A Collection of Math Competition Problems

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A Collection of Math Competition Problems

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Preface

The International Mathematics Competition (IMC) for university students is an annual mathematics competition open to all undergraduate students of mathematics. The IMC is primarily a competition for individuals, although most participating universities select and send one or more teams of students.

Another prestige international mathematical competition for university students is the Vojtěch Jarník International Mathematical Competition. It is a competition with varied international participation, known as the oldest mathematics competition for university students in the European Union. The Vojtěch Jarník competition held each year since 1991 in Ostrava, Czech Republic.

In the last decade the students from University of Primorska took active participation in both competitions. In the period 2013-2023, at IMC, our students won 1 golden, 5 silver, 8 bronze medals and 13 honorable mentions.

In front of you there is a collection of math competition problems, given on IMC and Vojtěch Jarník, which took place between 2013 and 2023. The solutions of the problems can be found on the links https://www.imc-math.org.uk/ and https://vjimc.osu.cz/.
In cases when a larger number of students are interested to participate on the competitions, the Faculty of Mathematics, Natural Sciences and Information Technologies organizes Team Selection Test. The problems posed on TST are also included in this textbook.
This collection of competition problems is completed with original authored problems, proposed by University of Primorska to the Problem Selection Committees of IMC and Vojtěch Jarník.

Slobodan Filipovski ; ; Koper, July, 2024

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ACADEMIC YEAR 2012/2013

INTERNATIONAL MATHEMATICS COMPETITION 2013

Between August 6 and 12, the twentieth anniversary edition of the International Mathematics Competition for University Students 2013 took place in Blagoevgrad, Bulgaria. The colors of FAMNIT and the University of Primorska were represented by five students: **Ratko Darda**, **Edin Husić** and **Anastasiya Tanana** took part in the competition from the first year, and **Radovan Krtolica** and **Bećo Merulić** from the second year. Our students excelled, as Anastasiya Tanana received the second prize, while Edin Husić and Radovan Krtolica received praise. It should be noted that all students competed in the same category, so our students were among the youngest.

IMC 2013

Day 1, August 8, 2013

Problem 1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\pi/2, \pi/2)$ such that

$$f''(\xi) = f(\xi)(1 + 2\tan^2 \xi).$$

Problem 3. There are $2n$ students in a school ($n \in \mathbb{N}, n \geq 2$). Each week n students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

Problem 4. Let $n \geq 3$ and let $x_1, \ldots, x_n$ be nonnegative real numbers.
Define $A = \sum_{i=1}^n x_i$, $B = \sum_{i=1}^n x_i^2$ and $C = \sum_{i=1}^n x_i^3$. Prove that

$$(n+1)A^2B + (n-2)B^2 \geq A^4 + (2n-2)AC.$$

Problem 5. Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^\infty a_n^p$ converges if and only if p is not a prime?

IMC 2013

Day 2, August 9, 2013

Problem 6. Let z be a complex number with $|z+1|>2$. Prove that $|z^3+1|>1$.

Problem 7. Let p and q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1}(-1)^{\lfloor\frac{k}{p}\rfloor+\lfloor\frac{k}{q}\rfloor}=\begin{cases}0, & \text{if pq is even;}\\ 1, & \text{if pq is odd.}\end{cases}$$

(Here $\lfloor x\rfloor$ denotes the integer part of x.)

Problem 8. Suppose that $v_1,\ldots,v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unit vector u such that

$$|u\cdot v_i|\le 1/\sqrt{d}$$

for $i=1,2,\ldots,d$.
(Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.)

Problem 9. Does there exist an infinite set M consisting of positive integers such that for any $a,b\in M$, with $a<b$, the sum $a+b$ is square-free?
(A positive integer is called square-free if no perfect square greater than 1 divides it.)

Problem 10. Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called *good*, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.
(Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.)

ACADEMIC YEAR 2013/2014

VOJTĚCH JARNÍK 2014

On April 4, 2014, the Vojtěch Jarník International Mathematical Competition was held in Ostrava, Czech Republic.

It is a competition with varied international participation and a long tradition. This year was the 24th edition. In category I, the colors of FAMNIT and the University of Primorska were successfully represented by Mathematics students, namely: **Marko Palangetić**, **Marko Rajković** and **Roman Solodukhin** from the 1st year and **Ratko Darda** and **Anastasiya Tanana** from the 2nd year. The team leader was our PhD student **István Estélyi**.

They scored a total of 66 points and thus ranked 2nd according to the total number of points scored. Ahead of them were only students from the University of Dolgoprudny.

The most successful among our students was Anastasiya Tanana, who collected as much as half of all points in this extremely demanding competition. At this year's competition, 86 students from 31 universities from 16 countries competed in the mentioned category.

VOJTĚCH JARNÍK 2014

April 4, 2014

Category I

Problem 1. Find all complex numbers z such that $|z^3 + 2 - 2i| + z\overline{z}|z| = 2\sqrt{2}$.
($\overline{z}$ is the conjugate of z.)

Problem 2. We have a desc of $2n$ cards. Each shuffling changes the order from $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ to $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. Determine all even numbers $2n$ such that after shuffling the desk 8 times the original order is restored.

Problem 3. Let $n \geq 2$ be an integer and let $x > 0$ be a real number. Prove that

$$\left(1 - \sqrt{\tanh(x)}\right)^n + \sqrt{\tanh(nx)} < 1.$$

Recall that $\tanh t = \frac{e^{2t}-1}{e^{2t}+1}$.

Problem 4. Let P_1, P_2, P_3, P_4 be the graphs of four quadratic polynomials drawn in the coordinate plane. Suppose that P_1 is tangent to P_2 at the point q_2, P_2 is tangent to P_3 at the point q_3, P_3 is tangent to P_4 at the point q_4, and P_4 is tangent to P_1 at the point q_1. Assume that all the points q_1, q_2, q_3, q_4 have distinct x-coordinates. Prove that q_1, q_2, q_3, q_4 lie on a graph of an at most quadratic polynomial.

VOJTĚCH JARNÍK 2014

April 4, 2014

Category II

Problem 1. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function. Assume that

$$\lim_{x \to \infty} \left(f(x) + \frac{f'(x)}{x} \right) = 0.$$

Prove that

$$\lim_{x \to \infty} f(x) = 0.$$

Problem 2. Let p be a prime number and let A be a subgroup of the multiplicative group $\mathbb{F}_p^\star$ of the finite field $\mathbb{F}_p$ with p elements. Prove that if the order of A is a multiple of 6, then there exist $x, y, z \in A$ satisfying $x + y = z$.

Problem 3. Let k be a positive even integer. Show that

$$\sum_{n=0}^{k/2} (-1)^n \binom{k+2}{n} \binom{2(k-n)+1}{k+1} = \frac{(k+1)(k+2)}{2}.$$

Problem 4. Let $0 < a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function with $\int_a^b f(t)dt = 0$. Show that

$$\int_a^b \int_a^b f(x)f(y) \ln(x+y) dx dy \le 0.$$

INTERNATIONAL MATHEMATICS COMPETITION 2014

Between July 29 and August 4, 2014, the 21st edition of the International Mathematics Competition for University Students 2014 took place in Blagoevgrad, Bulgaria. The colors of FAMNIT and the University of Primorska were represented by three undergraduate students. **Marko Rajković** and **Marko Palangetić** took part in the competition from the first year, and **Anastasiya Tanana** from the second year. A leader of our team was **István Estélyi**.

All three students return with a prize: Anastasiya Tanana won the first prize, and Marko Rajković and Marko Palangetić won the third prize.

IMC 2014

Day 1, July 31, 2014

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\mathrm{trace}(M) = a$ and $\det(M) = b$.

Problem 2. Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \ldots).$$

Find all pairs (α, β) of positive real numbers such that $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \alpha_k}{n^{\alpha}} = \beta$.

Problem 3. Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \ldots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \ldots \pm a_1 x + a_0$$

has n distinct real roots.

Problem 4. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorization with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number.
A number n is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

Problem 5. Let $A_1 A_2 \ldots A_{3n}$ be a closed broken line consisting of $3n$ line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \ldots, 3n$, the triangle $A_i A_{i+1} A_{i+2}$ has counterclockwise orientation and $\angle A_i A_{i+1} A_{i+2} = 60^\circ$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$.

IMC 2014

Day 2, August 1, 2014

Problem 6. For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \ldots, 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x) 10^{n-1}$. Suppose that for some sequence $(a_n)_{n=1}^{\infty}$ there are only finitely many zeros in the sequence $(d_n(a_n))_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

Problem 7. Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \le i < j \le n} a_{ii} a_{jj} \ge \sum_{1 \le i < j \le n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

Problem 8. Let $f(x) = \frac{\sin(x)}{x}$, for $x > 0$, and let n be a positive integer. Prove that $|f^{(n)}(x)| < \frac{1}{n+1}$, where $f^{(n)}$ denotes the n^{th} derivative of f.

Problem 9. We say that a subset of $\mathbb{R}^n$ is k-*almost contained* by a hyperplane if there are less than k points in that set which do not belong to the hyperplane. We call a finite set of points k-*generic* if there is no hyperplane that k-almost contains the set. For each pair of positive integers k and n, find the minimal number $d(k, n)$ such that every finite k-generic set in $\mathbb{R}^n$ contains a k-generic subset with at most $d(k, n)$ elements.

Problem 10. For every positive integer n, denote by D_n the number of permutations $(x_1, \ldots, x_n)$ of $(1, 2, \ldots, n)$ such that $x_j \ne j$ for every $1 \le j \le n$. For $1 \le k \le \frac{n}{2}$, denote by $\Delta(n, k)$ the number of permutations $(x_1, \ldots, x_n)$ of $(1, 2, \ldots, n)$ such that $x_i = k + i$ for every $1 \le i \le k$ and $x_j \ne j$ for every $1 \le j \le n$. Prove that

$$\Delta(n, k) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}.$$

ACADEMIC YEAR 2014/2015

VOJTĚCH JARNÍK 2015

On March 27, 2015, the Vojtěch Jarník International Mathematical Competition was held in Ostrava, Czech Republic.

It is a competition with varied international participation and a long tradition. This year was the 25th edition. In category I, the colors of FAMNIT and the University of Primorska were successfully represented by Mathematics students, namely: **Anes Valentić** from the 1st year and **Marko Palangetić**, **Marko Rajković** and **Roman Solodukhin** from the 2nd year. A leader of the team was **István Estélyi**.

They scored a total of 70 points and thus ranked in 7th place. The most successful among our students were Marko Palangetić and Roman Solodukhin, who scored 24 points each in this extremely demanding competition. At this year's competition, 74 students from 31 universities competed in the mentioned category. Students from the University of Maribor also performed from Slovenia.

In the picture, from left to right: Marko Rajković, István Estélyi, Roman Solodukhin, Marko Palangetić and Anes Valentić.

VOJTĚCH JARNÍK 2015

March 27, 2015

Category I

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Prove that there exists $x \in [0,1]$ such that

$$\frac{4}{\pi}(f(1) - f(0)) = (1 + x^2)f^{'}(x).$$

Problem 2. Consider the infinite chessboard whose rows and columns are indexed by positive integers. Is it possible to put a single positive rational number into each cell of the chessboard so that each positive rational number appears exactly once and the sum of every row and of every column is finite?

Problem 3. Let $P(x) = x^{2015} - 2x^{2014} + 1$ and $Q(x) = x^{2015} - 2x^{2014} - 1$. Determine for each of the polynomials P and Q whether it is a divisor of some nonzero polynomial $c_0 + c_1x + \ldots + c_nx^n$ whose coefficients c_i are all in the set $\{1, -1\}$.

Problem 4. Let m be a positive integer and let p be a prime divisor of m. Suppose that the complex polynomial $a_0 + a_1x + \ldots + a_nx^n$ with $n < \frac{p}{p-1}\phi(m)$ and $a_n \neq 0$ is divisible by the cyclotomic polynomial $\Phi_m(x)$. Prove that there are at least p nonzero coefficients a_i.

The cyclotomic polynomial $\Phi_m(x)$ is the monic polynomial whose roots are the m-th primitive complex roots of unity. Euler's totient function $\phi(m)$ denotes the number of positive integers less than or equal to m which are coprime to m.

VOJTĚCH JARNÍK 2015

March 27, 2015

Category II

Problem 1. Let A and B be two 3×3 matrices with real entries. Prove that

$$A - (A^{-1} + (B^{-1} - A)^{-1})^{-1} = ABA,$$

provided all the inverses appearing on the left-hand side of the equality exist.

Problem 2. Determine all pairs (n, m) of positive integers satisfying the equation

$$5^n = 6m^2 + 1.$$

Problem 3. Determine the set of real values of x for which the following series converges, and find its sum:

$$\sum_{n=1}^{\infty} \left(\sum_{\substack{k_1,\ldots,k_n \geq 0 \\ 1\cdot k_1 + 2\cdot k_2 + \cdots + n\cdot k_n = n}} \frac{(k_1 + \ldots + k_n)!}{k_1! \cdot \ldots \cdot k_n!} x^{k_1+\ldots+k_n} \right).$$

Problem 4. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$, such that for every $a \geq 0$ the following relation holds:

$$\iiint\limits_{D(a)} x f\left(\frac{ay}{\sqrt{x^2 + y^2}}\right) dxdydz = \frac{\pi a^3}{8}(f(a) + \sin a - 1),$$

where $D(a) = \{(x, y, z) : x^2 + y^2 + z^2 \leq a^2, |y| \leq \frac{x}{\sqrt{3}}\}$.

INTERNATIONAL MATHEMATICS COMPETITION 2015

Between July 27 and August 2, 2015, the 22nd International Mathematics Competition for University Students 2015 took place in Blagoevgrad, Bulgaria . The colors of FAMNIT and the University of Primorska were represented by four second-year undergraduate students **Marko Palangetić**, **Ivan Bartulović**, **Roman Solodukhin** and **Vladan Jovičić**. Team leader was **Slobodan Filipovski**.

Marko Palangetić received the second prize, Ivan Bartulović the third prize, and Roman Solodokhin and Vladan Jovičić were commended.

In the picture, from left to right: Roman Solodukhin, Ivan Bartulović, Slobodan Filipovski (team leader), Vladan Jovičić and Marko Palangetić.

IMC 2015

Day 1, July 29, 2015

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.
Does the same conclusion follow for matrices with complex entries?

Problem 2. For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

Problem 3. Let $F(0) = 0, F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$. Determine whether or not $\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$ is a rational number.

Problem 4. Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$

Problem 5. Let $n \geq 2$, let $A_1, A_2, \ldots, A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of $A_1, A_2, \ldots, A_{n+1}$. Prove that $\angle A_i B A_j > 90^\circ$ holds for at least n pairs (i, j) with $1 \leq i < j \leq n+1$.

IMC 2015

Day 2, July 30, 2015

Problem 6. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

Problem 7. Compute
$$\lim_{A \to +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}}\, dx.$$

Problem 8. Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

Problem 9. An $n \times n$ complex matrix A is called *t-normal* if $AA^t = A^t A$ where A^t is the transpose of A. For each n, determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

Problem 10. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that
$$\max_{0 \le x \le 1} |p(x)| > \frac{1}{e^n}.$$

ACADEMIC YEAR 2015/2016

VOJTĚCH JARNÍK 2016

On April 4, the 26th Vojtěch Jarník International Mathematical Competition took place in Ostrava, Czech Republic - the oldest mathematics competition for students in the European Union.

This year's performance was attended by 154 students from 35 universities. In the first category (for 1st and 2nd year students or under 22 years old), Famnit's colors were represented by **Anes Valentić** and **Marija Tepegjozova** (2nd year MA), while in the second category **Marko Palangetić** and **Roman Solodukhin** (3rd year MA).

Marko Palangetić received a certificate of successful participant for his 13 points (the winner in this category collected 29 points out of a possible 40). At the same time, we point out that the task of Famit's doctoral student and assistant **Slobodan Filipovski**, who was a delegate of Famnit's team in the Czech Republic, was also included among the competing tasks for the second category (problem 1).

In the picture, from left to right: Anes Valentić, Marija Tepegjozova, Slobodan Filipovski (team leader), Marko Palangetić and Roman Solodukhin.

VOJTĚCH JARNÍK 2016

April 8, 2016

Category I

Problem 1. Let $f : \mathbb{R} \to (0, \infty)$ be a continuously differentiable function. Prove that there exists $\xi \in (0, 1)$ such that

$$e^{f'(\xi)} f(0)^{f(\xi)} = f(1)^{f(\xi)}.$$

Problem 2. Find all positive integers n such that $\phi(n)$ divides $n^2 + 3$.
($\phi(n)$ denotes Euler's totient function, i.e. the number of positive integers $k \leq n$ coprime to n.)

Problem 3. Let $d \geq 3$ and let $A_1 \dots A_{d+1}$ be a simplex in $\mathbb{R}^d$. (A simplex is the convex hull of $d+1$ points not lying in a common hyperline.) For every $i = 1, \dots, d+1$ let O_i be the circumcentre of the face $A_1 \dots A_{i-1} A_{i+1} \dots A_{d+1}$, i.e. O_i lies in the hyperplane $A_1 \dots A_{i-1} A_{i+1} \dots A_{d+1}$ and it has the same distance from all points $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{d+1}$. For each i draw a line through A_i perpendicular to the hyperplane $O_1 \dots O_{i-1} O_{i+1} \dots O_{d+1}$. Prove that either these lines are parallel or they have a common point.

Problem 4. Find the value of the sum $\sum_{n=1}^{\infty} A_n$, where

$$A_n = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{1}{k_1^2} \frac{1}{k_1^2 + k_2^2} \cdots \frac{1}{k_1^2 + \cdots + k_n^2}.$$

VOJTĚCH JARNÍK 2016

April 8, 2016

Category II

Problem 1. Let a, b and c be positive real numbers such that $a + b + c = 1$. Show that

$$\left(\frac{1}{a}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{ca}\right)\left(\frac{1}{c}+\frac{1}{ab}\right) \geq 1728.$$

Problem 2. Let X be a set and let $\mathcal{P}(X)$ be the set of all subsets of X. Let $\mu : \mathcal{P}(X) \to \mathcal{P}(X)$ be a map with the property that $\mu(A \cup B) = \mu(A) \cup \mu(B)$ whenever A and B are disjoint subsets of X. Prove that there exists a set $F \subset X$ such that $\mu(F) = F$.

Problem 3. For $n \geq 3$ find the eigenvalues (with their multiplicities) of the $n \times n$ matrix

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & \dots & \dots & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & \dots & \dots & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & \dots & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1
\end{pmatrix}$$

Problem 4. Let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function satisfying

$$f(x) = \int_{x-1}^{x} f(t)dt$$

for all $x \geq 1$. Show that f has bounded variation on $[1, \infty)$, i.e.

$$\int_{1}^{\infty} |f'(x)|dx < \infty.$$

Proposed problems for Vojtěch Jarník 2016 by University of Primorska

Problem 1. Let a, b and c be positive real numbers such that $a+b+c=1$. Show that

$$\left(\frac{1}{a}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{ca}\right)\left(\frac{1}{c}+\frac{1}{ab}\right) \geq 1728.$$

Solution 1. By using the inequality between arithmetic and geometric means we get: $\frac{1}{a}+\frac{1}{bc}=\frac{1}{a}+\frac{1}{3bc}+\frac{1}{3bc}+\frac{1}{3bc} \geq 4\frac{1}{\sqrt[4]{27ab^3c^3}}$ and $\frac{1}{27}=\left(\frac{a+b+c}{3}\right)^3 \geq abc$. Thus

$$\left(\frac{1}{a}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{ca}\right)\left(\frac{1}{c}+\frac{1}{ab}\right) \geq 64\cdot\frac{1}{\sqrt[4]{27ab^3c^3}}\frac{1}{\sqrt[4]{27a^3bc^3}}\frac{1}{\sqrt[4]{27a^3b^3c}}=\frac{64}{\sqrt[4]{3^9(abc)^7}}$$

$$\geq \frac{64}{\sqrt[4]{3^9(3^{-3})^7}}=64\sqrt[4]{3^{12}}=64\cdot 27=1728.$$

Solution 2. If we replace 1 with $a+b+c$ and $k=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ we get

$$\left(\frac{1}{a}+\frac{a+b+c}{bc}\right)\left(\frac{1}{b}+\frac{a+b+c}{ca}\right)\left(\frac{1}{c}+\frac{a+b+c}{ab}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{bc}\right)\cdot$$

$$\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{b}{ca}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{c}{ab}\right)=\left(k+\frac{a}{bc}\right)\left(k+\frac{b}{ca}\right)\left(k+\frac{c}{ab}\right)=$$

$$k^3+k^2\left(\frac{c}{ab}+\frac{b}{ca}+\frac{a}{bc}\right)+k\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{1}{abc}.$$

From the inequality between arithmetic and harmonic means for the positive numbers a, b and c follows that

$$\frac{1}{3}=\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{3}{k} \Leftrightarrow k \geq 9.$$

From Solution 1 we already know that $\frac{1}{abc} \geq 27$. In the end we have

$$L.H.S \geq 9^3+9^2\cdot\frac{3}{\sqrt[3]{abc}}+\frac{27}{\sqrt[3]{(abc)^2}}+\frac{1}{abc} \geq 9^3+9^3+27\cdot 9+27=1728.$$

TEAM SELECTION TEST FOR THE IMC 2016

June 2016

FAMNIT

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function continuous on $[0, 1]$ and differentiable on $(0, 1)$. Let also $f(0) = 0$ and $f(1) = 0$. Prove that there exists a point c in $(0, 1)$ such that $f'(c) = f(c)$.

Problem 2. Show that if A and B are real $n \times n$ matrices that commute, i.e., $AB = BA$, then

$$\det(A^2 + B^2) \geq 0.$$

Problem 3. Find the value of the following summation

$$\sum_{j=0}^{n} \binom{2n}{2j} (-3)^j.$$

Problem 4. Let $A_1, \ldots, A_n$ be n points on a given ellipsoid $\mathcal{F} = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} = 1\}$, such that the vectors OA_i are pairwise perpendicular. Show that the distance of the hyperplane $A_1A_2 \ldots A_n$ from the origin O is independent of the choice of points.

INTERNATIONAL MATHEMATICS COMPETITION 2016

The 23rd International Mathematics Competition IMC (International Mathematics Competition for University Students 2016), which takes place every year in Blagoevgrad, Bulgaria, took place between July 25 and 31. The colors of FAMNIT and the University of Primorska were represented by three of our undergraduate students: **Marko Palangetić**, **Roman Solodukhin** and **Mirza Krbezlija**. A team leader was **Slobodan Filipovski**.

Marko Palangetić won the second prize, and Roman Solodukhin the third.

In the picture, from left to right: Roman Solodukhin, Slobodan Filipovski (team leader), Marko Palangetić and Mirza Krbezlija.

IMC 2016

Day 1, July 27, 2016

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$.

(a) Prove that $f(a)f(b) = 0$.

(b) Give an example of such a function on $[0, 1]$.

Problem 2. Let k and n be positive integers. A sequence $(A_1, \ldots, A_k)$ of $n \times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_iA_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

Problem 3. Let n be a positive integer. Also let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_ib_i - b_i^2}{a_i + b_i} \leq \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n}(a_i + b_i)}.$$

Problem 4. Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

(i) $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements;

(ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$.

Prove that $\mathcal{F}$ contains at least three sets with at least n elements.

Problem 5. Let S_n denote the set of permutations of the sequence $(1, 2, \ldots, n)$. For every permutation $\pi = (\pi_1, \ldots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \leq i < j \leq n$ with $\pi_i > \pi_j$; i.e. the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n + 1$.
Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

IMC 2016

Day 2, July 28, 2016

Problem 6. Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leq 2.$$

Problem 7. Today, Ivan the Confessor prefers continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geq |x - y|$ for all pairs $x, y \in [0, 1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

Problem 8. Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo n. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x + 1) + 1) + 1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

Problem 9. Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + \ldots + |x_k| \leq n$. Prove that for every $n \geq 1$, we have $f(n-1)f(n+1) \leq f(n)^2$.

Problem 10. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2} \|A\|^{n-1}.$$

(Here $\|B\| = \sup_{\|x\| \leq 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

Proposed problems for the IMC 2016
by University of Primorska

Problem 1. Let a, b, c be positive real numbers greater than 1 such that $(a-1)(b-1)(c-1) = 1$. Prove that

$$\sqrt[5]{\frac{a+b}{4}} + \sqrt[5]{\frac{b+c}{4}} + \sqrt[5]{\frac{c+a}{4}} \geq 3.$$

Solution. Let x, y, z be real numbers such that $x = \sqrt[5]{a-1}, y = \sqrt[5]{b-1}$ and $z = \sqrt[5]{c-1}$. From the conditions, it follows that x, y, z are positive real numbers for which $xyz = 1$. It suffices to prove the equivalent inequality

$$\sqrt[5]{\frac{x^5+y^5+2}{4}} + \sqrt[5]{\frac{y^5+z^5+2}{4}} + \sqrt[5]{\frac{z^5+x^5+2}{4}} \geq 3.$$

By using the inequality between power means of order 5 and 1 for the positive numbers $x, y, 1$ and 1 we get $\sqrt[5]{\frac{x^5+y^5+2}{4}} = \sqrt[5]{\frac{x^5+y^5+1^5+1^5}{4}} \geq \frac{x+y+2}{4}$. Analogously we get $\sqrt[5]{\frac{y^5+z^5+2}{4}} \geq \frac{y+z+2}{4}$ and $\sqrt[5]{\frac{z^5+x^5+2}{4}} \geq \frac{z+x+2}{4}$. Summing the last three inequalities we obtain

$$\text{L.H.S} \geq \frac{x+y+z}{2} + \frac{3}{2} \geq \frac{3\sqrt[3]{xyz}}{2} + \frac{3}{2} = 3.$$

Problem 2. Let A and B be 3×3 matrices over the field of complex numbers. Prove that

$$\|(AB - BA)^3\|_2 \geq |det(AB - BA)|.$$

Solution. By using Cayley-Hamilton theorem for the matrix $AB - BA$ we have

$$(AB - BA)^3 - a(AB - BA)^2 + b(AB - BA) - cI_3 = O_3 \qquad (1)$$

where $a = \text{trace}(AB - BA) = 0$ and $c = det(AB - BA)$. Computing the trace of the matrices in both hand sides of (1), we get $\text{trace}((AB - BA)^3) = 3 \cdot det(AB - BA)$.
Now, from $\|A\|_2 \geq |\lambda_{max}(A)|$ we obtain

$$\|(AB - BA)^3\|_2 \geq |\lambda_{max}((AB - BA)^3)| \geq \left|\frac{\text{trace}((AB - BA)^3)}{3}\right| = |det(AB - BA)|.$$

ACADEMIC YEAR 2016/2017

TEAM SELECTION TEST FOR VOJTĚCH JARNÍK 2017

March 2017

FAMNIT

Problem 1. Let $n, p > 1$ be positive integers and p be prime. Given that $n \mid p-1$ and $p \mid n^3 - 1$, prove that $4p - 3$ is a perfect square.

Solution. Since $n \mid p-1$, let $p - 1 = kn$ for some positive integer k, therefore $p = kn + 1$. This satisfies the first condition of the requirement. We now look at the second condition, which is $p \mid n^3 - 1 = (n-1)(n^2+n+1)$. Note since $p = kn+1$, we have $p \geq n-1$, and because p is a prime, $gcd(p, n-1) = 1$:

$$p \mid (n-1)(n^2+n+1) \Rightarrow p = kn+1 \mid n^2+n+1.$$

In order for this to be true, $kn+1 \leq n^2+n+1 \Rightarrow k \leq n+1$. Since $n^2+n+1 \mid k(n^2+n+1)$, we also have

$$p = kn+1 \mid kn^2+kn+k$$

$$\Rightarrow kn+1 \mid kn^2+kn+k-n(kn+1) = kn+k-n.$$

Similarly, to have this divisibility, $kn+k-n \geq kn+1 \Rightarrow k \geq n+1$. However, above we found that $k \leq n+1$, therefore, $k = n+1$. Substituting this in for p gives $p = (n+1)n+1 = n^2+n+1$, giving

$$4p - 3 = 4n^2 + 4n + 4 - 3 = 4n^2 + 4n + 1 = (2n+1)^2.$$

Problem 2. Let $n \geq 2$ be an integer. Let A be an $n \times n$ matrix with coefficients in a field $\mathbb{F}$.

1. Assume that A is a strictly upper triangular matrix, that is, its entries satisfy $a_{ij} = 0$ for all $i \geq j$. If I is the identity matrix of size n, show that $I - A$ is an invertible matrix with
$$I + A + A^2 + \cdots + A^{n-1} \tag{2}$$
as its inverse $(I-A)^{-1}$.
2. Does there exist a matrix A with zero diagonal such that neither A nor $A^\top$ is strictly upper triangular and $I - A$ is invertible with (2) as its inverse? Provide an answer for all integers $n \geq 2$ and all fields $\mathbb{F}$.

Solution. (i) Matrix $I - A$ is invertible with (2) as its inverse if and only if

$$(I-A)(I + A + A^2 + \cdots + A^{n-1}) = I = (I + A + A^2 + \cdots + A^{n-1})(I-A),$$

which holds if and only if

$$A^n = 0. \tag{3}$$

To prove (3) it suffices to show that for each $k \geq 1$,

$$[A^k]_{ij} = 0 \quad \text{whenever} \quad i \geq j - k + 1. \tag{4}$$

Here, $[A^k]_{ij}$ denotes the (i,j)-th entry of A^k. We prove (4) by an induction on k.

For $k = 1$ we have $[A^1]_{ij} = a_{ij} = 0$ whenever $i \geq j = j - 1 + 1$. Assume now that (4) holds for some $k \geq 1$. Let $i \geq j - (k+1) + 1 = j - k$. Then

$$\begin{aligned} [A^{k+1}]_{ij} &= \sum_{t=1}^{n} [A^k]_{it} a_{tj} \\ &= \sum_{t=i+k}^{n} [A^k]_{it} a_{tj} = 0, \end{aligned}$$

which end the induction step. Here, we applied facts that $[A^k]_{it} = 0$ for $t \leq i + k - 1$ (i.e. $i \geq t - k + 1$) and $a_{tj} = 0$ for $t \geq i + k \geq j - k + k = j$.

(ii) Let E_{ij} be the matrix with 1 as the (i,j)-th entry. If $n \geq 3$, then $A = E_{12} + E_{32}$ has zero diagonal, and both A, $A^\top$ are not strictly upper triangular, while $0 = A^2 = A^n$, so (2) is the inverse $(I - A)^{-1}$. If $n = 2$, then any such A is of the form

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

for some nonzero $a, b \in \mathbb{F}$. Consequently, A is invertible and the same hold for A^n. In particular, $A^n \neq 0$ for all n, so $I - A$ cannot have the inverse given by (2).

Problem 3. Let a, b, c be non-negative real numbers. Prove that

$$\frac{a}{\sqrt{4b^2 + bc + 4c^2}} + \frac{b}{\sqrt{4c^2 + ca + 4a^2}} + \frac{c}{\sqrt{4a^2 + ab + 4b^2}} \geq 1.$$

Solution. We may assume that $a + b + c = 1$. Since $f(x) = \frac{1}{\sqrt{x}}$ is a convex function, according to Jansen inequality, we obtain

$$a \cdot f(4b^2 + bc + 4c^2) + b \cdot f(4c^2 + ca + 4a^2) + c \cdot f(4a^2 + ab + 4b^2) \geq f(K),$$

where

$$K = a(4b^2 + bc + 4c^2) + b(4c^2 + ca + 4a^2) + c(4a^2 + ab + 4b^2) = 4 \sum_{cyc} ab(a+b) + 3abc.$$

Now, it is enough to prove that $f(K) \geq 1$ or $K \leq 1$. It is certainly true because

$$1 - K = \left(\sum_{cyc} a \right)^3 - 4 \sum_{cyc} ab(a+b) - 3abc = \sum_{cyc} a^3 - \sum_{cyc} ab(a+b) + 3abc =$$

$$= \prod_{cyc} abc - \prod_{cyc} (a + b - c) \geq 0.$$

Equality holds for $a = b = c$ and $a = 0, b = c$ up to permutation.

Problem 4. Let $a < b$ be real numbers. Suppose that a real function f is smooth on some open neighborhood of the interval $[a, b]$. That is, there exists $a_1, b_1 \in \mathbb{R}$ such that $[a, b] \subseteq (a_1, b_1)$ and the derivative $f^{(n)}$ of arbitrary order n exists on (a_1, b_1). If there exists $r \in \mathbb{R}$ such that the set

$$\{x \in [a, b] : f(x) = r\}$$

is infinite, show that for every integer $n \geq 1$ there exists $y \in (a, b)$ such that $f^{(n)}(y) = 0$.

Is function f necessarily constant on some nonempty subinterval $(a_2, b_2) \subseteq [a, b]$? Explain your answer in details.

Solution. Let $g(x) := f(x) - r$. Then f and g have the same derivatives, and

$$S := \{x \in [a, b] : g(x) = 0\}$$

is an infinite set. Choose a sequence $(x_k)_k$ in S such that $a \leq x_1 < x_2 < \cdots \leq b$. By Rolle's Theorem there exist a sequence $(y_k)_k$ in (a, b) such that $x_k < y_k < x_{k+1}$ and $g'(y_k) = 0$ for all k. We may now repeat the process for functions $g', g'', \ldots$ instead of g. By an induction we show that for every $n \geq 1$ there exists a sequence $(z_k)_k$ in (a, b) such that $f^{(n)}(z_k) = g^{(n)}(z_k) = 0$ for all k, so we have an infinite number of candidates for y.

The function f is not necessarily constant on any subinterval. For example, let $a' = -2$, $b' = 2$, $a = -1$, $b = 1$, $r = 0$ and let $f : (-2, 2) \to \mathbb{R}$ be defined by $f(x) = e^{-1/x} \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Then $f(1/(k\pi)) = 0$ for all $k \in \{1, 2, \ldots\}$, so the set $\{x \in [a, b] : f(x) = r\}$ is infinite. Clearly, f is nonconstant on any nonempty open subinterval $(a_2, b_2) \subseteq [-1, 1]$. It now suffices to show that f is smooth on $(-2, 2)$. Clearly, it is smooth on $(-2, 2)\backslash\{0\}$. By an induction we see that any derivative $f^{(n)}$ on $(-2, 2)\backslash\{0\}$ is of the form

$$\sum_{i \in I} \frac{\cos(1/x)}{e^{1/x} p_i(x)} + \frac{\sin(1/x)}{e^{1/x} q_i(x)},$$

where the set I is finite and $p_i(x)$, $q_i(x)$ are some polynomials of the form ax^t for some $a \in \mathbb{R}$ and integer $t \geq 0$. Consequently, we can use the induction, with the induction step given by

$$\lim_{x \to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0} = 0,$$

to see that $f^{(n)}(0)$ exists and equals 0 for all n. Hence, f is smooth on whole $(-2, 2)$.

VOJTĚCH JARNÍK 2017

On Friday, March 31, the 27th Vojtěch Jarník International Mathematical Competition took place in Ostrava, Czech Republic - the oldest mathematics competition for students in the European Union, in which UP FAMNIT students traditionally participate.

132 students from 36 universities applied for this year's event. In the first category (for 1st and 2nd year students, respectively under the age of 22), Famnit's colors were represented by **Arbër Avdullahu** and **Daniil Baldouski** (both 1st year Mathematics), while in the second category **Roman Solodukhin** and **Anes Valentić** (graduate and 3. year of Mathematics). Avdullahu, Baldouski and Solodukhin received a certificate of successful participant, and Solodukhin with 20 points achieved the best ranking of Famnit students in the second category so far. **Alejandra Ramos-Rivera** was the leader of the FAMNIT's team.

In the picture, from left to right: Roman Solodukhin, Arbër Avdullahu, Alejandra Ramos-Rivera (team leader), Daniil Baldouski and Anes Valentić.

VOJTĚCH JARNÍK 2017

March 31, 2017

Category I

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$f(x + 2y) = 2f(x)f(y)$$

for every $x, y \in \mathbb{R}$. Prove that f is constant.

Problem 2. We say that we extend a finite sequence of positive integers $(a_1, \dots, a_n)$ if we replace it by

$$(1, 2, \dots, a_1 - 1, a_1, 1, 2, \dots, a_2 - 1, a_2, 1, 2, \dots, a_3 - 1, a_3, \dots, 1, 2, \dots, a_n - 1, a_n),$$

i.e., each element k of the original sequence is replaced by $1, 2, ..., k - 1, k$. Géza takes the sequence $(1, 2, \dots, 9)$ and he extends it 2017 times. Then he chooses randomly one element of the resulting sequence. What is the probability that the chosen element is 1?

Problem 3. Let P be a convex polyhedron. Jaroslav writes a non-negative real number to every vertex of P in such a way that the sum of these numbers is 1. Afterwards, to every edge he writes the product of the numbers at the two endpoints of that edge. Prove that the sum of the numbers at the edges is at most $\frac{3}{8}$.

Problem 4. Let $f : (1, \infty) \to \mathbb{R}$ be a continuously differentiable function satisfying $f(x) \leq x^2 \log(x)$ and $f'(x) > 0$ for every $x \in (1, \infty)$. Prove that

$$\int_1^\infty \frac{1}{f'(x)} dx = \infty.$$

VOJTĚCH JARNÍK 2017

March 31, 2017

Category II

Problem 1. Let $(a_n)_{n=1}^{\infty}$ be a sequence with $a_n \in \{0,1\}$ for every n. Let $F : (-1,1) \to \mathbb{R}$ be defined by

$$F(x) = \sum_{n=1}^{\infty} a_n x^n$$

and assume that $F\left(\frac{1}{2}\right)$ is rational. Show that F is the quotient of two polynomials with integer coefficients.

Problem 2. Prove or disprove the following statement. If $g : (0,1) \to (0,1)$ is an increasing function and satisfies $g(x) > x$ for all $x \in (0,1)$, then there exists a continuous function $f : (0,1) \to \mathbb{R}$ satisfying $f(x) < f(g(x))$ for all $x \in (0,1)$, but f is not an increasing function.

Problem 3. Let $n \geq 2$ be an integer. Consider the system of equations

$$x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = \ldots = x_n + \frac{2}{x_1}.$$

1. Prove that (1) has infinitely many real solutions $(x_1, \ldots, x_n)$ such that the numbers $x_1, \ldots, x_n$ are distinct.
2. Prove that every solution $(x_1, \ldots, x_n)$ of (1), such that the numbers $x_1, \ldots, x_n$ are not all equal, satisfies $|x_1 x_2 \cdots x_n| = 2^{n/2}$.

Problem 4. A positive integer is called a Jane's integer if $t = x^3 + y^2$ for some positive integers x and y. Prove that for every integer $n \geq 2$ there exist infinitely many positive integers m such that the set of n^2 consecutive integers $\{m+1, m+2, \ldots, m+n^2\}$ contains exactly $n+1$ Jane's integers.

Proposed problems for Vojtěch Jarník 2017 by University of Primorska

Problem 1. Let p and q be odd prime numbers such that $p^2 > 2q$ and $q \mid p^2+1$. Prove that at least one of the numbers $4p^2-4q$ and $8p^2-16q$ is a sum of three or fewer squares.

Solution. Note that $4p^2-4q$ and $8p^2-16q$ are positive integers. Since the odd prime divisors of x^2+1 are of the form $4k+1$, we have $q \equiv 1 \pmod 4$. Moreover, since p is an odd number holds $p^2 \equiv 1 \pmod 4$. Let us suppose that $4p^2-4q$ is not a sum of three or fewer squares. Using Legendre's theorem on sums of three squares we have that $4p^2-4q$ is of form $4^a(8b+7)$, for some non-negative integers a and b. Since $4p^2-4q$ is not a sum of three or fewer squares, it follows that p^2-q is not a sum of three or fewer squares. Thus $p^2-q = 4^\alpha(8\beta+7)$, for some non-negative integers α and β. Since $4 \mid p^2-q$ we have that $\alpha \geq 1$. Now, if we suppose that $8p^2-16q$ is not a sum of three or fewer squares, then there exist non-negative integers α_1 and β_1 such that $8p^2-16q = 4^{\alpha_1}(8\beta_1+7)$. From $p^2-q = 4^\alpha(8\beta+7)$ we obtain

$$4^{\alpha_1}(8\beta_1+7) = 8(p^2-q) - 8q = 8(4^\alpha(8\beta+7)-q).$$

Since $\alpha \geq 1$, it follows that $4^\alpha(8\beta+7)-q$ is odd, which implies that $\alpha_1 = 1$. Considering the last equation modulo 4 we have $2q \equiv 1 \pmod 4$, which is not possible because $q \equiv 1 \pmod 4$.

INTERNATIONAL MATHEMATICS COMPETITION 2017

The 24th International mathematical competition for University Students – IMC 2017, which takes place every year in Blagoevgrad in Bulgaria, was held between 31 July and 6 August.

FAMNIT and the University of Primorska were represented by three of our first year Mathematics students: **Filip Božić**, **Daniil Baldouski** and **Arbër Avdulahu**, who received a honourable mention. Their team leader was **Slobodan Filipovski**.

In the picture, from left to right: Filip Božić, Daniil Baldouski and Arbër Avdulahu.

IMC 2017

Day 1, August 2, 2017

Problem 1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

Problem 2. Let $f : \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$|f'(x) - f'(y)| \leq L|x - y|$$

for all x, y. Prove that

$$(f'(x))^2 < 2Lf(x)$$

holds for all x.

Problem 3. For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence

$$a_1(n) = n, \quad a_{k+1}(n) = P(a_k(n)) \ \ (k = 1, 2, \ldots, 2016).$$

Determine whether for every set $S \subseteq \{1, 2, \ldots, 2017\}$, there exists a positive integer n such that the following condition is satisfied:

For every k with $1 \leq k \leq 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

Problem 4. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $n/2017$ persons in S have exactly two friends in S.

Problem 5. Let k and n be positive integers with $n \geq k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \ldots = c_{n-2}c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

IMC 2017

Day 2, August 3, 2017

Problem 6. Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)dx = L.$$

Problem 7. Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer n, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

Problem 8. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$

where I_m is the $m \times m$ identity matrix.
Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, respectively.

Problem 9. Define the sequence $f_1, f_2, \ldots : [0,1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \quad f_{n+1}' = f_n f_{n+1} \quad \text{on} \quad (0,1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1)$ and determine the limit function.

Problem 10. Let K be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\epsilon > 0$ with the following property: If n is a positive integer, and $T_1, \ldots, T_n$ are non-overlapping triangles inside K such that each of them is homothetic to K with a negative ratio, and

$$\sum_{l=1}^{n} \text{area}(T_l) > \text{area}(K) - \epsilon,$$

then

$$\sum_{l=1}^{n} \text{perimeter}(T_l) > p.$$

Proposed problems for the IMC 2017 by University of Primorska

Problem 1. Let A be a symmetric matrix of size n and let $k(\geq 3)$ be an odd number. The set of the eigenvalues of A consists of $\pm k$ (with multiplicity 1) and the roots of the polynomial $H_m(x) - 2, (3 \leq m \leq n-2)$. Prove that $m \mid n-2$.

(Here $H_m(x)$ is the Dickson polynomial defined as follows: $H_0(x) = 1, H_1(x) = x, H_{i+2}(x) = xH_{i+1}(x) - (k-1)H_i(x)$, with $i \geq 0$.)

Solution. We will use the following lemma.

Lemma 1. *Let A be a rational symmetric matrix, $q(x) \in \mathbb{Q}[x]$ be its characteristic polynomial and $p(x) \in \mathbb{Q}[x]$ an irreducible polynomial with $p(x) \mid q(x)$. If $\lambda, \mu \in \mathbb{R}$ are roots of $p(x)$, then the multiplicities $m_A(\lambda)$ and $m_A(\mu)$ of λ and μ as eigenvalues of A fulfill $m_A(\lambda) = m_A(\mu)$.*

According to the above lemma, it is enough to prove that the polynomial $H_m(x) - 2$ is irreducible. Then, $m_A(\lambda) = m_A$, for every root λ of $H_m(x) - 2$. It will leads to $2 + mm_A = n$, that is, $m \mid n-2$.

We prove, using induction on $m \geq 3$, that $H_m(x) = x^m + (k-1)P_{m-2}(x)$, where $P_{m-2}(x)$ is an integer polynomial of degree $m-2$. We calculate $H_3(x) = x^3 - 2(k-1)x$. Let us suppose that the above formula holds for $H_{m-1}(x)$ and $H_{m-2}(x)$. That yields

$$H_m(x) = x(x^{m-1} + (k-1)P_{m-3}(x)) - (k-1)(x^{m-2} + (k-1)P_{m-4}(x)) = x^m + (k-1)P_{m-2}(x).$$

Therefore, $H_m(x) - 2 = x^m + (k-1)P_{m-2}(x) - 2$. By the induction hypothesis, it follows that $H_m(0) = (-1)^{\frac{m}{2}}(k-1)^{\frac{m}{2}}$ for an even m, and $H_m(0) = 0$ for an odd m. Hence, for an even $m(\geq 4)$ $|(-1)^{\frac{m}{2}}(k-1)^{\frac{m}{2}} - 2|$ is not divisible by 2^2, and clearly for an odd $m(\geq 3)$, -2 is not divisible by 2^2. Since $k-1$ is even, it follows that every coefficient on $H_m(x) - 2$ except for the coefficient 1 of x^m is divisible by 2. Thus, the conditions of the Eisenstein's criterion are satisfied, and $H_m(x) - 2$ is irreducible.

ACADEMIC YEAR 2017/2018

TEAM SELECTION TEST FOR VOJTĚCH JARNÍK 2018

March 2018,

FAMNIT

Problem 1. The definite integrals between 0 and 1 of the squares of the continuous real functions $f(x)$ and $g(x)$ are both equal to 1. Prove that there is a real number c such that

$$f(c) + g(c) \leq 2.$$

Problem 2. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(f(f(x))) + 4f(f(x)) + f(x) = 6x.$$

Problem 3. Prove that given two matrices $A \in M_m(\mathbb{R})$ and $B \in M_n(\mathbb{R})$ have a common eigenvalue if and only if there exists a non-zero matrix $C \in M_{m \times n}(\mathbb{R})$ such that $AC = CB$.

Problem 4. In $\mathbb{R}^{2018}$, a ball B_0 centered at $P_0(1, 1, \ldots, 1)$ touches all the axis (of $\mathbb{R}^{2018}$). For how many points P from $\mathbb{R}^{2018}$ does there exist a ball centered at P which touches all the axis, and moreover touches the ball B_0?

VOJTĚCH JARNÍK 2018

Also in 2018, our math students participated at the oldest mathematics competition for students within EU: the 28th Vojtěch Jarník International Mathematical Competition, which was held in Ostrava (Czech Republic) on Friday, April 13th. At this year's event, 150 students from 33 European universities competed. In the first category (for 1st and 2nd year students or under 22 years of age), Famnit was represented by **Đorđe Mitrović** (1st year of Mathematics), **Arbër Avdullahu** and **Daniil Baldouski** (both 2nd year of Mathematics), in the second category our Faculty was represented by **Mirza Krbezlija** (3rd year of Mathematics). **Slobodan Filipovski** was a team leader.

Mitrović, Avdullah, and Baldouski received the certificate of a successful participant.

In this picture, from left to right: Mirza Krbezlija, Đorđe Mitrović, Arbër Avdullahu and Daniil Baldouski.

VOJTĚCH JARNÍK 2018

April 13, 2018

Category I

Problem 1. Every point of the rectangle $R = [0,4] \times [0,40]$ is coloured using one of four colours. Show that there exist four points in R with the same colour that form a rectangle having integer side lengths.

Problem 2. Find all prime numbers p such that p^3 divides the determinant

$$\begin{vmatrix} 2^2 & 1 & 1 & \dots & 1 \\ 1 & 3^2 & 1 & \dots & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & (p+7)^2 \end{vmatrix}$$

Problem 3. Let n be a positive integer and let $x_1, \dots, x_n$ be positive real numbers satisfying $|x_i - x_j| \le 1$ for all pairs (i,j) with $1 \le i \le j \le n$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \dots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \ge \frac{x_2+1}{x_1+1} + \frac{x_3+1}{x_2+1} + \dots + \frac{x_n+1}{x_{n-1}+1} + \frac{x_1+1}{x_n+1}.$$

Problem 4. Determine all possible (finite or infinite) values of

$$\lim_{x\to-\infty} f(x) - \lim_{x\to+\infty} f(x),$$

if $f : \mathbb{R} \to \mathbb{R}$ is a strictly decreasing continuous function satisfying

$$f(f(x))^4 - f(f(x)) + f(x) = 1$$

for all $x \in \mathbb{R}$.

VOJTĚCH JARNÍK 2018

April 13, 2018

Category II

Problem 1. Find all real solutions of the equation

$$17^x + 2^x = 11^x + 2^{3x}.$$

Problem 2. Let n be positive integer and let $a_1 \le a_2 \le \ldots \le a_n$ be real numbers such that

$$a_1 + 2a_2 + \cdots + na_n = 0.$$

Prove that

$$a_1[x] + a_2[2x] + \cdots + a_n[nx] \ge 0$$

for every real number x. (Here $[t]$ denotes the integer satisfying $[t] \le t < [t] + 1$.)

Problem 3. In $\mathbb{R}^3$ some n points are coloured. In every step, if four coloured points lie on the same line, Vojtěch can colour any other point on this line. He observes that he can colour any point $P \in \mathbb{R}^3$ in a finite number of steps (possibly depending on P). Find the minimal value of n for which this could happen.

Problem 4. Compute the integral

$$\iint\limits_{\mathbb{R}^2} \left(\frac{1 - e^{-xy}}{xy} \right)^2 e^{-x^2-y^2} dxdy.$$

Proposed problems for Vojtech Jarnik 2018 by University of Primorska

Problem 1. Let $k \geq 3$ and $\delta \geq 1$ be positive integers. Prove that the roots of the polynomial $p(x) = x^k + x^{k-1} + \ldots + x + 1 - \delta$ are simple. Moreover, if $\delta > k + 1$, prove that there exists at least one complex root with negative real part.

Solution. In the proof we use Descartes' rule of signs: the number of positive roots of a single-variable polynomial with real coefficients is equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number; similarly, the number of negative roots is the number of sign changes after multiplying the coefficients of odd-power terms by -1, or fewer than it by an even number. According to Descartes' rule of signs, the polynomial $p(x)$ has exactly one positive real root. If θ is a positive real root of $p(x)$ with multiplicity greater than 1, then θ also is a root of its derivative $kx^{k-1} + (k-1)x^{k-2} + \ldots + 2x + 1$, which is impossible. Thus, the unique positive real root of the polynomial $p(x)$ is simple. If $\delta \neq k+1$, then the polynomial $p(x)$ has the same roots as the equation

$$x^{k+1} - \delta x + \delta - 1 = 0, \tag{5}$$

except for the extra root of (5) $x = 1$; if $\delta = k + 1$, then $x = 1$ is a root of $p(x)$ with multiplicity 1 and a root of (5) with multiplicity 2. Using Descartes' rule of signs we have that the equation $x^{k+1} - \delta x + \delta - 1 = 0$ has at most two positive real roots (one of them is $x = 1$) and at most one negative real root. If we suppose that θ is a root of (5) with multiplicity greater than 1, we deduce that θ also satisfies the first derivative of (5), that is, $(k + 1)x^k - \delta$. Combining (5) and the identity $(k + 1)x^k - \delta = 0$ we obtain $\theta = \frac{(\delta-1)(k+1)}{\delta k} \geq 0$, which yields that there exist no negative real root nor complex roots of $p(x)$ with multiplicity greater than 1.

Now, let $\delta > k + 1 \geq 4$. Clearly $p(1) = k + 1 - \delta < 0$, and therefore, the unique positive root of $p(x)$ belongs to the interval $(1, \infty)$, and we denote it by θ_1. Descartes' rule asserts that the polynomial $p(x)$ has no negative real root when k is an odd number. We will prove the existence of a complex root of $p(x)$ with negative real part when $p(x)$ has a negative real root θ_2; in such case k must be an even number. The case when $p(x)$ has no negative root can be handled similarly.

Let $\theta_j = p_j + q_j i$, with $3 \leq j \leq k$, be the complex roots of $p(x)$. By way of contradiction we assume that $p_j \geq 0$, for all $3 \leq j \leq k$. Using the fact that the complex roots come in conjugate pairs and applying Vieta's formulas to the polynomial $p(x)$, we obtain

$$-1 = \theta_1 + \theta_2 + \ldots + \theta_k = \theta_1 + \theta_2 + (p_3 + \ldots + p_k) \geq \theta_1 + \theta_2.$$

On the other hand, since k is an even number and θ_1 is a positive, using the inequality $(1 + \theta_1)^t > (1 + \theta_1)^{t-1} + \theta_1^t + \theta_1^{t-1}$, for $2 \leq t \leq k$, we can deduce

$$p(-1 - \theta_1) = (-1 - \theta_1)^k + (-1 - \theta_1)^{k-1} + \ldots + (-1 - \theta_1) + 1 - \delta =$$

$$= (1+\theta_1)^k - (1+\theta_1)^{k-1} + \ldots + (1+\theta_1)^2 - (1+\theta_1) + 1 - \delta > \theta_1^k + \theta_1^{k-1} + \ldots + 1 - \delta = 0.$$

Since $p(0) < 0$ and $p(-1-\theta_1) > 0$, it follows that the negative root of $p(x)$ belongs to the interval $(-1-\theta_1, 0)$, that is, $-1-\theta_1 < \theta_2 < 0$. Thus $\theta_1+\theta_2 > -1$, which is in contradiction to $\theta_1 + \theta_2 \leq -1$.

INTERNATIONAL MATHEMATICAL COMPETITION 2018

From July 22 to 28, the 25th International Mathematics Competition IMC (International Mathematics Competition for University Students 2018) took place in Blagoevgrad, Bulgaria, in which Famnit's students traditionally participate. The colors of the faculty and the University of Primorska were represented this year by: **Đorđe Mitrović**, **Daniil Baldouski** and **Arbër Avdulahu**. They returned home with two bronze medals (Avdullahu – 28 points, Baldouski 23 – points), while Mitrović received the honorable mention of the competition. Team leader of our team was **Slobodan Filipovski**.

In this picture, from left to right: Daniil Baldouski, Đorđe Mitrović, Arbër Avdulahu and Slobodan Filipovski (team leader).

IMC 2018

Day 1, July 24, 2018

Problem 1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

(1) There is sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge;

(2) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

Problem 2. Does there exist a field such that its multiplicative group is isomorphic to its additive group?

Problem 3. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

Problem 4. Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b - a)f'(\sqrt{ab}) \text{ for all } a, b > 0$$

.

Problem 5. Let p and q be prime numbers with $p < q$. Suppose that in a convex polygon $P_1P_2 \ldots P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \geq \frac{k^3 + k}{2}$$

holds for every integer k with $1 \leq k \leq p$.

IMC 2018

Day 2, July 25, 2018

Problem 6. Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal.

Problem 7. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \dots$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

Problem 8. Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geq x \geq y \geq z \geq 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n, n, n) starting from $(0, 0, 0)$ in exactly $3n$ jumps.

Problem 9. Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

Problem 10. For $R > 1$ let $D_R = \{(a, b) \in \mathbb{Z}^2 : 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R \to \infty} \sum_{(a,b) \in D_R} \frac{(-1)^{a+b}}{a^2 + b^2}.$$

ACADEMIC YEAR 2018/2019

VOJTĚCH JARNÍK 2019

Also in 2019, our math students participated at the oldest mathematics competition for students within EU: the 29th Vojtěch Jarník International Mathematical Competition, which was held in Ostrava (Czech Republic) on March 29th. At this year's event, 145 students from 35 European universities competed. In the first category (for 1st and 2nd year students or under 22 years of age), Famnit was represented by **Đorđe Mitrović** (2nd year of Mathematics), **Eva Siladji** and **Besfort Shala** (both 1st year of Mathematics), in the second category our Faculty was represented by **Roman Solodukhin** (2nd year of Mathematical Sciences). Team leader of FAMNIT' team was **Slobodan Filipovski**.

Mitrović, Shala and Solodukhin received the certificate of a successful participant.

In this picture, from left to right: Roman Solodukhin, Eva Siladji, Besfort Shala and Đorđe Mitrović.

VOJTĚCH JARNÍK 2019

March 29, 2019

Category I

Problem 1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence given recursively by $a_0 = 1$ and

$$a_{n+1} = \frac{7a_n + \sqrt{45a_n^2 - 36}}{2}, \quad n = 0, 1, \ldots$$

Show that the following statements hold for all positive integers n:

a) a_n is a positive integer.

b) $a_n a_{n+1} - 1$ is the square of an integer.

Problem 2. A triplet of polynomials $u, v, w \in \mathbb{R}[x, y, z]$ is called smart if there exist polynomials $P, Q, R \in \mathbb{R}[x, y, z]$ such that the following polynomial identity holds:

$$u^{2019}P + v^{2019}Q + w^{2019}R = 2019.$$

a) Is the triplet of polynomials

$$u = x + 2y + 3, \quad v = y + z + 2, \quad w = x + y + z$$

smart?

b) Is the triplet of polynomials

$$u = x + 2y + 3, \quad v = y + z + 2, \quad w = x + y - z$$

smart?

Problem 3. For an invertible $n \times n$ matrix M with integer entries we define a sequence $S_M = \{M_i\}_{i=0}^{\infty}$ by the recurrence

$$M_0 = M$$
$$M_{i+1} = (M_i^T)^{-1} M_i, \quad i = 0, 1, \ldots$$

Find the smallest integer $n \geq 2$ for which there exists a normal $n \times n$ matrix M with integer entries such that its sequence S_M is non-constant and has period $P = 7$, i.e., $M_{i+7} = M_i$ for all $i = 0, 1, \ldots$

(M^T means the transpose of a matrix M. A square matrix M is called normal if $M^T M = M M^T$ holds.)

Problem 4. Determine the largest constant $K \geq 0$ such that

$$\frac{a^a(b^2 + c^2)}{(a^a - 1)^2} + \frac{b^b(c^2 + a^2)}{(b^b - 1)^2} + \frac{c^c(a^2 + b^2)}{(c^c - 1)^2} \geq K \left(\frac{a + b + c}{abc - 1} \right)^2$$

holds for all positive real numbers a, b, c such that $ab + bc + ca = abc$.

VOJTECH JARNIK 2019

March 29, 2019

Category II

Problem 1.

a) Is it true that for every non-empty set A and every associative operation $*: A \times A \to A$ the conditions

$$x * x * y = y \quad \text{and} \quad y * x * x = y \quad \text{for every} \quad x, y \in A$$

imply commutativity of $*$?

b) Is it true that for every non-empty set A and every associative operation $*: A \times A \to A$ the condition

$$x * x * y = y \quad \text{for every} \quad x, y \in A$$

imply commutativity of $*$?

Problem 2. Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f''(x) \cos(f(x)) \geq (f'(x))^2 \sin(f(x)); \quad \text{for every} \quad x \in \mathbb{R}.$$

Problem 3. Let p be an even non-negative continuous function with $\int_{\mathbb{R}} p(x)dx = 1$ and let n be a positive integer. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent identically distributed random variables with density function p. Define

$$\begin{aligned} X_0 &= 0, \\ X_1 &= X_0 + \xi_1, \\ X_2 &= X_1 + \xi_2, \\ &\vdots \\ X_n &= X_{n-1} + \xi_n. \end{aligned}$$

Prove that the probability that all the random variables $X_1, X_2, \ldots, X_{n-1}$ lie between X_0 and X_n equals $\frac{1}{n}$.

Problem 4. Let $D = \{z \in \mathbb{C} : \text{Im}z > 0, \text{Re}z > 0\}$. Let $n \geq 1$ and let $a_1, \ldots, a_n \in D$ be distinct complex numbers. Define

$$f(z) = z \cdot \prod_{j=1}^{n} \frac{z - a_j}{z - \overline{a_j}}.$$

Prove that f' has at least one root in D.

INTERNATIONAL MATHEMATICS COMPETITION 2019

The 26th International mathematical competition for University Students – IMC 2019, which takes place every year in Blagoevgrad in Bulgaria, was held between 28 July and 3 August.

FAMNIT and the University of Primorska were represented by **Besfort Shala**, **Đorđe Mitrović** and **Roman Solodukhin**. All three were very successfull and brought home three medals: two silvers (Shala in Solodukhin, both 37 points) and one bronze (Mitrović – 27 points).

In the picture, from left to right: Đorđe Mitrović, Besfort Shala and Roman Solodukhin.

IMC 2019

Day 1, July 30, 2019

Problem 1. Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Problem 2. A four-digit number YEAR is called *very good* if the system

$$Yx + Ey + Az + Rw = Y$$

$$Rx + Yy + Ez + Aw = E$$
$$Ax + Ry + Yz + Ew = A$$
$$Ex + Ay + Rz + Yw = R$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.
(The 21st century starts in 2001 and ends in 2100.)

Problem 3. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f^{'}(x) + xf^{''}(x) \geq 1 \text{ for } x \in (-1, 1).$$

Prove that

$$\int_{-1}^{1} xf(x)dx \geq \frac{1}{3}.$$

Problem 4. Define the sequence $a_0, a_1, \ldots$ of numbers by the following recurrence:

$$a_0 = 1, \quad a_1 = 2, \quad (n+3)a_{n+2} = (6n+9)a_{n+1} - na_n \text{ for } n \geq 0.$$

Prove that all terms of this sequence are integers.

Problem 5. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

1. $\det(B) = 1$;
2. $AB = BA$;
3. $A^4 + 4A^2B^2 + 16B^4 = 2019I$.
   (Here I denotes the $n \times n$ identity matrix.)

IMC 2019

Day 2, July 31, 2019

Problem 6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $(f(0) - g^{'}(0))(g^{'}(1) - f(1)) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g^{'}(c)$.

Problem 7. Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n.$$

Problem 8. Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum_{i \in I} x_i$. Assume that the function $I \to s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

Problem 9. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

Problem 10. 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

ACADEMIC YEAR 2019/2020

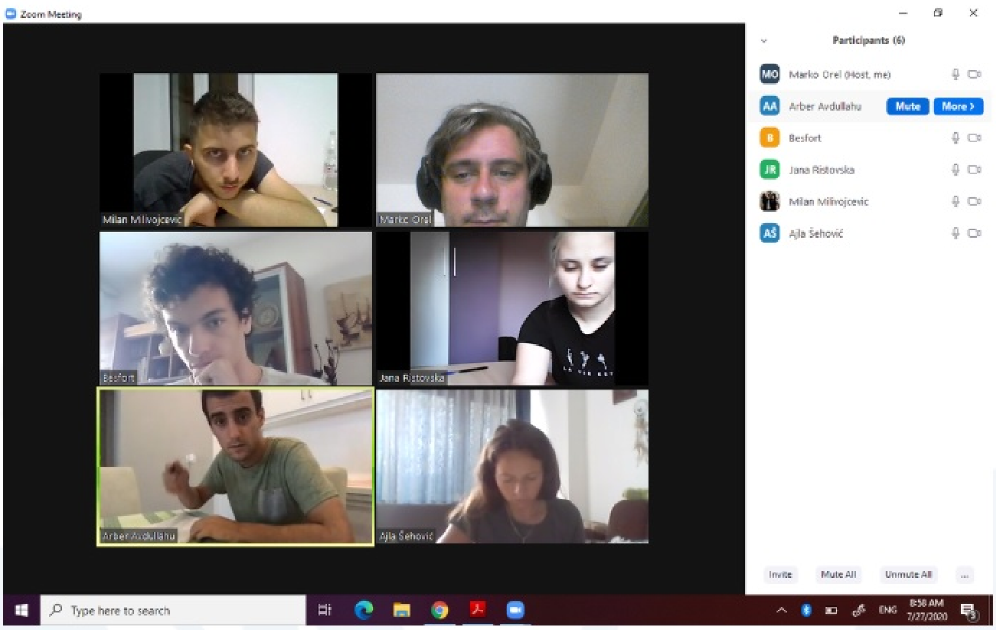
INTERNATIONAL MATHEMATICS COMPETITION 2020

The 27th IMC - International Mathematical Student Competition) took place this year from 25 to 30 July. The IMC is one of the most renowned competitions of its kind worldwide.

As the IMC was held online due to pandemic COVID -19, the participation was extremely high, with more than 560 students from all over the world taking part. The University of Primorska is the only institution in Slovenia that organised the preparations for its students well in advance and registered them for the competition.

The UP FAMNIT team was represented by **Arbër Avdullahu** (1st year, Mathematical Sciences), **Besfort Shala** (2nd year, Mathematics), **Ajla Šehović** (1st year, Mathematics), **Milan Milivojčević** (1st year, Computer Science), **Jana Ristovska** (1st year, Computer Science) and team leader Assoc. Prof. **Marko Orel.**

The opening ceremony took place on Sunday 25 July, followed by two days of competitions, during which the students had four hours a day to solve four problems. After the second day, the team leaders evaluated the students' tasks 24 hours a day, as the competition took place simultaneously all over the world. The results were announced on the last day at the closing ceremony. Among the UP FAMNIT students, Arbër Avdullahu and Besfort Shala were the best participants, both received an Honourable Mention certificate.



(UP FAMNIT students just before the start of the 2nd day of competition)

IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

(1) w consists of n letters, all of them are from the alphabet $\{a, b, c, d\}$;

(2) w contains an even number of letters a;

(3) w contains an even number of letters b.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.
Prove that

$$\mathrm{trace}(ABAB) - \mathrm{trace}(A^2B^2) = \frac{1}{2}n(n-1).$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. $\mathrm{trace}(M)$ denotes the trace of M, that is the sum of diagonal elements in M.)

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\epsilon \in (0, 1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\epsilon^{1-d}$ vertices such that

$$(1 - \epsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset R^d$ of at most α points, i.e, $T = \{\sum_{x\in X} t_x x \mid t_x \geq 0, \sum_{x\in X} t_x = 1.\}$ For a real λ put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x + 1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1 - t) \geq p(t)$ for $0 \leq t \leq 1/2$.

IMC 2020 Online

Day 2, July 27, 2020

Problem 5. Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying

$$f^{''}(x)f(x) \geq 2(f^{'}(x))^2$$

for all $x \in \mathbb{R}$.

Problem 6. Find all prime numbers p for which there exists a unique $a \in \{1, 2, \ldots, p\}$ such that $a^3 - 3a + 1$ is divisible by p.

Problem 7. Let G be a group and $n \geq 2$ be an integer. Let H_1 and H_2 be two subgroups of G that satisfy

$$[G : H_1] = [G : H_2] = n \quad \text{and} \quad [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that H_1 and H_2 are conjugate in G.
(Here $[G : H]$ denotes the *index* of the subgroup H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1 and H_2 are *conjugate* if there exists an element $g \in G$ such that $g^{-1}H_1g = H_2$.)

Problem 8. Compute

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k.$$

(Here log denotes the natural logarithm.)

ACADEMIC YEAR 2020/2021

INTERNATIONAL MATHEMATICS COMPETITION 2021

This year, the 28th International Mathematics Competition for University Students took place between 2 and 7 August. This year's competition, where 590 students from all over the world took part as 113 teams, was held online.

Exceptional result was achieved by Besfort Shala, winning second prize with 23 points. Todor Antić (7 points) and Dorotea Redžepi (4 points) received an honorary mention.

The UP FAMNIT team were represented by **Ajla Šehović** (2nd year, Mathematics), **Besfort Shala** (graduate, Mathematics), **Dorotea Redžepi** (1st year, Mathematics), **Lazar Marković** (1st year Mathematics), **Milan Milivojčević** (2nd year, Computer Science), **Todor Antić** (2nd year, Mathematics) and team leader, Assist. Prof. **Slobodan Filipovski**.

IMC 2021 Online

Day 1, August 3, 2021

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k-element subset X of $\{1, 2, \ldots, k + a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose a random n-element subset Y of $\{1, \ldots, k + n + a\}$ uniformly.

Prove that the probability

$$\mathsf{P}\Big(\min(Y) > \max(X)\Big)$$

does not depend on a.

Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \ldots \in (0, d)$ such that for each n, the points $a_1, \ldots, a_n$ partition the interval $[0, d]$ into segments of length at most $1/n$ each. Find

$$\sup\Big\{d \;\Big|\; d \text{ is good}\Big\}.$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if} \quad |x - y| < \min\big\{g(x), g(y)\big\}, \quad \text{then} \quad \big|f(x) - f(y)\big| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim\limits_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

IMC 2021 Online

Day 2, August 4, 2021

Problem 5. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \le 1$.

Problem 6. For a prime number p, let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

Problem 7. Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \le 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$\big|f(0)\big| \le \max_{|z|=1} \big|f(z)p(z)\big|.$$

Problem 8. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

ACADEMIC YEAR 2021/2022

TEAM SELECTION TEST FOR VOJTĚCH JARNÍK 2022

March 4, 2022

FAMNIT

Problem 1. Let A, B and X be $n \times n$ matrices over the same field, with X being nonsingular. Prove that $AB = AX + X^{-1}B$ if and only if $BA = XA + BX^{-1}$.

Problem 2. Let $\gamma_1 < \gamma_2 < \ldots < \gamma_{2022}$ be real numbers,

$$f(x) = \frac{1}{\gamma_1 - x} + \cdots + \frac{1}{\gamma_{2022} - x} + x,$$

and let $\alpha, \beta \in \mathbb{R}, \alpha < \beta$. Compute the total length of the preimage $f^{-1}([\alpha, \beta])$. (The length of the set consisting of intervals is the sum of their lengths.)

Problem 3. Let n, k be positive integers with $n \geq 3$; let $p(x) = x^n + x^{n-1} + \cdots + x - k$.

(a) Prove that the roots of $p(x)$ are simple.

(b) Prove that if $k \geq n + 1$, then $p(x)$ has at least one root with nonzero imaginary part and negative real part.

Problem 4. Let

$$S(N) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \cdots \sum_{k_N=1}^{k_{N-1}} 1.$$

Find the limit $\lim_{N\to\infty} \sqrt[N]{S(N)}$.

VOJTĚCH JARNÍK 2022

Between 31 March and 3 April, the 30th traditional Vojtěch Jarník International Mathematical Competition, the oldest mathematics competition for students in the European Union, which UP FAMNIT students have traditionally participated in for many years, took place in Ostrava (Czech Republic). This year's competition was held after a two-year break, and was again attended by UP FAMNIT students, who achieved outstanding results. In the first category (for 1st and 2nd year students or under 22 years of age) Famnit was represented by **Dren Neziri**, **Diar Gashi** (both 1st year of Mathematics) and **Dorotea Redžepi** (2nd year of Mathematics). In the second category our faculty was represented by **Ajla Šehović** (3rd year of Mathematics). Among the competitive problems was also included the problem proposed by dr. **Slobodan Filipovski** (Problem 3, Category 2), who was their team leader.

In the picture, from left to right: Dren Neziri, Diar Gashi, Dorotea Redžepi and Ajla Šehović.

Student Dren Neziri reached the best results, finishing 10th with 15 points, for which he also received a certificate.

VOJTĚCH JARNÍK 2022

April 2, 2022

Category I

Problem 1. Assume that a real polynomial $P(x)$ has no real roots. Prove that the polynomial

$$Q(x) = P(x) + \frac{P''(x)}{2!} + \frac{P^{(4)}(x)}{4!} + \dots$$

also has no real roots.

Problem 2. Let $n \geq 1$. Assume that A is a real $n \times n$ matrix which satisfies the equality

$$A^7 + A^5 + A^3 + A - I = 0.$$

Show that $\det(A) > 0$.

Problem 3. Let $f : [0,1] \to \mathbb{R}$ be a given continuous function. Find the limit

$$\lim_{n\to\infty} (n+1) \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} f(x) dx.$$

Problem 4. In a box there are 31, 41 and 59 stones coloured, respectively, red, green and blue. Three players, having t-shirts of these three colours, play the following game. They sequentially make one of two moves:

(I) either remove three stones of one colour from the box,

(II) or replace two stones of different colours by two stones of the third colour.

The game ends when all the stones in the box have the same colour and the winner is the player whose t-shirt has this colour. Assuming that the players play optimally, is it possible to decide whether the game ends and who will win, depending on who the starting player is?

VOJTĚCH JARNÍK 2022

April 2, 2022

Category II

Problem 1. Determine whether there exists a differentiable function $f : [0,1] \to \mathbb{R}$ such that

$$f(0) = f(1) = 1, \ |f^{'}(x)| \leq 2 \text{ for all } x \in [0,1] \text{ and } |\int_0^1 f(x)dx| \leq \frac{1}{2}.$$

Problem 2. For any given pair of positive integers $m > n$ find all $a \in \mathbb{R}$ for which the polynomial $x^m - ax^n + 1$ can be expressed as a quotient of two nonzero polynomials with real nonnegative coefficients.

Problem 3. Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, \ldots, x_n\}$. The mean μ and the variance σ^2 of X are defined as

$$\mu(X) = \frac{x_1 + \ldots + x_n}{n} \text{ and } \sigma^2(X) = \frac{(x_1 - \mu(X))^2 + \ldots + (x_n - \mu(X))^2}{n}.$$

By X^2 denote the discrete random variable uniformly distributed on $\{x_1^2, \ldots, x_n^2\}$. Prove that

$$\sigma^2(X) \geq \left(\frac{m}{2M^2}\right)^2 \sigma^2(X^2).$$

Problem 4. A function $f : \mathbb{Z}^+ \to \mathbb{R}$ is called multiplicative if for every $a, b \in \mathbb{Z}^+$ with $\gcd(a,b) = 1$ we have $f(ab) = f(a)f(b)$. Let g be the multiplicative function given by

$$g(p^\alpha) = \alpha p^{\alpha - 1},$$

where $\alpha \in \mathbb{Z}^+$ and $p > 0$ is a prime. Prove that there exist infinitely many positive integers n such that

$$g(n) + 1 = g(n+1).$$

Proposed problems for Vojtěch Jarník 2022 by University of Primorska

Problem 1. Let $x_1, \ldots, x_n$ be given real numbers with $0 < m \leq x_i \leq M$ for each $i \in \{1, \ldots, n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1, \ldots, x_n\}$. The mean μ and the variance σ^2 of X are defined as

$$\mu(X) = \frac{x_1 + \ldots + x_n}{n} \text{ and } \sigma^2(X) = \frac{(x_1 - \mu(X))^2 + \ldots + (x_n - \mu(X))^2}{n}.$$

By X^2 denote the discrete random variable uniformly distributed on $\{x_1^2, \ldots, x_n^2\}$. Prove that

$$\sigma^2(X) \geq \left(\frac{m}{2M^2}\right)^2 \sigma^2(X^2).$$

Solution. First we prove the following lemma:

Lemma 2. *If x and y are strictly positive real numbers, then*

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{(x-y)^2}{2(x^2+y^2)}.$$

Proof. We prove the following equivalent inequality

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{\left(\frac{x}{y}\right)^2 - 2\left(\frac{x}{y}\right) + 1}{2\left(\left(\frac{x}{y}\right)^2 + 1\right)}.$$

Let $t^2 = \frac{x}{y}, t > 0$. The required inequality is equivalent to the inequalities

$$t + \frac{1}{t} \geq 2 + \frac{t^4 - 2t^2 + 1}{2(t^4+1)} \Leftrightarrow 2t^6 - 5t^5 + 2t^4 + 2t^3 + 2t^2 - 5t + 2 \geq 0.$$

Now we easily show $2t^6 - 5t^5 + 2t^4 + 2t^3 + 2t^2 - 5t + 2 = (t-1)^4(2t^2 + 3t + 2) \geq 0.$ □

Let $a_i = \frac{x_i^2}{x_1^2 + \ldots + x_n^2}$ and $b_i = \frac{1}{n}$ for $i = 1, \ldots, n$. Applying the above lemma for $x = a_i$ and $y = b_i$ we obtain

$$\frac{x_i^2}{x_1^2 + \ldots + x_n^2} + \frac{1}{n} \geq \left(2 + \frac{(x_i^2 n - (x_1^2 + \ldots + x_n^2))^2}{2(x_i^4 n^2 + (x_1^2 + \ldots + x_n^2)^2)}\right) \frac{x_i}{\sqrt{(n(x_1^2 + \ldots + x_n^2)}}. \tag{6}$$

Now if we sum up the obtained n inequalities in (6) we get

$$2 \geq \frac{2}{\sqrt{n(x_1^2 + \ldots + x_n^2)}} \sum_{i=1}^{n} x_i + \frac{m}{\sqrt{n(x_1^2 + \ldots + x_n^2)}} \cdot \frac{1}{2(M^4 + \mu^2(X^2))} \cdot \sum_{i=1}^{n} (x_i^2 - \frac{x_1^2 + \ldots + x_n^2}{n})^2 \Leftrightarrow$$

$$\sqrt{\frac{x_1^2+\ldots+x_n^2}{n}} \geq \frac{\sum_{i=1}^n x_i}{n} + \frac{m\cdot\sigma^2(X^2)}{4(M^4+\mu^2(X^2))} = \mu(X) + \frac{m\cdot\sigma^2(X^2)}{4(M^4+\mu^2(X^2))} \Leftrightarrow$$

$$\sqrt{\mu(X^2)} \geq \mu(X) + \frac{m\cdot\sigma^2(X^2)}{4(M^4+M^4)} = \mu(X) + \frac{m\cdot\sigma^2(X^2)}{8M^4}.$$

In the end we get

$$\sigma^2(X) = (\sqrt{\mu(X^2)}-\mu(X))(\sqrt{\mu(X^2)}+\mu(X)) \geq \frac{m\sigma^2(X^2)}{8M^4}\cdot 2m = \left(\frac{m}{2M^2}\right)^2\cdot\sigma^2(X^2).$$

Problem 2. Let $p \geq 3$ be a prime number and let $n \geq 1$ be a natural number. Prove that for any k such that $2 \leq k \leq p^n$ holds

$$\gcd\left(\binom{p^n+2}{k}, \binom{p^n+1}{2}\right) > 1.$$

Solution. Let $k = 2$. Then

$$\gcd\left(\binom{p^n+2}{k}, \binom{p^n+1}{2}\right) = \gcd\left((p^n+2)\cdot\frac{(p^n+1)}{2}, \frac{(p^n+1)}{2}\cdot p^n\right) \geq \frac{p^n+1}{2} > 1.$$

Now, without loss of generality let $3 \leq k \leq \frac{p^n+2}{2}$. Let $\gcd(p^n, k(k-1)(k-2)) = d$. There exists integers x and y such that $p^n x + k(k-1)(k-2)y = d$. We have

$$d\binom{p^n+2}{k} = (p^n x + k(k-1)(k-2)y)\binom{p^n+2}{k} =$$

$$= p^n x\binom{p^n+2}{k} + k(k-1)(k-2)y\cdot\frac{p^n+2}{k}\cdot\frac{p^n+1}{k-1}\cdot\frac{p^n}{k-2}\cdot\binom{p^n-1}{k-3} =$$

$$= p^n\left(x\binom{p^n+2}{k} + (p^n+2)(p^n+1)y\binom{p^n-1}{k-3}\right).$$

Thus $\frac{p^n}{d}|\binom{p^n+2}{k}$. We consider two cases:

1. If $d < p^n$, then $\frac{p^n}{d}$ is a divisor of $\binom{p^n+2}{k}$. On the other hand, $\binom{p^n+1}{2} = \frac{(p^n+1)p^n}{2} = \frac{p^n+1}{2}\cdot p^n$. Thus $p^n|\binom{p^n+1}{2}$, that is, $\frac{p^n}{d}$ is a divisor of $\binom{p^n+1}{2}$. In this case the claim holds.
2. Let $d = p^n$. Then from $p^n|k(k-1)(k-2)$ and $\gcd(k, k-1, k-2) = 1, \gcd(k, k-2) \leq 2$ we have $p^n|k$ or $p^n|k-1$ or $p^n|k-2$. Clearly, these three divisibilities are not possible since $k \leq \frac{p^n+2}{2}$.

INTERNATIONAL MATHEMATICS COMPETITION 2022

The 29. International Mathematics Competition for University Students 2022 (IMC) was held between 1st and 7th August 2022, where, now traditionally, FAMNIT students competed among 667 participants from all over the world. This year, UP FAMNIT students performed with exceptional results. Ranking best was **Diar Gashi** (1st year, Mathematics), who received a bronze medal with 21 points, while the honorary mention went to **Dren Neziri** (1st year, Mathematics), **Mirza Redzić** (1st year, Mathematical Sciences) and **Todor Antić** (3rd year, Mathematics).

The students **Dorotea Redžepi** (2nd year, Mathematics) and **Ajla Šehović** (3rd year, Mathematics) who received a certificate of participation. Team leader of FAMNIT's team was **Slobodan Filipovski**.

After two years the IMC was finally held live. The competition was organized under the auspices of University College London and hosted by the American University in Bulgaria. Hybrid participation also allowed for online competition, where the final results are not published yet.

In the picture, from left to right: Diar Gashi, Dren Neziri, Ajla Šehivić, Dorotea Redžepi, Mirza Redzič and Todor Antić.

IMC 2022

Day 1, August 3, 2022

Problem 1. Let $f : [0, 1] \to (0, \infty)$ be an integrable function such that $f(x) \cdot f(1 - x) = 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x)dx \geq 1.$$

Problem 2. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

for some integer $k \geq n$.
(A^T denotes the transpose of A.)

Problem 3. Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p - 1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.

Problem 4. Let $n > 3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1, 2, \ldots, n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1 \leq a < b < c < d \leq n$, the triples $\{a, b, c\}$ and $\{b, c, d\}$ have different colours. Prove that

$$\frac{1}{100} \log \log n \leq m \leq 100 \log \log n$$

.

IMC 2022

Day 2, August 4, 2022

Problem 5. We colour all the sides and diagonals of a regular polygon P with 43 vertices either red or blue in such a way that every vertex is an endpoint of 20 red segments and 22 blue segments. A triangle formed by vertices of P is called monochromatic if all of its sides have the same colour. Suppose that there are 2022 blue monochromatic triangles. How many red monochromatic triangles are there?

Problem 6. Let $p > 2$ be a prime number. Prove that there is a permutation $(x_1, x_2, \ldots, x_{p-1})$ of the numbers $(1, 2, \ldots, p-1)$ such that

$$x_1x_2 + x_2x_3 + \ldots + x_{p-2}x_{p-1} \equiv 2 \pmod{p}.$$

Problem 7. Let $A_1, A_2, \ldots, A_k$ be $n \times n$ idempotent complex matrices such that

$$A_iA_j = -A_jA_i \text{ for all } i \neq j.$$

Prove that at least one of the given matrices has rank $\leq \frac{n}{k}$.
(A matrix A is called idempotent if $A^2 = A$.)

Problem 8. Let$n, k \geq 3$ be integers, and let S be a circle. Let n blue points and k red points be chosen uniformly and independently at random on the circle S. Denote by F the intersection of the convex hull of the red points and the convex hull of the blue points. Let m be the number of vertices of the convex polygon F (in particular, $m = 0$ when F is empty). Find the expected value of m.

Proposed problems for IMC 2022 by University of Primorska

Problem 1. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 3&0&1&1&0&0&1&1&1&1\\ 0&3&0&1&1&1&0&1&1&1\\ 1&0&3&0&1&1&1&0&1&1\\ 1&1&0&3&0&1&1&1&0&1\\ 0&1&1&0&3&1&1&1&1&0\\ 0&1&1&1&1&3&1&0&0&1\\ 1&0&1&1&1&1&3&1&0&0\\ 1&1&0&1&1&0&1&3&1&0\\ 1&1&1&0&1&0&0&1&3&1\\ 1&1&1&1&0&1&0&0&1&3 \end{pmatrix}.$$

Solution. Note that

$$A = \begin{pmatrix} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&1&1&0\\ 0&1&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&1\\ 0&0&0&1&0&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&0&0 \end{pmatrix}^2 = \begin{pmatrix} A_1 & I_5 \\ I_5 & A_2 \end{pmatrix}^2 = B^2,$$

where $A_1 = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix}$ and $I_5 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$.

Lemma 3. *If λ is an eigenvalue of B, then λ^2 is an eigenvalue of B^2.*

Based on the above lemma, it suffices to calculate the eigenvalues of the block matrix B. Since the matrices I_5 and $A_2 - \lambda I_5$ commute, we have

$$\det(B - \lambda I_{10}) = \det \begin{pmatrix} A_1 - \lambda I_5 & I_5 \\ I_5 & A_2 - \lambda I_5 \end{pmatrix} = \det((A_1 - \lambda I_5) \cdot (A_2 - \lambda I_5) - I_5 \cdot I_5) =$$

$$= \det(A_1 \cdot A_2 - \lambda(A_1 + A_2) + (\lambda^2 - 1)I_5).$$

It is easy to verify that $A_1 \cdot A_2 = A_1 + A_2 = J_5 - I_5$, where J_5 is all-ones matrix. Thus

$$\det(B - \lambda I_{10}) = 0 \Leftrightarrow \det((1-\lambda)J_5 + (\lambda^2 + \lambda - 2)I_5) = 0.$$

Now, it remains to find all values for λ such that

$$\det \begin{pmatrix} \lambda^2-1 & 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda & \lambda^2-1 & 1-\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda & 1-\lambda & \lambda^2-1 & 1-\lambda & 1-\lambda \\ 1-\lambda & 1-\lambda & 1-\lambda & \lambda^2-1 & 1-\lambda \\ 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda & \lambda^2-1 \end{pmatrix} = 0. \tag{7}$$

We reduce the matrix in (7) to an upper triangular matrix.

$$\det \begin{pmatrix} \lambda^2-1 & 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda & \lambda^2-1 & 1-\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda & 1-\lambda & \lambda^2-1 & 1-\lambda & 1-\lambda \\ 1-\lambda & 1-\lambda & 1-\lambda & \lambda^2-1 & 1-\lambda \\ 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda & \lambda^2-1 \end{pmatrix} =$$

$$= \det \begin{pmatrix} -1-\lambda & 1 & 1 & 1 & 1 \\ 1 & -1-\lambda & 1 & 1 & 1 \\ 1 & 1 & -1-\lambda & 1 & 1 \\ 1 & 1 & 1 & -1-\lambda & 1 \\ 1 & 1 & 1 & 1 & -1-\lambda \end{pmatrix} \cdot (1-\lambda)^5 =$$

$$= (1-\lambda)^5 \cdot \det \begin{pmatrix} -1-\lambda & 1 & 1 & 1 & 1 \\ 0 & \frac{\lambda+2}{\lambda+1} & \frac{\lambda+2}{\lambda+1} & \frac{\lambda+2}{\lambda+1} & -\frac{\lambda^2+2\lambda}{\lambda+1} \\ 0 & 0 & \lambda+2 & \lambda+2 & (\lambda+2)(1-\lambda) \\ 0 & 0 & 0 & \lambda+2 & -\lambda^2+4 \\ 0 & 0 & 0 & 0 & -\lambda^2+\lambda+6 \end{pmatrix} = (1-\lambda)^5(\lambda+2)^3(\lambda^2-\lambda-6).$$

Solving the equation $(1-\lambda)^5(\lambda+2)^3(\lambda^2-\lambda-6) = 0$ we get that $-2, 1$ and 3 are the eigenvalues of B (with multiplicity $4, 5$ and 1, respectively).
From the lemma we conclude that $1, 4$ and 9 are eigenvalues of A, with multiplicity $5, 4$ and 1, respectively.

Problem 2. Let $p > 3$ be a prime number. Prove that, if p is a primitive root of $4p+1$, then $2p+1$ is a composite number.

Solution. Since p is a primitive root of $4p+1$ we have $4p+1 = q^k$, where q is odd prime and $k \geq 1$. Let $k > 1$. We get $4p = q^k - 1 = (q-1)(q^{k-1} + \ldots + q + 1)$. Thus $q = 3$ or $q = 5$. If $q = 3$ we get $4p + 1 = 3^k$. From $3^k \equiv 1 \pmod 4$ we get that k is an even number, $k = 2k_1$. Thus $p = \frac{(3^{k_1}-1)(3^{k_1}+1)}{4}$. Since both of the numbers $3^{k_1} - 1$ and $3^{k_1} + 1$ are even, and exactly one of them is divisible by 4 we get that p is an even number.
Let $q = 5$ and $4p + 1 = 5^k$. From $\phi(5^k) = 4 \cdot 5^{k-1}$ we get:

$$4p \equiv -1 \pmod{5^k} \Leftrightarrow (4p)^{\frac{\phi(5^k)}{2}} \equiv (-1)^{\frac{\phi(5^k)}{2}} \equiv 1 \pmod{5^k}.$$

By induction we can prove that $4^{2\cdot 5^{k-1}} \equiv 1 \pmod{5^k}$. Thus we get $p^{\frac{\phi(5^k)}{2}} \equiv 1 \pmod{5^k}$, which implies that p is not a primitive root of $4p+1$. Hence $4p+1$ is a prime number. Since $4p+1$ is prime we have $p \equiv 1 \pmod 3$. In this case $2p+1$ is divisible by 3, that is, $2p+1$ is a composite number.

ACADEMIC YEAR 2022/2023

INTERNATIONAL MATHEMATICS COMPETITION 2023

At the end of July 2023, the 30th International Mathematics Competition for University Students (IMC) took place in Blagoevgrad, Bulgaria. The competition brought together more than 400 students from numerous universities worldwide to showcase their mathematical knowledge and problem-solving skills.

Traditionally were part of the competition also UP FAMNIT students, who together with their team leader **Blas Fernández** from the Department of Mathematics, faced all the challenges presented by the competition.

This time, the University of Primorska (UP FAMNIT) was represented by **Dren Neziri** (a 2nd-year student of the Mathematics) and **Diar Gashi** (a 3rd-year student of the Mathematics), who received an honorable mention, and **Yllkë Jashari** (a 1st-year student of the Mathematics) and **Dmytro Tupkalenko** (a 1st-year student of the Mathematics), who received a certificate of achievement.

The IMC spanned two days of intense mathematical examinations. Each day, participants were tasked with solving five exercises, with each exercise proposed by different team leaders. On the second day of the competition, a particular problem proposed by **Slobodan Filipovski** from the University of Primorska, was selected for the participants to tackle.

In the picture, from left to right: Yllkë Jashari, Dmytro Tupkalenko, Blas Fernández (team leader), Dren Neziri and Diar Gashi.

IMC 2023

Day 1 , August 2, 2023

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

Problem 2. Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \text{ and } B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

Problem 3. Find all polynomials P in two variables with real coefficients satisfying the identity

$$P(x,y)P(z,t) = P(xz - yt, xt + yz).$$

Problem 4. Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \ldots, p-1$ form a complete residue system modulo p. What is the set of possible remainders of a_2 upon division by p?

Problem 5. Fix positive integers n and k such that $2 \le k \le n$ and a set M consisting of n fruits. A *permutation* is a sequence $x = (x_1, x_2, \ldots, x_n)$ such that $\{x_1, \ldots, x_n\} = M$. Ivan *prefers* some (at least one) of these permutations. He realized that for every preferred permutation x, there exist k indices $i_1 < i_2 < \ldots < i_k$ with the following property: for every $1 \le j < k$, if he swaps x_{i_j} and $x_{i_{j+1}}$, he obtains another preferred permutation.
Prove that he prefers at least $k!$ permutations.

IMC 2023

Day 2 , August 3, 2023

Problem 6. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

1. he chooses a row or a column of the matrix, and
2. he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

Problem 7. Let V be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$, differentiable on $(0, 1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0, 1)$ such that

$$f(\xi) + \alpha = f^{'}(\xi).$$

Problem 8. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u, v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u and v. Consider the sums

$$W(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} d(u, v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} \frac{1}{d(u, v)}.$$

Prove that

$$W(T) \cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

Problem 9. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in $\mathbb{R}^3$, with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint. Find $\sup\{V \mid V \text{ is good}\}$.

Problem 10. For every positive integer n, let $f(n), g(n)$ be the minimal positive integers such that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} = \frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n) > n^{0.999n}$.

Proposed problems for IMC 2023
by University of Primorska

Problem 1. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u, v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u and v. Consider the sums

$$W(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} d(u,v) \quad\text{and}\quad H(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} \frac{1}{d(u,v)}.$$

Prove that

$$W(T)\cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

Solution: Let $k = \binom{n}{2}$ and let $x_1 \leq x_2 \leq \ldots \leq x_k$ be the distances between the pairs of vertices in the tree T_n. Thus

$$W(T_n)\cdot H(T_n) = (x_1 + x_2 + \ldots + x_k)\cdot\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k}\right).$$

Since the tree has exactly $n-1$ edges, there are exactly $n-1$ pairs of vertices at distance one, that is, $x_1 = x_2 = \ldots = x_{n-1} = 1$. Thus

$$W(T_n)\cdot H(T_n) = (n-1+x_n+x_{n+1}+\ldots+x_k)\cdot\left(n-1+\frac{1}{x_n}+\frac{1}{x_{n+1}}+\ldots+\frac{1}{x_k}\right) =$$

$$= (n-1)^2 + (n-1)\left(\left(x_n+\frac{1}{x_n}\right)+\ldots+\left(x_k+\frac{1}{x_k}\right)\right) +$$

$$+(x_n+\ldots+x_k)\left(\frac{1}{x_n}+\ldots+\frac{1}{x_k}\right).$$

From Cauchy inequality we have

$$(x_n+\ldots+x_k)\left(\frac{1}{x_n}+\ldots+\frac{1}{x_k}\right) \geq (1+1+\ldots+1)^2 = (k-n+1)^2 = \frac{(n-1)^2(n-2)^2}{4}.$$

The equality holds if and only if $x_n = x_{n+1} = \ldots = x_k$.

Now we minimize the expression $\left(x_n+\frac{1}{x_n}\right)+\ldots+\left(x_k+\frac{1}{x_k}\right)$, where $x_i \in [2, n-1]$.

It is clear that the minimal value is achieved for $x_n = x_{n+1} = \ldots = x_k = 2$. Therefore we get

$$W(T_n)\cdot H(T_n) \geq (n-1)^2+(n-1)\left(\left(2+\frac{1}{2}\right)(k-n+1)\right)+\frac{(n-1)^2(n-2)^2}{4} = \frac{(n-1)^3(n+2)}{4}.$$

The equality holds for $x_1 = \ldots = x_{n-1} = 1$ and $x_n = x_{n+1} = \ldots = x_k = 2$, that is, the smallest value is achieved for the tree where $n-1$ pairs are at distance one, and the remaining $k-(n-1) = \frac{(n-1)(n-2)}{2}$ pairs are at distance two. The unique tree which satisfies these conditions is the star graph S_n. In this case it holds

$$W(S_n) \cdot H(S_n) = (n-1)^2 \cdot \frac{(n-1)(n+2)}{4} = \frac{(n-1)^3(n+2)}{4}.$$

Problem 2. Let A be a real square matrix such that the sum of each row is equal to $d > 1$, $d \in \mathbb{N}$, and $\mathrm{tr}(A^i) = 0$, for each $i = 1, \ldots, 2023$. Prove that

$$A^{2023} + A^{2022} + \ldots + A + I \neq J,$$

where J is the all-ones matrix.

Solution 1. Let us suppose that there is a square matrix A of size n for which $A^{2023} + A^{2022} + \ldots + A + I = J$. Clearly, the spectrum of A consists of d and some of the roots of the equation $x^{2023} + x^{2022} + \ldots + x + 1 = 0$. Since $\mathrm{tr}(A^k) = 0$ for each $k = 1, \ldots, 2023$ we get

$$d^k + \sum_{i=1}^{n-1} x_i^k = 0. \tag{8}$$

From $x_i = \frac{1}{x_i} = \frac{1}{x_i^{2023}}$ and (8) we get

$$-d = \sum_{i=1}^{n-1} x_i = \sum_{i=1}^{n-1} \frac{1}{x_i} = \sum_{i=1}^{n-1} x_i^{2023} = -d^{2023}.$$

Thus we get $d = 0$ or $d = 1$.

Solution 2. Let us suppose that there is a matrix A for which $A^{2023}+A^{2022}+\ldots+A+I = J$. Clearly, d is an eigenvalue of A and $d^{2023} + d^{2022} + \ldots + d + 1 = n$ is an eigenvalue of J. Thus, the eigenvalues of J are n (with multiplicity 1) and 0 (with multiplicity $n-1$). Moreover, the eigenvalues of A are d and the roots of the polynomial $p(x) = x^{2023}+\ldots+x+1$. The roots of $p(x)$ are the 2024-th roots of unity, (except $x = 1$), so they are simple roots. We use the following lemma derived by Feit and Higman.

Lemma 4. *Let θ be a simple root of the polynomial $f(x)$ and let $f_\theta(x) = \frac{f(x)}{x-\theta}$. If M is a matrix satisfying $f(M) = O$, then $\frac{\mathrm{tr}(f_\theta(M))}{f_\theta(\theta)}$ is the multiplicity of θ as a characteristic root of M.*

Since $f(x) = (x-d)(x^{2023} + \ldots + x + 1)$ is the minimal polynomial of A we have $f(A) = O_n$.
Let θ be an eigenvalue of A different than d and 1. We use the above lemma to compute the multiplicity of θ, $m(\theta)$. Let $g(x) = x^{2024} - 1$. We have

$$f_\theta(x) = \frac{f(x)}{x-\theta} = \frac{x-d}{x-\theta} \cdot \frac{g(x)}{x-1} \tag{9}$$

$$f_\theta(\theta) = \frac{\theta - d}{\theta - 1} \cdot D(g(\theta)) = \frac{\theta - d}{\theta - 1} \cdot 2024\theta^{2023}. \tag{10}$$

From (9) we have $f_\theta(0) = \frac{d}{\theta}$. Since $\mathrm{tr}(A^i) = 0$ for $i = 1, \ldots, 2023$ we get $\mathrm{tr}(f_\theta(A)) = f_\theta(0)\mathrm{tr}(I) = \frac{d}{\theta} \cdot n$. By using the above lemma we get

$$m(\theta) = \frac{nd}{2024} \cdot \frac{\theta - 1}{\theta - d}. \tag{11}$$

Since θ is a root of $p(x)$, we pick θ to be a complex number. From $d > 1$, we get that $\frac{\theta-1}{\theta-d}$ is a complex number as well, thus $m(\theta)$ is a complex number, which is not possible. Therefore, for each matrix A which satisfies the given conditions holds $A^{2023} + A^{2022} + \ldots + A + I \neq J$.

Problem 3. Let $\{a_i\}_{i=1}^{\infty}$ be a geometric progression of natural numbers which quotient has exactly k prime divisors. Prove that the $(k-1)$-th differences of the sequence $\{\tau(a_i)\}_{i=1}^{\infty}$ is an arithmetic progression.

Solution. In the solution we will use the following two well-known identities:

Lemma 5. *1.* $\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i} = 0$.

2. For each $1 \leq m \leq k$ *it holds* $\sum_{i=1}^{k+1}(-1)^i i^m\binom{k+1}{i} = 0$.

Let $q = p_1^{\alpha_1}p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ be the canonical form of the quotient of the progression and let $a_1 = p_1^{\beta_1}p_2^{\beta_2} \cdot \ldots \cdot p_k^{\beta_k} \cdot b$, where $\beta_i \geq 0$ and $gcd(p_i, b) = 1$ for $i = 1, 2, \ldots, k$. Then for any $i \geq 1$ we have $a_i = a_1 \cdot q^{i-1} = p_1^{(i-1)\alpha_1+\beta_1} \cdot p_2^{(i-1)\alpha_2+\beta_2} \cdot \ldots \cdot p_k^{(i-1)\alpha_k+\beta_k} \cdot b$ and

$$\tau(a_i) = ((i-1)\alpha_1 + \beta_1 + 1)((i-1)\alpha_2 + \beta_2 + 1) \cdot \ldots \cdot ((i-1)\alpha_k + \beta_k + 1)\tau(b). \tag{12}$$

It is easy to prove that the $(k-1)$th differences of $\{\tau(a_i)\}_{i=1}^{\infty}$ is the sequence

$$\{s_i\}_{i=1}^{\infty} = \{\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\tau(a_{k+i-j-1})\}_{i=1}^{\infty}. \tag{13}$$

We will prove that $s_{n+1} - 2s_n + s_{n-1} = 0$ for any $n \geq 2$, which is a sufficient condition to assert that the sequence $\{s_i\}_{i=1}^{\infty}$ is arithmetic. We have

$$s_{n+1} - 2s_n + s_{n-1} = 0 \Leftrightarrow$$

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\tau(a_{k+n-j}) - 2\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\tau(a_{k+n-j-1}) + \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\tau(a_{k+n-j-2}) = 0. \tag{14}$$

The identity in (14) is equivalent to the identity

$$\binom{k-1}{0}\tau(a_{k+n}) - \left(\binom{k-1}{1} + 2\binom{k-1}{0}\right)\tau(a_{k+n-1}) +$$

$$+\sum_{j=2}^{k-1}(-1)^j\left(\binom{k-1}{j} + 2\binom{k-1}{j-1} + \binom{k-1}{j-2}\right)\tau(a_{k+n-j}) +$$

$$+(-1)^k\left(2\binom{k-1}{k-1}+\binom{k-1}{k-2}\right)\tau(a_n)+(-1)^{k+1}\binom{k-1}{k-1}\tau(a_{n-1})=0.$$

Using $\binom{k-1}{0}=\binom{k+1}{0}$, $\binom{k-1}{1}+2\binom{k-1}{0}=\binom{k+1}{1}$, $\binom{k-1}{j}+2\binom{k-1}{j-1}+\binom{k-1}{j-2}=\binom{k+1}{j}$, for $2\leq j\leq k-1$, $2\binom{k-1}{k-1}+\binom{k-1}{k-2}=\binom{k+1}{k}$ and $\binom{k-1}{k-1}=\binom{k+1}{k+1}$, the above identity is equivalent to

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\tau(a_{k+n-j})=0. \tag{15}$$

Let $b_t=\alpha_t(k+n-1)+\beta_t+1$ for $1\leq t\leq k$. From the formula (12) and since $\tau(b)\neq 0$, the identity in (15) is equivalent to

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(b_1-j\alpha_1)(b_2-j\alpha_2)\cdot\ldots\cdot(b_k-j\alpha_k)=0. \tag{16}$$

Now we have $(b_1-j\alpha_1)(b_2-j\alpha_2)\cdot\ldots\cdot(b_k-j\alpha_k)=C_0-jC_1+j^2C_2+....+(-1)^kj^kC_k$ where C_i are constants which depends on α's and b's. Finally the identity in (16) is equivalent to

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}(C_0+\sum_{i=1}^{k}(-1)^iC_ij^i)=0. \tag{17}$$

We rearrange the last expression and we get

$$C_0\left(\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}\right)-C_1\left(\sum_{i=1}^{k+1}(-1)^i i\cdot\binom{k+1}{i}\right)+$$

$$+C_2\left(\sum_{i=1}^{k+1}(-1)^i i^2\cdot\binom{k+1}{i}\right)-\ldots+(-1)^kC_k\left(\sum_{i=1}^{k+1}(-1)^i i^k\cdot\binom{k+1}{i}\right)=0.$$

Based on the above lemma we verify that the last identity is valid, which complete the proof.