is a bijection, the desired conclusion follows.

(2) Assume that $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$ and $\alpha(S) = S^{-1}$. Then if $\alpha(x^{-1})y \in S$, we know that $\alpha(\alpha(x^{-1})y) \in \alpha(S) = S^{-1}$. In particular, $x^{-1}\alpha(y) \in S^{-1}$ and consequently, $(\alpha(y))^{-1}x = \alpha(y^{-1})x \in S$. This is exactly the condition Definition 8.13(3).

(3) Let $x, y \in G$. Then by definition, x and y are adjacent if and only if $\alpha(x^{-1})y \in S$ if and only if $\alpha(y) \in x\alpha(S)$. Applying (1), we get that this is further equivalent to $y \in \alpha(x)S$. It follows that $N_X(x) = \alpha(x)S$, as desired. □

In [22, Theorem 3.1], the authors claimed that for a connected, non-bipartite graph X, it holds that BX is a Cayley graph if and only if X is a generalized Cayley graph. This statement has been proven to be false in [11]. To compensate for this, an even wider class of graphs, named *extended Generalized Cayley graphs* has been introduced in [11, Definition 4.1]. The author then proves that for a connected, non-bipartite graph X, the canonical double cover BX is a Cayley graph if and only if X is an extended Generalized Cayley graph.

However, we will only need the following weaker result, that serves as a correction of [22, Theorem 3.1].

Recall that for a graph X, $\tau\colon V(BX) \to V(BX)$ is an automorphism of BX given by $\tau(x,i) = (x, i+1)$ for $x \in V(X), i \in \mathbb{Z}_2$.

Theorem 8.17 (Hujdurović [11, Theorem 3.2]). *Let X be a connected, non-bipartite graph. Then BX is a Cayley graph with a regular subgroup containing τ if and only if X is a generalized Cayley graph.*

We are now ready to introduce the main tool of this section for producing unstable graphs. The proof of this result given in [22] uses the result [22, Theorem 3.1] which contains an error, but only when concluding that for a generalized Cayley graph X, its canonical double cover BX is a Cayley graph as well. As this is true by Theorem 8.17, we present both the result and the original proof below.

Proposition 8.18 ([22, Proposition 3.3]). *Let X be a Generalized Cayley graph. If X is stable, then X is a Cayley graph.*

Proof. Assume that BX is a Cayley graph. Then by Proposition 2.25(1), $\mathrm{Aut}(BX)$ contains a regular subgroup, which we will denote by G. As X is stable, we know that $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X) \times S_2$. Define $H \coloneqq \{\varphi \mid \overline{\varphi} \in G\}$. As $\varphi \mapsto \overline{\varphi}$ is an injective homomorphism from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(BX)$ (see Lemma 3.4(3)), it follows that $H \cong \overline{H} = \overline{\mathrm{Aut}(X)} \cap G$.

As $\overline{H} = \overline{\mathrm{Aut}(X)} \cap G$ has exactly two orbits on $V(BX)$, given by the colours classes of BX, and is semi-regular (because G is semi-regular), it follows that H is a regular subgroup of $\mathrm{Aut}(X)$. By Theorem 2.26, this is equivalent to saying that X is a Cayley graph (on H). □

It follows from Proposition 8.18 that graphs that are Generalized Cayley graphs, but not Cayley, are necessarily unstable. Moreover, since [11, Theorem 4.3] states that for a connected, non-bipartite graph X, BX is a Cayley graph if X is an extended generalized Cayley graph, the same argument as in the proof of Proposition 8.18 applies. Therefore, we conclude that connected, non-bipartite extended generalized Cayley graphs that are not Cayley, are also unstable.

We will use the Proposition 8.18 to show that the following family contains non-trivially unstable graphs.

Definition 8.19. Let $n \geq 4$ and denote by $\mathcal{X}(n)$ the Generalized Cayley graph $\mathrm{GCay}(G, S, \alpha)$ with

- $G = \mathbb{Z}_n \times \mathbb{Z}_n$,
- $S = \{(1,0), (0, n-1), (1,1), (n-1, n-1)\}$,
- $\alpha(i,j) = (j,i)$ for $i, j \in \mathbb{Z}_n$.

Remark 8.20. *We show that parameters given in Definition* 8.19 *indeed define a Generalized Cayley graph.*

It is clear that α is an automorphism of G and moreover, $\alpha^2 = 1$. Hence, (1) *is satisfied.*

Condition (2) *holds because none of the elements of S are of the form $(k, -k)$ for $k \in \mathbb{Z}_n$.*

To prove that condition(3) *holds, we note that $\alpha(S) = S^{-1}$ by a direct computation and apply Lemma* 8.16(2)*.*

Lemma 8.21. *Let $n \geq 5$ be a positive integer such that* $\gcd(n, 4) = 1$*. Let $\mathcal{X}(n)$ be the graph defined in Definition* 8.19*. Then*

1. *$\mathcal{X}(n)$ is not vertex-transitive,*
2. *$\mathcal{X}(n)$ is connected, non-bipartite and twin-free.*

Proof. (1) Let $(i, j) \in V(\mathcal{X}(n))$. Then by Lemma 8.16(3), it holds that

$$\begin{aligned} N_{\mathcal{X}(n)}(i,j) = \alpha(i,j)S = (j,i)S = \\ = \{(j+1,i),(j,i-1),(j+1,i+1),(j-1,i-1)\}. \end{aligned} \tag{8.1}$$

Note that this implies that

$$N_{\mathcal{X}(n)}(0,0) = \{(1,0),(0,n-1),(1,1),(n-1,n-1)\} = S,$$

$$N_{\mathcal{X}(n)}(1,0) = \{(1,1),(0,0),(1,2),(n-1,0)\}.$$

A direct computation will show that $(1, 0)$ only lies on one triangle, namely the one formed by vertices $(0, 0), (1, 0)$ and $(1, 1)$.

However, since $\alpha(-(0, n-1)) + (n-1, n-1) = (1, 0) + (n-1, n-1) = (0, n-1) \in S$, it follows that $(0, 0)$ also lies on a triangle formed by the vertices $(0, 0), (0, n-1), (n-1, n-1)$.

In particular, there cannot exist an automorphism of $\mathcal{X}(n)$ that would map $(0, 0)$ to $(1, 0)$, proving that $\mathcal{X}(n)$ is not vertex-transitive.

(2) Connectedness of $\mathcal{X}(n)$ follows from a more general result given in [22, Proposition 3.5].

As we have already established that $\mathcal{X}(n)$ contains triangles, we know it is non-bipartite.

Note that from Eq. (8.1), it follows that sum of all elements in $N_{\mathcal{X}(n)}(i, j)$ equals $(4j + 1, 4i + 1)$. Hence, if $N_{\mathcal{X}(n)}(i, j) = N_{\mathcal{X}(n)}(k, \ell)$ for $i, j, k, \ell \in \mathbb{Z}_n$, it follows that $4j + 1 = 4\ell + 1$ and $4i + 1 = 4k + 1$. As $\gcd(n, 4) = 1$, 4 has a multiplicative inverse modulo n and it follows that $j = \ell, i = k$. In particular, $\mathcal{X}(n)$ is twin-free. $\square$

Theorem 8.22. *The graphs $\mathcal{X}(n)$ are non-trivially unstable for all $n \geq 5, \gcd(n, 4) = 1$. Moreover, the canonical double cover $B\mathcal{X}(n)$ is a Cayley graph.*

Proof. As $\mathcal{X}(n)$ is not vertex-transitive by Lemma 8.21(1), it cannot be isomorphic to a Cayley graph, which are always vertex-transitive (see Proposition 2.25(2)). It follows by Proposition 8.18 that $\mathcal{X}(n)$ is unstable. By Lemma 8.21(2), $\mathcal{X}(n)$ is also connected, non-bipartite and twin-free. Therefore, $\mathcal{X}(n)$ is non-trivially unstable. Finally, each $\mathcal{X}(n)$ is a Generalized Cayley graph, so $B\mathcal{X}(n)$ is a Cayley graph by Theorem 8.17. $\square$

8.3 DOUBLE GRAPHS OF PALEY GRAPHS

In his article [30], Surowski constructs three infinite families of arc-transitive unstable graphs. These constructions are surprising, because at the time the article appeared, it was thought that most arc-transitive graphs are stable, with the dodecahedron as one of the only non-trivially unstable examples available.

The three constructions Surowski describes are based on the hyperbolic space $\Omega^+(2n, 2)$, double graphs of self-complementary graphs and complex character theory of the finite simple group $\mathrm{PSL}_2(q)$, respectively. We will study the second construction in detail and provide brief remarks on the third one at the end of the subsection. We start by describing the second construction. The following idea is based on the work of E. Shult in [29].

We start with a graph X. Let X' denote an isomorphic copy of X with the underlying graph isomorphism being the map $x \mapsto x'$ for $x \in V(X)$. We construct a new graph which we denote by X^*.

- $V(X^*) \coloneqq \{+\infty\} \cup V(X) \cup V(X') \cup \{-\infty\}$, where $+\infty$ and $+\infty$ are new vertices satisfying $\pm\infty \notin V(X) \cup V(X')$.
- We define the edges of X^* in the following manner.

$$\begin{aligned} E(X^*) \coloneqq{}& E(X) \cup E(X') \cup \\ &\cup \{\{+\infty, x\} \mid x \in V(X)\} \cup \{\{-\infty, x'\} \mid x' \in V(X')\} \cup \\ &\cup \{\{x, y'\} \mid x, y \in V(X), x \neq y, \{x, y\} \notin E(X)\}. \end{aligned}$$

  In words, $E(X^*)$ is defined via the following incident relations:

  1. $+\infty$ is adjacent to every $x \in V(X)$ and no other vertices.
  2. Two vertices $x, y \in V(X)$ are adjacent in X^* if and only if they are adjacent in X. This is to say that X is the subgraph of X^* induced by $V(X) \subseteq V(X^*)$.

3. Two vertices $x \in V(X)$ and $y' \in V(X')$ are adjacent in X^* if and only if $x \neq y$ and x and y are non-adjacent in X i.e., $\{x, y\} \notin E(X)$.

4. Two vertices $x', y' \in V(X')$ are adjacent in X^* if and only if they are adjacent in X'. This is to say that X' is the subgraph of X^* induced by $V(X') \subseteq V(X^*)$.

5. $-\infty$ is adjacent to every $x' \in V(X')$ and no other vertices.

Definition 8.23. The graph X^* constructed above is called the *double graph of* X.

Proposition 8.24. *Let* X *be a graph. Then its double graph* X^* *has the following properties:*

1. *If* X *is not a complete graph, then* X^* *is connected.*

2. *If* X *is not an empty graph, then* X^* *is non-bipartite.*

3. X^* *is twin-free.*

4. *The map* $\gamma\colon V(X^*) \to V(X^*)$ *that swaps* $+\infty$ *and* $-\infty$ *as well as* x *and* x' *for all* $x \in V(X)$ *is an automorphism of* X^* *of order* 2.

Proof. (1) We first note that $N_{X^*}(+\infty) = V(X)$ and $N_{X^*}(-\infty) = V(X')$, so X^* has at most two connected components, namely C, with vertex set $\{+\infty\} \cup V(X)$ and C', with vertex set $\{-\infty\} \cup V(X')$. Since X is not a complete graph, it follows that X contains a pair of distinct non-adjacent vertices $x, y \in V(X)$. By definition, $\{x, y'\} \in E(X^*)$ with $x \in V(C)$ and $y' \in V(C')$. It follows that X has exactly one connected component $C \cup C'$ and is connected.

(2) Note that, if X is not an empty graph, it must contain at least one edge $\{x, y\} \in E(X)$. Then x, y and $+\infty$ form a 3-cycle in X^*. Hence X^* is non-bipartite.

(3) Note that the vertices $+\infty$ and $-\infty$ do not have twins. Furthermore, if $x \in V(X) \subseteq V(X^*)$ and $y \in V(X') \subseteq V(X^*)$, then $+\infty$ is a neighbour of x, but not y. In particular, x and y are not twins.

The only remaining case is when $x, y \in V(X) \subseteq V(X^*)$ (the case $x, y \in V(X') \subseteq V(X^*)$ is entirely symmetric). If x and y are twins, then they are not adjacent. Hence, x is adjacent to $y' \in V(X')$, while y is not, a contradiction.

(4) We first note that γ is clearly a non-trivial permutation of $V(X^*)$ and is its own inverse. Hence, it is of order 2.

Note that

- γ swaps the edges $\{+\infty, x\}$ and $\{-\infty, x'\}$ for $x \in V(X)$,

- γ swaps the edges $\{x, y\}$ and $\{x', y'\}$ with $x, y \in V(X)$, and

- γ swaps the edges $\{x, y'\}$ and $\{x', y\}$ for $x, y \in V(X), x \neq y$ and $\{x, y\} \notin E(X)$.

This shows that γ preserves the set of edges of X and is consequently an automorphism of X. □

Corollary 8.25. *Let X be a graph that is neither empty nor complete. Then X^* is either stable or non-trivially unstable.*

It follows from Corollary 8.25 that to produce non-trivially unstable graphs via this construction, we just need to ensure that X^* is unstable.

It turns out that when X is a self-complementary graph, this is exactly what happens. Recall that a graph is self-complementary if it is isomorphic to its complement - see Definition 2.38.

Definition 8.26 ([30, p. 103])**.** Let X be a self-complementary graph. A graph automorphism $t\colon V(X) \to V(\overline{X}) = V(X)$ between X and its complement $\overline{X}$ is called a *complementing permutation.*

Note that complementing permutations are characterized by the following property:

$$\{x, y\} \in E(X) \text{ if and only if } \{t(x), t(y)\} \notin E(X), \forall x, y \in V(X).$$

In turns out that we can use Wilson's criteria from Section 5.1 to establish that double graphs of self-complementary graphs are unstable.

Theorem 8.27. *Let X be a self-complementary graph with at least two vertices. Then X^* is non-trivially unstable with an anti-symmetry. In particular, instability of X^* can be explained by Theorem* 5.14.

Proof. Let γ be the map swapping $+\infty$ and $-\infty$ as well as all x and x' of X^* for $x \in V(X)$. Then by Proposition 8.24(4), γ is an automorphism of X^* of order 2.

As X is self-complementary, there exists a complementing permutation $t\colon V(X) \to V(X)$. Recall that this means that

$$\{x, y\} \in E(X) \text{ if and only if } \{t(x), t(y)\} \notin E(X). \tag{8.2}$$

Define the following map.

$$\alpha\colon V(X^*) \to V(X^*)$$
$$\alpha(y) = \begin{cases} +\infty, & y = +\infty \\ t(x)', & y = x, x \in V(X) \\ t(x), & y = x', x \in V(X) \\ -\infty, & y = -\infty \end{cases}$$

As t is permutation of $V(X)$, it follows that α is a permutation of $V(X)$. We will show that α is anti-symmetry of X^* with respect to γ by proving they satisfy both conditions listed in Definition 5.13.

Let $y \in V(X^*)$. We will show that $(\alpha\gamma)(y) = (\gamma\alpha)(y)$. If $y \in \{\pm\infty\}$, the claim clearly holds as α fixes both of these vertices, while γ swaps them.

Let $y = x \in V(X) \subseteq V(X^*)$. Then $(\alpha\gamma)(x) = \alpha(x') = t(x)$, while $(\gamma\alpha)(x) = \gamma(t(x)') = t(x)$. Similarly, if $y = x', x \in V(X)$, then $(\alpha\gamma)(x') = \alpha(x) = t(x)'$, while $(\gamma\alpha)(x') = \gamma(t(x)) = t(x)'$. In conclusion, α and γ commute, so they satisfy condition (1) of Definition 5.13.

Let $\{x, y\} \in E(X^*)$. We show that $\{\alpha(x), (\alpha\gamma)(y)\} \in E(X^*)$.

Case 1. $x = +\infty$ *and* $y \in V(X) \subseteq V(X^*)$ *(or vice-versa)*

Then $\alpha(x) = +\infty$ and $(\alpha\gamma)(y) = \alpha(y') = t(y)$. As $t(y) \in V(X)$, the condition is satisfied.

Case 2. $x, y \in V(X) \subseteq V(X^*)$

Then $\alpha(x) = t(x)'$ and $(\alpha\gamma)(y) = \alpha(y') = t(y)$. Because $\{x, y\} \in E(X)$, we conclude that $t(x) \neq t(y)$ (as $x \neq y$) and by Eq. (8.2), also $\{t(x), t(y)\} \notin E(X)$. Hence, $\{t(x)', t(y)\} \in E(X^*)$.

Case 3. $x \in V(X) \subseteq V(X^*), y \in V(X') \subseteq V(X^*)$ *(or vice-versa).*

We can find $z \in V(X)$ such that $y = z'$. Note that then $x \neq z$ and $\{x, z\} \notin E(X)$. It follows that $\alpha(x) = t(x)'$ and $(\alpha\gamma)(y) = (\alpha\gamma)(z') = \alpha(z) = t(z)'$. As $\{x, z\} \notin E(X)$ and t is a complementing permutation, it follows by Eq. (8.2) that $\{t(x), t(z)\} \in E(X)$. In particular, $\{t(x)', t(z)'\} \in E(X^*)$.

Case 4. $x, y \in V(X') \subseteq V(X^*)$

Then we can find $z, w \in V(X)$ such that $x = z'$ and $y = w'$ and $\{z, w\} \in E(X)$. It follows that $\alpha(x) = \alpha(z') = t(z)$ and $(\alpha\gamma)(y) = (\alpha\gamma)(w') = \alpha(w) = t(w)'$ As $x \neq y$, also $z \neq w$ and consequently, $t(z) \neq t(w)$. Moreover, as $\{z, w\} \in E(X)$ and t is a complementing permutation, Eq. (8.2) implies that $\{t(z), t(w)\} \notin E(X)$. Hence, $\{t(z), t(w)'\} \in E(X^*)$.

Case 5. $x = +\infty, y \in V(X') \subseteq V(X^*)$ *(or vice-versa)*

We can find $z \in V(X)$ such that $y = z'$. Then $\alpha(x) = +\infty$ and $(\alpha\gamma)(y) = (\alpha\gamma)(z') = \alpha(z) = t(z)'$. As $t(z) \in V(X)$, the condition is clearly satisfied.

It follows that α and γ satisfy condition (2). Hence, X^* has an anti-symmetry and is unstable by Theorem 5.14. Moreover, it is non-trivially unstable by Corollary 8.25. This finishes the proof. □

Remark 8.28. *In [30, Proposition* 4.1*], Surowski shows that the double graph* X^* *of a self-complementary graph* X *with a complementing permutation* $t\colon V(X) \to V(X)$ *is* *unstable by constructing an unexpected automorphism* $T\colon V(BX^*) \to V(BX^*)$ *of* BX^* *given by*

a) $T(+\infty, 0) = (+\infty, 0)$, $T(-\infty, 0) = (-\infty, 0)$;

b) $T(x, 0) = (t(x)', 0), T(x', 0) = (t(x), 0), x \in V(X)$;

c) $T(x, 1) = (t(x), 1), T(x', 1) = (t(x)', 1), x \in V(X)$;

d) $T(+\infty, 1) = (-\infty, 1)$, $T(-\infty, 1) = (+\infty, 1)$.

It is easy to see that T *is actually of the following form, where* α *and* γ *are permutation of* $V(BX^*)$ *defined in the proof of Theorem* 8.27.

$$T(y, i) = \begin{cases} (\alpha(y), 0), & i = 0 \\ ((\gamma\alpha)(y), 1), & i = 1 \end{cases}$$

Corollary 8.29. *Let* X *be a self-complementary graph with at least two vertices. Then* X^* *is non-trivially unstable.*

Proof. Note that the complement of the complete graph K_n is the empty graph $\overline{K_n}$. Furthermore, these graphs are isomorphic if and only if $n = 1$ (for example, $E(\overline{K_n}) = \emptyset$, while $E(K_n)$ is non-empty when $n \geq 2$).

As X is not K_1 and is assumed to be self-complementary, it follows that X is neither complete nor empty. Then by Corollary 8.25, X^* is either stable or non-trivially unstable.

As X is self-complementary, Theorem 8.27 implies that X^* is unstable.

In conclusion, X^* is non-trivially unstable, as desired. □

Example 8.30. Let X be a 3-edge path. Then X is self-complementary on 4 vertices, so by Corollary 8.29, its double graph X^* is non-trivially unstable. However, calculations in MAGMA show that it is not arc-transitive.

Example 8.30 shows that even though double graphs of self-complementary graphs are always non-trivially unstable (by Corollary 8.29), they are not necessarily arc-transitive. Therefore, we need to impose further structural restrictions on X, so that X^* will be arc-transitive. The following proposition suggests one such condition.

Proposition 8.31 (Surowski [30, Proposition 4.2])**.** *Let* X *be a vertex-transitive graph, and assume that given some vertex* $x \in V(X)$ *there exist bijections* $h_1\colon X_1(x) \to X_1(x)$ *and* $h_2\colon \overline{X}_1(x) \to \overline{X}_1(x)$ *such that* h_1 *and* h_2 *are automorphisms of subgraphs of* X *induced by* $X_1(x)$ *and* $\overline{X}_1(x)$*, respectively, and that*

$$\{y, z\} \in E(X) \text{ if and only if } \{h_1(y), h_2(z)\} \notin E(X), y \in X_1(x), z \in \overline{X}_1(x).$$

Then X^* *is arc-transitive.*

All that is left now is to find examples of self-complementary, vertex-transitive graphs that satisfy the condition in Proposition 8.31.

Luckily, Paley graphs come to our rescue. Let q be a prime power such that $q \equiv 1 \pmod 4$ and let S and NS denote the sets of non-zero squares and non-squares of the Galois field $\mathrm{GF}(q)$, respectively.

Lemma 8.32. *The Paley graph* $\mathrm{P}(q)$ *satisfies the condition listed in Proposition* 8.31 *for* $x = 0$ *and* h_1, h_2 *being restrictions of the inversion map* $\iota\colon y \mapsto y^{-1}$ *to* $\mathrm{P}(q)_1(0)$ *and* $\overline{\mathrm{P}(q)}_1(0)$, *respectively.*

Proof. Let $x, y \in \mathrm{GF}(q)$ be non-zero. Then an easy computation shows that

$$y^{-1} - x^{-1} = (xy)^{-1}(x - y). \tag{8.3}$$

Note that $\mathrm{P}(q)_1(0) = S$ and $\overline{\mathrm{P}(q)}_1(0) = (\mathrm{GF}(q) \setminus \{0\}) \setminus S = NS$ (compare with Proposition 2.40(3)). Since for $x \in \mathrm{GF}(q), x \neq 0$ it holds that x is a square if and only if x^{-1} is a square, it follows that the maps $h_1\colon\ \mathrm{P}(q)_1(0) \to \mathrm{P}(q)_1(0)$ and $h_2\colon \overline{\mathrm{P}(q)}_1(0) \to \overline{\mathrm{P}(q)}_1(0)$ are well-defined.

Let $\{x, y\} \in E(X)$ be an edge of X. Then $y - x$ is a square. If x, y are both squares or both non-squares, it follows that $(xy)^{-1}$ is a square. It follows by Eq. (8.3) that $y^{-1} - x^{-1}$ is a square, equivalently $\{\iota(x), \iota(y)\} = \{x^{-1}, y^{-1}\} \in E(X)$. This proves that h_1 and h_2 are automorphisms of the subgraphs of X induced by S and NS, respectively.

Finally, if $x \in S$ and $y \in NS$ are adjacent, then $h_1(x) = x^{-1}$ and $h_2(y) = y^{-1}$ are not adjacent, again by Eq. (8.3), since $(xy)^{-1}$ is a non-square. The converse holds by an analogous argument. □

Definition 8.33. The *double Paley graph*, denoted $\mathrm{P}^*(q)$, is the double graph $(\mathrm{P}(q))^*$ of the Paley graph $\mathrm{P}(q)$.

Theorem 8.34 (Surowski, [30, Theorem 4.3])**.** *Let* q *be a prime power such that* $q \equiv 1 \pmod 4$. *Then the double Paley graph* $\mathrm{P}^*(q)$ *is arc-transitive and non-trivially unstable.*

Proof. By Definition 8.33, $\mathrm{P}^*(q)$ is just the double graph of $\mathrm{P}(q)$. By Proposition 2.40(3), $\mathrm{P}(q)$ is self-complementary and as its order is at least 5, Corollary 8.29 implies that $\mathrm{P}^*(q)$ is non-trivially unstable. By Proposition 2.40(1) and Lemma 8.32, we can apply Proposition 8.31 to $\mathrm{P}(q)$ to conclude that $\mathrm{P}^*(q)$ is arc-transitive. □

In [31], Surowski actually computes the automorphism group of $\mathrm{P}^*(q)$ and its bipartite double cover.

Theorem 8.35 (Surowski [31, Theorem 1.5])**.** *Let* $q \equiv 1 \pmod 4$ *be a prime power. Let* $\mathrm{P}(q)$ *be the Paley graph of order* q *and* $\mathrm{P}^*(q)$ *its double graph.*

Then it holds that

$$\mathrm{Aut}(\mathrm{P}^*(q)) = \mathbb{Z}_2 \times P\Sigma L_2(q).$$

Moreover, the automorphism group of the double cover of $\mathrm{P}^*(q)$ *has the following form.*

$$\mathrm{Aut}(B(\mathrm{P}^*(q))) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes P\Gamma L_2(q)$$

Here $P\Sigma L_2(q)$ *centralizes* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *and any element of* $P\Gamma L_2(q) \setminus P\Sigma L_2(q)$ *acts as the non-trivial involutary automorphism of* $\mathbb{Z}_2 \times \mathbb{Z}_2$*.*

Corollary 8.36 (Surowski [31, p. 4])**.** *The double Paley graph* $\mathrm{P}^*(q)$ *is non-trivially unstable with index of instability* 2*.*

Proof. We know that $\mathrm{P}^*(q)$ is non-trivially unstable by Theorem 8.34. By Theorem 8.35, it follows that the index of instability of $\mathrm{P}^*(q)$ equals to the index of $P\Sigma L_2(q)$ in $P\Gamma L_2(q)$, which is 2. □

We now comment on one additional constructions Surowski considers in [30].

Definition 8.37 (Surowski [30, p. 105])**.** Let q a power of an odd prime p. Let $\mathcal{C}$ be the conjugacy class of the element represented by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the finite simple group $\mathrm{PSL}_2(q)$. We define the graph $\Gamma(q)$ with the vertex set $V(\Gamma(q)) \coloneqq \mathcal{C}$ and $x, y \in \mathcal{C}$ adjacent if and only if xy has an order 3.

Note that conjugation by a fixed element of $\mathrm{PSL}_2(q)$ induces an automorphism of $\Gamma(q)$.

Proposition 8.38 (Surowski [30, Proposition 5.2])**.** *Let q be a power of an odd prime such that* $q \equiv 5$ *or* $7 \pmod{12}$*. Then the graph* $\Gamma(q)$ *is q-regular and arc-transitive.*

Theorem 8.39 (Surowski [30, Theorem 5.6])**.** *Let q be a power of an odd prime such that* $q \equiv 5$ *or* $7 \pmod{12}$*. Then the graph* $\Gamma(q)$ *is (non-trivially) unstable.*

In [31], Surowski describes the automorphism groups of $\Gamma(q)$ and $B\Gamma(q)$, as well as determines the index of instability of $\Gamma(q)$, for the case when $q = p$ is an odd prime.

Theorem 8.40 (Surowski [31, Theorem 2.2])**.** *Let p be an odd prime and consider the graph* $\Gamma(p)$*. Then it holds that*

$$\mathrm{Aut}(\Gamma(p)) \cong \begin{cases} \mathbb{Z}_2 \times \mathrm{PSL}_2(q), & p \equiv 5 \pmod{12} \\ \mathrm{PGL}_2(q), & p \equiv 7 \pmod{12} \end{cases}$$

Moreover, in either case, it holds that

- *the structure of* $\mathrm{Aut}(B\Gamma(q))$ *is isomorphic to* $\mathrm{PGL}_2(q)$ *acting on the dihedral group* D_{p-1} *of order* $p-1$*, and*
- *the index of instability of* $\Gamma(p)$ *is* $\frac{p-1}{2}$*.*

8.4 NON-TRIVIALLY UNSTABLE CIRCULANT GRAPHS WITH NO WILSON TYPE

We recall that Example 5.51 shows that the Wilson types are not sufficient for explaining instability of all non-trivially unstable circulants. In the following example, we will construct an infinite family of non-trivially unstable circulant graphs, that do not have a Wilson type, but whose instability can be explained by the generalizations of Wilson types introduced in Section 5.4.

Example 8.41 (Hujdurović-Mitrović-Morris [13, Example 3.9])**.** Let $n = 2p^2$, where p is prime and $p \equiv 1 \pmod 4$, and choose $c \in \mathbb{Z}$, such that $c^2 \equiv -1 \pmod p$. Fix some $a \in \mathbb{Z}_n$ of order p, and let $S = S_e \cup S_o$, where

$$\begin{aligned} S_e &= \big(\pm 2 + \langle a \rangle\big) \cup \{\pm a\} \subseteq 2\mathbb{Z}_n, \\ S_o' &= \big(\pm 2 + \langle a \rangle\big) \cup \{\pm ca\} \subseteq 2\mathbb{Z}_n, \\ S_o &= n/2 + S_o' \subseteq 1 + 2\mathbb{Z}_n. \end{aligned}$$

Then

1. $\operatorname{Cay}(\mathbb{Z}_n, S) \cong \operatorname{Cay}\big(\mathbb{Z}_n, S + (n/2)\big)$, so Proposition 5.56 implies that $\operatorname{Cay}(\mathbb{Z}_n, S)$ is (non-trivially) unstable, but
2. $\operatorname{Cay}(\mathbb{Z}_n, S)$ does not have a Wilson type.

Proof. (1) Choose a set $\mathcal{R}$ of coset representatives for $\langle a \rangle$ in $\mathbb{Z}_n$, such that $\mathcal{R} + n/2 = \mathcal{R}$, and define $\alpha \colon \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$\alpha(r + x) = r + cx \quad \text{for } r \in \mathcal{R} \text{ and } x \in \langle a \rangle.$$

It suffices to show that if $v, w \in \mathbb{Z}_n$, such that $v - w \in S$, then $\alpha(v) - \alpha(w) \in S + n/2$.

First, consider two vertices $v = r + x$ and $w = r + y$ that are in the same coset of $\langle a \rangle$. Then the definition of S implies that $v - w = \pm a$, so $y = x \pm a$, so

$$\alpha(v) - \alpha(w) = \big(r + cx\big) - \big(r + c(x \pm a)\big) = \pm ca \in S_o' = S_o + n/2.$$

Next, suppose $v \in w + n/2 + \langle a \rangle$. Assume, without loss of generality, that $v \in 1 + 2\mathbb{Z}_n$ and $w \in 2\mathbb{Z}_n$. Write $v = r + n/2 + x$ and $w = r + y$ with $r \in \mathcal{R}$ and $x, y \in \langle a \rangle$. The definition of S implies that $v - w = n/2 \pm ca$, so $y = x \pm ca$, so (using the fact that $c^2 \equiv -1 \pmod p$) we have

$$\alpha(v) - \alpha(w) = \big(r + n/2 + cx\big) - \big(r + c(x \pm ca)\big) = n/2 \pm c^2 a = n/2 \mp a \in n/2 + S_e.$$

We may now assume that v and w are in two different cosets of $\langle a, n/2 \rangle$. Then, from the definition of S, we see that every vertex in $v + \langle a \rangle$ is adjacent to every vertex in $w + \langle a \rangle$. Since $\alpha(v) \in v + \langle a \rangle$ and $\alpha(w) \in w + \langle a \rangle$, it is therefore obvious that $\alpha(v)$ is adjacent to $\alpha(w)$.

(2) The proof is by contradiction.

Suppose, first, that $\mathrm{Cay}(\mathbb{Z}_n, S)$ has Wilson type (C.1), (C.2′), or (C.3′). Then Remark 5.55 tells us that the graph actually has Wilson type (C.1), so $h + S_e = S_e$ for some non-zero $h \in 2\mathbb{Z}_n$. Since $2\mathbb{Z}_n \cong \mathbb{Z}_{p^2}$, we know that $|h|$ is divisible by p. Since S_e is a union of cosets of $\langle h \rangle$, this implies that $|S_e|$ is divisible by p, which contradicts the fact that $|S_e| = 2|a| + 2 = 2p + 2$.

We may now assume that the graph is of Wilson type (C.4). Then we can find $m \in \mathbb{Z}_{2p^2}^{\times}$, such that $mS + n/2 = S$. Since $n/2$ is odd, this implies $mS_e + n/2 = S_o$, so $mS_e = S_o'$. By passing to the quotient group $2\mathbb{Z}_n/\langle a \rangle$, we conclude that $m \equiv \pm 1 \pmod{p}$. So $ma = a \notin S_o'$. This contradicts the fact that $mS_e = S_o'$. □

As we already remarked, we have seen in Chapter 7 that every non-trivially unstable circulant graph of valency at most 7 has a Wilson type, so the following examples have minimal valency among those that do not have a Wilson type. Their instability follows by Proposition 5.56.

Example 8.42 (Hujdurović-Mitrović-Morris [13, Example 3.10])**.** Let $n \coloneqq 3 \cdot 2^{\ell}$, where $\ell \geq 4$ is even, and let

$$S \coloneqq \left\{ \pm 3, \pm 6, \pm \frac{n}{12}, \frac{n}{2} \pm 3 \right\}.$$

Then the circulant graph $X \coloneqq \mathrm{Cay}(\mathbb{Z}_n, S)$ has valency 8 and is non-trivially unstable, but does not have a Wilson type.

9 CONCLUSION

In this thesis, we reviewed known and developed new results on canonical bipartite double covers of graphs and their automorphisms. A graph is said to be stable if all automorphisms of its canonical double cover are expected (that is, inherited from its factors). If the canonical double cover has additional automorphisms, referred to as unexpected automorphisms, the graph is called unstable. We have paid special attention to non-trivially unstable graphs, which are not only unstable but also connected, non-bipartite and twin-free.

New results establishing stability of graphs have been discussed in Chapter 4 (in particular, Proposition 4.15, Proposition 4.18 and Lemma 4.32). Besides the results on stability of Cayley graphs of abelian groups of odd order and arc-transitive circulants, that have already been known, we discussed the classifications of non-trivially unstable circulants of order $2p$ (see Theorem 6.18) and non-trivially unstable circulants of low valency (in Chapter 7). We were able to classify unstable Andrásfai graphs (Proposition 4.10) and Johnson graphs (Theorem 4.30). We proved infinitely many Kneser graphs are stable (Corollary 4.23 and Corollary 4.24). We also considered strongly regular graphs.

We gave a summary of Steve Wilson's results on unexpected automorphisms. In particular, we have studied his characterization of non-trivially unstable graphs in terms of graphs with anti-symmetry, (generalized) cross-covers and twists of graphs. We have derived Wilson types and demonstrated some of their limitations. New results in this area include the generalizations of Wilson types (Theorem 5.52, Proposition 5.56 and Proposition 5.58).

Finally, we have presented different methods from the literature, borrowing ideas from the theory of TF-automorphisms, double graphs and Generalized Cayley graphs, for generating non-trivially unstable graphs with surprising properties such as asymmetry or arc-transitivity. In Example 8.41 and Example 8.42, we constructed infinite families of non-trivially unstable circulants whose instability can be explained by generalizations of Wilson types, but not by the original four types.

Despite the progress that has been made, many of the questions regarding unstable graphs and their canonical double covers still remain open. Even in the most well-studied case of circulant graphs, the classification of all non-trivially unstable members is incomplete. It seems likely that additional conditions explaining instability remain to be discovered.

10 DALJŠI POVZETEK V SLOVENSKEM JEZIKU

V tem magisterskem delu smo pregledali znane in razvili nove rezultate o kanoničnih dvojnih krovih grafov in njihovih avtomorfizmih.

Na začetku magisterske naloge, v poglavju 2, smo se spomnili različnih pomembnih pojmov iz teorije permutacijskih grup in (algebraične) teorije grafov. Predstavili smo produkte grafov, Cayleyjeve grafe, Paleyeve grafe, grafe brez dvojčkov in kartezične skelete grafov ter določili nekatere njihove lastnosti.

V poglavju 3 smo kanonični dvojni krov grafa X definirali kot graf $BX \coloneqq X \times K_2$. Opazili smo nekaj njegovih pomembnih lastnosti. Na primer, BX je povezan, če in samo če je X povezan in ni dvodelen. Poleg tega smo pokazali, da je podgrupa $\mathrm{Aut}(BX)$, ki jo generirajo dvigi avtomorfizmov X in avtomorfizem $\tau\colon (x,i) \mapsto (x,i{+}1)$, izomorfna $\mathrm{Aut}(X)\times S_2$. Elemente te podgrupe smo imenovali pričakovani avtomorfizmi BX. Stabilne grafe smo definirali kot grafe, za katere so vsi avtomorfizmi pričakovani, to je $\mathrm{Aut}(BX) \cong \mathrm{Aut}(X)\times S_2$. Za grafe, katerih kanonični dvojni krov ima dodatne avtomorfizme, imenovane nepričakovani avtomorfizmi, pravimo, da so nestabilni. Indeks nestabilnosti je definiran kot indeks podgrupe pričakovanih avtomorfizmov $\mathrm{Aut}(X)\times S_2$ v grupi avtomorfizmov dvojnega krova BX. Opazili smo, da je indeks nestabilnosti 1 takrat in samo takrat, ko je graf stabilen, in navedli smo primere stabilnih in nestabilnih grafov. Ugotovili smo, da je graf nestabilen, če je nepovezan, dvodelen (z netrivialno grupo avtomorfizmov) ali vsebuje različne točke z enakimi množicami sosedov. To nas je pripeljalo do definicije netrivialno nestabilnih grafov kot nestabilnih grafov, ki so tudi povezani, niso dvodelni in nimajo dvojčkov.

Obravnavali smo več zadostnih pogojev za stabilnost različnih vrst grafov, ki so na voljo v literaturi. Ti pogoji so segali od algebrskih lastnosti skupine avtomorfizmov dvojnega krova do kombinatoričnih omejitev, ki jih izpolnjuje sam graf. Poleg tega nam je uspelo posplošiti nekatere prej pridobljene rezultate, kar je privedlo do novih močnejših rezultatov, zlasti za grafe z lastnostjo, da je njihova vsaka povezava vsebovana v trikotniku. Ti rezultati so bili uporabljeni za preučevanje stabilnosti različnih družin grafov. V poglavju 4 smo se ukvarjali z krepko regularnimi grafi. Klasificirali smo tudi vse nestabilne Johnsonove grafe in pokazali, da je neskončno veliko Kneserjevih grafov stabilnih.

V poglavju 5 smo povzeli rezultate Steva Wilsona o nepričakovanih avtomorfizmih grafov. Obravnavali smo vsakega od njegovih štirih kriterijev nestabilnosti za splošne grafe. Še posebej smo preučili njegovo karakterizacijo netrivialno nestabilnih grafov v smislu grafov z anti-simetrijo, (posplošenih) križnih krovov in zasukov grafov. Z uporabo teh rezultatov smo izpeljali Wilsonove tipe za cirkulante. Nato smo obravnavali popravke Wilsonovih tipov. Njihove omejitve smo pokazali s primeri netrivialno nestabilnih cirkulantov brez Wilsonovega tipa. Nato smo preučili posplošitve Wilsonovih tipov.

V poglavju 6 smo, začenši s cirkulantimi praštevilskega reda in prehajajoč na cirkulante poljubnega lihega reda, na koncu pokazali, da ni netrivialno nestabilnih Cayleyjevih grafov na abelskih grupah lihega reda. Enak rezultat smo ugotovili za ločnotranzitivne cirkulante. V istem poglavju smo dobili popolno klasifikacijo netrivialno nestabilnih cirkulantov reda $2p$, za vsa praštevila p.

V poglavju 7 pa smo klasificirali vse netrivialno nestabilne cirkulante z valenco največ 7.

Nazadnje smo predstavili štiri različne metode za generiranje netrivialno nestabilnih grafov s presenetljivimi lastnostmi. Z uporabo TF-avtomorfizmov smo ustvarili neskončno družino netrivialno nestabilnih asimetričnih grafov. S konstrukcijo dvojnega grafa smo zgradili dvojne Paleyeve grafe, ki so netrivialno nestabilni ločno-trazitivni grafi z indeksom nestabilnosti, enakim 2. Konstruirali smo tudi družino netrivialno nestabilnih posplošenih Cayleyjevih grafov, kateri niso točkovno-trazitivni ampak imajo dvojni krov ki je Cayleyjev graf, ter družino netrivialno nestabilnih cirkulantov brez Wilsonovega tipa.

Kljub doseženemu napredku so številna vprašanja v zvezi z nestabilnimi grafi in njihovimi kanoničnimi dvojnimi krovi še vedno odprta. Celo v najbolj raziskanem primeru cirkulantov je klasifikacija vseh netrivialno nestabilnih členov nepopolna. Zdi se verjetno, da je treba odkriti še dodatne pogoje, ki pojasnjujejo nestabilnost.

11 REFERENCES

[1] Young Gheel Baik, Yanquan Feng, Hyo Seob Sim, and Mingyao Xu. On the normality of cayley graphs of abelian groups. *Algebra Colloq.*, 1:126–133, 1998.

[2] Richard A Brualdi, Frank Harary, and Zevi Miller. Bigraphs versus digraphs via matrices. *Journal of Graph Theory*, 4(1):51–73, 1980.

[3] John D Dixon and Brian Mortimer. *Permutation groups*, volume 163. Springer Science & Business Media, 1996.

[4] Ted Dobson, Štefko Miklavič, and Primož Šparl. On automorphism groups of deleted wreath products. *Mediterranean Journal of Mathematics*, 16(6):1–27, 2019.

[5] Shao-Fei Du, Ru-Ji Wang, and Ming-Yao Xu. On the normality of cayley digraphs of groups of order twice a prime. *Australasian Journal of Combinatorics*, 18:227–234, 1998.

[6] Yan-Quan Feng, Klavdija Kutnar, Aleksander Malnič, and Dragan Marušič. On 2-fold covers of graphs. *Journal of Combinatorial Theory, Series B*, 98(2):324–341, 2008.

[7] Blas Fernandez and Ademir Hujdurović. Canonical double covers of circulants. *Journal of Combinatorial Theory, Series B*, 154:49–59, 2022.

[8] Chris Godsil and Gordon F Royle. *Algebraic graph theory*, volume 207. Springer Science & Business Media, 2001.

[9] Richard H Hammack and Wilfried Imrich. Vertex-transitive direct products of graphs. *The Electronic Journal of Combinatorics*, pages P2–10, 2018.

[10] Richard H Hammack, Wilfried Imrich, Sandi Klavžar, Wilfried Imrich, and Sandi Klavžar. *Handbook of product graphs*, volume 2. CRC press Boca Raton, 2011.

[11] Ademir Hujdurović. Graphs with cayley canonical double covers. *Discrete Mathematics*, 342(9):2542–2548, 2019.

[12] Ademir Hujdurović, Đorđe Mitrović, and Dave Witte Morris. Automorphisms of the double cover of a circulant graph of valency at most 7, 2021. `arXiv:2108.05164 [math.CO]`.

[13] Ademir Hujdurović, Đorđe Mitrović, and Dave Witte Morris. On automorphisms of the double cover of a circulant graph. *The Electronic Journal of Combinatorics*, 28(4)(P4.43), 2021.

[14] Andrew Kotlov and László Lovász. The rank and size of graphs. *Journal of Graph Theory*, 23(2):185–189, 1996.

[15] István Kovács. Classifying arc-transitive circulants. *Journal of Algebraic Combinatorics*, 20(3):353–358, 2004.

[16] Matjaž Krnc and Tomaž Pisanski. Characterization of generalized petersen graphs that are kronecker covers. *arXiv preprint arXiv:1802.07134*, 2018.

[17] Josef Lauri, Russell Mizzi, and Raffaele Scapellato. Unstable graphs: A fresh outlook via tf-automorphisms. *ARS MATHEMATICA CONTEMPORANEA*, 8(1):115–131, 2014.

[18] Josef Lauri, Russell Mizzi, and Raffaele Scapellato. The construction of a smallest unstable asymmetric graph and a family of unstable asymmetric graphs with an arbitrarily high index of instability. *Discrete Applied Mathematics*, 266:85–91, 2019.

[19] Cai Heng Li. On isomorphisms of finite cayley graphs—a survey. *Discrete Mathematics*, 256(1):301–334, 2002.

[20] Caiheng Li, Dragan Marušič, and Joy Morris. Classifying arc-transitive circulants of square-free order. *Journal of Algebraic Combinatorics*, 14(2):145–151, 2001.

[21] Dragan Marušič, Raffaele Scapellato, and N Zagaglia Salvi. A characterization of particular symmetric (0, 1) matrices. *Linear Algebra and its Applications*, 119:153–162, 1989.

[22] Dragan Marušič, Raffaele Scapellato, and Norma Zagaglia Salvi. Generalized cayley graphs. *Discrete mathematics*, 102(3):279–285, 1992.

[23] Dave Witte Morris. On automorphisms of direct products of cayley graphs on abelian groups. *The Electronic Journal of Combinatorics*, page P3.5, 2021.

[24] Roman Nedela and Martin Škoviera. Regular embeddings of canonical double coverings of graphs. *Journal of Combinatorial Theory, Series B*, 67(2):249–277, 1996.

[25] Walter Pacco and Raffaele Scapellato. Digraphs having the same canonical double covering. *Discrete Mathematics*, 173(1):291–296, 1997.

[26] Yan-Li Qin, Binzhou Xia, and Sanming Zhou. Stability of circulant graphs. *Journal of Combinatorial Theory, Series B*, 136:154–169, 2019.

[27] Gert Sabidussi. On a class of fixed-point-free graphs. *Proceedings of the American Mathematical Society*, 9(5):800–804, 1958.

[28] Gert Sabidussi. The composition of graphs. *Duke Mathematical Journal*, 26(4):693–696, 1959.

[29] Ernest Shult. The graph extension theorem. *Proceedings of the American Mathematical Society*, 33(2):278–284, 1972.

[30] David B Surowski. Stability of arc-transitive graphs. *Journal of Graph Theory*, 38(2):95–110, 2001.

[31] David B Surowski. Automorphism groups of certain unstable graphs. *Mathematica Slovaca*, 53(3):215–232, 2003.

[32] Derek A Waller. Double covers of graphs. *Bulletin of the Australian Mathematical Society*, 14(2):233–248, 1976.

[33] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[34] Steve Wilson. Unexpected symmetries in unstable graphs. *Journal of Combinatorial Theory, Series B*, 98(2):359–383, 2008.

[35] Ming-Yao Xu. Automorphism groups and isomorphisms of cayley digraphs. *Discrete Mathematics*, 182(1):309–319, 1998.

[36] Bohdan Zelinka. On double covers of graphs. *Mathematica Slovaca*, 32(1):49–54, 1982.