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ON CERTAIN PROBLEMS IN VERTEX-TRANSITIVE GRAPHS

PhD Thesis

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Abstract

On Certain Problems in Vertex-transitive Graphs

In this PhD Thesis four related topics from algebraic graph theory are considered. The first one considers non-Cayley vertex-transitive graphs. In 1983, Marušič [Ars Combinatorial 16B (1983), 297–302] asked for which positive integers n there exists a non-Cayley vertex-transitive graph on n vertices. (The term non-Cayley numbers has later been given to such integers.) Motivated by this problem, Feng [Discrete Math. 248 (2002), 265–269] asked to determine the smallest valency $\vartheta(n)$ *among valencies of non-Cayley vertex-transitive graphs of order* n*. As cycles are clearly Cayley graphs,* $\vartheta(n) \geq 3$ *for any non-Cayley number* n*. In this PhD Thesis a goal is set to determine those non-Cayley numbers* n *for which* $\vartheta(n) = 3$*, and among the latter to determine those for which the generalized Petersen graphs are the only non-Cayley vertex-transitive graphs of order* n*. It is known that for a prime* p *every vertex-transitive graph of order* p*,* p^2 *or* p^3 *is a Cayley graph, and that, with the exception of the Coxeter graph, every cubic non-Cayley vertex-transitive graph of order* $2p$*,* $4p$ *or* $2p^2$ *is a generalized Petersen graph. In this PhD Thesis the next natural step is taken by proving that every cubic non-Cayley vertex-transitive graph of order* $4p^2$*,* $p > 7$ *a prime, is a generalized Petersen graph. In addition, cubic non-Cayley vertex-transitive graphs of order* $2p^k$*, where* $p > 7$ *is a prime and* $k \leq p$*, are characterized.*

The second topic considers automorphism groups of certain vertex-transitive graphs. In particular, in this PhD Thesis it is proved that connected cubic non-symmetric Cayley graphs on ten particular finite non-abelian simple groups are normal, where these ten groups are M_{11}*,* M_{22}*,* M_{23}*,* J_2*, Suz,* $\mathrm{PSL}(2,11)$*, and* A_n*,* $n \in \{5, 11, 23, 47\}$*. This result solves an open problem posed in [Discrete Math. 244 (2002), 67–75].*

The third topic considers one-regular graphs, a special class of vertex-transitive graphs that have received particular attention over the last decade. The main contribution of this PhD Thesis is a complete classification of tetravalent one-regular graphs of order $4p^2$*, where* p *is a prime.*

The last topic of this PhD Thesis considers the famous open problem in algebraic graph theory, asking about existence of Hamilton paths in vertex-transitive graphs. This problem is considered for graphs of order $10p$*,* $p \neq 7$ *a prime. In particular, it is proved that every connected vertex-transitive graph of order* $10p$*,* $p \neq 7$ *a prime, which is not isomorphic to a quasiprimitive graph arising from the action of* $\mathrm{PSL}(2,k)$ *on cosets of* $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$*, contains a Hamilton path.*

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Key words: graph, vertex-transitive, Cayley graph, non-Cayley graph, simple group, one-regular graph, Hamilton path, regular action, regular cover, automorphism group.

Izvleček

O nekaterih odprtih problemih iz točkovno tranzitivnih grafov

Disertacija obravnava štiri med seboj povezane teme s področja algebraične teorije grafov. Prva od štirih tem so ne-Cayleyjevi točkovno tranzitivni grafi. Leta 1983 je Marušič [Ars Combinatorial 16B (1983), 297–302] postavil vprašanje, za katera pozitivna števila n *obstaja ne-Cayleyjev točkovno tranzitiven graf reda* n*. (Takim številom pravimo ne-Cayleyjeva števila.) To vprašanje je bilo kasneje motivacija za Fengovo vprašanje [Discrete Math. 248 (2002), 265–269], ki sprašuje po najmanjši valenci* $\vartheta(n)$ *med valencami ne-Cayleyjevih točkovno tranzitivnih grafov reda* n*. Ker so cikli seveda Cayleyjevi grafi, za vsako ne-Cayleyjevo število* n *velja* $\vartheta(n) \geq 3$*. V disertaciji obravnavamo tista ne-Cayleyjeva števila* n*, za katera je* $\vartheta(n) = 3$*, in med njimi iščemo tiste, za katere so posplošeni Petersenovi grafi edini ne-Cayleyjevi točkovno tranzitivni grafi reda* n*. Znano je, da je za praštevilo* p *vsak točkovno tranzitiven graf reda* p*,* p^2 *ali* p^3 *Cayleyjev graf in da je z izjemo Coxeterjevega grafa vsak kubični ne-Cayleyjev točkovno tranzitiven graf reda* $2p$*,* $4p$ *ali* $2p^2$ *posplošeni Petersenov graf. V disertaciji je narejen naslednji korak. Dokazano je, da je vsak kubični ne-Cayleyjev točkovno tranzitiven graf reda* $4p^2$*,* $p > 7$ *praštevilo, posplošeni Petersenov graf. Poleg tega je narejena karakterizacija kubičnih ne-Cayleyjevih točkovno tranzitivnih grafov reda* $2p^k$*, kjer je* $p > 7$ *praštevilo in* $k \leq p$*.*

Druga v disertaciji obravnavana tema so grupe avtomorfizmov posebne družine točkovno tranzitivnih grafov. Dokazano je, da je vsak povezan kubični ne-simetrični Cayleyjev graf grupe G*, kjer je* $G \in \{\mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_2, \mathrm{Suz}, \mathrm{PSL}(2,11), \mathrm{A}_n \mid n \in \{5, 11, 23, 47\}\}$*, normalen graf. Ta rezultat delno reši problem, ki je bil postavljen v [Discrete Math. 244 (2002), 67–75].*

Tretja v disertaciji obravnavana tema so ena-regularni grafi, posebna družina točkovno tranzitivnih grafov, ki so bili v zadnjem desetletju predmet številnih raziskav. Glavni prispevek disertacije je popolna klasifikacija štirivalentnih ena-regularnih grafov reda $4p^2$*, kjer je* p *praštevilo.*

Zadnja tema obravnava enega izmed najpomembnejših odprtih problemov v algebraični teoriji grafov, problem obstoja hamiltonskih poti v točkovno tranzitivnih grafih. V disertaciji je problem obravnavan za točkovno tranzitivne grafe reda $10p$*,* $p \neq 7$ *praštevilo. Dokazano je, da vsak povezan točkovno tranzitiven graf reda* $10p$*,* $p \neq 7$ *praštevilo, ki ni izomorfen kvaziprimitivnemu grafu glede na delovanje grupe* $\mathrm{PSL}(2,k)$ *na odsekih po* $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$*, premore hamiltonsko pot.*

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Ključne besede: graf, točkovno tranzitiven graf, Cayleyjev graf, ne-Cayleyjev graf, enostavne grupa, ena-regularen, hamiltonska pot, regularno delovanje, regularni krov, grupa avtomorfizmov.

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Chapter 1

Introduction

The PhD Thesis contains four related topics from algebraic graph theory: non-Cayley vertex-tansitive graphs, automorphism groups of certain vertex-transitive graphs, one-regular graphs and Hamilton paths in vertex-transitive graphs. All these topics concern *vertex-transitive graphs*, that is, graphs whose automorphism groups act transitively on the corresponding vertex sets, which justifies the title of the PhD Thesis. In the literature vertex-transitive graphs are also called *vertex symmetric graphs*. Throughout the thesis graphs are finite, simple and undirected, and groups are finite, unless specified otherwise.

The motivation for these studies comes from four open problems. They are introduced at the beginning of each chapter, where also an overview of known partial results is given (see Chapters 3, 4, 5 and 6).

A graph is said to be a *Cayley graph* if its automorphism group admits a subgroup acting regularly on its vertex set. Thus every Cayley graph is vertex-transitive. However, there are vertex-transitive graphs which are not Cayley graphs, the smallest example is the well-known Petersen graph. Non-Cayley vertex-transitive graphs, considered in Chapter 3, have been an active topic of research for a long time. Since connected vertex-transitive graphs of valency 2 are just cycles, which are clearly Cayley graphs on cyclic groups, the first interesting family of non-Cayley vertex-transitive graphs are cubic graphs. Consequently, articles found in the literature study these graphs frequently. The contribution of this PhD Thesis to the topic is the result that every cubic non-Cayley vertex-transitive graph of order $4p^2$, $p > 7$ a prime, is a generalized Petersen graph, see Section 3.1. In addition, cubic non-Cayley vertex-transitive graphs of order $2p^k$, where $p > 7$ is a prime and $k \leq p$, are characterized in the PhD Thesis, see Section 3.2.

Chapter 4 deals with automorphism groups of cubic Cayley graphs on finite non-abelian simple groups. Li [66] showed that a cubic arc-transitive Cayley graph on a finite non-abelian simple group G is normal if G is not one of the following seven groups: A_5, $\mathrm{PSL}(2,11)$, $\mathrm{M}_{11}, \mathrm{A}_{11}, \mathrm{M}_{23}, \mathrm{A}_{23}$ and A_{47}. (A Cayley graph X on a group G is said to be *normal* if the right regular representation of G is normal in the automorphism group of X.) For these seven remaining groups, Xu, Fang, Wang and Xu [123] proved that a cubic arc-transitive Cayley graph on G is normal if $G \neq \mathrm{A}_{47}$ and it is not normal if $G = \mathrm{A}_{47}$. However, normality of cubic Cayley graphs on finite non-abelian simple groups which are not arc-transitive is still an open problem. In particular, Fang, Li, Wang and Xu [34] proved that a connected cubic non-arc-transitive Cayley graph on a finite non-abelian simple group G is normal or G is one of the groups listed in [34]. In the PhD Thesis the normality of cubic Cayley graphs on ten groups from this list is proved. In particular, it is shown that a connected non-arc-transitive cubic Cayley graph on G, where $G \in \{\mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_2, \mathrm{Suz}, \mathrm{PSL}(2,11), \mathrm{A}_5, \mathrm{A}_{11}, \mathrm{A}_{23}, \mathrm{A}_{47}\}$, is a normal Cayley graph, see Chapter 4.

Chapter 5 deals with one-regular graphs, that is, graphs whose automorphism groups act regularly on the corresponding arc sets. Clearly, a one-regular graph with no isolated vertices is connected, and it is of valency 2 if and only if it is a cycle. The first example of a cubic one-regular graph was constructed by Frucht [40], and later on lots of work have been done along this line as part of a more general problem dealing with the investigation of cubic arc-transitive graphs (see [19, 25, 35, 36, 37, 38, 95]). Tetravalent one-regular graphs have also received considerable attention in recent decades, see Chapter 5. This PhD Thesis contributes to this topic with a complete classification of tetravalent one-regular graphs of order $4p^2$, p a prime, see Theorem 5.4.1.

The results in Chapters 3, 4 and 5 are obtained by detailed study of the automorphism groups of the graphs in question. In this study basic group-theoretic results, covering graph techniques and combinatorial techniques are used.

Chapter 6 deals with the problem of existence of Hamilton paths in finite connected vertex-transitive graphs. In particular, in 1969 Lovász [71] asked whether every finite, connected vertex-transitive graph has a Hamilton path. Although this problem received considerable attention over the years it remains open. However, all known connected vertex-transitive graphs have a Hamilton path and with the exception of K_2, only four connected vertex-transitive graphs that do not have a Hamilton cycle are known to exist. These four graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. In the PhD Thesis it is shown that every connected vertex-transitive graph of order $10p$, $p \neq 7$ a prime, which is not isomorphic to a quasiprimitive graph arising from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, contains a Hamilton path, see Theorem 6.0.1. Our strategy in the search for Hamilton paths in connected vertex-transitive graphs of order $10p$ is based on the so-called lifting Hamilton cycles approach, a frequently used approach for constructing Hamilton paths and cycles in vertex-transitive graphs. This approach is based on a quotienting/reduction with respect to an imprimitivity block system of the corresponding automorphism group or with respect to a suitable semiregular automorphism.

In Chapter 2 notions concerning the thesis are introduced together with the notation and some auxiliary results that are needed throughout the thesis.

The results of this PhD Thesis are published in the following articles:

- C. Zhang and X. G. Fang, A note on the automorphism groups of cubic Cayley graphs of finite simple groups, *Discrete Math.* **310** (2010), 3030-3032.
- K. Kutnar, D. Marušič and C. Zhang, On cubic non-Cayley vertex-transitive graphs, *J. Graph Theory*, DOI 10.1002/jgt.20573, in press.

- Y.-Q. Feng, K. Kutnar, D. Marušič and C. Zhang, Tetravalent one-regular graphs of order $4p^2$, submitted.
- K. Kutnar, D. Marušič and C. Zhang, Hamilton paths and cycles in vertex-transitive graphs of order $10p$, submitted.

Chapter 2

Background

2.1 Groups

For group-theoretic terms not defined here we refer the reader to [97, 107, 114]. We will use the symbol $\mathbb{Z}_r$, both for cyclic group of order r and the ring of integers modulo r. In the latter case, $\mathbb{Z}_r^*$ will denote the multiplicative group of units of $\mathbb{Z}_r$. By D_{2n} we denote the dihedral group of order $2n$.

2.1.1 Group action

Let G be a group and let Ω be a nonempty set. Then a *(right) group action* of G on Ω is a binary function

$$\Omega \times G \to \Omega$$

denoted

$$(\omega, g) \mapsto \omega^g$$

which satisfies the following two axioms: 1. $\omega^1 = \omega$ for every ω in Ω (where 1 denotes the identity element of G) and 2. $(\omega^g)^h = \omega^{gh}$ for all $g, h \in G$ and $\omega \in \Omega$. This group action we denote by (Ω, G). In a similar way we can define *(left) group action* (G, Ω).

A right group action of a group G on the set Ω gives rise to a group homomorphism $\chi\colon G \to \mathrm{Sym}_r(\Omega)$ defined by the rule:

$$g \mapsto \chi_g,\ \ \chi_g(\omega) := \omega^g,$$

where $\mathrm{Sym}_r(\Omega)$ is the *right symmetric group* of the set Ω (that is, the set of all bijective maps $\phi\colon \Omega \to \Omega$, which are called *permutations of the set* Ω, together with the composition of the permutations). Conversely: if $f\colon G \to Sym_r(\Omega)$ is a group homomorphism then it gives rise to the right group action of the group G on the set Ω in such a way that $\chi = f$. The group homomorphism χ is called the *representation of the right action* (Ω, G). The image $G_R = G^\chi$ is the *right representation* of the group G. The *degree* of the right representation G_R of the group G is the cardinality of the set Ω. The *kernel of the homomorphism* χ is the set of all those elements of the group G that act on a trivial way: $Ker\chi = \{g \in G \mid \chi_g = id\} = \{g \in G \mid \chi_g(\omega) = \omega, \forall \omega \in \Omega\} = \{g \in G \mid \omega^g = \omega, \forall \omega \in \Omega\}$. If the kernel is trivial then the action is said to be *faithful*.

Let a group G act on the set Ω. Then the set

$$Orb_G(\omega) = \omega^G = \{\omega^g \mid g \in G\},$$

where $\omega \in \Omega$, is called a G-*orbit* (in short an *orbit* if the group G is clear from the context) of the element ω with respect to the action of the group G. If an orbit $Orb_G(\omega)$ is equal to the entire set Ω for some element ω in Ω, then G is *transitive*.

For $\omega \in \Omega$ the set

$$G_\omega = \{g \in G \mid \omega^g = \omega\},$$

the *stabilizer of the element* ω, is a subgroup of G. The orbits of G_ω on Ω are called *suborbits of the group* G (with respect to the element ω). Suborbit $\{\omega\}$ is *trivial suborbit*. If $|G_\omega| = 1$ for every element $\omega \in \Omega$ then we say that G acts *semiregularly*. If G acts on Ω transitively and $|G_\omega| = 1$ for every element $\omega \in \Omega$ we say that G acts *regularly* (G is *regular*).

The following well-known fact is known as the *Orbit - stabilizer* property:

$$|Orb_G(\omega)| = |G : G_\omega| \quad \text{for every } \omega \in \Omega.$$

Given a transitive group G acting on a set Ω, we say that a partition $\mathcal{B}$ of Ω is G-*invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. In other words, a nonempty subset $B \subseteq \Omega$ is a block for the group G if for every $g \in G$

$$\text{either } B^g = B \quad \text{or} \quad B^g \cap B = \emptyset.$$

If the trivial partitions $\{\Omega\}$ and $\{\{\omega\} : \omega \in \Omega\}$ are the only G-invariant partitions of Ω, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. In the latter case we shall refer to a corresponding G-invariant partition as to an *imprimitivity block system* of G (see also [10]). An imprimitive group G with an imprimitivity block system formed by the orbits of a proper normal subgroup of G is called a *genuinely imprimitive* group. If G is imprimitive, but there exists no transitive subgroup $H \leq G$ having a nontransitive normal subgroup, then G is said to be *quasiprimitive*.

Let a group H act transitively on the set Ω and let $G = H_R$ be the right representation of the group H. Then in a natural way G also acts on the set $\Omega \times \Omega = \Omega^2$. G-orbits on the set $\Omega \times \Omega = \Omega^2$ are called *orbitals*. The orbital $\{(\omega,\omega) \mid \omega \in \Omega\}$ is a *trivial orbital*. If $\Delta = \{(\omega,\nu) \mid \omega,\nu \in \Omega\} \subseteq \Omega^2$ is an orbital, then also $\Delta^* = \{(\nu,\omega) \mid (\omega,\nu) \in \Delta\}$ is an orbital, it is called the *paired orbital* of the orbital Δ. If $\Delta = \Delta^*$ then Δ is said to be *selfpaired orbital*. There exists a natural correspondence between suborbits and orbitals. Let G_ω be a stabilizer of an element $\omega \in \Omega$, let Γ be the suborbit with respect to ω and let $\nu \in \Gamma$. Then the corresponding orbital is the G-orbit on Ω^2 that contains (ω,ν). It is left to the reader to check up that this orbital is well defined. Moreover, if Δ is any orbital then the corresponding suborbit Γ is the suborbit that contains ν where ν is an arbitrary element from Ω for which: $(\omega,\nu) \in \Delta$.

2.1.2 Simple groups

A *simple group* is a nontrivial group whose only normal subgroups are the trivial group and the group itself. The study of (non-abelian) finite simple groups can be traced back at least as far as Galois (see [115]) who understood their fundamental significance as obstacles to the solution of polynomial equations by radicals (square roots, cube roots, etc.). However, the classification of finite simple groups was not completed until 1980's.

The Classification Theorem of Finite Simple Groups [50]: *Every finite simple group is cyclic of prime order, an alternating group, a finite simple group of Lie type, or one of the twenty-six sporadic finite simple groups.*

Finite simple groups will be considered throughout the thesis. In particular, we will frequently use the following result.

Proposition 2.1.1 [53] *Let G be a non-abelian simple group, $H < G$ and $|G:H| = p^n$, where p is a prime and n is a positive integer. Then one of the following holds:*

(i) $G = \mathrm{A}_m$ *and* $H \cong \mathrm{A}_{m-1}$, *where* $m = p^n$;

(ii) $G = \mathrm{PSL}(m, q)$, *H is the stabilizer of a line or hyperplane, and* $|G : H| = \frac{q^m-1}{q-1} = p^n$;

(iii) $G = \mathrm{PSL}(2, 11)$ *and* $H \cong \mathrm{A}_5$;

(iv) $G = \mathrm{M}_{23}$ *and* $H \cong \mathrm{M}_{22}$, *or* $G = \mathrm{M}_{11}$ *and* $H \cong \mathrm{M}_{10}$;

(v) $G = U_4(2) \cong S_4(3)$ *and H is the parabolic subgroup of index* 27.

We wrap up this subsection by a result on imprimitive groups of degree $5p$, $p \geq 7$ a prime, which will be needed later on in Chapter 6. In the proof of this result Proposition 2.1.1 and the following result will be needed.

Proposition 2.1.2 [104] *Let G be a finite group and $H \leq G$. If $|G : H| = n$, then G/H_G is isomorphic to a subgroup of the symmetric group S_n, where H_G is the largest normal subgroup of G that is contained in H.*

Proposition 2.1.3 *Let G be a transitive non-abelian simple group of degree $5p$, $p \geq 7$ a prime, and let H be a maximal subgroup of G such that $G_\alpha < H < G$. Then G is quasiprimitive and one of the following holds:*

(i) $G = \mathrm{PSL}(2, 11)$, $H = \mathrm{A}_5$, $|G : H| = 11$ *and* $G_\alpha = \mathrm{A}_4$;

(ii) $G = \mathrm{PSL}(m, q)$, *H is the stabilizer of a line or hyperplane, m is a prime, q is a prime power and* $|G : H| = \frac{q^m-1}{q-1} = p$.

Proof. Let G be a transitive non-abelian simple group of degree $5p$, where $p \geq 7$ is a prime. Since the stabilizer G_α of a point α is not maximal the group G is imprimitive, and consequently, since it is a simple group, it is quasiprimitive.

By the Praeger's classification of quasiprimitive groups [99], one can see that G is in class AS. Let H be a maximal subgroup of G such that $G_\alpha < H < G$. Since $|G : G_\alpha| = 5p$, we can conclude that either $|G : H| = 5$ or $|G : H| = p$. If $|G : H| = 5$, then $|H : G_\alpha| = p$, and since G is a non-abelian simple group, Proposition 2.1.2 implies that G is isomorphic to a subgroup of S_5. We can conclude that $G \cong \mathrm{A}_5$ and thus $H \cong \mathrm{A}_4$. But since $p \geq 7$, A_4 has no subgroup of index p, a contradiction. If, however, $|G : H| = p$, then $|H : G_\alpha| = 5$, and thus G is one of the groups listed in Proposition 2.1.1(i)-(iv).

Suppose that G is the group from Proposition 2.1.1(i). Then $G = \mathrm{A}_p$ and $H \cong \mathrm{A}_{p-1}$. For $p - 1 \leq 4$ the group $H \cong \mathrm{A}_{p-1}$ has no subgroup of index 5. On the other hand, if $p - 1 \geq 5$, then $H \cong \mathrm{A}_{p-1}$ is a simple group, and it has a subgroup of index 5 if and only if $p - 1 = 5$. But then $p = 6$ is not a prime, a contradiction. It follows that G is not a group from Proposition 2.1.1(i). Further, since M_{22} and M_{10} have no subgroup of index 5, G cannot be a group from Proposition 2.1.1(iv) either, and we can conclude that G is a group from Proposition 2.1.1(ii) or (iii). ∎

2.1.3 Group-theoretic results

In this section, we will introduce various group-theoretic results that will be used in the proofs throughout this PhD Thesis. For a subgroup H of a group G, we denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G.

Proposition 2.1.4 [114, Proposition 4.4] *A transitive abelian group G on a set Ω is regular and the centralizer $C_{S_\Omega}(G)$ of G in the symmetric group S_Ω is equal to G.*

Proposition 2.1.5 [56, Chapter I, Theorem 4.5] *Let G be a group and H a subgroup of G. Then the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of H.*

Proposition 2.1.6 [104, Theorem 8.5.3] *Every group of order $p^m q^n$, where p and q are primes, and m and n are non-negative integers, is solvable.*

Proposition 2.1.7 [114, Theorem 3.4] *Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. If p^m divides the length of the G-orbit containing $\omega \in \Omega$, then p^m also divides the length of the P-orbit containing ω.*

The following result can be extracted from [24, pp. 285].

Proposition 2.1.8 [24] *Let* $G = \mathrm{PSL}(2,7)$. *Then Sylow* 2*-subgroups of G are isomorphic to D_8. Moreover, all involutions of G are conjugate, and G has no subgroup of order* 14.

A transitive group G acting on a set Ω is said to be *doubly transitive* if it acts transitively on the set of ordered pairs of points from Ω. Further, G is said to be *simply primitive* if it is primitive but not doubly transitive. The following result is due to Burnside [9].

Proposition 2.1.9 [9] *Let G be a transitive group of prime degree p. Then either G is doubly transitive or G contains a normal Sylow p-subgroup.*

The following result on primitive groups of degree $2p$ may be deduced from [70].

Proposition 2.1.10 [70] *A primitive group G of degree $2p$, p a prime, is one of the following:*

(i) $p = 5$, *and* $G = \mathrm{A}_5$ *or* $G = S_5$*;*

(ii) $G = \mathrm{A}_{2p}$ *or* $G = S_{2p}$*;*

(iii) $p = 11$ *and* $G = \mathrm{M}_{22}$*;*

(iv) $p = \frac{1+q^{2^t}}{2}$, *and G is a subgroup of* $\mathrm{Aut}(\mathrm{PSL}(2,k))$ *containing* $\mathrm{PSL}(2,k)$, *where* $k = q^{2^t}$ *and q is an odd prime.*

Moreover, G is simply primitive in case (i) and is doubly transitive in all other cases.

We wrap up this subsection by the result that may be extracted from [28, Theorem 2.10].

Proposition 2.1.11 [28] *Let G be a transitive permutation group of degree $10p$, $p \geq 5$ a prime, with an imprimitivity block system $\mathcal{B}$ formed by a (proper, intransitive) minimal normal subgroup N of G. Then N^B is simple for all blocks $B \in \mathcal{B}$.*

2.2 Graphs

A *graph* or *undirected graph* X is an ordered pair $X = (V, E)$ where $V = V(X)$ is a set, whose elements are called *vertices* and $E = E(X)$ is a set of pairs (unordered) of distinct vertices, called *edges*. The vertices belonging to an edge are called the *endvertices* of the edge. The *order of a graph* X is the cardinality of its vertex set $|V(X)|$. A *multigraph* is a generalization of a graph in which we allow multiedges and loops.

For adjacent vertices u and v in X, we write $u \sim v$ and denote the corresponding edge by uv. If $u \in V(X)$ then $N(u)$ denotes the set of neighbors of u. The *valency* (or *degree*) of a vertex u in X is the number of edges incident to the vertex u, that is $|N(u)|$. If each vertex of the graph has the same valency d the graph is called a *regular graph of valency d*. A graph X is *cubic* if it is regular of valency 3. A graph X is *tetravalent* if it is regular of valency 4.

A *walk* is a sequence of graph vertices and graph edges such that the vertices and the edges are adjacent. A *path* on a graph is a sequence $v_1, v_2, \ldots, v_n$ such that v_1v_2, v_2v_3, …, $v_{n-1}v_n$ are edges of the graph and the vertices v_i are distinct. A closed path $v_1, v_2, \ldots, v_n, v_1$ on a graph is called a *cycle*. A *Hamilton path* in a graph X is a path which visits each vertex of X exactly once. A *Hamilton cycle* in a graph X is a cycle that visits each vertex of X exactly once (except the vertex which is both the start and the end, and so is visited twice). A graph is *hamiltonian* if it possess a Hamilton cycle. By an n-cycle we shall always mean a cycle with n vertices.

The *distance* $d(u,v)$ between two vertices u and v in a graph is the number of edges in a shortest path connecting them. With $N_i(u)$ we denote the set of vertices at distance $i > 1$ from a vertex u. A graph is *connected* if there exists a path between all pairs of vertices in the graph, otherwise the graph is *disconnected*. A *bipartite graph* is a graph X whose vertices can be divided into two disjoint sets U and U' ($U, U' \subseteq V(X)$, $U \cup U' = V(X)$) such that every edge connects a vertex in U and one in U'; that is, there is no edge between two vertices in the same set.

An ordered pair (u,v) of adjacent vertices u and v in a graph X is called an *arc*. If $e = (u,v)$ is an arc in X then $e^{\lambda} = (v,u)$ denotes the same edge but with the opposite direction. The arc e^{λ} is called the *reverse* of the arc e. A sequence $(u_0, u_1, u_2, \dots, u_k)$ of distinct vertices in X is called a *k-arc* if u_i is adjacent to u_{i+1} for every $i \in \{0, 1, \dots, k-1\}$.

The complement of a graph X will be denoted by X^c, that is, $V(X^c) = V(X)$ and there is an edge between two different vertices u and v in X^c if and only if $uv \notin E(X)$. Let U and W be disjoint subsets of $V(X)$. The subgraph of X induced by U will be denoted by $X\langle U\rangle$. Similarly, we let $X[U,W]$ (in short $[U,W]$) denote the bipartite subgraph of X induced by the edges having one endvertex in U and the other endvertex in W.

2.2.1 Action of groups on graphs

An *automorphism* α of a graph $X = (V,E)$ is an isomorphism of X with itself. Thus each automorphism α of X is a permutation of the vertex set V which preserves adjacency. The set of all automorphisms of X together with the composition of the permutations form the *automorphism group* $\mathrm{Aut}(X)$ of the graph X. Any subgroup G of the automorphism group $\mathrm{Aut}(X)$ of the graph X in a natural way acts on the set of vertices $V(X)$, set of edges $E(X)$ and set of arcs $A(X)$ of X. A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the set of vertices, edges and arcs of X, respectively. A graph X is said to be *G-vertex-transitive*, *G-edge-transitive*, and *G-arc-transitive* if G is vertex-transitive, edge-transitive and arc-transitive, respectively. If $G = \mathrm{Aut}(X)$ we simply say that X is *vertex-transitive*, *edge-transitive* or *arc-transitive*. An arc-transitive graph is also called a *symmetric graph*. A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *s-regular* if it acts transitively on the set of s-arcs and the stabilizer of an s-arc in G is trivial. If $G = \mathrm{Aut}(X)$ the graph X is said to be *s-regular*.

If the subgroup G of the automorphism group $\mathrm{Aut}(X)$ of a graph X acts (im)primitively on the vertex set $V(X)$ we say that X is *G-(im)primitive*. A vertex-transitive graph for which each transitive subgroup of its automorphism group is primitive is called a *primitive graph*. Otherwise it is called an *imprimitive graph*. If X is imprimitive with an imprimitivity block system which is formed by the orbits of a proper normal subgroup of some transitive subgroup $G \leq \mathrm{Aut}(X)$, then the graph X is said to be *genuinely imprimitive*. If X is imprimitive, but there exists no transitive subgroup $G \leq \mathrm{Aut}(X)$ having a nontransitive normal subgroup, then X is said to be *quasiprimitive*. Note that if $\mathcal{B}$ is an imprimitivity block system of some vertex-transitive graph, then any two blocks $B, B' \in \mathcal{B}$ induce isomorphic vertex-transitive subgraphs.

Imprimitive vertex-transitive graphs of order $2p$, p a prime, were described in [75]. Among other things it was proved there that, provided a vertex-transitive graph X of order $2p$ admits an imprimitive group G (with blocks of size p or 2), one can always find an imprimitive subgroup of G which has blocks of size p. In particular, the following result is proved in [75] and will be used later on.

Proposition 2.2.1 *Let X be a vertex-transitive graph of order $2p$, p a prime. If $G \leq \mathrm{Aut}(X)$ is an imprimitive subgroup of $\mathrm{Aut}(X)$ on X with blocks of size 2, then there exists an imprimitive subgroup H of G with blocks of size p.*

2.2.2 Orbital (di)graphs

Let G be a transitive permutation group of a set Ω and let $\Delta \subseteq \Omega^2$ be a nontrivial orbital. Then the *orbital digraph* $\vec{X}$ is an directed graph with vertex set Ω and edge set Δ.

The group G is a subgroup of the automorphism group $\mathrm{Aut}(\vec{X})$ of the graph $\vec{X}$ and it acts transitively on the vertex set and on the set of directed edges. If instead of directed edges we take undirected edges we get *orbital graph* X with vertex set Ω and edge set $\{\{\omega, \nu\} \mid (\omega, \nu) \in \Delta\}$. This graph is vertex- and edge-transitive but it is not necessarily arc-transitive. It is symmetric (arc-transitive) if and only if the orbital Δ is selfpaired.

2.2.3 Cayley graphs and normal Cayley graphs

Given a group G and a generating set S of G such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S has vertex set G and edge set $\{g \sim sg \mid g \in G, s \in S\}$. Then $X = \mathrm{Cay}(G, S)$ admits a right regular action of G on $V(X) = G$, hereafter identified with G (this should cause no confusion). Sabidussi characterized [105] Cayley graphs as follows. A graph is a Cayley graph on a group G if and only if its automorphism group contains a regular subgroup isomorphic to G.

Let $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ be a group of all those automorphisms of G fixing S setwise. By Godsil [49, Lemma 2.1], for a connected Cayley graph $X = \mathrm{Cay}(G, S)$ we have $N_{\mathrm{Aut}(X)}(G) = G \rtimes \mathrm{Aut}(G, S)$.

Following Xu [121], $X = \mathrm{Cay}(G, S)$ is called a *normal Cayley graph* if G is normal in $\mathrm{Aut}(X)$, that is, if $\mathrm{Aut}(G, S)$ coincides with the vertex stabilizer of $1 \in G$, that is, $\mathrm{Aut}(G, S) = A_1$. So, for a normal Cayley graph $X = \mathrm{Cay}(G, S)$ we have that $|\mathrm{Aut}(G, S)|$ divides $|S|!$. And, for a tetravalent one-regular normal Cayley graph $X = \mathrm{Cay}(G, S)$, the set S consists of four elements, say s_1, s_2, s_3 and s_4, and either there exists $\alpha \in \mathrm{Aut}(G, S)$ cyclically permuting s_1, s_2, s_3 and s_4 (in which case $A_1 \cong \mathbb{Z}_4$), or there exist $\alpha, \beta \in \mathrm{Aut}(G, S)$ such that $s_1^\alpha = s_2$ and $s_3^\beta = s_4$ (in which case $A_1 \cong \mathbb{Z}_2^2$).

2.2.4 Generalized Petersen graphs

Let $n \geq 3$ be a positive integer, and let $r \in \{1, \ldots, n-1\} \setminus \{n/2\}$. The *generalized Petersen graph* $\mathrm{GP}(n, r)$ is a graph with

$$V(\mathrm{GP}(n, r)) = \{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\}$$

and

$$E(\mathrm{GP}(n, r)) = \{u_i u_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{v_i v_{i+r} \mid i \in \mathbb{Z}_n\} \cup \{u_i v_i \mid i \in \mathbb{Z}_n\}.$$

Note that $\mathrm{GP}(n, r)$ is cubic, and it is easy to see that $\mathrm{GP}(n, r) \cong \mathrm{GP}(n, n-r)$. In 1971 Frucht, Graver and Watkins [42] proved that a generalized Petersen graph $\mathrm{GP}(n, t)$ is vertex-transitive if and only if $t^2 \equiv \pm 1 \pmod{n}$ or $(n, t) = (10, 2)$, and that there are only seven symmetric generalized Petersen graphs: $\mathrm{GP}(4, 1)$, $\mathrm{GP}(5, 2)$, $\mathrm{GP}(8, 3)$, $\mathrm{GP}(10, 2)$, $\mathrm{GP}(10, 3)$, $\mathrm{GP}(12, 5)$, and $\mathrm{GP}(24, 5)$. Moreover, the following proposition holds.

Proposition 2.2.2 [42, 96] *The generalized Petersen graph* $\mathrm{GP}(n, t)$ *is a non-Cayley vertex-transitive graph if and only if either* $t^2 \equiv -1 \pmod{n}$ *or* $(n, t) = (10, 2)$. *In addition, if* $t^2 \equiv -1 \pmod{n}$ *and* $(n, t) \notin \{(10, 3), (5, 2)\}$, *then*

$$\mathrm{Aut}(\mathrm{GP}(n, t)) = \langle a, b | a^n = b^4 = 1, b^{-1}ab = a^t \rangle.$$

2.2.5 Semiregular automorphisms and quotient graphs

Given a graph X and a partition $\mathcal{W}$ of its vertex set we let the *quotient graph corresponding to* $\mathcal{W}$ be the graph $X_{\mathcal{W}}$ whose vertex set equals $\mathcal{W}$ with $W, W' \in \mathcal{W}$ adjacent if there exist vertices $a \in W$ and $b \in W'$, such that $a \sim b$ in X. When $\mathcal{W}$ is the set of orbits of a subgroup H in $\mathrm{Aut}(X)$ the quotient graph $X_{\mathcal{W}}$ we will denoted by X_H.

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism of a graph is called (m, n)-*semiregular* if it has m orbits of length n and no other orbit. Let now X be a graph admitting an (m, n)-semiregular automorphism ρ and denote the set of the orbits of ρ by $\mathcal{S}$. Let $S, S' \in \mathcal{S}$. We let $d(S)$ and $d(S, S')$ denote the valency of $X\langle S \rangle$ and $X[S, S']$, respectively. (Clearly, the graph $X[S, S']$ is regular.) We let the *quotient multigraph corresponding to* ρ be the multigraph X_ρ whose vertex set is $\mathcal{S}$ and in which $S, S' \in \mathcal{S}$ are joined by $d(S, S')$ edges. Observe that $\mathcal{S}$ is a partition of $V(X)$, so we can also consider the quotient graph $X_{\mathcal{S}}$ which is precisely the underlying graph of X_ρ.

In the subsequent sections some of the graphs will be represented in Frucht's notation [41]. For the sake of completeness we include the definition. Let X be a connected graph of order mn admitting an (m, n)-semiregular automorphism ρ. Let $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_m\}$ be the set of orbits of ρ. Denote the vertices of X by v_i^j, where $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$, in such a way that $S_i = \{v_i^j \mid j \in \mathbb{Z}_n\}$ with $v_i^j = (v_i^0)^{\rho^j}$. Then X may be represented by the notation of Frucht [41] emphasizing the m orbits of ρ in the following way. The m orbits of ρ are represented by m circles. The symbol n/R, where $R \subseteq \mathbb{Z}_n \setminus \{0\}$, inside a circle corresponding to the orbit S_i indicates that for each $j \in \mathbb{Z}_n$, the vertex v_i^j is adjacent to all the vertices v_i^{j+r}, where $r \in R$. When $X\langle S_i \rangle$ is an independent set of vertices we simply write n inside its circle. Finally, an arrow pointing from the circle representing the orbit S_i to the circle representing the orbit S_k, $k \neq i$, labeled by the set $T \subseteq \mathbb{Z}_n$ indicates that for each $j \in \mathbb{Z}_n$, the vertex $v_i^j \in S_i$ is adjacent to all the vertices v_k^{j+t}, where $t \in T$. When the label is 0, the arrow on the line may be omitted. An example illustrating this notation is given in Figure 2.1.

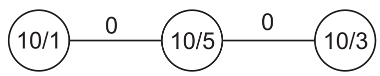

Figure 2.1: The Levi graph given in Frucht's notation with respect to a $(3, 10)$-semiregular automorphism.

A graph X admitting an (m, n)-semiregular automorphism is completely determined by the so-called *symbol*. However, we define it here only for graphs admitting a $(10, p)$-semiregular automorphism. Let ρ be a $(10, p)$-semiregular automorphism and let S_i, $i \in \mathbb{Z}_{10}$, be its orbits. Choose $s_i \in S_i$ and define the following subsets of $\mathbb{Z}_p$. For $i, j \in \mathbb{Z}_{10}$, we let $R_{i,j} = \{r \in \mathbb{Z}_p \mid s_i \sim s_j^{\rho^r}\}$. Note that $R_{j,i} = -R_{i,j}$. It is clear that the collection of all $R_{i,j}$ completely determines X. The 10×10-matrix $\mathrm{M}_\rho(X) = (R_{i,j})_{i,j}$, whose (i, j)-th entry is the set $R_{i,j}$, is the *symbol* of X relative to $(\rho, s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9)$. The symbols will be used in Chapter 6 to give relevant quasiprimitive and primitive graphs of order $10p$, p a prime.

The following proposition which is a generalization of [75, Theorem 3.4] is given in [64, Lemma 2.1] .

Proposition 2.2.3 *Let X be a vertex-transitive graph of order mp, where p is a prime and $m < p$, and let $G \leq \mathrm{Aut}(X)$ be a transitive subgroup of automorphisms of X. Then there exists some (m, p)-semiregular automorphism ρ of X, such that $\rho \in G$.*

We wrap up this subsection with two results about imprimitive graphs of certain orders which will be useful later on. The first result is a reformulation of [64, Lemma 2.1], and the second result may be deduced from [26, Lemma 2].

Proposition 2.2.4 *Let X be a G-imprimitive graph of order mq, q a prime, with a G-invariant partition $\mathcal{B}$ and let $H \leq G$ have m orbits of length q. Let S be an orbit of H and let $B \in \mathcal{B}$ be such that $B \cap S \neq \emptyset$. Then one of the following holds:*

(i) $|B \cap S| = 1$, in which case $|B \cap S'| = 1$ for every orbit S' of H which meets B, or

(ii) $B \cap S = S$, in which case q divides $|B|$.

Proposition 2.2.5 *Let X be a vertex-transitive graph of order mq, q a prime, let G be an imprimitive subgroup of automorphisms of X and let N be a normal subgroup* *of G with orbits of length q. Then X has an (m, q)-semiregular automorphism whose orbits coincide with the orbits of N.*

2.2.6 Graphs covers

We give here a formal definition of various concepts associated with graph covers. (For terms not defined here we refer the reader to [52].) An epimorphism $\wp\colon \tilde{X} \to X$ of connected graphs is a *regular covering projection* if it arises essentially as a factorization $\tilde{X} \to \tilde{X}_K \cong X$, where the action of $K \leq \mathrm{Aut}(\tilde{X})$ is semiregular on both vertices and edges of $\tilde{X}$. Note that the graph X may not be simple even if $\tilde{X}$ is. The graph $\tilde{X}$ is called the *covering graph* (or *regular K-cover*) and X is the *base graph*. The preimage $\wp^{-1}(v)$, $v \in V(X)$, corresponds to an orbit of K on $V(\tilde{X})$ and is called the (*vertex*)-*fibre* over v. Similarly, *edge-fibres* correspond to orbits of K on $E(\tilde{X})$. It is well-known that a regular covering projection $\tilde{X} \to X = \tilde{X}_K$ can be reconstructed in terms of voltage assignments valued in K as follows (see [52]). First label arbitrarily a vertex in each fibre by $1 \in K$, and then label all other vertices by the right regular action of $K \leq \mathrm{Aut}(\tilde{X})$ on each fibre. Consequently, given an arc $e \in A(X)$, the origins and termini of arcs in $\wp^{-1}(e)$ are labeled, respectively, by g and ag ($g \in K$) for some $a \in K$. This fact is recorded by assigning the voltage $\zeta(e) = a \in K$ to the corresponding arc e, with inverse arcs carrying inverse voltages. Observe that a voltage assignment on arcs extends to an assignment on walks in a natural way. By connectedness of $\tilde{X}$, the voltages of all fundamental closed walks at any vertex $v \in V(X)$ generate the whole voltage group K. It is also well known that a given voltage assignment can be modified so that the arcs of an arbitrarily prescribed spanning tree receive the trivial voltage, and that the modified assignment is associated with the same covering projection, see [52]. (A voltage assignment ζ such that the arcs of a spanning tree T carry the trivial voltage, is said to be *T-reduced*.)

We say that $\alpha \in \mathrm{Aut}(X)$ *lifts* to an automorphism of $\tilde{X}$ if there exists an automorphism $\tilde{\alpha} \in \mathrm{Aut}(\tilde{X})$, called a *lift* of α, such that $\tilde{\alpha}\wp = \wp\alpha$. The problem whether an automorphism α of X lifts or not is expressed in terms of voltages as follows. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages of fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by the rule $\zeta_C^{\bar{\alpha}} = \zeta_{C^\alpha}$ where C ranges over all fundamental closed walks at v, and ζ_C and ζ_{C^α} are the voltages of C and C^α, respectively. The next proposition, taken from [74, Theorem 4.2, Corollary 4.3], provides information about the relationship between automorphisms of graph covers and their base graphs.

Proposition 2.2.6 [74] *Let $\tilde{X}$ be a regular cover of $X = \tilde{X}_K$ with respect to the voltage assignment ζ. Then, an automorphism α of X lifts if and only if $\bar{\alpha}$ extends to an automorphism α^* of K.*

Chapter 3

Non-Cayley vertex-transitive graphs

Results of this chapter are published in [63].

Every Cayley graph is vertex-transitive. However, there are vertex-transitive graphs which are not Cayley graphs, the smallest example is the well-known Petersen graph. The order of a non-Cayley vertex-transitive graph is called a *non-Cayley number*.

In 1983, Marušič [77] asked for which positive integers n there exists a non-Cayley vertex-transitive graph on n vertices. Several articles directly or indirectly related to this subject (see [4, 5, 20, 55, 69, 73, 78, 86, 87, 88, 89, 91, 92, 100, 102, 108, 109, 111, 131] for some of the relevant references), have appeared in the literature, answering this question for particular positive integers. For example, in [78] it is proved that every vertex-transitive graph of order p^k, where p is an odd prime and $k \leq 3$, is a Cayley graph. Further, a family of non-Cayley vertex-transitive graphs of order p^k, where $p \geq 5$ is a prime and $k \geq 4$, was constructed by McKay and Praeger in [91]. In 1971 Frucht, Graver and Watkins [42] gave a construction of a family of non-Cayley vertex-transitive graphs of order $2p$, when $p \equiv 1 \pmod 4$ is a prime, and in 1979 Alspach and Sutcliffe [5] showed that $2p$, p a prime, is a non-Cayley number if and only if $p \equiv 1 \pmod 4$. Results from [4], [86] and [102] combined together give a classification of all non-Cayley vertex-transitive graphs of order a product of two primes, and results from [57] and [94] give a characterization of non-Cayley numbers of the form $2pq$, p and q odd primes. In 1996 McKay and Praeger [92] proved that any positive integer n divisible by a square of a prime number p, with the exception of $n \in \{12, p^2, p^3\}$, is a non-Cayley number. Non-Cayley properties of products of three distinct primes have been addressed by Seress in [108]. Most recently, Li and Seress [69] determined those square-free numbers n for which there exist non-Cayley vertex-transitive graphs of order n with a primitive automorphism group.

We say that an *m-Cayley graph X* on a group G is a graph admitting a semiregular automorphism subgroup G having m orbits, all of equal length, say n. In view of the well-known polycirculant conjecture [75, 83] regarding existence of semiregular automorphisms in vertex-transitive digraphs it seems natural to pose the following problem.

Problem 3.0.1 *Given a non-Cayley vertex-transitive graph X determine the smallest integer m such that X is an m-Cayley graph on a cyclic group.*

Motivated by above mentioned research, Feng [40] asked to determine the smallest valency $\vartheta(n)$ among valencies of non-Cayley vertex-transitive graphs of order n. In particular, he settled the problem for graphs of odd prime power order. Clearly a non-Cayley number n with $\vartheta(n) = 3$ is an even integer. As is well known, a generalized Petersen graph $\mathrm{GP}(n,t)$ is a non-Cayley vertex-transitive graph if and only if $t^2 \equiv -1 \pmod{n}$ or $(n,t) = (10,2)$ (see [42, 72, 96]), and thus for every positive integer n such that 4 divides $\phi(n)$ we have that $\vartheta(2n) = 3$. In view of these comments we propose to continue with the investigation of non-Cayley numbers along the following lines.

Problem 3.0.2 *Classify all non-Cayley numbers n for which $\vartheta(n) = 3$.*

Problem 3.0.3 *For non-Cayley numbers n with $\vartheta(n) = 3$ classify all connected cubic non-Cayley vertex-transitive graphs of order n. In particular, when is it true that every such graph is a generalized Petersen graph.*

Problem 3.0.3 has already been solved for integers of the form $2p$, $4p$, $2p^2$ and $2pq$, where p and q are odd primes (see, respectively, [5, 42, 78], [127], [128] and [131]). In particular, the generalized Petersen graphs are the only connected cubic non-Cayley vertex-transitive graphs of order $2p$, the Coxeter graph is the only connected cubic non-Cayley vertex-transitive graph of order $4p$ which is not a generalized Petersen graph, and every connected cubic non-Cayley vertex-transitive graph of order $2p^2$ is a generalized Petersen graph. By [131], there exists an infinite family of cubic non-Cayley vertex-transitive graphs of order $2pq$ which are not generalized Petersen graphs, the smallest is the Tutte graph (see [8, 17]). One of the aims of this chapter is to solve Problem 3.0.3 for $n = 4p^2$, $p > 7$ a prime. (Throughout this chapter p will always denote a prime number.) In particular, using group-theoretic and combinatorial techniques we show that the generalized Petersen graphs are the only examples of cubic non-Cayley vertex-transitive graphs of order $4p^2$ (see Theorem 3.1.4). In other words, we show that every cubic non-Cayley vertex-transitive graph of order $4p^2$ is a 2-Cayley graph on a cyclic group, thus solving Problem 3.0.1 for this particular class of non-Cayley vertex-transitive graphs. Furthermore, we show that every cubic non-Cayley vertex-transitive graph of order $2p^n$, where $p > 7$ is a prime and $n \leq p$, is a 2-Cayley graph on a p-group P generated by two elements a and b of the same order and admitting an automorphism $\phi \in \mathrm{Aut}(P)$ of order 4 such that $a^\phi = b$ and $b^\phi = a^{-1}$ (see Theorem 3.2.3).

The following six results about vertex-transitive graphs will be needed in the remainder of the chapter. The second one is obtained by combining together results from [37], [126] and [127]. The third one is obtained by combining together [35, Theorem 3.2 and Corollary 3.6] and [128, Theorem 2.1].

Proposition 3.0.4 [37, Theorem 6.2] *The generalized Petersen graph* $\mathrm{GP}(8,3)$*, also known as the Moebius-Kantor graph, is the only cubic symmetric graph of order* $4p^2$*, where p is a prime.*

The graph $\mathcal{NNC}(4n)$ in the following proposition is a Cayley graph $\mathrm{Cay}(G, S)$ on the group $G = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ with respect to the generating set $S = \{b, ab, a^n b\}$ (see Figure 3.1). In [126] it is shown that $\mathrm{Aut}(\mathcal{NNC}(4p)) \cong \mathbb{Z}_2^p \rtimes D_{2p}$.

Proposition 3.0.5 *Let X be a connected cubic vertex-transitive graph of order $4p$, where $p > 7$ is a prime. Then either a Sylow p-subgroup of $\mathrm{Aut}(X)$ is normal in $\mathrm{Aut}(X)$ or $X \cong \mathcal{NNC}(4p)$.*

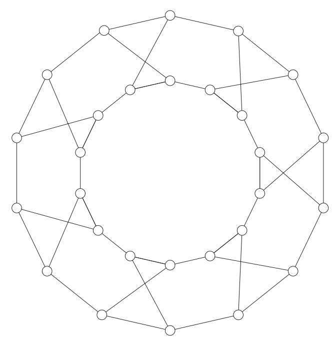

Figure 3.1: The graph $\mathcal{NNC}(4p)$ for $p = 7$.

Proposition 3.0.6 *Let X be a connected cubic vertex-transitive graph of order $2p^2$, where p is a prime. Then a Sylow p-subgroup of $\mathrm{Aut}(X)$ is normal in $\mathrm{Aut}(X)$.*

Proposition 3.0.7 [35, Theorem 3.2 and Corollary 3.4] *Let $p > 5$ be a prime and let n be a positive integer. Then every connected cubic symmetric graph X of order $2p^n$ is a Cayley graph. In addition, if $p \neq 7$ then a Sylow p-subgroup of $\mathrm{Aut}(X)$ is normal in $\mathrm{Aut}(X)$.*

Proposition 3.0.8 [128] *Every connected cubic non-Cayley vertex-transitive graph of order $2p^2$, p a prime, is a generalized Petersen graph.*

Proposition 3.0.9 [39, Corollary 3.2] *Let $p \neq 5$ be a prime and let G be a p-group of order p^n with $n \leq p$. Then all connected tetravalent Cayley graphs on G are normal.*

3.1 Cubic non-Cayley vertex-transitive graphs of order $4p^2$

In this section cubic non-Cayley vertex-transitive graphs of order $4p^2$, where $p > 7$ is a prime, are classified. In particular, we solve Problem 3.0.1 for this particular class of non-Cayley vertex-transitive graphs. The following three results will be needed in this respect.

Proposition 3.1.1 *Let $p > 7$ be a prime and let X be a connected cubic vertex-transitive graph of order $4p^2$ such that its automorphism group* $\mathrm{Aut}(X)$ *admits a normal subgroup $M \cong \mathbb{Z}_p$. If $X_M \cong \mathcal{NNC}(4p)$ then X is a Cayley graph.*

Proof. Let X be a connected vertex-transitive graph of order $4p^2$ such that there exists a normal subgroup M of $\mathrm{Aut}(X)$ isomorphic to $\mathbb{Z}_p$ and that the corresponding quotient graph X_M is isomorphic to $\mathcal{NNC}(4p)$. Then X is a regular M-cover of the graph $Y = \mathcal{NNC}(4p)$ and thus it can be derived from Y through a suitable voltage assignment. Let

$$V(Y) = \{x_i, y_i, z_i, w_i \mid i \in \mathbb{Z}_p\} \text{ and}$$

$$E(Y) = \{x_i y_i, x_i y_{i-1}, z_i w_i, z_i w_{i-1}, x_i w_i, z_i y_i \mid i \in \mathbb{Z}_p\},$$

and let T be a spanning tree of Y consisting of the edges $x_i y_i$, $x_i w_i$, $z_i y_i$, where $i \in \mathbb{Z}_p$, and $x_i y_{i-1}$, where $i \in \mathbb{Z}_p \setminus \{0\}$, see Figure 3.2. Let $\zeta\colon A(Y) \to \mathbb{Z}_p$ be a T-reduced voltage assignment giving rise to $X = \tilde{Y} = \mathrm{Cov}(Y, \zeta)$, where the voltages on the cotree arcs are $\zeta(z_i, w_i) = a_i$, $\zeta(w_{i-1}, z_i) = b_i$, $i \in \mathbb{Z}_p$ and $\zeta(y_{p-1}, x_0) = c$, see also Figure 3.2. There are $2p + 1$ fundamental cycles

$$\begin{aligned}
C_c &= x_0 y_0 x_1 y_1 \ldots x_{p-1} y_{p-1} x_0,\\
C_{a_i} &= x_i y_i z_i w_i x_i, \quad i \in \mathbb{Z}_p,\\
C_{b_i} &= z_i y_i x_i y_{i-1} x_{i-1} w_{i-1} z_i, \quad i \in \mathbb{Z}_p \setminus \{0\},\\
C_{b_0} &= z_0 y_0 x_1 y_1 \ldots x_{p-1} w_{p-1} z_0,
\end{aligned}$$

which are generated, respectively, by the cotree arcs (y_{p-1}, x_0), (z_i, w_i) and (w_{i-1}, z_i).

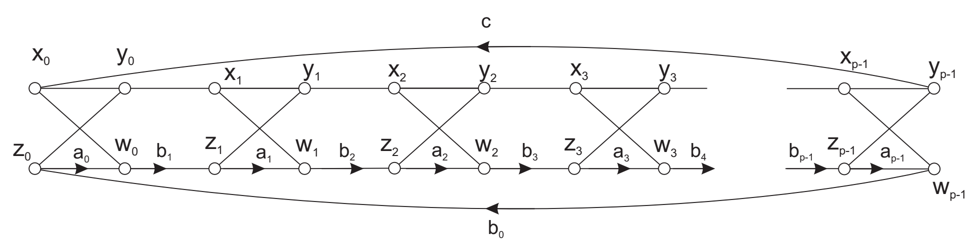

Figure 3.2: The graph $\mathcal{NNC}(4p)$ with voltage assignment ζ. The spanning tree consists of edges without labels, each of them carries voltage 0.

Observe that

$$\mathrm{Aut}(Y) = \langle \epsilon_i \mid i \in \mathbb{Z}_p \rangle \rtimes \langle \rho, \tau \rangle \cong \mathbb{Z}_2^p \rtimes D_{2p}$$

where $\epsilon_i = (y_i\, w_i)(x_{i+1}\, z_{i+1})$, $\rho = (x_0\, x_1 \ldots x_{p-1})(y_0\, y_1 \ldots y_{p-1})(z_0\, z_1 \ldots z_{p-1})$ $(w_0\, w_1 \ldots w_{p-1})$ and $\tau = \prod_{i \in \mathbb{Z}_p} (x_i\, y_{-i})(z_i\, w_{-i})$. It is convenient to view elements ϵ in $E = \langle \epsilon_i \mid i \in \mathbb{Z}_p \rangle$ as vectors in $\mathbb{Z}_2^p$. Namely, we write $\epsilon = (e_0, \ldots, e_{p-1})$ where $e_i = 1$ if and only if ϵ_i actually appears in ϵ.

Since the orbits of M form an $\mathrm{Aut}(X)$-invariant partition, the whole automorphism group $\mathrm{Aut}(X)$ of X projects to a subgroup of $\mathrm{Aut}(Y)$. In other words, the largest subgroup G of $\mathrm{Aut}(Y)$ which lifts with respect to the natural projection

$X = \mathrm{Cov}(Y, \zeta) \to Y$ is a vertex-transitive subgroup isomorphic to $\mathrm{Aut}(X)/M$. Since G is a vertex-transitive subgroup of $\mathrm{Aut}(Y)$ it contains a Sylow p-subgroup of $\mathrm{Aut}(Y)$. In addition, by conjugation, we may, without loss of generality, assume that $\langle\rho\rangle \leq G$. If $\epsilon = (1, 1, 1, \ldots, 1) \in G$ then $\langle\epsilon, \rho, \alpha\tau\rangle$, for some $\alpha \in E$, is a regular subgroup of G, and consequently X is a Cayley graph. Thus we may assume that $(1, 1, 1, \ldots, 1) \notin G$. Then, by [75, Theorem 6.2] (see also [98, Theorem 1]) and since $\rho \in G$, there exists $\epsilon = (1, 1, 0, \ldots) \in E \cap G$.

Now let us consider the mappings $\bar{\rho}$ and $\bar{\epsilon}$ from the set $\{a_i, b_i, c \mid i \in \mathbb{Z}_p\}$ of voltages of the fundamental cycles of Y to the cyclic group $\mathbb{Z}_p$ which are defined by $\zeta_C^{\bar{\rho}} = \zeta_{C^\rho}$ and $\zeta_C^{\bar{\epsilon}} = \zeta_{C^\epsilon}$, respectively. Proposition 2.2.6 implies that the mappings $\bar{\rho}$ and $\bar{\epsilon}$ are extended to automorphisms of $\mathbb{Z}_p$. Denote these extended automorphisms by ρ^* and ϵ^*, respectively. Considering the fundamental cycle C_c we see that $c^{\rho^*} = c$ which implies that either $c = 0$ or ρ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto 1$.

Case 1. $c = 0$.

Considering $C_{a_1}^\epsilon$ we see that either $a_1 = 0$ or ϵ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto 1$ and therefore the identity automorphism. However, considering $C_{a_2}^\epsilon$ we get that either $a_2 = 0$ or ϵ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto -1$. Since $a_0^{\rho^{*i}} = a_i$ we can conclude that $a_i = 0$ for every $i \in \mathbb{Z}_p$. Now, since $C_{b_2}^\epsilon = x_2y_2z_2w_1z_1y_1x_2$ we have that either $b_2 = 0$ or ϵ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto -1$.

Consider the forth component of $\epsilon = (1, 1, 0, \ldots) \in E \cap G$. If $\epsilon = (1, 1, 0, 1, \ldots)$ then $C_{b_3}^\epsilon = z_3w_3x_3y_2z_2w_2z_3$ and thus either $b_3 = 0$ or ϵ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto 1$. If, however, $\epsilon = (1, 1, 0, 0, \ldots)$ then $C_{b_3}^\epsilon = z_3y_3x_3y_2z_2w_2z_3$ and thus either $b_3 = 0$ or ϵ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto 1$. Since $b_1^{\rho^{*i}} = b_{i+1}$ we can, in all cases, conclude that $b_i = 0$ for every $i \in \mathbb{Z}_p$. But then all the voltages are equal to $0 \in \mathbb{Z}_p$ and thus X is disconnected, a contradiction.

Case 2. ρ^* is an automorphism of $\mathbb{Z}_p$ induced by $1 \mapsto 1$.

Considering the images of the fundamental cycles under the automorphism $\rho \in G$ we get that $a_0 = a_1 = \cdots = a_{p-1}$ and $b_1 = b_2 = \cdots = b_{p-1} = b_0 - c$. Moreover, when considering the images of the fundamental cycles under the automorphism $\epsilon \in G$ one can get, using the same method as in Case 1, that $a_i = 0$ for every $i \in \mathbb{Z}_p$, $b_i = 0$ for every $i \in \mathbb{Z}_p \setminus \{0\}$ and $b_0 = c$. But then we may assume that $b_0 = c = 1$ and thus $X \cong \mathcal{NNC}(4p^2)$ is a Cayley graph on the dihedral group D_{4p^2}. ■

Proposition 3.1.2 *Let X be a connected cubic vertex-transitive graph, let N be a normal subgroup of $\mathrm{Aut}(X)$, and let K be the kernel of $\mathrm{Aut}(X)$ in its action on the quotient graph X_N. If X_N is cubic then the stabilizer K_v, $v \in V(X)$, is trivial.*

Proof. The set of orbits $\mathcal{B}$ of N is an imprimitivity block system for $\mathrm{Aut}(X)$. Since X and $X_N = X_\mathcal{B}$ are both cubic graphs every induced subgraph $\langle B\rangle$, $B \in \mathcal{B}$, is an independent set of vertices. In addition, any vertex $v \in V(X)$ has neighboring vertices in three different orbits of N, all different from the orbit of N that contains v. But then, since K fixes each orbit of N setwise, the stabilizer K_v of v in K fixes every neighbor of v. By connectedness of X, we have $K_v = 1$ and consequently

$K = NK_v = N$. ■

The next result will be crucial in the proof of the main result of this section.

Lemma 3.1.3 *Let X be a cubic non-Cayley vertex-transitive graph of order $4p^2$, where $p > 7$ is a prime. Then a Sylow p-subgroup of* $\mathrm{Aut}(X)$ *is normal in* $\mathrm{Aut}(X)$.

Proof. Let $A = \mathrm{Aut}(X)$. Since X is a non-Cayley graph, no subgroup of A is regular on X. Proposition 3.0.4 implies that X is not arc-transitive, and consequently the stabilizer A_v of $v \in V(X)$ in A is a 2-group. Thus $|A| = 2^n p^2$, where $n > 2$. Let P be a Sylow p-subgroup of A, and let M be a minimal normal subgroup of A. By Proposition 2.1.6, A is solvable and therefore M is an elementary abelian group of order p, p^2 or 2^k, where $1 \leq k \leq n$. If $|M| = p^2$ then clearly $M = P$ and so P is normal in A. The other two possibilities need a more detailed analysis.

Case 1. $|M| = p$.

Let X_M be the quotient graph of X relative to M and let K be the kernel of A acting on X_M. Then, $M \leq K$ and A/K is transitive on X_M. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_m\}$ be the set of orbits of M. Clearly every orbit of M on $V(X)$ has length p, and thus since $|V(X)| = 4p^2$, we have that $|V(X_M)| = m = 4p$. Since X is cubic and M is normal in $\mathrm{Aut}(X)$, it follows that X_M is regular and of valency 2 or 3. Moreover, for each $B_i \in \mathcal{B}$, $\langle B_i \rangle$ is an independent set of vertices. (Namely, since each $B_i \in \mathcal{B}$ is of odd order the valency of each $\langle B_i \rangle$ has to be an even number.)

Suppose first that X_M is of valency 2. Then X_M is isomorphic to C_{4p} with alternating single and double edges. Let the orbits of M be labeled in such a way that B_i is adjacent to B_{i+1} for every $i \in \mathbb{Z}_m$, and that, in particular, $\langle B_{2i} \cup B_{2i+1} \rangle \cong C_{2p}$ and $\langle B_{2i-1} \cup B_{2i} \rangle \cong pK_2$, where $i \in \mathbb{Z}_m$. Since K fixes each $B \in \mathcal{B}$ setwise and X is connected it follows that K acts faithfully on $\langle B_0 \cup B_1 \rangle$, and so $K \leq \mathrm{Aut}(\langle B_0 \cup B_1 \rangle) \cong D_{4p}$. It follows that for any vertex v in B_0, we have $|K_v| \leq 4$, implying that $|K| \leq 4p$. Since $X_M = C_{4p}$ we have that A/K is a subgroup of $\mathrm{Aut}(X_M) = D_{8p}$. Clearly, D_{8p} has a normal Sylow p-subgroup, and thus also A/K has a normal Sylow p-subgroup, which is isomorphic to PK/K. Since K is normal in A it now follows that PK is normal in A, and since $|PK : P| \leq 4$ we have that P is normal in PK. Moreover, since P is a Sylow p-subgroup of PK, it is characteristic in PK, and so P is normal in A.

Suppose now that X_M is cubic. Then, by Proposition 3.1.2, $K_v = 1$ for every vertex $v \in V(X)$, and consequently $K = MK_v = M \cong \mathbb{Z}_p$. Proposition 3.0.5 and Proposition 3.1.1 combined together imply that $A/K \leq \mathrm{Aut}(X_M)$ has a normal Sylow p-subgroup. Hence it follows that $P/K \triangleleft A/K$, which implies that P is normal in A.

Case 2. $|M| = 2^k$, $k \leq n$.

Let $N = O_2(A)$ be a maximal normal 2-subgroup of A. Clearly $M \leq N$ and $|N| > 1$ since $|M| > 1$. Let X_N be the quotient graph of X relative to the orbits of N, let $\mathcal{B}$ be the set of orbits of N, and let K be the kernel of A acting on X_N. Then, $N \leq K$, $A/K \leq \mathrm{Aut}(X_N)$ and A/K acts transitively on X_N. In view of the fact that N is a

2-group and X is of order $4p^2$ we get that X_N is either of order p^2 or of order $2p^2$. In addition, since X is cubic and N is normal, the valency of X_N is either 2 or 3.

SUBCASE 2.1. X_N is cubic.

Then, since p^2 is odd, X_N is of order $2p^2$. By Proposition 3.0.6, $\mathrm{Aut}(X_N)$ has a normal Sylow p-subgroup, and consequently the subgroup A/K of $\mathrm{Aut}(X_N)$ also has a normal Sylow p-subgroup. It follows that PK/K is normal in A/K, and thus $PK \lhd A$. Since X is of order $4p^2$ and X_N is of order $2p^2$, we have $|B| = 2$ for every $B \in \mathcal{B}$. Applying Proposition 3.1.2 we get that $K = NK_v = N \cong \mathbb{Z}_2$. But then $|PK : P| = 2$, and thus $P \lhd PK$. Since P is a Sylow p-subgroup of PK, it is characteristic in PK, and so P is normal in A.

SUBCASE 2.2. X_N is of valency 2.

Then X_N is an m-cycle, where $m = p^2$ or $2p^2$. Without loss of generality we may assume that the orbits in $\mathcal{B}$ are labeled in such a way that $X_N = (B_0, B_1, \ldots, B_{m-1}, B_0)$, that is, B_i is adjacent to B_{i+1} for every $i \in \mathbb{Z}_m$.

Suppose first that $d(B) \neq 0$ for an orbit $B \in \mathcal{B}$. Then, since $\mathcal{B}$ is the set of orbits of a normal subgroup, there is an edge inside each orbit in $\mathcal{B}$. The fact that X is cubic implies that the graphs induced on the orbits of N are either all isomorphic to $2K_2$ or all isomorphic to K_2. Clearly, a vertex $v \in B_i$, $i \in \mathbb{Z}_m$, has one neighbor in B_{i-1}, one in B_i and one in B_{i+1}. Since K fixes each orbit of N setwise, the stabilizer K_v of $v \in V(X)$ fixes all neighbors of v. Applying the connectivity of X we get that $K_v = 1$, and so $K = NK_v = N$. Since A/K is transitive on X_N and X_N is a cycle, A/K contains a subgroup, say H/K, acting regularly on X_N. Then H acts transitively on X, and $|H| = |V(X_N)| \cdot |K| = m \cdot |K|$. If X_N is of order $2p^2$ then $|B| = 2$ for every $B \in \mathcal{B}$, whereas if X_N is of order p^2 then $|B| = 4$ for every $B \in \mathcal{B}$. In both cases the fact that $K = N$ implies that H is of order $4p^2$. But then H acts regularly on X, contradicting the assumption that X is a non-Cayley graph.

Suppose now that $d(B) = 0$ for every $B \in \mathcal{B}$. Since X is cubic and X_N is a cycle we have that the subgraph $\langle B_i \cup B_{i+1} \rangle$ of X induced on two neighboring orbits B_i and B_{i+1} of N is a regular graph with $d(B_i \cup B_{i+1}) = 1$ or $d(B_i \cup B_{i+1}) = 2$. Without loss of generality we may assume that $d(B_0 \cup B_1) = 1$. Then $\langle B_0 \cup B_1 \rangle = 4p^2/mK_2$ and $d(B_0 \cup B_{m-1}) = d(B_1 \cup B_2) = 2$, and by induction,

$$d(B_i \cup B_{i+1}) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases} .$$

Since $d(B_0 \cup B_{m-1}) = 2$ we must have that $d(B_{m-2} \cup B_{m-1}) = 1$, and consequently, m is even. It follows that $m = 2p^2$, $|B_i| = 2$ for every $i \in \mathbb{Z}_m$, and $\langle B_i \cup B_{i+1} \rangle \cong 2K_2$ if i is even whereas $\langle B_i \cup B_{i+1} \rangle \cong C_4$ if $i \in \mathbb{Z}_m$ is odd. Let $B_i = \{x_i, y_i\}$, $i \in \mathbb{Z}_m$. Then we may, without loss of generality, assume that

$$E(\langle B_i \cup B_{i+1} \rangle) = \begin{cases} \{x_i x_{i+1}, y_i y_{i+1}\} & \text{if } i \text{ is even} \\ \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1}, y_i x_{i+1}\} & \text{if } i \text{ is odd} \end{cases} .$$

(see also Figure 3.3). Thus $X \cong \mathcal{NC}(4p^2)$, which is not possible as X is a non-Cayley graph. ∎

We are now ready to prove the main result of this section.

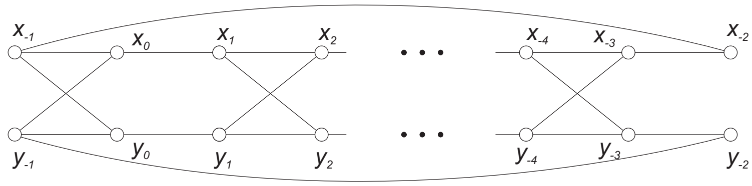

Figure 3.3: The structure of the graph X when N has orbits of length 2 and X_N is a cycle.

Theorem 3.1.4 *Let X be a cubic non-Cayley vertex-transitive graph of order $4p^2$, where $p > 7$ is a prime. Then X is a non-symmetric generalized Petersen graph $\mathrm{GP}(2p^2, t)$, where $t^2 \equiv -1 \pmod{2p^2}$.*

Proof. Let $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. Then, by Lemma 3.1.3, P is normal in A. Let $\mathcal{B}$ be the set of orbits of P, and let K be the kernel of A acting on X_P, the quotient graph X_P of X relative to P. Then $P \leq K$ and A/K is transitive on X_P. Proposition 3.0.4 implies that X is not symmetric, and consequently the stabilizer A_v of $v \in V(X)$ in A is a 2-group. Thus A is a $\{2, p\}$-group and A/K is a 2-group. Since X is of order $4p^2$, Proposition 2.1.7 implies that every orbit of P on X is of length p^2, and thus X_P is of order 4. Let $V(X_P) = \mathcal{B} = \{B_i \mid i \in \mathbb{Z}_4\}$. The normality of P implies that X_P is of valency 2 or 3.

Case 1. X_P is of valency 3.

Then $X_P \cong K_4$, and so $\mathrm{Aut}(X_P) \cong S_4$. Since A is a $\{2, p\}$-group we have that $A/K \leq D_8$. Moreover the fact that A/K is transitive on X_P implies that 4 divides the order of A/K. Therefore $|A/K|$ is either 4 or 8. In the first case, A/K acts regularly on X_P. In the second case, $A/K \cong D_8$ and so A/K is a Sylow 2-subgroup of $\mathrm{Aut}(X_P)$. Since $\mathrm{Aut}(X_P) \cong S_4$ it follows that A/K has a subgroup acting regularly on X_P. Thus in both cases A/K has a subgroup, call it H/K, acting regularly on X_P. It follows that H acts transitively on X and $|H| = 4|K|$. Applying Proposition 3.1.2 we get $K = PK_v = P$, and hence H is a transitive subgroup of A of order $4p^2$. But then H is regular on X, a contradiction.

Case 2. X_P is of valency 2.

Then $X_P = C_4 = (B_0, B_1, B_2, B_3, B_0)$. Clearly $d(B_i) = 0$, for every $i \in \mathbb{Z}_4$. Namely if there is some edge inside an orbit of P then the fact that the orbits are of odd length implies that the subgraphs induced on the orbits of P are of valency 2, which is impossible since X is a connected cubic graph and X_P is a 4-cycle. Without loss of generality, we may therefore assume that $d(B_0, B_1) = d(B_2, B_3) = 2$ and $d(B_1, B_2) = d(B_3, B_0) = 1$. In particular, $\langle B_1, B_2 \rangle = \langle B_3, B_0 \rangle = p^2 K_2$, and either $\langle B_0, B_1 \rangle = \langle B_2, B_3 \rangle = pC_{2p}$ or $\langle B_0, B_1 \rangle = \langle B_2, B_3 \rangle = C_{2p^2}$.

Subcase 2.1. $\langle B_0, B_1 \rangle = \langle B_2, B_3 \rangle = pC_{2p}$.

(Observe that in this case $P \cong \mathbb{Z}_p^2$.) Let the vertices of X be labeled in such a way that $B_i = \{(i, j, k) \mid j, k \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_4$. Then, since X is connected, the action of

P on the blocks B_i, $i \in \mathbb{Z}_4$, implies that we may, without loss of generality, assume that either

$$\begin{aligned} E(X) &= \{(i,j,k)(i+1,j,k) \mid i \in \mathbb{Z}_4 \setminus \{3\}, j,k \in \mathbb{Z}_p\} \cup \\ &\cup \{(i,j,k)(i+1,j,k-1) \mid i \in \mathbb{Z}_4 \setminus \{1,3\}, j,k \in \mathbb{Z}_p\} \cup \\ &\cup \{(3,j,k)(0,j-1,k) \mid j,k \in \mathbb{Z}_p\} \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} E(X) &= \{(i,j,k)(i+1,j,k) \mid i \in \mathbb{Z}_4 \setminus \{3\}, j,k \in \mathbb{Z}_p\} \cup \\ &\cup \{(0,j,k)(1,j,k-1) \mid j,k \in \mathbb{Z}_p\} \cup \\ &\cup \{(2,j,k)(3,j-1,k) \mid j,k \in \mathbb{Z}_p\} \cup \\ &\cup \{(3,j,k)(0,j+a,k+b) \mid j,k \in \mathbb{Z}_p\}, \end{aligned} \tag{3.2}$$

where $a, b \in \mathbb{Z}_p$.

Suppose first that the adjacencies in X are as given in (3.1). Observe that the sets $\Delta_{i,j} = \{(i,j,k) \mid k \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_4$ and $j \in \mathbb{Z}_p$, form an imprimitivity block system Ω for A such that $X_\Omega \cong C_{4p}$. Therefore we may relabel the blocks in Ω and the vertices of X in such a way that $\Omega = \{\Delta_i \mid i \in \mathbb{Z}_{4p}\}$, $\Delta_i = \{x_i^j \mid j \in \mathbb{Z}_p\}$, $X_\Omega = C_{4p} = (\Delta_0, \Delta_1, \ldots, \Delta_{4p-1})$, and that $X[\Delta_{2j} \cup \Delta_{2j+1}] \cong C_{2p}$ and $X[\Delta_{2j-1} \cup \Delta_{2j}] \cong pK_2$ for every $j \in \mathbb{Z}_{4p}$. But then, however, one can easily see that $\alpha\colon x_i^j \mapsto x_{i+2}^j$, $\beta\colon x_i^j \mapsto x_i^{j+1}$ and $\gamma\colon x_i^j \mapsto x_{4p-1-i}^j$, where $i \in \mathbb{Z}_{4p}$ and $j \in \mathbb{Z}_p$, are automorphisms of X that generate a transitive subgroup

$$G = \langle \alpha, \beta, \gamma \mid \alpha^{2p} = \beta^p = \gamma^2 = 1, \alpha^\beta = \alpha, \gamma^\beta = \gamma, \alpha^\gamma = \alpha^{-1} \rangle$$

of $\mathrm{Aut}(X)$ isomorphic to the group $D_{4p} \times \mathbb{Z}_p$ of order $4p^2$, which is impossible in view of the assumption that X is a non-Cayley graph.

Suppose now that the adjacencies in X are as given in (3.2). Then we first need to find a and b. Since X is a non-Cayley vertex-transitive graph there exists a non-identity automorphism $\alpha \in \mathrm{Aut}(X)$ that fixes the vertex $(2,0,0)$, that is, $\alpha \in A_{(2,0,0)}$. Using the fact that B_i, $i \in \mathbb{Z}_4$, are blocks of imprimitivity for A, one can see that α fixes $(2,i,0)$ and $(3,i,0)$ for every $i \in \mathbb{Z}_p$. Observe that these vertices induce a $2p$-cycle between B_2 and B_3, and thus all the vertices of this particular $2p$-cycle are fixes by α. Consequently, we get that α fixes $(1,i,0)$ for every $i \in \mathbb{Z}_p$. In addition, since the sets B_i, $i \in \mathbb{Z}_4$, form an imprimitivity block system, α also fixes vertices antipodal to vertices $(1,i,0)$, $i \in \mathbb{Z}_p$, on the $2p$-cycles between blocks B_0 and B_1, that is, α fixes

$$(0, i, \frac{p+1}{2}) \text{ for every } i \in \mathbb{Z}_p.$$

This shows that the vertex $(3,0,0)$ is adjacent to a vertex $(0,i,\frac{p+1}{2})$ for some $i \in \mathbb{Z}_p$. Namely, if the vertex $(3,0,0)$ is adjacent to a vertex in B_0 different from $(0,i,\frac{p+1}{2})$, then it can be seen that α fixes all the vertices of the graph, and consequently, that $\alpha = 1$, which contradicts our assumption that $\alpha \neq 1$. Now observe that the sets $\Delta_{0,1,i} = \{(0,i,j),(1,i,j) \mid j \in \mathbb{Z}_p\}$ and $\Delta_{2,3,i} = \{(2,j,i),(3,j,i) \mid j \in \mathbb{Z}_p\}$ form an imprimitivity block system for A. Since $\alpha \in A_{(\Delta_{2,3,0})}$ it follows that there exists a non-identity automorphism $\beta \in A_{(\Delta_{0,1,0})}$. Considering the action of β on X we have that

$$(3, i, j) \sim (0, i + \frac{p+1}{2}, j + \frac{p+1}{2}) \text{ for every } i, j \in \mathbb{Z}_p,$$

that is, $(a, b) = (\frac{p+1}{2}, \frac{p+1}{2})$. But then

$$\begin{array}{rcl} \alpha & : & (k,i,j) \mapsto (k,i+1,j), \\ \beta & : & (k,i,j) \mapsto (k,i,j+1), \\ \gamma & : & (0,i,j) \mapsto (1,i,j+\frac{p-1}{2}),\ (1,i,j) \mapsto (0,i,j+\frac{p+1}{2}), \\ & & (2,i,j) \mapsto (3,i+\frac{p-1}{2},j),\ (3,i,j) \mapsto (2,i+\frac{p+1}{2},j), \\ \delta & : & (0,i,j) \mapsto (2,j-\frac{p+1}{2},i),\ (1,i,j) \mapsto (3,j-\frac{p+1}{2},i), \\ & & (2,i,j) \mapsto (0,j,i+\frac{p+1}{2}),\ (3,i,j) \mapsto (1,j,i+\frac{p+1}{2}), \end{array}$$

where $i, j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4$, are automorphisms of X that generate a regular subgroup $G = \langle \alpha,\, \beta,\, \gamma,\, \delta \mid [\alpha, \beta] = [\gamma, \delta] = [\alpha, \gamma] = [\beta, \gamma] = 1,\, \alpha^\delta = \beta,\, \beta^\delta = \alpha \rangle$ of $\mathrm{Aut}(X)$, a contradiction.

Subcase 2.2. $\langle B_0, B_1 \rangle = \langle B_2, B_3 \rangle = C_{2p^2}$.

Let $C = \langle B_0, B_1 \rangle$ and $D = \langle B_2, B_3 \rangle$. Then $V(X) = V(C) \cup V(D)$. Let $V(C) = \{x_i \mid i \in \mathbb{Z}_{2p^2}\}$ and $V(D) = \{y_i \mid i \in \mathbb{Z}_{2p^2}\}$. We may, without loss of generality, assume that x_i is adjacent to x_{i+1}, $i \in \mathbb{Z}_{2p^2}$. Clearly every vertex in C is adjacent to some vertex of D, say x_i is adjacent to y_i, $i \in \mathbb{Z}_{2p^2}$. Let L be the kernel of A acting on $\{C\,, D\}$. Then $A/L \cong \mathbb{Z}_2$ and since every vertex in C is adjacent to some vertex in D, we have that L acts faithfully on $V(C)$. This shows that $L \leq \mathrm{Aut}(C) \cong D_{4p^2}$. If L acts non-transitively on $V(C)$, then $|L| \leq 2p^2$, forcing $|A| = 4p^2$, contradicting the fact that X is a non-Cayley graph. Thus L acts transitively on $V(C)$, and so there exists $g \in L$ such that $x_i^g = x_{i+1}$ for every $i \in \mathbb{Z}_{2p^2}$. Moreover, $\{x_i\,, y_i\}^g = \{x_{i+1}\,, y_{i+1}\}$, $i \in \mathbb{Z}_{2p^2}$, and so $g = (x_0, x_1, \ldots, x_{2p^2-1})(y_0, y_1, \ldots, y_{2p^2-1})$. Since $D = C_{2p^2}$ we may assume that y_0 is adjacent to y_t for some $t \in \mathbb{Z}_{2p^2}$. Then $\{y_0, y_t\}^{g^i} = \{y_i, y_{t+i}\}$ for every $i \in \mathbb{Z}_{2p^2}$. Consequently, $E(X) = \{\{x_i\, x_{i+1}\}, \{x_i, y_i\}, \{y_i, y_{i+t}\} | i \in \mathbb{Z}_{2p^2}\}$, which shows that X is isomorphic to the generalized Petersen graph $\mathrm{GP}(2p^2, t)$. Proposition 2.2.2 now implies that $t^2 \equiv -1(\mod 2p^2)$, completing the proof of Theorem 3.1.4. ■

3.2 Cubic non-Cayley vertex-transitive graphs of order $2p^n$

In this section we consider cubic non-Cayley vertex-transitive graphs of order twice an odd prime power. Examples of such graphs may be found among the generalized Petersen graphs. In particular, for every prime $p \equiv 1 \pmod 4$, the equation $x^2 = -1(\mod p^n)$ has a solution $t \in \mathbb{Z}^+$ and thus giving rise to a cubic non-Cayley vertex-transitive generalized Petersen graph $\mathrm{GP}(p^n, t)$ (see Proposition 2.2.2). We are not aware of any other examples of non-Cayley vertex-transitive graphs of order twice an odd prime-power p^n, $p > 7$ a prime, even though in view of Theorem 3.2.3 below such examples might indeed exist.

We first show that for a prime $p > 7$ the automorphism group of a cubic vertex-transitive graph of order $2p^n$, $n \leq p$, has a normal Sylow p-subgroup. We note that the next result may be extracted from the classification of vertex-transitive graphs of order $2p$, p a prime (see [75, Theorem 6.2]). However, for the sake of completeness we give here a direct self-contained proof.

Lemma 3.2.1 *Let X be a connected cubic vertex-transitive graph of order $2p$, where $p > 7$ is a prime. Then a Sylow p-subgroup of* $\mathrm{Aut}(X)$ *is normal in* $\mathrm{Aut}(X)$.

Proof. Let $A = \mathrm{Aut}(X)$. By Proposition 3.0.7, we may assume that X is not arc-transitive, and consequently that the stabilizer A_v of $v \in V(X)$ in A is a 2-group. Thus A is of order $|A| = 2^m p$, where $m \geq 1$. Let P be a Sylow p-subgroup of A, and let M be a minimal normal subgroup of A. By Proposition 2.1.6, A is solvable and thus M is an elementary abelian 2-group or p-group of order p.

Suppose that P is not normal in A. Then M is an elementary abelian 2-group with orbits of size 2. This implies that X_M is of odd order p and as such it cannot be cubic. Therefore X_M is a cycle of length p. Without loss of generality we may assume that the orbits in $\mathcal{B}$ are labeled in such a way that $X_M = (B_0, B_1, \ldots, B_{p-1}, B_0)$, that is, B_i is adjacent to B_{i+1} for every $i \in \mathbb{Z}_p$. Observe that $d(B) \neq 0$ for every $B \in \mathcal{B}$. Namely, if $d(B) = 0$ for some $B \in \mathcal{B}$ then $d(B') = 0$ for every orbit $B' \in \mathcal{B}$ and $d(B_i \cup B_{i+1}) \in \{1, 2\}$ for every $i \in \mathbb{Z}_p$. In addition, whenever $d(B_i \cup B_{i+1}) = 1$ (respectively, 2), we must have that $d(B_{i-1} \cup B_i) = d(B_{i+1} \cup B_i) = 2$ (respectively, 1), which is clearly impossible since X_M is of odd order. Therefore the subgraphs of X induced on the orbits of M are all isomorphic to the complete graph K_2, and X is isomorphic to the Cartesian product $C_p \square K_2$ of a p-cycle and the complete graph K_2. Since the automorphism group of $C_p \square K_2$ is isomorphic to $D_{2p} \times \mathbb{Z}_2$, we can conclude that P is normal in A. ■

Lemma 3.2.2 *Let X be a connected cubic vertex-transitive graph of order $2p^n$, where $p > 7$ is a prime and $n \leq p$. Then a Sylow p-subgroup of* $\mathrm{Aut}(X)$ *is normal in* $\mathrm{Aut}(X)$.

Proof. Let $A = \mathrm{Aut}(X)$. By Proposition 3.0.7 we may assume that X is not arc-transitive, and consequently that the stabilizer A_v of a vertex $v \in V(X)$ in A is a 2-group. Thus A is a $\{2, p\}$-group. Let $|A| = 2^m p^n$, where $m \geq 1$, $1 \leq n \leq p$ and let P be a Sylow p-subgroup of A. We proof this lemma by induction on n. If $n = 1$ then, by Lemma 3.2.1, P is normal in A. Thus we may assume that $n \geq 2$.

By Proposition 2.1.6, A is solvable. We will distinguish two different cases depending on whether a minimal normal subgroup M (an elementary abelian group) of A is a 2-group or a p-group. Let $\mathcal{B}$ be the set of orbits of M.

Case 1. M is a 2-group.

Then, the orbits of M are of length 2 and therefore X_M is of odd order p^n. This implies that X_M cannot be cubic, and hence X_M is a cycle of odd length p^n. Using the same argument as in the proof of Lemma 3.2.1 we get that X is isomorphic to the Cartesian product $C_{p^n}\Box K_2$ of a p^n-cycle and the complete graph K_2. Since the automorphism group of $C_{p^n}\Box K_2$ is isomorphic to $D_{2p^n} \times \mathbb{Z}_2$, we conclude that P is normal in A.

Case 2. M is a p-group.

Let $|M| = p^d$, $d \le n$. If $d = n$ then $M = P$ and thus P is normal in A. We may therefore assume that $d < n$. Let X_M be the quotient graph of X relative to the orbits $\mathcal{B}$ of M, and let K be the kernel of A acting on X_M. Then $M \le K$, and $A/K \le \mathrm{Aut}(X_M)$ acts transitively on X_M. Since M is a p-group and X is of order $2p^n$, we have that X_M is of order $2p^k$ where $k < n$. In addition, since X is cubic and M is normal in $\mathrm{Aut}(X)$, it follows that X_M is regular of valency 2 or 3.

Subcase 2.1. X_M is of valency 2.

Then $X_M = (B_0, B_1, \ldots, B_{2p^k})$ is a cycle of length $2p^k$, $k > 0$, and $A/K \le \mathrm{Aut}(X_M) = D_{4p^k}$. The facts that X is cubic, that $k > 0$ and that orbits of M are of odd length combined together imply that there is no edge inside the orbits of M, and consequently we may assume that

$$d(B_i, B_{i+1}) = \begin{cases} 1 & \text{if } i \text{ even} \\ 2 & \text{if } i \text{ odd} \end{cases},$$

where $i \in \mathbb{Z}_{2p^k}$. Clearly X is bipartite. Let $G \le A$ be the index 2 subgroup of $\mathrm{Aut}(X)$ that preserves the bipartition of $V(X)$. Applying Proposition 2.1.7 we get that P has two orbits, say U_0 and U_1 on $V(X)$ which coincide with the bipartition of $V(X)$, and therefore $P \le G$ is a Sylow p-subgroup of G. Clearly P acts regularly on both U_0 and U_1. Since $M \le P$ we may assume that $U_0 = \{B_i \mid i \in \mathbb{Z}_{2p^k} \text{ even}\}$ and $U_1 = \{B_i \mid i \in \mathbb{Z}_{2p^k} \text{ odd}\}$. Now collapsing every edge in the matchings $\langle B_i, B_{i+1}\rangle = p^d K_2$, where $i \in \mathbb{Z}_{2p^d}$ is even, to a vertex gives a 4-valent vertex-transitive graph Y of order p^k. Moreover $G \le \mathrm{Aut}(Y)$ and P acts regularly on $V(Y)$. Thus Y is a Cayley graph on P. Since $n \le p$, Proposition 3.0.9 implies that P is normal in G. Finally, since P is a Sylow p-subgroup of G it is characteristic in $G \lhd A$, and hence $P \lhd A$.

Subcase 2.2. X_M is cubic.

Then, by Proposition 3.1.2, $K_v = 1$ and therefore $K = MK_v = M$. By induction, $\mathrm{Aut}(X_M)$ has a normal Sylow p-subgroup and so $A/K = A/M$ has a normal Sylow p-subgroup too. In particular, P/M is normal in A/M, giving us that $P \trianglelefteq A$. ■

Let Dip_3 denote the 3-dipole (that is, the graph with two vertices and three parallel edges), and let $\mathcal{I}$ denote the graph with two vertices, with one edge between these two vertices and with a loop at each of the two vertices. We are now ready to prove the main result of this section.

Theorem 3.2.3 *Let X be a connected cubic non-Cayley vertex-transitive graph of order $2p^n$, where $p > 7$ is a prime and $n \le p$. Then X is a non-symmetric regular P-cover of $\mathcal{I}$, where P is a Sylow p-subgroup of* $\mathrm{Aut}(X)$. *In addition, either* *(i) X is isomorphic to a generalized Petersen graph* $\mathrm{GP}(p^n, t)$*, where* $t^2 \equiv -1 \pmod{p^n}$*, or*

(ii) X is of order $|V(X)| > 2p^2$ and $P = \langle a, b \mid a^{p^k} = b^{p^k} = 1, \ldots \rangle$ is a non-cyclic p-group generated by two elements a and b of the same order and admitting an automorphism $\phi \in \mathrm{Aut}(P)$ of order 4 such that $a^\phi = b$ and $b^\phi = a^{-1}$.

Proof. Proposition 3.0.7 implies that X is a non-symmetric graph. If $n = 1$ then X is of order $2p$ and isomorphic to a generalized Petersen graph. If $n = 2$ then Proposition 3.0.8 implies that X is a generalized Petersen graph. We may therefore assume that $n \geq 3$.

By Lemma 3.2.2, a Sylow p-subgroup P of $A = \mathrm{Aut}(X)$ is normal in A. Observe that $|P| = p^n$ and that, by Proposition 2.1.7, P has two orbits of length p^n on $V(X)$, which implies that X is a regular P-cover of $X_P \in \{\mathrm{Dip}_3, \mathcal{I}\}$. In addition, since the orbits of P form an A-invariant partition, A projects to a subgroup of $\mathrm{Aut}(X_P)$. On the other hand, the graph X can be viewed as a regular P-cover of X_P, and it can therefore be derived from X_P through a suitable voltage assignment.

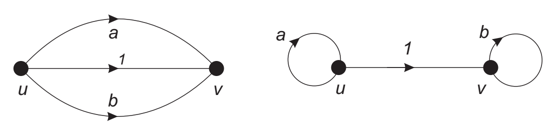

Figure 3.4: The voltage graphs.

Case 1. $X_P \cong \mathrm{Dip}_3$.

Let $V(X_P) = \{u, v\}$ and let $\zeta\colon A(X_P) \to P$ be a voltage assignment giving rise to $X = \tilde{X}_P = \mathrm{Cov}(X_P, \zeta)$, and assigning $1 \in P$ to the two arcs of the underlying edge uv in the spanning tree T, whereas the voltages on the cotree arcs are as shown in the left-hand side picture in Figure 3.4. The connectedness of X implies that $P = \langle a, b \rangle$. Moreover, since X is vertex-transitive an automorphism of X_P interchanging u and v must lift along this covering projection. In particular, in view of Proposition 2.2.6, there is an automorphism $\alpha \in \mathrm{Aut}(X_P)$ which gives rise to an automorphism α^* of P such that $\{a, b\}^{\alpha^*} = \{a^{-1}, b^{-1}\}$. If $a^{\alpha^*} = a^{-1}$ then $b^{\alpha^*} = b^{-1}$, and consequently $\tilde{\alpha} \in \mathrm{Aut}(X)$ is of order 2. But then $P\langle \tilde{\alpha} \rangle$ is a regular subgroup of A, contradicting our assumption that X is a non-Cayley graph. We may therefore assume that $a^{\alpha^*} = b^{-1}$ and $b^{\alpha^*} = a^{-1}$. Then $a^{\alpha^{*2}} = (b^{-1})^{\alpha^*} = a$ and $b^{\alpha^{*2}} = (a^{-1})^{\alpha^*} = b$, showing that α^{*2} fixes all the voltages, and consequently that the lift $\tilde{\alpha}$ is of order 2. This implies that in this case too, $P\langle \tilde{\alpha} \rangle$ is a regular subgroup of A, a contradiction.

Case 2. $X_P \cong \mathcal{I}$.

Let $V(X_P) = \{u, v\}$ and let $\zeta\colon A(X_P) \to P$ be a voltage assignment giving rise to $X = \tilde{X}_P = \mathrm{Cov}(X_P, \zeta)$, and assigning $1 \in P$ to the two arcs of the underlying edge uv in the spanning tree T, whereas the voltages on the cotree arcs are as shown in the right-hand side picture in Figure 3.4. The connectedness of X implies that

$P = \langle a, b\rangle$. Moreover, since X is vertex-transitive an automorphism of X_P interchanging u and v must lift along this covering projection. In particular, in view of Proposition 2.2.6, there is an automorphism $\alpha \in \mathrm{Aut}(X_P)$ which gives rise to an automorphism α^* of P such that $a^{\alpha^*} \in \{b, b^{-1}\}$. Without loss of generality we may assume that $a^{\alpha^*} = b$. If $b^{\alpha^*} = a$ then $\tilde{\alpha} \in \mathrm{Aut}(X)$ is of order 2, and consequently $P\langle\tilde{\alpha}\rangle$ is a regular subgroup of A, a contradiction. Therefore $b^{\alpha^*} = a^{-1}$ and $\alpha^* \in \mathrm{Aut}(P)$ is of order 4. Finally, P is non-cyclic since in the cyclic case X is a generalized Petersen graph. ■

Example 3.2.4 Using program package MAGMA [7] one can see that for a prime $p \in \{3, 5\}$ a regular G-cover of the graph $\mathcal{I}$, where

$$G = \langle a, b, c | a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = b^p\rangle,$$

arising from voltage assignment as indicated in the right-hand side picture in Figure 3.4 is a connected non-Cayley vertex-transitive graph of order $2 \cdot p^5$ which is not isomorphic to a generalized Petersen graph.

3.2.1 Observations and conclusions

In this chapter Problem 3.0.1 is solved for cubic non-Cayley vertex-transitive graphs of orders $4p^2$ and $2p^n$, where $p > 7$ is a prime and $n \leq p$. In addition, a complete classification of non-Cayley vertex-transitive graphs of order $4p^2$ is given in Theorem 3.1.4 whereas Theorem 3.2.3 characterizes non-Cayley vertex-transitive graphs of order $2p^n$, $p > 7$ a prime and $n \leq p$. If, however, one is to obtain a complete classification of non-Cayley vertex-transitive graphs of order $2p^n$, $p > 7$ a prime and $n \leq p$, then p-groups $P = \langle a, b \mid a^{p^k} = b^{p^k} = 1, \ldots\rangle$ generated by two elements a and b of the same order and admitting an automorphism $\phi \in \mathrm{Aut}(P)$ of order 4 such that $a^\phi = b$ and $b^\phi = a^{-1}$ need to be characterized. By [106, Theorem 3.5] the automorphism group of a nonsplit metacyclic p-group (p odd prime) is a p-group which implies that the group P in Theorem 3.2.3 is not a nonsplit metacyclic p-group. Also, if P is non-abelian and $\langle a\rangle \cap \langle b\rangle = \langle 1\rangle$ then $\mathrm{core}_P(\langle a\rangle) \neq \langle a\rangle$ and $\mathrm{core}_P(\langle b\rangle) \neq \langle b\rangle$. Namely, if, say, $\langle a\rangle$ is normal in P then $[a, b] = a^{-1}b^{-1}ab = a^{-1+l}$ for some $l \in \mathbb{Z}_{p^k} \setminus \{0\}$. Applying the automorphism ϕ we get that $b^{-1+l} = (a^{-1+l})^\phi = [a, b]^\phi = b^{-1}aba^{-1} = a^{-1+l}$, which shows that $l = 1$ and consequently P is abelian.

Let us also remark that each group P satisfying Theorem 3.2.3(ii) gives rise to a tetravalent arc-transitive Cayley graph $\mathrm{Cay}(P, \{a, a^{-1}, b, b^{-1}\})$ of order p^n. Such Cayley graphs belong to one of four families of tetravalent edge-transitive Cayley graphs with odd number of vertices given in [68]. However, no further information on the existence of such graphs is given there. This gives additional motivation for the following problem.

Problem 3.2.5 *Characterize non-cyclic p-groups $P = \langle a, b \mid a^{p^k} = b^{p^k} = 1, \ldots\rangle$ generated by two elements a and b of the same order and admitting an automorphism $\phi \in \mathrm{Aut}(P)$ of order 4 such that $a^\phi = b$ and $b^\phi = a^{-1}$.*

Chapter 4

On automorphism groups of certain vertex-transitive graphs

Results of this chapter are published in [125].

In this chapter we consider normality of connected cubic Cayley graphs on finite non-abelian simple groups. Li [66] proved that a connected cubic symmetric Cayley graph on a finite non-abelian simple group different from

$$\mathrm{A}_5, \mathrm{PSL}(2,11), \mathrm{M}_{11}, \mathrm{A}_{11}, \mathrm{M}_{23}, \mathrm{A}_{23} \text{ and } \mathrm{A}_{47}$$

is normal. Later on Xu and Xu [124] proved that a connected cubic symmetric Cayley graph on the alternating group A_5 is normal. Further, using results from [33], Xu [123] proved that connected cubic symmetric Cayley graphs on A_{47} are non-normal whereas connected cubic symmetric Cayley graphs on the remaining six groups listed above are all normal. In addition, Fang [34] proved that most of connected cubic non-symmetric Cayley graphs on finite non-abelian simple groups are normal. In particular, the following proposition holds.

Proposition 4.0.1 [34] *Let X be a connected cubic Cayley graph on one of the following finite simple groups*

(i) *a sporadic simple group different from* M_{11}, M_{22}, M_{23}, J_2, *and* Suz;

(ii) A_n, *where* $n \notin \{5, 11, 23, 47\} \cup \{2^m - 1 \mid m \geq 3\}$;

(iii) *a simple group of Lie type of odd characteristic with a possible exception* $\mathrm{PSL}(2, 11)$;

(iv) $\mathrm{PSL}(2, 2^e)$, $\mathrm{PSL}(3, 2^e)$, $\mathrm{U}_3(2^e)$, $\mathrm{PSp}_4(2^e)$, $\mathrm{E}_8(2^e)$, $\mathrm{F}_4(2^e)$, ${}^2\mathrm{F}_4(2^e)'$, $\mathrm{G}_2(2^e)$ *or* $\mathrm{Sz}(2^e)$.

Then $\mathrm{Aut}(X) = G \rtimes \mathrm{Aut}(G, S)$ *and* $\mathrm{Aut}(G, S) \leq S_3$.

In the PhD Thesis the normality of connected cubic non-symmetric Cayley graphs on groups given in Proposition 4.0.1(i), (iii) and (ii) with $n \in \{5, 11, 23, 47\}$ is considered. In particular, we show that each such graph is normal, see Theorem 4.0.4. This result depends on the classification of finite simple groups.

The following two lemmas will be needed in the proof of Theorem 4.0.4. The first lemma gives a general description of the full automorphism group of a connected Cayley graph on a finite non-abelian simple group.

Lemma 4.0.2 [33, Theorem 1.1] *Let $X = Cay(G, S)$ be a connected Cayley graph on a finite non-abelian simple group G. Let M be a subgroup of* $\mathrm{Aut}(X)$ *containing* $G \rtimes \mathrm{Aut}(G, S)$. *Then either $M = G \rtimes \mathrm{Aut}(G, S)$ or one of the following holds:*

(i) M is almost simple with $\mathrm{Soc}(M)$ *containing G as a proper subgroup;*

(ii) $G \rtimes \mathrm{Inn}(G) \leq M \leq G \rtimes \mathrm{Aut}(G, S).2$ and S is a self-inverse union of G-conjugacy classes;

(iii) M is not quasiprimitive on $V(X)$ and there is a maximal intransitive normal subgroup K of M such that one of the following holds:

(a) M/K is almost simple, and $\mathrm{Soc}(M/K)$ *contains $GK/K \cong G$ and is transitive on $V(X_K)$;*

(b) $M/K = \mathrm{AGL}_3(2)$, $G = \mathrm{PSL}(2, 7)$ and $X_K \cong K_8$;

(c) $\mathrm{Soc}(M/K) \cong T \times T$ *and $GK/K \cong G$ is a diagonal subgroup of* $\mathrm{Soc}(M/K)$ *(see* Table 1 *of* [34] *for T and G).*

For a finite group L, let $m(L)$ denote the minimal index of a proper subgroup of L. The following lemma records an upper bound on the order of a Sylow subgroup of a finite simple group G in terms of $m(G)$.

Lemma 4.0.3 [32, Lemma 2.1] *Let G be a finite non-abelian simple group and let p be a prime divisor of $|G|$. Suppose that a Sylow p-subgroup of G has order p^d. Then $d \leq (m(G) - 1)/(p - 1)$ if $p \neq 2$ and $d \leq m(G) - 2$ if $p = 2$.*

Theorem 4.0.4 *Let $X = Cay(G, S)$ be a connected cubic non-symmetric Cayley graph on G, where $G \in \{\mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_2, \mathrm{Suz}, \mathrm{PSL}(2, 11), \mathrm{A}_5, \mathrm{A}_{11}, \mathrm{A}_{23}, \mathrm{A}_{47}\}$. Then* $\mathrm{Aut}(X) = G \rtimes \mathrm{Aut}(G, S)$.

Proof. For $X = \mathrm{Cay}(G, S)$ given in Theorem 4.0.4, write $A = \mathrm{Aut}(X)$ and let A_1 denote the stabilizer of $\mathrm{Aut}(X)$ fixing $v = 1 \in V(X)$. Then $\mathrm{Aut}(X) = GA_1$ with $G \cap A_1 = 1$. Since X is a cubic non-symmetric graph, A_1 is a 2-group, and so $|A_1| = 2^s$ for some integer $s \geq 0$. If G is not normal in A then there is a subgroup M of A such that $G \rtimes \mathrm{Aut}(G, S)$ is maximal in M. Then either Lemma 4.0.2(i) or 4.0.2(iii)(a) occurs. In the former case M is almost simple with $\mathrm{Soc}(M)$ properly containing G, which implies that G is not normal in M. It follows that $|\mathrm{Soc}(M) : G| = 2^n$, for some integer $n \geq 2$, which by Proposition 2.1.1 is not the case for G given in Theorem 4.0.4. So we need only to consider Lemma 4.0.2(iii)(a).

Then M has a maximal intransitive normal subgroup K such that M/K is an almost simple group containing $GK/K \cong G$. Since $K \cap G = 1$, K is a 2-group. Let $|K| = 2^m$, for some integer $m \geq 1$. If $\mathrm{Soc}(M/K) \neq GK/K$, again by Proposition 2.1.1, a similar argument as above yields a contradiction. So $\mathrm{Soc}(M/K) = GK/K \cong G$, which implies that $GK \lhd M$. If K is centralized by G, then G is a characteristic subgroup of GK and hence $G \lhd M$, which is not the case since $\mathrm{N}_{\mathrm{Aut(X)}}(\mathrm{G}) = \mathrm{G} \rtimes \mathrm{Aut}(\mathrm{G},\mathrm{S})$ is a maximal subgroup of M. Thus by conjugation G acts nontrivially on K and hence G is isomorphic to a subgroup of $\mathrm{Aut}(K)$. Then, by Hall [54], we conclude further that G is isomorphic to an irreducible subgroup of $\mathrm{PSL}(d,2)$, for some integer $d \leq m$. On the other hand, K must be semiregular on $V(X)$ (otherwise, the quotient graph X_K has valency two and $\mathrm{Aut}(X_K)$ is a dihedral group, which contradicts to the fact that $GK/K \leq M/K \leq \mathrm{Aut}(X_K)$. This implies that $|G|_2$ is divisible by $|K|$ and hence by 2^d, where $|G|_2$ stands for the 2-part of $|G|$. Moreover, if $|G|_2 = 2^d$, then X_K is a cubic graph of odd order, which is impossible. So 2^d is a proper divisor of $|G|_2$. Now we check the ten groups case by case.

If $G = \mathrm{M}_{11}$, then $|G|_2 = 2^4$. However, by Kleidman and Liebeck [60, Proposition 5.3.8], $d \geq 5$, a contradiction. A similar argument shows that also $G = \mathrm{M}_{23}$ does not occur.

For $G = \mathrm{M}_{22}$, we have $|G|_2 = 2^7$, and, by Kleidman and Liebeck [60, Proposition 5.3.8], $d \geq 6$. Thus $d = 6$ and $|K| = 2^6$. It follows that $K \cong \mathbb{Z}_2^6$, and hence G is isomorphic to a proper subgroup of $\mathrm{PSL}(6,2)$. On the other hand, $|\mathrm{M}_{22}|$ is divisible by 11 and $(11, |\mathrm{PSL}(6,2)|) = 1$, a contradiction. So $G = \mathrm{M}_{22}$ is not the case. A similar argument shows that $G = J_2$ is not the case either.

For $G = \mathrm{Suz}$ we have $|G|_2 = 2^{13}$, and, by Kleidman and Liebeck [60, Proposition 5.3.8], $d \geq 12$. A similar argument as used for M_{22} shows that G is isomorphic to a proper subgroup of $\mathrm{PSL}(12,2)$. On the other hand, it is straightforward to verify that $(13, |\mathrm{PSL}(12,2)|) = 1$. Since 13 is a prime divisor of $|\mathrm{Suz}|$, it follows that $|\mathrm{PSL}(12,2)|$ is not divisible by $|\mathrm{Suz}|$, a contradiction.

For $G = \mathrm{PSL}(2,11)$ we have $|G|_2 = 2^2$, and, by Kleidman and Liebeck [60, Table 5.3.A], $d \geq 5$, a contradiction.

For $G = \mathrm{A}_5$ we have $|G|_2 = 2^2$, and, by Kleidman and Liebeck [60, Proposition 5.3.7], $d = 2$, a contradiction.

For $G = \mathrm{A}_n$ with $n \in \{11, 23, 47\}$ we have, by Kleidman and Liebeck [60, (i) of Proposition 5.3.7], $d \geq n-2$. However, by Lemma 4.0.3, $|G|_2$ is at most 2^{n-2} which contradicts that 2^d is a proper divisor of $|G|_2$. ■

4.1 Observations and conclusions

In this chapter connected cubic non-symmetric Cayley graphs on finite simple groups are considered. In particular, it is proved that connected cubic non-symmetric Cayley graphs on a group G, where

$$G \in \{\mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_2, \mathrm{Suz}, \mathrm{PSL}(2,11), \mathrm{A}_5, \mathrm{A}_{11}, \mathrm{A}_{23}, \mathrm{A}_{47}\},$$

are normal. This result improves results given in Proposition 4.0.1. However, cubic non-symmetric Cayley graphs on finite simple groups are still not completely classified, see Proposition 4.0.1. Unfortunately, the methods used in this chapter do not work for the remaining cases. So, to solve these remaining cases a new method must be developed.

Chapter 5

One-regular graphs

Not surprisingly arc-transitive graphs, and one-regular graphs in particular, have received considerable attention over the years, the aim being to obtain structural results and possibly a classification of such graphs of particular orders or satisfying certain additional properties. Research in one-regular graphs is interesting for two reasons, the first being their connection to regular maps, a lively area of research. Namely, the underlying graphs of chiral maps admit one-regular group actions with a cyclic vertex stabilizers (see, for example, [18, 21, 22, 23]). Second, one may argue that one-regular graphs are interesting in their own right if one's goal is a description of all arc-transitive graphs. For some classes of Cayley graphs, for example circulants, this has been achieved, whereas for others, such as Cayley graphs on dihedral groups, all 2-arc-transitive graphs have been completely classified [30], but arc-transitivity remains an open problem.

Clearly, a one-regular graph with no isolated vertices is connected, and it is of valency 2 if and only if it is a cycle. The first example of a cubic one-regular graph was constructed by Frucht [40]. Further research in cubic one-regular graphs has been part of a more general project dealing with the investigation of cubic arc-transitive graphs (see [19, 25, 35, 36, 37, 38, 95]). Tetravalent one-regular graphs have also received considerable attention. In [11] tetravalent one-regular graphs of prime order were constructed, and in [82] an infinite family of tetravalent one-regular Cayley graphs on alternating groups was given. Tetravalent one-regular circulant graphs were classified in [122], and tetravalent one-regular Cayley graphs on abelian groups were classified in [120]. Next, one may extract a classification of tetravalent one-regular Cayley graphs on dihedral groups from [65, 111, 112]. Let p and q be primes. Clearly every tetravalent one-regular graph of order p is a circulant graph. Also, by [16, 100, 102, 113, 120, 122], every tetravalent one-regular graph of order pq or p^2 is a circulant graph. Furthermore, the classification of tetravalent one-regular graphs of order $2pq$ was given in [129]. The aim of this chapter is to classify tetravalent one-regular graphs of order $4p^2$, see Theorem 5.4.1. (For more results on tetravalent arc-transitive graphs, see [44, 45, 67, 101].)

5.1 Tetravalent arc-transitive graphs

In this section we gather known results about tetravalent arc-transitive graphs that will be needed in subsequent sections of this chapter. The first two propositions can be deduced from [120, Theorem 3.5].

Proposition 5.1.1 [120] *Let p be a prime, and $G \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$ or $G \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$. Then there exists a tetravalent one-regular Cayley graph on G if and only if $p-1$ is a multiple of 4. In particular, in this case, exactly one such graph exists.*

Proposition 5.1.2 [120] *Let p be a prime and $G \cong \mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$. Then there is no tetravalent one-regular Cayley graph on G.*

The following proposition is a 'reduction' theorem which is deduced from [44, Theorem 1.1].

Proposition 5.1.3 [44, Theorem 1.1] *Let X be a tetravalent connected symmetric graph and let $G \leq \mathrm{Aut}(X)$ be an arc-transitive subgroup of $\mathrm{Aut}(X)$. Then for each normal subgroup N of G one of the following holds:*

(1) N is transitive on $V(X)$;

(2) X is bipartite and N acts transitively on each of the two bipartition sets;

(3) N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} of X_N;

(4) N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-symmetric graph and X is a regular cover of X_N.

To state the next result we need to introduce three families of tetravalent graphs that were first defined in [45]. First, let $\mathcal{C}^{\pm 1}(p;4,2)$ be a graph with vertex set $\mathbb{Z}_p^2 \times \mathbb{Z}_4$, and adjacencies in $\mathcal{C}^{\pm 1}(p;4,2)$ satisfying the following conditions: for $i, j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4$

$$(i,j,k) \sim \begin{cases} (i \pm 1, j, k+1) & \text{if } k \text{ is even} \\ (i, j \pm 1, k+1) & \text{if } k \text{ is odd} \end{cases} .$$

Second, for a prime $p \equiv \pm 1 (\mod 8)$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv 2 \pmod p$ the graph $\mathcal{NC}^0_{4p^2}$ is defined as a graph with

$$\begin{aligned} V(\mathcal{NC}^0_{4p^2}) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x,y,z) \mid x, y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}^0_{4p^2}) &= \{(x,y,0)(x \pm 1, y, 1) \mid x, y \in \mathbb{Z}_p\} \cup \{(x,y,1)(x, y \pm 1, 2) \mid x, y \in \mathbb{Z}_p\} \cup \\ &\quad \{(x,y,2)(x \mp 1, y \pm k, 3) \mid x, y \in \mathbb{Z}_p\} \cup \{(x,y,3)(x \mp k, y \pm 1, 0) \mid x, y \\ &\quad \in \mathbb{Z}_p\}. \end{aligned}$$

And third, for a prime p, $p \equiv 1(\mod 8)$ or $p \equiv 3(\mod 8)$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv -2 \pmod{p}$ the graph $\mathcal{NC}^1_{4p^2}$ is defined as a graph with

$$\begin{aligned} V(\mathcal{NC}^1_{4p^2}) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x,y,z) \mid x,y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}^1_{4p^2}) &= \{(x,y,0)(x\pm 1,y,1) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,1)(x,y\pm 1,2) \mid x,y \in \mathbb{Z}_p\} \cup \\ & \quad \{(x,y,2)(x\pm 1,y\pm k,3) \mid x,y \in \mathbb{Z}_p\} \cup \{(x,y,3)(x\pm k,y\mp 1,0) \mid x,y \\ & \quad \in \mathbb{Z}_p\}. \end{aligned}$$

The graphs $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are extracted from [45, Lemma 8.4, Lemma 8.7]. Now we can state the result of Gardiner and Praeger [45, Theorem 1.2] about connected tetravalent graphs admitting arc-transitive subgroups of automorphisms with a normal elementary abelian p-group N such that the corresponding quotient graph X_N is a cycle.

Proposition 5.1.4 [45, Theorem 1.2] *For an odd prime p let X be a connected, G-symmetric, tetravalent graph of order $4p^2$, let $N = \mathbb{Z}_p^2$ be a minimal normal subgroup of G with orbits of size p^2, and let K be the kernel of the action of G on $V(X_N)$. If $X_N = C_4$ and $K_v = \mathbb{Z}_2$ then X is isomorphic to one of the following graphs: $\mathcal{C}^{\pm 1}(p;4,2)$, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$.*

In [45] it is proven that the three graphs in the above proposition all admit a one-regular subgroup of automorphisms. In the following two lemmas we improve this result by showing that $\mathcal{C}^{\pm 1}(p;4,2)$ is not one-regular whereas $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are.

Lemma 5.1.5 *Let p be a prime. Then $\mathcal{C}^{\pm 1}(p;4,2)$ is not one-regular graph.*

Proof. First recall that the vertex set of $\mathcal{C}^{\pm 1}(p;4,2)$ is equal to $V(X) = \{(i,j,k) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_p, k \in \mathbb{Z}_4\}$ and the edges are of the form

$$\begin{aligned} (i,j,2l) &\sim (i\pm 1,j,2l+1), \quad \text{where } i,j \in \mathbb{Z}_p \text{ and } l \in \{0,1\} \\ (i,j,2l-1) &\sim (i,j\pm 1,2l), \quad \text{where } i,j \in \mathbb{Z}_p \text{ and } l \in \{0,1\}. \end{aligned}$$

Then the reader can check that a permutation α of $V(X)$ defined by $(i,j,k)^\alpha = (-i,j,k)$ maps edges to edges, and hence α is an automorphism of X. Since α fixes the arc $(0,0,1)(0,1,2) \in A(X)$ it follows that X is not one-regular. ■

Lemma 5.1.6 *Let p be a prime, then $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ are both one-regular graphs.*

Proof. Let $X \in \{\mathcal{NC}^0_{4p^2}, \mathcal{NC}^1_{4p^2}\}$ and let X^2 be the distance-2-graph of X, that is, $V(X^2) = V(X)$ with two vertices being adjacent in X^2 if and only if they are at distance 2 in X. Let

$$\Delta_i = \{(x,y,i) \mid x,y \in \mathbb{Z}_p, \quad i \in \mathbb{Z}_p\}.$$

Then for every $i \in \mathbb{Z}_4$ the subgraph $X^2[\Delta_i]$ of X^2 induced by the vertices in Δ_i is a 2-dimensional grid $C_p \times C_p$ whereas any edge uv in X^2 with endvertices $u \in \Delta_i$ and

$v \in \Delta_j$, where $i \neq j$, is contained in an induced subgraph of X^2 isomorphic to the complete graph K_4. Moreover this induced subgraph isomorphic to K_4 containing the edge uv is unique. Take four vertices $u_1, u_2, u_3, u_4 \in \Delta_i$ such that the subgraph Y of X^2 induced on these four vertices is isomorphic to a 4-cycle C_4. Then Y^g for any $g \in \mathrm{Aut}(X^2)$ is an induced subgraph of X^2 isomorphic to C_4. Since there is no set of four vertices containing vertices from different sets Δ_i such that the induced subgraph of X^2 is isomorphic to C_4 it follows that Y^g is a subgraph of $X^2[\Delta_j]$ for some $j \in \mathbb{Z}_4$. This shows that the sets Δ_i, $i \in \mathbb{Z}_4$, are blocks of imprimitivity for $\mathrm{Aut}(X)$. Since Δ_i, $i \in \mathbb{Z}_4$, are blocks of imprimitivity for $\mathrm{Aut}(X)$, any automorphism $g \in \mathrm{Aut}(X)$ that fixes the vertices $(0,0,0)$ and $(1,0,1)$ (and thus it fixes the arc $(0,0,0),(1,0,1))$ also fixes the vertices $(2,0,0)$ and $(-1,0,1)$. Now looking at the action of g on X^2 we get that g fixes both Δ_0 and Δ_1 pointwise. Since all the vertices in Δ_1 are fixed by g and the induced bipartite subgraph $X[\Delta_1, \Delta_2]$ is a disjoint union of p $2p$-cycles it follows that also Δ_2 is fixed pointwise by g. Using the same argument for $X[\Delta_0, \Delta_3]$ one can see that g also fixes the vertices in Δ_3 and thus $g = 1$, which shows that X is one-regular. ■

To state the next result we need to introduce two more families of tetravalent graphs that were first defined in [45]. The graph $\mathcal{C}^{\pm 1}(p; 4p, 1)$ is defined to have the vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ and the edge set $\{(i,j)(i \pm 1, j+1) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_{4p}\}$. The graph $\mathcal{C}^{\pm\varepsilon}(p; 4p, 1)$ is a graph with vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ with adjacencies in $\mathcal{C}^{\pm\varepsilon}(p; 4p, 1)$ satisfying the following conditions:

$$(i,j) \sim \begin{cases} (i \pm \varepsilon, j+1) & \text{if } j \text{ is odd} \\ (i \pm 1, j+1) & \text{if } j \text{ is even} \end{cases},$$

where $i \in \mathbb{Z}_p$, $j \in \mathbb{Z}_{4p}$ and ε is an element of order 4 in $\mathbb{Z}_p^*$.

Proposition 5.1.7 [45, Theorem 1.1] *Let X be a connected, G-symmetric, tetravalent graph of order $4p^2$, and let $N = \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p, where p is an odd prime. Let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_{4p}$ and $K_v = \mathbb{Z}_2$ then X is isomorphic either to $\mathcal{C}^{\pm 1}(p; 4p, 1)$ or to $\mathcal{C}^{\pm\varepsilon}(p; 4p, 1)$.*

We end this section with a result on tetravalent arc-transitive graphs of order $4p$, where p is a prime. In order to state the result, first recall that the *lexicographic product* $X[Y]$ (sometimes also called the *wreath product*) of two graphs X and Y has vertex set $V(X) \times V(Y)$, and two vertices (a,u) and (b,v) are adjacent in $X[Y]$ if $ab \in E(X)$ or if $a = b$ and $uv \in E(Y)$. Second, following [130], for a prime p congruent to 1 modulo 4, an element w of order 4 in $\mathbb{Z}_p^*$ and the group $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, we use notation $\mathcal{CA}^0_{4p} = \mathrm{Cay}(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})$ and $\mathcal{CA}^1_{4p} = \mathrm{Cay}(G, \{a, a^{-1}, a^{w}b, a^{-w}b\})$. Then, for the definition of the graph $\mathcal{C}(2,p,2)$ stated in the sixth row of Table 5.1 see Section 5.3. Finally, by [130, Example 3.7], $\mathfrak{g}_{28} = \mathrm{Cos}(G, T, TaT)$ is a coset graph of the group $G = \mathrm{PGL}(2,7)$ with respect to a subgroup T isomorphic to A_4 and an involution a from the center of the normalizer of a Sylow 3-subgroup of T in G.

Proposition 5.1.8 [130, Theorem 4.1] *Let s be a positive integer and let p be a prime. Then a connected tetravalent graph of order $4p$ is s-arc-transitive if and only if it is isomorphic to one of the graphs listed in Table 5.1. Furthermore, all graphs listed in Table 5.1 are pairwise non-isomorphic.*

X	s	$\mathrm{Aut}(X)$	comments
$K_{4,4}$	3	$\mathbb{Z}_2 \ltimes (S_4 \times S_4)$	p=2
$C_{2p}[2K_1]$	1	$D_{4p} \ltimes \mathbb{Z}_2^{2p}$	$p > 2$
$\mathcal{CA}_{4p}^0$	1	$\mathbb{Z}_2^2 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2),$	$p \equiv 1(\mod 4)$
$\mathcal{CA}_{4p}^1$	1	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2),$	$p \equiv 1(\mod 4)$
$\mathcal{C}(2,p,2)$	1	$D_{2p} \ltimes \mathbb{Z}_2^{2p}$	$p > 2$
$\mathfrak{g}_{28}$	3	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$	$p = 7$

Table 5.1: Tetravalent s-arc-transitive graphs of order $4p$.

5.2 Examples

In this section, we give examples of tetravalent one-regular graphs of order $4p^2$, where p is a prime.

Example 5.2.1 Introduced by Wilson [116] the *bicycle wheels* are defined in the following way, given natural numbers n, a, r and s, the graph $X = \mathcal{BW}_n(a,r,s)$ is defined to be a graph of order $3n$ with the vertex set $V(X) = \{A_i, B_i, C_i \mid i \in \mathbb{Z}_n\}$ and the edge set

$$E(X) = \{A_iB_i, B_iA_{i+1}, B_iC_i, C_iB_{i+a}, A_iA_{i+r}, C_iC_{i+s} \mid i \in \mathbb{Z}_n\}.$$

With the help of the computer software package Magma [7] one can see that $\mathcal{BW}_{12}(5,1,5)$ is one-regular. In addition, it is a Cayley graph $\mathrm{Cay}(G_{36}, S)$ on the group $G_{36} = \langle a,b,c,d \mid a^2 = b^2 = c^3 = d^3 = 1 = [a,b] = [a,c] = [b,c] = [c,d], d^{-1}ad = b, d^{-1}bd = ab\rangle$ with respect to the generating set $S = \{ad, (ad)^{-1}, bdc, (bdc)^{-1}\}$, and $\mathrm{Aut}(\mathcal{CA}_{36}^2) \cong G_{36} \rtimes \mathbb{Z}_2^2$.

Remark: The automorphism group of the graph $\mathcal{BW}_{12}(5,1,5)$ has a non-normal Sylow 3-subgroup. Since, by Theorem 5.4.1, the automorphism groups of the graphs $\mathcal{CA}_{4p^2}^i$, $i \in \{0,1,2\}$, given in Examples 5.2.3 and 5.2.4 and Lemma 5.2.6, all have normal Sylow p-subgroups, the graph $\mathcal{BW}_{12}(5,1,5)$ is not isomorphic to any of these graphs.

Example 5.2.2 Given natural numbers k and m, and a 2×2 matrix M over $\mathbb{Z}_n$ the 2-dimensional generalized power spidergraph $\mathcal{GPS}2(k,n,M)$ is defined to be a graph with the vertex set $\mathbb{Z}_k \times \mathbb{Z}_n \times \mathbb{Z}_n$, and the edge set $\{(i,x)(i+1, x+a_i), (i,x)(i+1, x+b_i) \mid i \in \mathbb{Z}_k, x \in \mathbb{Z}_n \times \mathbb{Z}_n\}$ where $a_i = (1,0)M^i$ and $b_i = (-1,0)M^i$ (see [116]). With the use of Magma [7] one can see that $\mathcal{GPS}2(4,3,(0\ 1):(1\ 2))$ is a one-regular graph. In addition, it is not a Cayley graph and the stabilizer of a vertex in the automorphism group is isomorphic to $\mathbb{Z}_4$.

Example 5.2.3 Let $p \equiv 1 \pmod 4$ be a prime and w an element of order 4 in $\mathbb{Z}_p^*$ with $1 \leq w \leq p-1$. Let $G^0_{4p^2} = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$. Then, by [120, Proposition 3.3(iv)], the Cayley graph $\mathcal{CA}^0_{4p^2} = \mathrm{Cay}(G^0_{4p^2}, \{a, a^{-1}, a^w b, a^{-w}b\})$ is a tetravalent one-regular graph. Furthermore, $\mathrm{Aut}(\mathcal{CA}^0_{4p^2}) \cong (\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2^2$.

Example 5.2.4 Let p be an odd prime and $G^1_{4p^2} = \langle a\,, b \,|\, a^{4p} = b^p = 1\,, ab = ba\rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$. Then, by [120, Proposition 3.3], the Cayley graph $\mathcal{CA}^1_{4p^2} = \mathrm{Cay}(G^1_{4p^2}, \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\})$ is a tetravalent one-regular graph. Furthermore, $\mathrm{Aut}(\mathcal{CA}^1_{4p^2}) \cong (\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$. The graph $\mathcal{DW}(12,3)$ of order 36 given in [116] is the smallest example of such graphs.

For an odd prime p, the tetravalent graph $\mathcal{C}^{\pm 1}(p\,; 4p, 1)$ is defined in the paragraph preceding Proposition 5.1.7. In the following lemma we prove that $\mathcal{C}^{\pm 1}(p\,; 4p, 1)$ is isomorphic to $\mathcal{CA}^1_{4p^2}$, and thus Example 5.2.4 implies that it is one-regular.

Lemma 5.2.5 *Let p be an odd prime, let $G^1_{4p^2} = \langle a\,, b \mid a^{4p} = b^p = 1\,, ab = ba\rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$, and let $S = \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\}$. Then $\mathcal{C}^{\pm 1}(p\,; 4p, 1) \cong$ Cay$(G^1_{4p^2}, S) = \mathcal{CA}^1_{4p^2}$.*

Proof. Recall that $\mathcal{C}^{\pm 1}(p\,; 4p, 1)$ has vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ and edge set $\{(i,j)(i \pm 1, j+1) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_{4p}\}$. The map defined by $(i,j) \mapsto a^j b^i$ is an isomorphism from $\mathcal{C}^{\pm 1}(p\,; 4p, 1)$ to the Cayley graph $\mathcal{CA}^1_{4p^2}$. We leave the details to the reader. ■

Let $p \equiv 1 \pmod 4$ be a prime and let $\varepsilon \in \mathbb{Z}_p$ be such that $\varepsilon^2 \equiv -1 \pmod p$. The following lemma shows that the graph $\mathcal{C}^{\pm\varepsilon}(p\,; 4p, 1)$ is a Cayley graph.

Lemma 5.2.6 *Let $p \equiv 1 \pmod 4$ be a prime, let $\varepsilon \in \mathbb{Z}_p$ be such that $\varepsilon^2 \equiv -1 \pmod p$, let $G^2_{4p^2} = \langle a, b \mid a^{4p} = b^p = 1, a^{-1}ba = b^{\varepsilon}\rangle$, and let*

$$S = \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\}.$$

Then $\mathcal{CN}^2_{4p^2} =$ Cay$(G^2_{4p^2}, S)$ is a symmetric graph isomorphic to $\mathcal{C}^{\pm\varepsilon}(p\,; 4p, 1)$.

Proof. Recall that the graph $\mathcal{C}^{\pm\varepsilon}(p\,; 4p, 1)$ has vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ with adjacencies defined as follows:

$$(i,j) \sim \begin{cases} (i \pm \varepsilon, j+1) & \text{if } j \text{ is odd} \\ (i \pm 1, j+1) & \text{if } j \text{ is even} \end{cases}$$

where $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{4p}$.

Let $G = G^2_{4p^2}$ and $X = \mathrm{Cay}(G, S)$. Then the map defined by $(i,j) \mapsto a^j b^i$ is an isomorphism from $\mathcal{C}^{\pm\varepsilon}(p\,; 4p, 1)$ to X. Since, by [45], the graph $\mathcal{C}^{\pm\varepsilon}(p\,; 4p, 1)$ is symmetric, the lemma holds. ■

5.3 Analysis of tetravalent one-regular graphs of order $4p^2$

Let p be a prime. Then define $C(2,p,2)$ to be a graph with $V(C(2,p,2)) = \mathbb{Z}_4 \times \mathbb{Z}_p$ and adjacencies in $C(2,p,2)$ satisfying the following conditions:

$$\begin{array}{lcl} (0,i) \sim (0,j) & \iff & j-i = \pm 1, \\ (0,i) \sim (1,j) & \iff & j-i = -1, \\ (0,i) \sim (2,j) & \iff & j-i = 1, \\ (1,i) \sim (2,j) & \iff & j-i = \pm 1, \\ (1,i) \sim (3,j) & \iff & j-i = -1, \\ (2,i) \sim (3,j) & \iff & j-i = 1, \\ (3,i) \sim (3,j) & \iff & j-i = \pm 1. \end{array}$$

Let $X = C(2,p,2)$ and let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0,i),(1,i),(2,i),(3,i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$. Observe that for each $j \in \mathbb{Z}_p$, $j \neq i$, the subgraph $X[B_i, B_j]$ induced on the union $B_i \cup B_j$ is not an independent set of vertices if and only if $j = i \pm 1$. Moreover, for each such j we have that $X[B_j, B_{j+1}] \cong 2C_4$, see also Figure 5.1. The following lemma shows that there is no one-regular $\mathbb{Z}_p$-cover of $C(2,p,2)$.

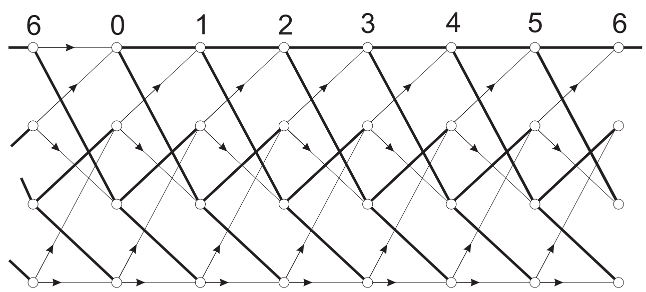

Figure 5.1: A spanning tree in the base graph $C(2,p,2)$ for $p = 7$.

Lemma 5.3.1 *Let Y be a tetravalent one-regular graph of order $4p^2$, $p > 3$ a prime, such that there exists a normal subgroup H of $\mathrm{Aut}(Y)$ of order p. Then Y is not a regular $\mathbb{Z}_p$-cover of the graph $C(2,p,2)$.*

Proof. Let $\mathcal{K} = \{1, \tau_1, \tau_2, \tau_3\}$ be the Klein 4-group acting on $\mathbb{Z}_4$ so that $\tau_1 = (0\,1)(2\,3)$, $\tau_2 = (0\,2)(1\,3)$ and $\tau_3 = (0\,3)(1\,2)$. Let $X = C(2,p,2)$, let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0,i),(1,i),(2,i),(3,i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$, and let K be the kernel of the action of $\mathrm{Aut}(X)$ on $\mathcal{B}$. We shall be sloppy and shall identify restrictions of elements of K to sets B_i by elements of $\mathcal{K}$. For instance, when we say that the restriction γ_i of $\gamma \in K$ to B_i is, for example, τ_1, we mean that $\gamma_i = ((0,i)(1,i))((2,i)(3,i))$. Now, the structure of X indicated in Figure 5.1 implies that the restrictions γ_i must satisfy the following conditions:

$$\gamma_i \in \{1, \tau_1\} \iff \gamma_{i+1} \in \{1, \tau_2\} \quad \forall i \in \mathbb{Z}_p. \tag{5.1}$$

Let the vertices of X be labeled in the following way: $a_i = (0,i)$, $b_i = (1,i)$, $c_i = (2,i)$ and $d_i = (3,i)$. Let $E = \langle \gamma_i \mid i \in \mathbb{Z}_p \rangle$. It is well known, see for instance [101, 130], that $\mathrm{Aut}(X) = E \rtimes \langle \rho, \tau \rangle \cong \mathbb{Z}_2^p \rtimes D_{2p}$ where

$$\rho = (a_0\ a_1\ \ldots\ a_{p-1})(b_0\ b_1\ \ldots\ b_{p-1})(c_0\ c_1\ \ldots\ c_{p-1})(d_0\ d_1\ \ldots\ d_{p-1})$$

and

$$\tau = (a_0)(b_0\ c_0)(d_0)\prod_{i=1}^{p-1}(a_i\ a_{-i})(b_i\ c_{-i})(c_i\ b_{-i})(d_i\ d_{-i}).$$

Now let Y be a tetravalent one-regular graph of order $4p^2$. Assume that $\mathrm{Aut}(Y)$ contains a normal subgroup H isomorphic to $\mathbb{Z}_p$ such that the corresponding quotient graph Y_H is isomorphic to $X = C(2,p,2)$. Then, since the orbits of H form an $\mathrm{Aut}(Y)$-invariant partition, the whole automorphism group $\mathrm{Aut}(Y)$ of Y projects to a subgroup of $\mathrm{Aut}(X)$. On the other hand, the graph Y can be viewed as an H-covering graph (that is, a $\mathbb{Z}_p$-covering) of X, and it can therefore be derived from X through a suitable voltage assignment ζ. To find this voltage assignment fixes the spanning tree T of X as indicated on Figure 5.1.

Let G be the largest subgroup of $\mathrm{Aut}(X)$ which lifts with respect to the natural projection $X \times_\zeta \mathbb{Z}_p \cong Y \to Y_H \cong X$, where ζ is as given in Figure 5.1. Clearly, since Y is arc-transitive, we may assume that $\rho, \tau \in G$. Let F denote the largest subgroup of E which lifts. Then $G = F \rtimes \langle \rho, \tau \rangle$ and thus $|G| = 2p|F|$. We will show that $|F| > 8$. This will then imply that the lift $\bar{G}$ of G is of order $|\bar{G}| = 2p^2|F| > 16p^2$, and consequently that Y is not one-regular.

Since $\rho, \tau \in G$, we have that

$$\text{if } \phi \in F \text{ then } \phi^\rho, \phi^\tau \in F. \tag{5.2}$$

It is convenient to view elements γ in E as vectors in $\mathbb{Z}_4^p$. Namely, we write $\gamma = (e_0, \ldots, e_{p-1})$ where $e_i = s$ if and only if $\gamma_i = \tau_s$ (where $e_i = 0$ means that $\gamma_i = \tau_0 = id$). Note that in this context (5.2) can be interpreted as follows: F is invariant under the "cyclic shift"

$$\phi = (f_0, f_1, \ldots, f_{p-1}) \mapsto (f_{p-1}, f_0, \ldots, f_{p-2}),$$

and under the "reflection around the first entry"

$$\phi = (f_0, f_1, \ldots, f_{p-1}) \mapsto (f'_0, f'_{p-1}, f'_{p-2}, \ldots, f'_2, f'_1),$$

where

$$f'_i = \begin{cases} 0 & , \text{ if } f_i = 0 \\ 1 & , \text{ if } f_i = 2 \\ 2 & , \text{ if } f_i = 1 \\ 3 & , \text{ if } f_i = 3 \end{cases}$$

Now choose $\phi \in F$. By (5.1) the first two components of ϕ can be one of the following pairs: $\phi = (0,0,\ldots)$, $\phi = (0,2,\ldots)$, $\phi = (1,0,\ldots)$, $\phi = (1,2,\ldots)$, $\phi = (2,1,\ldots)$, $\phi = (2,3,\ldots)$, $\phi = (3,1,\ldots)$, or $\phi = (3,3,\ldots)$. Since the lift of G acts arc-transitively on Y the group G must be of order $|G| = 2p|F| \geq 16p$ and thus $|F| \neq 1$.

Suppose first that there exists $\psi \in F$ such that $\psi \notin \{id, (3,3,\ldots,3)\}$. Since ρ is of prime order, the conjugacy class of ψ under $\langle\rho\rangle$ is of size p. But then, by (5.2), we have that $|F| > 8$, which implies that $\bar{G}$ is not acting one-regularly on Y.

Suppose now that $(3,3,\ldots,3)$ belongs to F. Then, since $\langle(3,3,\ldots,3)\rangle \leq F$ is of order 2 and $|G| = 2p|F| = 16p$, we have that there must also exist a non-identity automorphism $\psi \in F$ which is different from $(3,3,\ldots,3)$. But then, as above, the conjugacy class of ψ is of size p, and consequently $|F| > 8$. This shows that $\bar{G}$ is not acting one-regularly on Y, and the proof is completed. ■

By the following lemma there are only two normal one-regular Cayley graphs on the group $G = \langle a,\, b,\, c,\, g \,|\, a^p = b^p = c^2 = g^2 = [a,b] = [c,g] = [a,c] = [b,c] = 1,\, a^g = b,\, b^g = a\rangle$.

Lemma 5.3.2 *Let p be a prime and $G = \langle a,\, b,\, c,\, g \,|\, a^p = b^p = c^2 = g^2 = [a,b] = [c,g] = [a,c] = [b,c] = 1,\, a^g = b,\, b^g = a\rangle$. Then a tetravalent normal Cayley graph X of order $4p^2$ on G is one-regular if and only if it is either isomorphic to*

$$\mathcal{CN}^3_{4p^2} = Cay(G, \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\}) \text{ or to}$$

$$\mathcal{CN}^4_{4p^2} = Cay(G, \{ag,\, b^{\varepsilon}cg,\, b^{-1}g,\, a^{-\varepsilon}cg\}).$$

Moreover, $\mathrm{Aut}(\mathcal{CN}^3_{4p^2}) \cong G \rtimes \mathbb{Z}_2^2$ *and* $\mathrm{Aut}(\mathcal{CN}^4_{4p^2}) \cong G \rtimes \mathbb{Z}_4$.

Proof. Let X be a tetravalent one-regular normal Cayley graph $\mathrm{Cay}(G,S)$ on the group G with respect to the generating set S. Since X is one-regular and normal, the stabilizer $A_1 = \mathrm{Aut}(G,S)$ of the vertex $1 \in G$ is transitive on S, and either $\mathrm{Aut}(G,S) \cong \mathbb{Z}_2^2$ or $\mathrm{Aut}(G,S) \cong \mathbb{Z}_4$. This implies that elements in S are all of the same order.

Observe that G contains elements of order 2, p and $2p$. In particular, elements of the form c, a^ib^jg and a^ib^jcg, where $p \mid i+j$, are of order 2; elements of the form a^ib^j are of order p; and elements of the form a^ib^jc, a^mb^ng and a^mb^ncg, where $p \nmid m+n$, are of order $2p$. In the following, we will show that up to isomorphism, there are only two generating sets of size 4 such that the corresponding Cayley graphs are normal and one-regular.

First, observe that neither four involutions nor two elements of order p can generate G. Moreover, G cannot be generated by the following pairs of elements of order $2p$: $a^{i_1}b^{j_1}c$ and $a^{i_2}b^{j_2}c$, $a^{m_1}b^{n_1}g$ and $a^{m_2}b^{n_2}g$, $a^{m_1}b^{n_1}cg$ and $a^{m_2}b^{n_2}cg$, where $m_i + n_i \neq 0\,(1 \leq i \leq 2)$. Second, $Z(G) = \langle ab,\, c\rangle = \langle ab\rangle \times \langle c\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_2$, and thus $\langle c\rangle$ char G. Also, since $\mathrm{Aut}(G,S)$ is transitive on S, we have that $S \neq \{a^ib^jc,\, a^mb^ng,\, (a^ib^jc)^{-1},\, (a^mb^ng)^{-1}\}$ and

$$S \neq \{a^ib^jc,\, a^mb^ncg,\, (a^ib^jc)^{-1},\, (a^mb^ncg)^{-1}\},$$

where $m+n \neq 0$. Now suppose that G is generated by

$$S_0 = \{a^ib^jg,\, a^{m'}b^{n'}cg,\, (a^ib^jg)^{-1},\, (a^{m'}b^{n'}cg)^{-1}\},$$

where $p \nmid i+j$ and $p \nmid m'+n'$.

Case 1. $\mathrm{Aut}(G, S_0) = \langle\alpha\rangle \times \langle\beta\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where α and β are such that $a^\alpha = a^{i_1}b^{j_1}$, $b^\alpha = a^{j_1}b^{i_1}$, $c^\alpha = c$, $g^\alpha = a^x b^{-x} cg$, $a^\beta = a^{i_2}b^{j_2}$, $b^\beta = a^{j_2}b^{i_2}$, $c^\beta = c$ and $g^\beta = a^y b^{-y} g$.

Subcase 1.1. Let $i = j$.

Since $ab \in Z(G)$, G can be generated by S_0 if and only if $m' \neq n'$. Now take an automorphism σ of G such that

$$a^\sigma = a^i,\ b^\sigma = b^i,\ c^\sigma = c,\ g^\sigma = g.$$

Then $(abg)^\sigma = a^i b^i g$, and hence

$$\begin{aligned} S &= {S_0}^{\sigma^{-1}} = \{abg,\, a^m b^n cg,\, (abg)^{-1},\, (a^m b^n cg)^{-1}\} \\ &= \{abg,\, a^m b^n cg,\, a^{-1}b^{-1}g, a^{-n}b^{-m}cg\}, \end{aligned}$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$. Moreover, it can be easily seen that $m \neq n$.

Suppose first that $(abg)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = abg$, $(a^{-1}b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = a^{-1}b^{-1}g$. It follows that either $m + n = 2$ or $m + n = -2$. If $m + n = 2$ then, since $m \neq n$, we have that $m \neq 1$ and

$$a^\alpha = b,\ b^\alpha = a,\ c^\alpha = c,\ g^\alpha = a^{m-1}b^{1-m}cg.$$

If $m + n = -2$, then since $m \neq n$, we have $n \neq -1$ and

$$a^\alpha = a^{-1},\ b^\alpha = b^{-1},\ c^\alpha = c,\ g^\alpha = a^{-1-n}b^{1+n}cg.$$

Suppose now that $(abg)^\beta = a^{-1}b^{-1}g$. Then $(a^{-1}b^{-1}g)^\beta = abg$, $(a^m b^n cg)^\beta = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\beta = a^m b^n cg$. By a similar argument as above, one can get that

$$a^\beta = b^{-1},\ b^\beta = a^{-1},\ c^\beta = c,\ g^\beta = g.$$

Consequently, either $S_0 = S_1 = \{abg,\, a^m b^{2-m} cg,\, a^{-1}b^{-1}g,\, a^{m-2}b^{-m}cg\}$, where $m \neq 1$, or $S_0 = S_2 = \{abg,\, a^{-2-n}b^n cg,\, a^{-1}b^{-1}g,\, a^{-n}b^{n+2}cg\}$, where $n \neq -1$. In addition, replacing $-n$ with m, it can be seen that $S_2 = S_1$. Moreover, it can be easily seen that G can indeed be generated by S_1. Namely, since $(abg)^p = g$ we have $g, ab \in \langle S_1\rangle$. Then, since $a^m b^{2-m} cg \in \langle S_1\rangle$, we get that $a^m b^{2-m} c \in \langle S_1\rangle$. Further, since $(a^m b^{2-m} c)^p = c$, also $c, a^m b^{2-m} \in \langle S_1\rangle$. Now, since $a^m b^{2-m} = a^m b^m b^{2-2m}$, $m \neq 1$, and $ab \in \langle S_1\rangle$, we get that $b^{2-2m} \in \langle S_1\rangle$. Finally, the fact that $b^g = a$ implies that $G = \langle S_1\rangle$.

Subcase 1.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^\sigma = a^i b^j$, $b^\sigma = a^j b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(ag)^\sigma = a^i b^j g$ and

$$S = {S_0}^{\sigma^{-1}} = \{ag,\, a^m b^n cg,\, (ag)^{-1},\, (a^m b^n cg)^{-1}\} = \{ag,\, a^m b^n cg,\, b^{-1}g,\, a^{-n}b^{-m}cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = ag$, $(b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = b^{-1}g$. In addition, either $m+n = 1$ or $m+n = -1$. If $m+n = 1$ then, since $\{ag,\, acg,\, b^{-1}g,\, b^{-1}cg\}$ cannot generate G, we have that $m \neq 1$. Thus α

is mapping according to the rule: $a^\alpha = b$, $b^\alpha = a$, $c^\alpha = c$, and $g^\alpha = a^m b^{-m} cg$. If on the other hand $m+n=-1$ then, since $\{ag,\, b^{-1}cg,\, b^{-1}g,\, acg\}$ cannot generate G, we have that $n \neq -1$, and hence α is mapping according to the rule: $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$, $c^\alpha = c$, and $g^\alpha = a^{-n}b^n cg$.

Suppose now that $(ag)^\beta = b^{-1}g$. Then we have that $(b^{-1}g)^\beta = ag$, $(a^m b^n cg)^\beta = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\beta = a^m b^n cg$. Whenever $m+n=1$ or $m+n=-1$, we can get that β is mapping according to the rule: $a^\beta = b^{-1}$, $b^\beta = a^{-1}$, $c^\beta = c$, and $g^\beta = g$. Thus, we can conclude that either $S_0 = S_3 = \{ag,\, a^m b^{1-m}cg,\, b^{-1}g,\, a^{m-1}b^{-m}cg\}$, where $m \neq 1$, or $S_0 = S_4 = \{ag,\, a^{-n-1}b^n cg,\, b^{-1}g, a^{-n}b^{n+1}cg\}$, where $n \neq -1$. Moreover, replacing $-n$ with m, it can be easily seen that $S_4 = S_3$. Also, since $(ag)^2 = ab$ and $aga^m b^{1-m}cg = a^{2-m}b^m c$, we get that $c, a^{2-m}b^m \in \langle S_3 \rangle$. Further, the facts that $a^{2-m}b^m = a^{2-2m}a^m b^m$, $m \neq 1$ and $ab \in \langle S_3 \rangle$ combined together imply that $a^{2-2m} \in \langle S_3 \rangle$. Since $ag \in \langle S_3 \rangle$, it follows that $g \in \langle S_3 \rangle$. Finally, since $a^g = b$, G is indeed generated by S_3.

Now considering the automorphism γ of G defined by $a^\gamma = a^{\frac{1}{2}}$, $b^\gamma = b^{\frac{1}{2}}$, $c^\gamma = c$, and $g^\gamma = a^{\frac{1}{2}}b^{-\frac{1}{2}}g$ we get that ${S_1}^\gamma = \{ag,\, a^{\frac{m+1}{2}}b^{1-\frac{m+1}{2}}cg,\, b^{-1}g,\, a^{\frac{m+1}{2}-1}b^{-\frac{m+1}{2}}cg\}$, where $m \neq 1$. Thus we only need to consider the generating set $S_3 = \{ag,\, a^m b^{1-m}cg,\, b^{-1}g,\, a^{m-1}b^{-m}cg\}$, where $m \neq 1$.

Case 2. $\mathrm{Aut}(G, S_0) = \langle \alpha \rangle \cong \mathbb{Z}_4$, where α is such that $a^\alpha = a^{i_1}b^{j_1}$, $b^\alpha = a^{j_1}b^{i_1}$, $c^\alpha = c$, and $g^\alpha = a^x b^{-x} cg$.

Subcase 2.1. Let $i = j$.

Since $ab \in Z(G)$, G can be generated by S_0 (where $p \nmid i$ and $p \nmid m' + n'$) if and only if $m' \neq n'$. Now take an automorphism σ of G such that $a^\sigma = a^i$, $b^\sigma = b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(abg)^\sigma = a^i b^i g$, and consequently

$$\begin{aligned} S &= {S_0}^{\sigma^{-1}} = \{abg,\, a^m b^n cg,\, (abg)^{-1},\, (a^m b^n cg)^{-1}\} \\ &= \{abg,\, a^m b^n cg,\, a^{-1}b^{-1}g, a^{-n}b^{-m}cg\}, \end{aligned}$$

where $a^m b^n cg = (a^{m'}b^{n'}cg)^{\sigma^{-1}}$, and $m \neq n$.

Suppose first that $(abg)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = a^{-1}b^{-1}g$, $(a^{-1}b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, $(a^{-n}b^{-m}cg)^\alpha = abg$. Hence either $m+n=\omega$ or $m+n=-\omega$, where $\omega^2 = -4$. If $m+n=\omega$ then since $m \neq n$, we have that $m \neq \frac{\omega}{2}$. It follows that $a^\alpha = a^i b^{\frac{\omega}{2}-i}$, $b^\alpha = a^{\frac{\omega}{2}-i}b^i$, $c^\alpha = c$, and $g^\alpha = a^{m-\frac{\omega}{2}}b^{\frac{\omega}{2}-m}cg$, where $i = \frac{(m+1)\omega+2-2m}{2(2m-\omega)}$. If on the other hand $m+n=-\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^\alpha = a^i b^{-\frac{\omega}{2}-i}$, $b^\alpha = a^{-\frac{\omega}{2}-i}b^i$, $c^\alpha = c$, and $g^\alpha = a^{-\frac{\omega}{2}-n}b^{\frac{\omega}{2}+n}cg$, where $i = \frac{2-2n-(n+1)\omega}{2(2n+\omega)}$.

Suppose now that $(abg)^\alpha = a^{-n}b^{-m}cg$. Then $(a^{-n}b^{-m}cg)^\alpha = a^{-1}b^{-1}g$, $(a^{-1}b^{-1}g)^\alpha = a^m b^n cg$, and $(a^m b^n cg)^\alpha = abg$. Hence, either $m+n=\omega$ or $m+n=-\omega$, where $\omega^2 = -4$. If $m+n=\omega$ then, since $m \neq n$, we have that $m \neq \frac{\omega}{2}$, and thus $a^\alpha = a^i b^{-\frac{\omega}{2}-i}$, $b^\alpha = a^{-\frac{\omega}{2}-i}b^i$, $c^\alpha = c$, and $g^\alpha = a^{m-\frac{\omega}{2}}b^{\frac{\omega}{2}-m}cg$, where $i = \frac{(1-m)\omega-2m-2}{2(2m-\omega)}$. If however $m+n=-\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^\alpha = a^i b^{\frac{\omega}{2}-i}$, $b^\alpha = a^{\frac{\omega}{2}-i}b^i$, $c^\alpha = c$, and $g^\alpha = a^{-\frac{\omega}{2}-n}b^{\frac{\omega}{2}+n}cg$, where $i = \frac{(n-1)\omega-2n-2}{2(2n+\omega)}$.

We can conclude that either $S_0 = S_5 = \{abg,\, a^m b^{\omega-m}cg,\, a^{-1}b^{-1}g,\, a^{m-\omega}b^{-m}cg\}$, where $m \neq \frac{\omega}{2}$, or $S_0 = S_6 = \{abg,\, a^{-\omega-n}b^n cg,\, a^{-1}b^{-1}g,\, a^{-n}b^{n+\omega}cg\}$, where $n \neq -\frac{\omega}{2}$. Moreover, replacing $-n$ with m, it can be easily seen that $S_5 = S_6$. Also, the group G is indeed generated by S_5. Namely, since $(abg)^p = g$ we have that $g, ab \in \langle S_5\rangle$. Further, since $a^m b^{\omega-m}cg \in \langle S_5\rangle$, also $a^m b^{\omega-m}c \in \langle S_5\rangle$, and the fact that $(a^m b^{\omega-m}c)^p = c$ implies that $c, a^m b^{\omega-m} \in \langle S_5\rangle$. Finally, since $a^m b^{\omega-m} = a^m b^m b^{\omega-2m}$, $m \neq \frac{\omega}{2}$, and $ab \in \langle S_5\rangle$, it follows that $b^{\omega-2m} \in \langle S_5\rangle$. Now this fact and $b^g = a$ combined together imply that $G = \langle S_5\rangle$.

Subcase 2.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^\sigma = a^i b^j$, $b^\sigma = a^j b^i$, $c^\sigma = c$, and $g^\sigma = g$. Then $(ag)^\sigma = a^i b^j g$, and consequently

$$S = {S_0}^{\sigma^{-1}} = \{ag,\, a^m b^n cg,\, (ag)^{-1},\, (a^m b^n cg)^{-1}\} = \{ag,\, a^m b^n cg,\, b^{-1}g,\, a^{-n}b^{-m}cg\},$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^\alpha = a^m b^n cg$. Then $(a^m b^n cg)^\alpha = b^{-1}g$, $(b^{-1}g)^\alpha = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^\alpha = ag$. Also, either $m+n = \varepsilon$ or $m+n = -\varepsilon$, where $\varepsilon^2 = -1$. If $m+n = \varepsilon$ then, since $\{ag,\, a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}}cg,\, b^{-1}g,\, a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}}cg\}$ cannot generate G (namely, for $\varphi \in \mathrm{Aut}(G)$ such that $a^\varphi = a^2$, $b^\varphi = b^2$, $c^\varphi = c$, and $g^\varphi = a^{-1}bg$ we have $\{ag,\, a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}}cg,\, b^{-1}g,\, a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}}cg\}^\varphi = \{abg,\, a^\varepsilon b^\varepsilon cg,\, a^{-1}b^{-1}g,\, a^{-\varepsilon}b^{-\varepsilon}cg\}$), we have that $m \neq \frac{\varepsilon+1}{2}$. It follows that

$$a^\alpha = a^i b^{\varepsilon-i},\ b^\alpha = a^{\varepsilon-i}b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{m-i}b^{i-m}cg,$$

where $i = \frac{m\varepsilon-m+1}{2m-\varepsilon-1}$. If on the other hand $m+n = -\varepsilon$ then, since G cannot be generated by $\{ag,\, a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}}cg, b^{-1}g,\, a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}}cg\}$, we have that $n \neq -\frac{\varepsilon+1}{2}$, and so

$$a^\alpha = a^i b^{-\varepsilon-i},\ b^\alpha = a^{-\varepsilon-i}b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{-\varepsilon-i-n}b^{\varepsilon+i+n}cg,$$

where $i = -\frac{(n+1)\varepsilon+n}{2n+\varepsilon+1}$.

Suppose now that $(ag)^\alpha = a^{-n}b^{-m}cg$. Then $(a^{-n}b^{-m}cg)^\alpha = b^{-1}g$, $(b^{-1}g)^\alpha = a^m b^n cg$, and $(a^m b^n cg)^\alpha = ag$. Also, either $m+n = \varepsilon$ or $m+n = -\varepsilon$, where $\varepsilon^2 = -1$. If $m+n = \varepsilon$ then, since $\{ag,\, a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}}cg,\, b^{-1}g,\, a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}}cg\}$ cannot generate G, we have that $m \neq \frac{\varepsilon+1}{2}$, and thus

$$a^\alpha = a^i b^{-\varepsilon-i},\ b^\alpha = a^{-\varepsilon-i}b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{m-\varepsilon-i}b^{\varepsilon+i-m}cg,$$

where $i = \frac{\varepsilon(1-m)-m}{2m-\varepsilon-1}$. If however $m+n = -\varepsilon$ then, since $\{ag,\, a^{\frac{1-\varepsilon}{2}} b^{-\frac{\varepsilon+1}{2}}cg,\, b^{-1}g,\, a^{\frac{\varepsilon+1}{2}} b^{\frac{\varepsilon-1}{2}}cg\}$ cannot generate G, we have that $n \neq -\frac{\varepsilon+1}{2}$, and consequently

$$a^\alpha = a^i b^{\varepsilon-i},\ b^\alpha = a^{\varepsilon-i}b^i,\ c^\alpha = c, \text{ and } g^\alpha = a^{-i-n}b^{i+n}cg,$$

where $i = \frac{n(\varepsilon-1)-1}{2n+\varepsilon+1}$.

We can conclude that either $S_0 = S_7 = \{ag,\, a^m b^{\varepsilon-m}cg,\, b^{-1}g,\, a^{m-\varepsilon}b^{-m}cg\}$, where $m \neq \frac{\varepsilon+1}{2}$, or $S_0 = S_8 = \{ag,\, a^{-n-\varepsilon}b^n cg,\, b^{-1}g, a^{-n}b^{n+\varepsilon}cg\}$, where $n \neq -\frac{\varepsilon+1}{2}$. Further, replacing $-n$ with m, one can see that $S_8 = S_7$. That G is indeed generated by S_7 can be seen in the following way. Since $(ag)^2 = ab$ and $aga^m b^{\varepsilon-m}cg =$

$a^{\varepsilon+1-m}b^m c$, we have that $c, a^{\varepsilon+1-m}b^m \in \langle S_7\rangle$. Then, since $a^{\varepsilon+1-m}b^m = a^{\varepsilon+1-2m}a^m b^m$, $m \neq \frac{\varepsilon+1}{2}$, and $ab \in \langle S_7\rangle$, we get that $a^{\varepsilon+1-2m} \in \langle S_7\rangle$. Finally, since $ag \in \langle S_7\rangle$, it follows that also $g \in \langle S_7\rangle$. Now the fact that $a^g = b$ implies that $G = \langle S_7\rangle$.

Now considering the automorphism γ of G defined by

$$a^\gamma = a^{\frac{1}{2}},\, b^\gamma = b^{\frac{1}{2}},\, c^\gamma = c, \text{ and } g^\gamma = a^{\frac{1}{2}}b^{-\frac{1}{2}}g,$$

gives that ${S_5}^\gamma = \{ag,\, a^{\frac{m+1}{2}}b^{\frac{\omega}{2}-\frac{m+1}{2}}cg,\, b^{-1}g,\, a^{\frac{m+1}{2}-\frac{\omega}{2}}b^{-\frac{m+1}{2}}cg\}$, where $m \neq \frac{\omega}{2}$. So we only need to consider the generating set $S_7 = \{ag,\, a^m b^{\varepsilon-m}cg,\, b^{-1}g,\, a^{m-\varepsilon}b^{-m}cg\}$, where $m \neq \frac{\varepsilon+1}{2}$ and $\varepsilon^2 = -1$. Observe also, that this implies that $p \equiv 1 \pmod 4$.

We have proved that when $\mathrm{Aut}(G, S_0) \cong \mathbb{Z}_2\times\mathbb{Z}_2$ there always exists an automorphism σ of G such that ${S_0}^\sigma = S = \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\}$. Moreover, $\mathrm{Aut}(G, S) = \langle\alpha,\, \beta\rangle$, where

$$a^\alpha = b,\, b^\alpha = a,\, c^\alpha = c,\, g^\alpha = cg\,, a^\beta = b^{-1},\, b^\beta = a^{-1},\, c^\beta = c, \text{ and } g^\beta = g.$$

On the other hand when $\mathrm{Aut}(G, S_0) \cong \mathbb{Z}_4$ there always exists an automorphism δ of G such that ${S_0}^\delta = S = \{ag,\, b^\varepsilon cg,\, b^{-1}g,\, a^{-\varepsilon}cg\}$. Moreover, in this case $\mathrm{Aut}(G, S) = \langle\rho\rangle$, where

$$a^\rho = a^{\frac{\varepsilon-1}{2}}b^{\frac{\varepsilon+1}{2}},\, b^\rho = a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}},\, c^\rho = c, \text{ and } g^\rho = a^{\frac{1-\varepsilon}{2}}b^{\frac{\varepsilon-1}{2}}cg.$$

Observe also that the following hold:

(1) If $\varepsilon^2 = -1$ then $\{ag,\, b^\varepsilon cg,\, b^{-1}g,\, a^{-\varepsilon}cg\}^\tau = \{ag,\, b^{-\varepsilon}cg,\, b^{-1}g,\, a^\varepsilon cg\}$, where τ is an automorphism of G mapping according to the rule $a^\tau = b^{-\varepsilon}$, $b^\tau = a^{-\varepsilon}$, $c^\tau = c$, and $g^\tau = cg$.

(2) Since $agbcg = a^2c$, $(a^2c)^2 = a^4$, $(a^2c)^p = c$, $a^g = b$ and p is an odd prime, we can conclude that $\langle\{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\}\rangle = \langle ag,\, bcg\rangle = \langle a,\, b,\, c,\, g\rangle = G$.

(3) Let $\varepsilon^2 = -1$. Then $agb^\varepsilon cg = a^{1+\varepsilon}c$, $(a^{1+\varepsilon}c)^2 = a^{2(1+\varepsilon)}$, and $(a^{1+\varepsilon}c)^p = c$. Since p is an odd prime and $a^g = b$, we can conclude that $\langle\{ag,\, b^\varepsilon cg,\, b^{-1}g,\, a^{-\varepsilon}cg\}\rangle = \langle ag,\, b^\varepsilon cg\rangle = \langle a,\, b,\, c,\, g\rangle = G$.

To finish the proof, it is sufficient to prove that the graphs $\mathrm{Cay}(G, \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\})$ and $\mathrm{Cay}(G, \{ag,\, b^\varepsilon cg,\, b^{-1}g,\, a^{-\varepsilon}cg\})$ are normal Cayley graphs.

First, let $X = \mathrm{Cay}(G, \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\})$, let $A = \mathrm{Aut}(X)$ and let A_1^* be the subgroup of the stabilizer A_1 fixing the set $S = \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\}$ pointwise. Then, since the 2-arc $(1, ag, a^{-1}bc)$ lies on a 6-cycle but the 2-arc $(1, ag, ab)$ does not, one can see that A_1^* fixes every vertex at distance 2 from 1 in X (see also Figure 5.2). By connectivity of X and transitivity of A on $V(X)$, A_1^* fixes every vertex in X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $\mathrm{Aut}(G, S) = \mathbb{Z}_2^2 \leq A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_2^2, D_8, A_4, S_4\}$. If $A_1 \in \{A_4, S_4\}$ then there exists a permutation δ in A_1 of order 3. We can, without loss of generality, assume that δ fixes ag, and cyclicly permutates the other three neighbors of 1. But, however, considering the images of the vertices at distance 2 from 1, one can see that this is impossible (see Figure 5.2). If $A_1 = D_8$ then we may, without loss of generality, assume that there

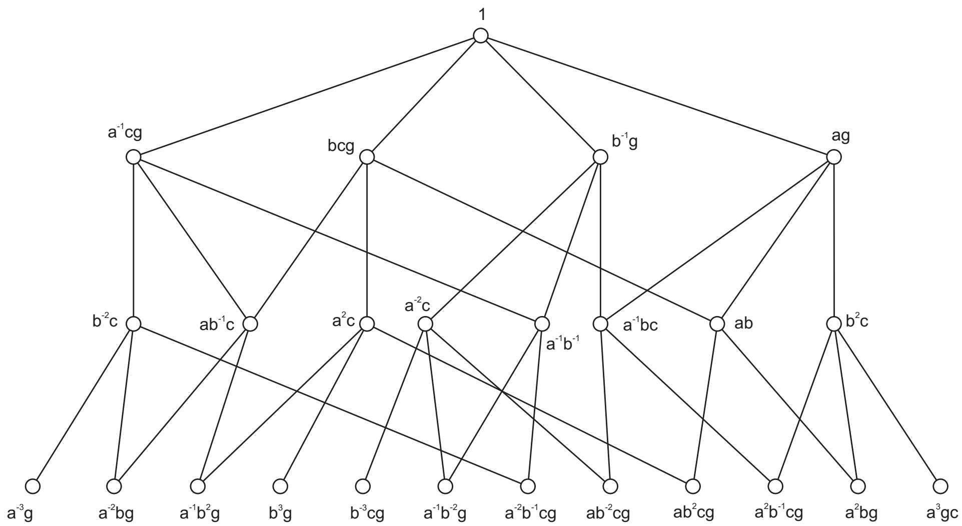

Figure 5.2: A local structure of the graph $\mathcal{CN}^3_{4p^2}$.

exists an involution $\gamma \in A_1$ such that $\gamma \notin \mathrm{Aut}(G,S)$, $(ag)^\gamma = ag$, $(b^{-1}g)^\gamma = b^{-1}g$, $(bcg)^\gamma = a^{-1}cg$ and $(a^{-1}cg)^\gamma = bcg$. However, ab is a common neighbor of ag and bcg in X, but there is no common neighbor of ag and $a^{-1}cg$, and thus this case cannot occur. It follows that $A_1 = \mathrm{Aut}(G,S) = \mathbb{Z}_2^2$, and so X is a normal one-regular Cayley graph as claimed.

Now let $X = \mathrm{Cay}(G, \{ag,\, b^\varepsilon cg,\, b^{-1}g,\, a^{-\varepsilon}cg\})$, let $A = \mathrm{Aut}(X)$ and let A_1^* be the subgroup of the stabilizer A_1 fixing S pointwise. Then considering 6-cycles passing through the vertex 1 one can see that A_1^* fixes all the vertices at distance 2 from 1 in X (see also Figure 5.3). Then, connectivity and vertex-transitivity of X combined together imply that A_1^* fixes every vertex of X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \le S_4$. Since $\mathrm{Aut}(G,S) \cong \mathbb{Z}_4 \lnsim A_1 \le S_4$, we have that $A_1 \in \{\mathbb{Z}_4, D_8, S_4\}$. If $A_1 \in \{D_8, S_4\}$ then, without loss of generality, we may assume that there exists an involution $\zeta \in A_1$ such that $\zeta \notin \mathrm{Aut}(G,S)$, $(ag)^\zeta = ag$, $(b^{-1}g)^\zeta = b^{-1}g$, $(b^\varepsilon cg)^\zeta = a^{-\varepsilon}cg$, and $(a^{-\varepsilon}cg)^\zeta = (b^\varepsilon cg)$. Since there is no 6-cycle passing through $b^{-1}g$, 1, ag and ab, it follows that ζ fixes ab. On the other hand, since ζ normalizes a Sylow p-subgroup P of G ($P \trianglelefteq A$, see Theorem 5.4.1), we have that $(xy)^\zeta = 1^{R(xy)\zeta} = 1^{\zeta^{-1}(R(x)R(y))\zeta} = 1^{R(x)^\zeta R(y)^\zeta} = R(x)^\zeta R(y)^\zeta = 1^{R(x)^\zeta} 1^{R(y)^\zeta} = x^\zeta y^\zeta$, for every $x, y \in \langle a, b\rangle$. In other words, ζ induces an automorphism on $\langle a, b\rangle$. Thus, ζ fixes $\langle ab\rangle$ pointwise, and, in particular, ζ fixes both $a^\varepsilon b^\varepsilon$ and $a^{-\varepsilon}b^{-\varepsilon}$, a contradiction. This means that $A_1 = \mathrm{Aut}(G,S) = \mathbb{Z}_4$, and thus X is a normal one-regular Cayley graph as claimed. ■

Lemma 5.3.3 $\mathcal{CA}^1_{4p^2} \cong \mathcal{CN}^3_{4p^2}$.

Proof. Let $G^1_{4p^2} = \langle a\,, b \,|\, a^{4p} = b^p = 1,\, ab = ba\rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$ and let $G^3_{4p^2} = \langle a,\, b,\, c,\, g |\, a^p = b^p = c^2 = g^2 = [a,b] = [c,g] = [a,c] = [b,c] = 1,\, a^g = b,\, b^g = a\rangle$.

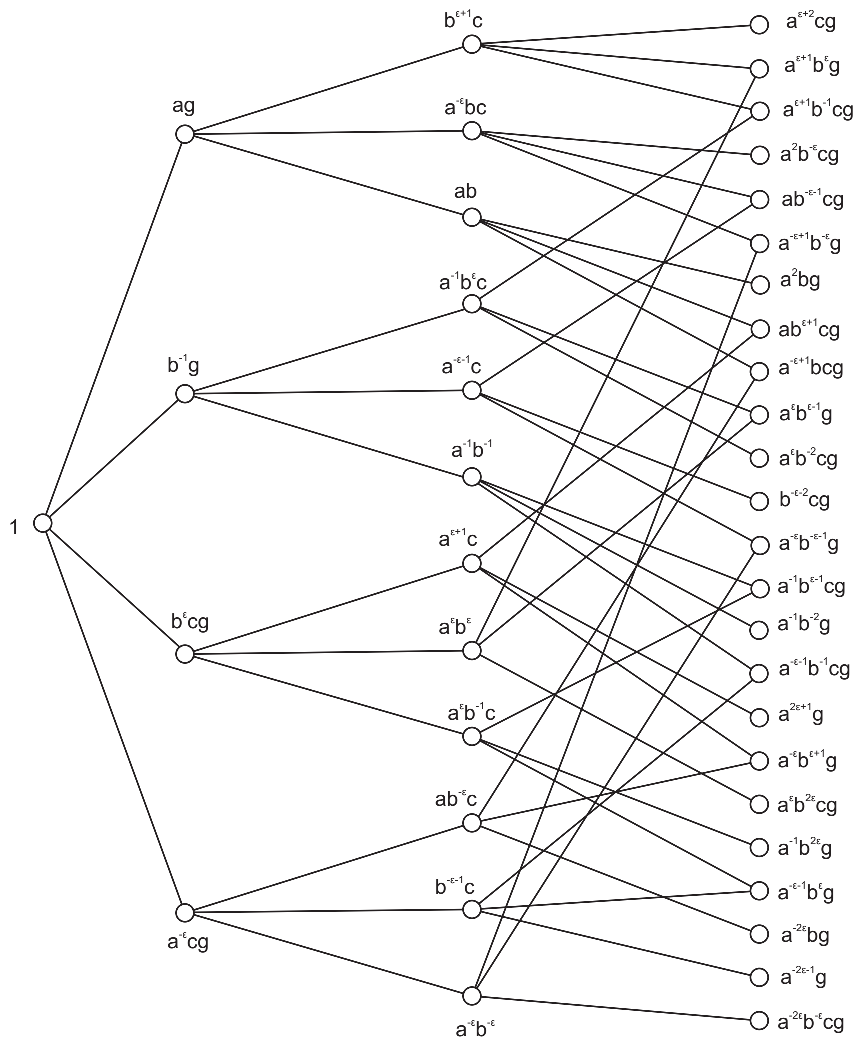

Figure 5.3: A local structure of the graph $\mathcal{CN}^4_{4p^2}$.

Then the automorphism group of $\mathcal{CN}^3_{4p^2} = \mathrm{Cay}(G^3_{4p^2}, \{ag,\, bcg,\, b^{-1}g,\, a^{-1}cg\})$, is equal to $\mathrm{Aut}(\mathcal{CN}^3_{4p^2}) = R(G^3_{4p^2}) \rtimes A_1 = R(G^3_{4p^2}) \rtimes \langle \alpha,\, \beta\rangle \cong G^3_{4p^2} \rtimes \mathbb{Z}_2^2$, where $a^\alpha = b$, $b^\alpha = a$, $c^\alpha = c$, $g^\alpha = cg$, $a^\beta = b^{-1}$, $b^\beta = a^{-1}$, $c^\beta = c$, $g^\beta = g$.

Let $H = \langle R(ag)\alpha,\, R(b)\rangle$. Then it is easy to see that $H = \langle R(ag)\alpha\rangle \times \langle R(b)\rangle \cong G^1_{4p^2}$. Since $H_1 \leq A_1 = \langle \alpha,\, \beta\rangle \cong \mathbb{Z}_2^2$ and subgroups of order 4 in H are cyclic, we have that $H_1 < A_1$. Moreover, since $(R(ag)\alpha)^{2p}$ is a unique element of order 2 in H and $1^{(R(ag)\alpha)^{2p}} \neq 1$, we have that $H_1 \notin \{\langle\alpha\rangle, \langle\beta\rangle, \langle\alpha\beta\rangle\}$. Thus $H_1 = 1$, that is, H is a regular subgroup of $\mathrm{Aut}(\mathcal{CN}^3_{4p^2})$. Now Proposition 5.1.1 and Example 5.2.4 combined together imply that $\mathcal{CA}^1_{4p^2} \cong \mathcal{CN}^3_{4p^2}$. ■

Lemma 5.3.4 $\mathcal{CN}^2_{4p^2} \cong \mathcal{CN}^4_{4p^2}$.

Proof. Let $G^2_{4p^2} = \langle a\,, b \,|\, a^{4p} = b^p = 1\,, a^{-1}ba = b^{\varepsilon}, \varepsilon^2 \equiv -1(\mod p)\rangle$, and let $G^3_{4p^2} = \langle a,\, b,\, c,\, g |\, a^p = b^p = c^2 = g^2 = [a,b] = [c,g] = [a,c] = [b,c] = 1,\, a^g = b,\, b^g = a\rangle$. Let 4^{-1} be the inverse of 4 in $\mathbb{Z}_p$ and let $r = 4^{-1}(\varepsilon - 1)$. Observe that $8r(\varepsilon + 1) + 4 \equiv 0 \pmod{4p}$ and that $4r \neq \varepsilon - 1$ in $\mathbb{Z}_{4p}$.

Now define a map α from the vertex set of $\mathcal{CN}^4_{4p^2} = \mathrm{Cay}(G^3_{4p^2}, \{ag,\, b^{\varepsilon}cg,\, b^{-1}g,\, a^{-\varepsilon}cg\})$ to the vertex set of $\mathcal{CN}^2_{4p^2} = \mathrm{Cay}(G^2_{4p^2}, \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\})$ in the following way:

$$\begin{array}{rcl} a^ib^j & \mapsto & a^{4r(i-j)}b^{i+j} \\ a^ib^jc & \mapsto & a^{4r(i-j+\varepsilon+1)+2}b^{i+j} \\ a^ib^jg & \mapsto & a^{4r(j-i+1)+1}b^{i+j} \\ a^ib^jgc & \mapsto & a^{4r(j-i-\varepsilon)-1}b^{i+j} \end{array}$$

where c and g are involutions in $G^3_{4p^2}$. Then

$$\begin{array}{rcl} (a^ib^j, ag\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^{j+1}b^ig)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-j-1+1)+1}b^{i+j+1}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)+1}b^{i+j+1}) \\ & = & (a^{4r(i-j)}b^{i+j}, ab\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, b^{\varepsilon}cg\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^jb^{i+\varepsilon}gc)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i+\varepsilon-j-\varepsilon)-1}b^{i+j+\varepsilon}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)-1}b^{i+j+\varepsilon}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{-1}b^{\varepsilon}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, b^{-1}g\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^jb^{i-1}g)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-1-j+1)+1}b^{i-1+j}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)+1}b^{i-1+j}) \\ & = & (a^{4r(i-j)}b^{i+j}, ab^{-1}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^ib^j, a^{-\varepsilon}cg\cdot a^ib^j)^{\alpha} & = & (a^ib^j, a^{j-\varepsilon}b^igc)^{\alpha} = (a^{4r(i-j)}b^{i+j}, a^{4r(i-j+\varepsilon-\varepsilon)-1}b^{i+j-\varepsilon}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{4r(i-j)-1}b^{i+j-\varepsilon}) \\ & = & (a^{4r(i-j)}b^{i+j}, a^{-1}b^{-\varepsilon}\cdot a^{4r(i-j)}b^{i+j}). \end{array}$$

Similarly, it can be checked that for any edge $(u, s\cdot u)$, we have that $(u, s\cdot u)^{\alpha} = (v, \bar{s}\cdot v)$, where $u \in \{a^ib^jc, a^ib^jg, a^ib^jgc\}$, $v \in \{a^{4r(i-j+\varepsilon+1)+2}b^{i+j}, a^{4r(j-i+1)+1}b^{i+j}, a^{4r(j-i-\varepsilon)-1}b^{i+j}\}$, $s \in \{ag,\, b^{\varepsilon}cg,\, b^{-1}g,\, a^{-\varepsilon}cg\}$, and $\bar{s} \in \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\}$. From this it follows that α is an isomorphism from $\mathcal{CN}^2_{4p^2}$ to $\mathcal{CN}^4_{4p^2}$. The details are omitted. ■

Lemma 5.3.5 *The graphs $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS}2(4,3,(0\ 1){:}(1\ 2))$, $\mathcal{CA}^i_{4p^2}$, $i \in \{0,1\}$, $\mathcal{CN}^2_{4p^2}$, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$, are pairwise non-isomorphic.*

Proof. First, by the remark subsequent to Example 5.2.1, the graph $\mathcal{BW}_{12}(5,1,5)$ is not isomorphic to any of the other graphs listed in the lemma. Next, Example 5.2.2 shows that $\mathcal{GPS}2(4,3,(0\ 1){:}(1\ 2))$ is not isomorphic to any of the other graphs listed in the lemma. Then, since the automorphism group of $\mathcal{CA}^0_{4p^2}$ has a cyclic Sylow p-subgroup, $\mathcal{CA}^0_{4p^2}$ is not isomorphic to $\mathcal{CA}^1_{4p^2}$ and $\mathcal{CN}^2_{4p^2}$. Also, Example 5.2.4 and Lemmas 5.3.3 and 5.3.4 combined together show that $\mathcal{CA}^1_{4p^2}$ and

$\mathcal{CN}^2_{4p^2}$ are not isomorphic. Namely, the stabilizer of a vertex in $\mathcal{CA}^1_{4p^2}$ is isomorphic to $\mathbb{Z}_2^2$ whereas the stabilizer of a vertex in $\mathcal{CN}^2_{4p^2}$ is isomorphic to $\mathbb{Z}_4$. Finally, since the automorphism groups of both $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ have a minimal normal Sylow p-subgroup and the automorphism groups of $\mathcal{CA}^1_{4p^2}$, $\mathcal{CN}^2_{4p^2}$, do not have a minimal normal Sylow p-subgroup, we have that none of $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathcal{CA}^1_{4p^2}$, $\mathcal{CN}^2_{4p^2}$. Moreover, since the automorphism groups of both $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ have an elementary abelian Sylow p-subgroup and the automorphism group of $\mathcal{CA}^0_{4p^2}$ has a cyclic Sylow p-subgroup, which follows that none of $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathcal{CA}^0_{4p^2}$. The result now follows from the fact that the stabilizer of a vertex in $\mathcal{NC}^0_{4p^2}$ is isomorphic to $\mathbb{Z}_2^2$ whereas the stabilizer of a vertex in $\mathcal{NC}^1_{4p^2}$ is isomorphic to $\mathbb{Z}_4$ (see [45, Lemmas 8.4 and 8.7] and Lemma 5.1.6). ■

5.4 The classification

X	$\lvert V(X)\rvert$	$\mathrm{Aut}(X)$	*References*
$\mathcal{BW}_{12}(5,1,5)$	36	$G_{36} \rtimes \mathbb{Z}_2^2$	Example 5.2.1
$\mathcal{GPS}2(4,3,(0\ 1){:}(1\ 2))$	36	$\lvert\mathrm{Aut}(X)\rvert = 144$	Example 5.2.2
$\mathcal{NC}^0_{4p^2}$	$4p^2$, $p>7$, $p \equiv \pm1 \pmod 8$	given in [45, Lemma 8.4]	Lemma 5.1.6
$\mathcal{NC}^1_{4p^2}$	$4p^2, p>7$, or $p \equiv 1$ or $3 \pmod 8$	given in [45, Lemma 8.7]	Lemma 5.1.6
$\mathcal{CA}^0_{4p^2}$	$4p^2, p \equiv 1 \pmod 4$	$(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	Example 5.2.3
$\mathcal{CA}^1_{4p^2}$	$4p^2, p>2$	$(\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$	Example 5.2.4
$\mathcal{CN}^2_{4p^2}$	$4p^2, p \equiv 1 \pmod 4$	$G^3_{4p^2} \rtimes \mathbb{Z}_4$	Lemmas 5.3.2 and 5.2.6

Table 5.2: Tetravalent one-regular graphs of order $4p^2$.

We are now ready to state the main theorem of this chapter.

Theorem 5.4.1 *Let p be a prime. Then a tetravalent graph X of order $4p^2$ is one-regular if and only if it is isomorphic to one of the graphs listed in Table 5.2. Furthermore, all the graphs listed in Table 5.2 are pairwise non-isomorphic.*

Proof. Let X be a tetravalent one-regular graph of order $4p^2$. Let $A = \mathrm{Aut}(X)$ and let A_v be the stabilizer of $v \in V(X)$ in A. By [116], there is no tetravalent one-regular graph of order 16, and $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS}2[4,3,(0\ 1){:}(1\ 2)]$ and $\mathcal{CA}^1_{36}$ are the only tetravalent one-regular graphs of order 36 (see also Examples 5.2.1, 5.2.2 and 5.2.4). Thus, we assume that $p>3$. Since X is one-regular we have that $|A| = 16p^2$, and thus A is a solvable group. Let P be a Sylow p-subgroup of A.

Claim: P is normal in A.

Since $|A| = 16p^2$ Sylow's theorems imply that the number of Sylow p-subgroups of A is equal to $|A{:}N_A(P)| = kp+1$. In addition, this number divides 16. Hence, if

$p > 7$ then we clearly have that P is normal in A as claimed. Now we will prove that P is normal in A also when $p \in \{5, 7\}$.

Let $N = O_2(A)$ be the largest normal 2-subgroup of A. Suppose first that $|N| = 16$ and consider the quotient graph X_N. Then $N \le K$, where K is the kernel of A acting on $V(X_N)$, X_N is a symmetric graph of valency 2 or 4, and, by Proposition 5.1.3, A/K acts arc-transitively on X_N. But then $2 \mid |A/K|$, which is clearly impossible since $|A| = 16p^2$. Therefore $|N||8$. Now we distinguish three different cases depending on the order of N. Let T be a minimal normal subgroup of A.

Case 1. $|N| = 1$.

Then either $|T| = p^2$ or $|T| = p$. In the former case we have that $T = P$ and thus $P \trianglelefteq A$ as claimed. We may therefore assume that $|T| = p$. Let X_T be the quotient graph of X relative to the orbits of T, and let K be the kernel of A acting on $V(X_T)$. Then $T \le K$ and A/K acts arc-transitively on X_T. If A/T is abelian then, since A/K is a quotient group of the group A/T, also A/K is abelian. But since A/K is vertex-transitive on X_T, Proposition 2.1.4 implies that it is regular on X_T, contradicting arc-transitivity of A/K on X_T. Thus A/T is a non-abelian group. Let $C = C_A(T)$. Then $T \le C$ and, by Proposition 2.1.5, A/C is isomorphic to a subgroup of $\mathrm{Aut}(T) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and consequently $T < C$. Let L/T be a minimal normal subgroup of A/T contained in C/T. Then $L/T \cong \mathbb{Z}_p$, and therefore $P = L \trianglelefteq A$.

Case 2. $|N| = 2$.

Then $|T| \in \{p^2, p, 2\}$. If $|T| = p^2$ then $P \trianglelefteq A$ as claimed. Suppose now that $|T| = 2$, and let $C = C_A(T)$. Then $T \le C$ and, moreover, by Proposition 2.1.5, $|A/C| = 1$ which implies that $T < C$. Let L/T be a minimal normal subgroup of C/T. Then either $|L/T| = p^2$ or $|L/T| = p$. In the former case it follows that $|L| = 2p^2$, and consequently P char $L \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed. In the later case we have $L = \mathbb{Z}_2 \times \mathbb{Z}_p$. Suppose first that A/L is abelian and consider the quotient graph X_L of X relative to the orbits of L. Let K be the kernel of A acting on $V(X_L)$. Then $L \le K$, A/K is a quotient group of A/L, and as such also abelian. But since A/K is vertex-transitive on X_L, Proposition 2.1.4 implies that A/K is regular on X_L, which is impossible since A/K acts arc-transitively on X_L. Thus, A/L is a non-abelian group. Let $C = C_A(L)$. Then $L \le C$ and, by Proposition 2.1.5, $A/C \lesssim \mathrm{Aut}(L) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and so $L < C$. Let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$ and thus $M \trianglelefteq A$ and $|M| = 2p^2$. In addition, since P char $M \trianglelefteq A$, we have that $P \trianglelefteq A$ as claimed.

Assume now that $|T| = p$. Then an argument similar to the one used above shows that A/T is a non-abelian group. Let $C = C_A(T)$. Then, by Proposition 2.1.5, we have that $A/C \lesssim \mathrm{Aut}(T) \cong \mathbb{Z}_{p-1}$. Thus A/C is abelian, which implies that $T < C$. Let L/T be a minimal normal subgroup A/T contained in C/T. Then either $L/T \cong \mathbb{Z}_p$ or $L/T \cong \mathbb{Z}_2$. If $L/T \cong \mathbb{Z}_p$, then clearly $L = P \trianglelefteq A$. If however $L/T \cong \mathbb{Z}_2$, then $L \cong \mathbb{Z}_{2p}$ and, by Proposition 2.1.5, $A/C \lesssim \mathrm{Aut}(L) \cong \mathbb{Z}_{p-1}$ where $C = C_A(L)$. Hence A/C is abelian, and consequently $L < C$. Now let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$, and so $|M| = 2p^2$. But then P char $M \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed.

Case 3. $|N| \in \{4, 8\}$.

Then either $|A/N| = 2p^2$ or $|A/N| = 4p^2$. Then, clearly PN/N is a Sylow p-subgroup of A/N and by Sylow's theorems, $PN/N \trianglelefteq A/N$. Moreover, $PN \trianglelefteq A$. If $|N| = 4$ then for $p \in \{5, 7\}$ we have that P is characteristic in PN, and hence normal in A. Also, if $|N| = 8$ and $p = 5$ then one can easily see that P is characteristic in PN and hence normal in A. Therefore we can now assume that $|N| = 8$ and $p = 7$. Then N is isomorphic to one of the following groups: D_8, Q_8 (the quaternion group), $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2^3$. Let $C = C_A(N)$. By Proposition 2.1.5, we have that $A/C \lesssim \mathrm{Aut}(N)$. If $N \ncong \mathbb{Z}_2^3$ then $7 \nmid |\mathrm{Aut}(N)|$ and hence $7^2 \mid |C|$, which implies that $P \leq C$. It follows that P is characteristic in PN and hence normal in A. If however $N \cong \mathbb{Z}_2^3$ then $N \leq C$ and $\mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$. Observe that $|A/N| = 98$ and $A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$. But $\mathrm{Aut}(N) = \mathrm{PSL}(2, 7)$ has no subgroup of order 98 since $|\mathrm{PSL}(2, 7)| = 168$, implying that $A/N \neq A/C$, and therefore $N < C$. Note also that $|C| > 8$, but $16 \nmid |C|$. Namely, if $16 \mid |C|$, the fact that A/K acts arc-transitively on X_C, where K is the kernel of A acting on $V(X_C)$, implies that $2 \mid |A/K|$. But this is impossible since $C \leq K$. Therefore $7 \mid |C|$. If $7^2 \nmid |C|$ then $|C| = 8 \cdot 7 = 56$. But then A/C is a group of order $|A/C| = 2 \cdot 7 = 14$ isomorphic to a subgroup of $\mathrm{Aut}(N) \cong \mathrm{PSL}(2, 7)$, which by Proposition 2.1.8 is impossible. Therefore $7^2 \mid |C|$, and consequently $P \leq C_A(N)$. It follows that P is characteristic in PN, and thus normal in A. This proves that A always has a normal Sylow p-subgroup as claimed.

Assume first that P is cyclic. Let X_P be the quotient graph of X relative to the orbits of P and let K be the kernel of A acting on $V(X_P)$. By Proposition 2.1.7, the orbits of P are of length p^2. Thus $|V(X_P)| = 4$, $P \leq K$ and A/K acts arc-transitively on X_P. By Proposition 5.1.3, we have that $X_P \cong C_4$ and hence $A/K \cong D_8$, forcing $|K| = 2p^2$. Since A/K is a quotient group of A/P, it follows that A/P is a non-abelian group. Moreover, $|K| = 2p^2$ and thus K is not semiregular on $V(X)$. Then $K_v \cong \mathbb{Z}_2$ where $v \in V(X)$. By Proposition 2.1.5, $A/C \lesssim \mathrm{Aut}(P) \cong \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since A/P is non-abelian, we have that P is a proper subgroup of C. If $C \cap K \neq P$ then $C \cap K = K$ ($|K| = 2p^2$). Since K_v is a Sylow 2-subgroup of K, K_v is characteristic in K and so normal in A, implying that $K_v = 1$, a contradiction. Thus, $C \cap K = P$ and $1 \neq C/P = C/(C \cap K) \cong CK/K \trianglelefteq A/K \cong D_8$. If $C/P \cong \mathbb{Z}_2$ then C/P is in the center of A/P and since $(A/P)/(C/P) \cong A/C$ is cyclic, A/P is abelian, a contradiction. It follows that $|C/P| \in \{4, 8\}$, and hence C/P has a characteristic subgroup of order 4, say H/P. Thus, $|H| = 4p^2$ and $H/P \trianglelefteq A/P$, implying that $H \trianglelefteq A$. In addition, since $H \leq C = C_A(P)$, we have that H is abelian. Clearly, $|H_v| \in \{1, 2, 4\}$. First, suppose that $|H_v| = 4$. Then H_v is a Sylow 2-subgroup of H, implying that H_v is characteristic in H. The normality of H in A implies that $H_v \trianglelefteq A$, forcing $H_v = 1$, a contradiction. Second, suppose that $|H_v| = 2$, and let Q be a Sylow 2-subgroup of H. Then $Q \trianglelefteq A$ and $Q_v = H_v$. Consider the quotient graph X_Q of X relative to the orbits of Q. Since $|Q| = 4$ and $Q_v \cong \mathbb{Z}_2$, Proposition 5.1.3 implies that $X_Q \cong C_{2p^2}$ and hence $X \cong C_{2p^2}[2K_1]$, contradicting one-regularity of X. Thus, we have that $H_v = 1$, and since $|H| = 4p^2$, H is regular on $V(X)$. It follows that X is a Cayley graph on an abelian group with a cyclic

Sylow p-subgroup P. By elementary group theory, we know that up to isomorphism $\mathbb{Z}_{4p^2}$ and $\mathbb{Z}_{2p^2} \times \mathbb{Z}_2$, where $p > 3$, are the only abelian groups with a cyclic Sylow p-subgroup. However, by Xu [122, Theorems 3], there is no tetravalent one-regular Cayley graph on $\mathbb{Z}_{4p^2}$, and so $H \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$. Proposition 5.1.1 and Example 5.2.3 combined together now imply that $X \cong \mathcal{CA}^0_{4p^2}$.

Now assume that P is elementary abelian. Suppose first that P is a minimal normal subgroup of A, and consider the quotient graph X_P of X relative to the orbits of P. Let K be the kernel of A acting on $V(X_P)$. By Proposition 2.1.7, we have that the orbits of P are of length p^2, and thus $|V(X_P)| = 4$. By Proposition 5.1.3, $X_P \cong C_4$, and hence $A/K \cong D_8$, forcing $|K| = 2p^2$ and thus $K_v = \mathbb{Z}_2$. Proposition 5.1.4 now implies that X is isomorphic to $C^{\pm 1}(p; 4, 2)$, $\mathcal{NC}^0_{4p^2}$ or $\mathcal{NC}^1_{4p^2}$. However, by Lemma 5.1.5, $C^{\pm 1}(p; 4, 2)$ is not one-regular whereas, by Lemma 5.1.6, $\mathcal{NC}^0_{4p^2}$ and $\mathcal{NC}^1_{4p^2}$ both are one-regular. Conditions on the prime p written in Table 5.2 follow from the definition of these graphs (see paragraphs above the Proposition 5.1.4).

Suppose now that P is not a minimal normal subgroup of A. Then a minimal normal subgroup N of A is isomorphic to $\mathbb{Z}_p$. Let X_N be the quotient graph of X relative to the orbits of N and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and A/K is transitive on $V(X_N)$. Moreover, we have that $|V(X_N)| = 4p$. By Proposition 5.1.3, X_N is a cycle of length $4p$, or N acts semiregularly on $V(X)$, the quotient graph X_N is a tetravalent connected G/N-arc-transitive graph and X is a regular cover of X_N. If $X_N \cong C_{4p}$, and hence $A/K \cong D_{8p}$, then $|K| = 2p$ and thus $K_v = \mathbb{Z}_2$. Applying Proposition 5.1.7 we get that X is either isomorphic to $\mathcal{C}^{\pm 1}(p; 4p, 1)$ or to $\mathcal{C}^{\pm \varepsilon}(p; 4p, 1)$. By Lemmas 5.2.5 and 5.2.6 and Example 5.2.4, these two graphs are both one-regular and they are, respectively, isomorphic to $\mathcal{CA}^0_{4p^2}$ and $\mathcal{CA}^1_{4p^2}$. If, however, X_N is a tetravalent connected G/N-symmetric graph, then, by Proposition 5.1.3, X is a covering graph of a symmetric graph of order $4p$. By Proposition 5.1.8, there are six tetravalent symmetric graphs of order $4p$: $K_{4,4}$, $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$, $\mathcal{CA}^1_{4p}$, $\mathcal{C}(2, p, 2)$ and $\mathfrak{g}_{28}$. But, since there is no tetravalent one-regular graph of order 16, the automorphism group of $\mathfrak{g}_{28}$ does not admit a one-regular subgroup, and since, by Lemma 5.3.1, there is no one-regular $\mathbb{Z}_p$-cover of $\mathcal{C}(2, p, 2)$, we only need to consider the covering graphs of $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$ and $\mathcal{CA}^1_{4p}$. Observe that in each of these three graphs a one-regular subgroup of automorphisms contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$. Let H be a one-regular subgroup of automorphisms of X_N. Since X is one-regular graph, A is the lift of H. Since H contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$ also A contains a normal regular subgroup. Therefore X is a normal Cayley graph of order $4p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_{2p} \times \mathbb{Z}_2 \trianglelefteq H$, there exists a normal subgroup G of A such that $G/\mathbb{Z}_p \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$. The classification of groups of order $4p^2$, given in [13, 14], and a detail analysis of all these groups give that G is either isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$ or to $G = \langle a, b, c, g \,|\, a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_{2p}) \rtimes \mathbb{Z}_2$. However, by Proposition 5.1.2, there is no tetravalent one-regular graph on $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$, whereas for the latter group, Lemmas 5.3.2, 5.3.3 and 5.3.4, combined together imply that X is either isomorphic to $\mathcal{CA}^1_{4p^2}$ or to $\mathcal{CN}^2_{4p^2}$. Since, by Lemma 5.3.3, graphs listed in Table 5.2 are pairwise non-isomorphic the proof is completed. $\blacksquare$

5.5 Observations and conclusions

In this chapter connected tetravalent one-regular graphs of order $4p^2$, p a prime, are classified. This result is obtained with the use of basic group theory results, combinatorial techniques, and covering techniques.

Chapter 6

The Hamiltonicity problem

In 1969, Lovász [71] asked if every finite, connected vertex-transitive graph has a Hamilton path, that is, a simple path going through all vertices of the graph. With the exception of K_2, only four connected vertex-transitive graphs that do not have a Hamilton cycle are known to exist. These four graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. The fact that none of these four graphs is a Cayley graph has led to a folklore conjecture that every Cayley graph with order greater than 2 has a Hamilton cycle.

Many articles directly and indirectly related to this subject have appeared in the literature (see [1, 2, 3, 6, 15, 27, 31, 46, 47, 48, 51, 59, 61, 64, 76, 78, 79, 80, 84, 85, 93, 103, 110, 117, 118, 119] for some of the relevant references), affirming the existence of such paths and, in some cases, even Hamilton cycles. For example, it is known that connected vertex-transitive graphs of order kp, where $k \leq 5$, (except for the Petersen graph and the Coxeter graph), of order p^j, where $j \leq 4$, and of order $2p^2$, where p is a prime, contain a Hamilton cycle. It is also known that connected vertex-transitive graphs of order pq, where p and q are primes, admitting an imprimitive subgroup of automorphisms contain a Hamilton cycle. A Hamilton path is known to exist in connected vertex-transitive graphs of order $6p$. In addition, it is known that every connected vertex-transitive graph whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime-power order, with the exception of the Petersen graph, has a Hamilton cycle (this result was obtained with a generalization of the method used in [31, 59, 76]). We refer the reader to [62] for the current status of this problem.

This chapter deals with the existence of Hamilton paths in connected vertex-transitive graphs of order $10p$, where p is a prime. The main object of this chapter is to show that, with the exception of a certain family of graphs arising from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, every connected vertex-transitive graph of order $10p$, $p \neq 7$, contains a Hamilton path.

Theorem 6.0.1 *Let X be a connected vertex-transitive graph of order $10p$, where $p \neq 7$ is a prime, not isomorphic to a quasiprimitive graph arising from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$. Then X contains a Hamilton path.*

The main tool in proving Theorem 6.0.1 is the so-called *lifting Hamilton cycles* *approach*, a frequently used approach for constructing Hamilton paths and cycles in vertex-transitive graphs. This approach is based on a quotienting/reduction with respect to an imprimitivity block system of the corresponding automorphism group or with respect to a suitable semiregular automorphism, preferably one of prime order. In particular, every vertex-transitive graph is either genuinely imprimitive, quasiprimitive or primitive. Following the method in [62] we divide our investigation depending on which of these three families the graph in question belongs to. There is no primitive graph of order $10p$ for $p > 19$. Also, there is no quasiprimitive graph of order $10p$ for $p > 31$ arising from a group action different from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$. For $p \leq 31$ all primitive and quasiprimitive graphs of order $10p$ are known and the existence of Hamilton cycles in such graphs (with the exception of the truncation of the Petersen graph) is proved with the help of program package MAGMA [7]. In particular, we construct all relevant graphs and in each of them we either find a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order (and thus the above mentioned result proved in [27] applies) or we find a semiregular automorphism of prime order such that the corresponding quotient graph contains such a Hamilton cycle that it can be, with the use of the lifting Hamilton cycle approach, lifted to a Hamilton cycle of the original graph. For the genuinely imprimitive graphs we use the lifting Hamilton cycle approach based on a quotienting/reduction with respect to an imprimitivity block system formed by the orbits of a minimal normal subgroup of a genuinely imprimitive group of automorphisms. In particular, the investigation depends on the size of the blocks in such imprimitivity block systems.

In Section 6.1 some auxiliary results that are needed in the subsequent sections are introduced. The rest of the chapter is devoted to proving Theorem 6.0.1. The genuinely imprimitive graphs are considered in Section 6.2, the quasiprimitive graphs are considered in Section 6.3, and the primitive graphs are considered in Section 6.4. Finally, the results are combined in Section 6.5, where the Theorem 6.0.1 is proved.

6.1 Existence of Hamilton cycles/paths in particular graphs

The following classical result, due to Jackson [58], giving a sufficient condition for the existence of Hamilton cycles in 2-connected regular graphs will be used throughout this chapter. (Note that every connected vertex-transitive graph is 2-connected.)

Proposition 6.1.1 [58, Theorem 6] *Every* 2*-connected regular graph of order* n *and valency at least* $n/3$ *contains a Hamilton cycle.*

A graph is *Hamilton-connected* if for every pair of vertices there is a Hamilton path between the two vertices, and it is *edge-hamiltonian* if each of its edges is contained in some Hamilton cycle. By the following proposition Cayley graphs on abelian groups are edge-hamiltonian graphs.

Proposition 6.1.2 [12, Theorem 6] *Let X be a connected Cayley graph on an abelian group of order at least three. Then each edge of X is contained in some Hamilton cycle of X.*

The following three results about the existence of Hamilton cycles in particular vertex-transitive graphs will be used in the proofs throughout this chapter.

Proposition 6.1.3 [1] *Let X be a connected vertex-transitive graph of order $2p$, p is a prime. Then X is the Petersen graph or X is hamiltonian.*

A detailed description of connected vertex-transitive graphs of order qp, q and p primes, whose automorphism groups contain imprimitive subgroups is given in [86, 88]. It was proved in [81] that with the exception of the Petersen graph every such graph has a Hamilton cycle. For $q = 5$ every connected vertex-transitive graph of order qp with a primitive automorphism group containing no imprimitive subgroups arises from one of primitive groups of degree qp without imprimitive subgroups given in [88, Theorem 2.1], and their hamiltonicity was proved in [29]. Therefore, the following proposition holds.

Proposition 6.1.4 *Let X be a connected vertex-transitive graph of order $5p$, p a prime. Then X is the Petersen graph or X is hamiltonian.*

Proposition 6.1.5 [27, Theorem 1.1] *Let X be a connected vertex-transitive graph of order at least 3. If there is a transitive group G of automorphisms of X such that the commutator subgroup of G is cyclic of prime-power order, then X is the Petersen graph or X is hamiltonian.*

We next introduce the following notion of a lift of a path in a graph with a semiregular automorphism. Let X be a graph that admits an (m, n)-semiregular automorphism ρ. Let $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ be the set of orbits of ρ, let $X_\mathcal{S}$ be the corresponding quotient graph and let $\wp : X \to X_\mathcal{S}$ be the corresponding projection. Let $W = S_{i_1} S_{i_2} \dots S_{i_k}$ be a path in $X_\mathcal{S}$. We let the *lift of the path* W be the set of all paths of X whose images under $\wp$ are W.

A frequently used approach to constructing Hamilton cycles in vertex-transitive graphs, which will also be used in this thesis, is based on a quotienting/reduction with respect to a suitable semiregular automorphism, preferably one of prime order. Provided the quotient graph contains a Hamilton cycle it is sometimes possible to lift this cycle to construct a Hamilton cycle in the original graph, consistently spiraling through the corresponding orbits (see Example 6.1.7). Lifts of Hamilton cycles from quotient graphs which themselves have a Hamilton cycle are always possible, for example, where the quotienting is done relative to a semiregular automorphism of prime order and where in the quotient graph there are at least two adjacent orbits on the Hamilton cycle joined by a double edge. In this case one can always lift the Hamilton cycle from the quotient graph because the double edge gives us the possibility to conveniently "change direction" so as to get a walk in the quotient that lifts to a full cycle above. In particular, the following lemma is straightforward and is just a reformulation of [84, Lemma 5].

Proposition 6.1.6 *Let X be a graph admitting an (m,p)-semiregular automorphism ρ, where p is a prime. Let C be a cycle of length k in the quotient graph $X_{\mathcal{S}}$, where $\mathcal{S}$ is the set of orbits of ρ. Then, the lift of C either contains a cycle of length kp or it consists of p disjoint k-cycles. In the latter case we have $d(S,S')=1$ for every edge SS' of C.*

Observe that for a given graph X admitting an (m,n)-semiregular automorphism ρ, the corresponding quotient graph X_ρ can be viewed as the graph whose vertices are circles in Frucht's notation of X with respect to ρ and edges are the edges between the circles. For an arc $e \in A(X_\rho)$ let $l(e)$ denote the label of the corresponding arc in Frucht's notation of X with respect to ρ. Similarly, for a walk W in X_ρ let $l(W)$ denote the sum of the labels of the arcs in Frucht's notation corresponding to the arcs belonging to the walk W. Throughout the chapter the following observation is used frequently: If there exists a Hamilton cycle C of X_ρ such that $(l(C),n)=1$ then X has a Hamilton cycle.

Example 6.1.7 The generalized orbital graph X arising from the action of the group $\mathrm{PSL}(2,11)$ on cosets of D_6 with respect to a union of a self-paired suborbit of length 1 and a self-paired suborbit of length 3 contains a $(10,11)$-semiregular automorphism ρ, and it can be nicely represented in Frucht's notation as shown in Figure 6.1. Since $C = S_0S_1S_5S_7S_3S_8S_9S_6S_2S_4S_0$ is a Hamilton cycle in the quotient graph $X_{\mathcal{S}} = X_\rho$, where $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_{10}\}$ is the set of orbits of ρ, such that the sum of the labels of the arcs lying on C is equal to 9 (which is coprime to 11) this cycle can be lifted to a Hamilton cycle in the original graph X (see Figure 6.1). This graph is one of the quasiprimitive graphs of order 110 arising from row 2 of Table 6.1, see Section 6.3.

We end this section with a result about the existence of Hamilton paths in vertex-transitive graphs admitting a semiregular automorphism of prime order such that the corresponding quotient graph is of order congruent to 2 modulo 4 and is either isomorphic to a complete bipartite graph or a complete bipartite graph minus a matching.

Proposition 6.1.8 *Let X be a connected vertex-transitive graph of order $2qm$, where q is a prime and m is odd, admitting a $(2m,q)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$ and let $\mathcal{O}$ be the set of orbits of ρ. If $X_{\mathcal{O}} \in \{K_{m,m}, K_{m,m} - mK_2\}$ then X has a Hamilton path.*

PROOF. Let $X_{\mathcal{O}} \in \{K_{m,m}, K_{m,m} - mK_2\}$ and let $\mathcal{O} = \{S_i, T_i \mid i \in \mathbb{Z}_m\}$ such that $\{S_i \mid i \in \mathbb{Z}_m\}$ and $\{T_i \mid i \in \mathbb{Z}_m\}$ are the two bipartite sets of $X_{\mathcal{O}}$. Since every edge of $X_{\mathcal{O}}$ belongs to some Hamilton cycle of $X_{\mathcal{O}}$, we may, by Proposition 6.1.6, assume that $X_\rho = X_{\mathcal{O}}$, that is, $d(S_i,T_j)=1$ for every $i,j \in \mathbb{Z}_m$. Since X is regular it follows that $d(S) = d(S')$ for any two orbits $S,S' \in \mathcal{O}$. Moreover, since q is a prime either $d(S)=0$ or $d(S) \geq 2$ is even. If $d(S)=2$ then a Hamilton cycle of X exists by [3, Theorem 3.9], and if $d(S) \geq 4$ then [12, Theorem 4] implies that for every $S \in \mathcal{O}$ the subgraph $X\langle S \rangle$ is Hamilton-connected, and so a Hamilton cycle of X clearly exists. We may therefore assume that $d(S)=0$ for every $S \in \mathcal{O}$, that is,

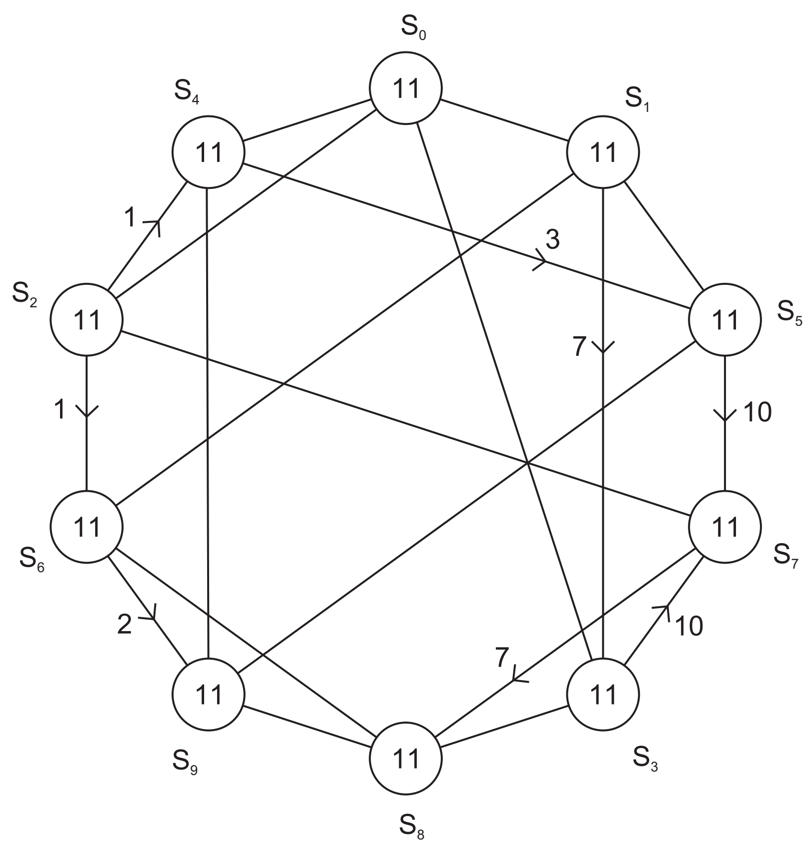

Figure 6.1: A vertex-transitive graph arising from the action of PSL$(2, 11)$ on cosets of D_6 given in Frucht's notation with respect to the $(10, 11)$-semiregular automorphism ρ where undirected lines carry label 0.

$X\langle S\rangle = qK_1$. We distinguish two different cases depending on whether $X_{\mathcal{O}} \cong K_{m,m}$ or $X_{\mathcal{O}} \cong K_{m,m} - mK_2$.

Case 1. $X_{\mathcal{O}} \cong K_{m,m}$.

Then $S_i \sim T_j$ for every $i, j \in \mathbb{Z}_m$. If there exists a Hamilton cycle C of $X_\rho = X_{\mathcal{S}}$ such that $l(C) \neq 0$ then X clearly has a Hamilton cycle. Thus we may assume that no such Hamilton cycle exists in X_ρ. Also, if there exist two disjoint cycles C_1 and C_2 such that $V(X_\rho) = V(C_1) \cup V(C_2)$ and $l(C_1)$ and $l(C_2)$ are both different from 0 then, since q is a prime, we have that C_1 and C_2 both lift to a single cycle in X, and consequently the connectedness of X implies that X has a Hamilton path. We may therefore assume that no such pair of cycles exists in X_ρ.

Since $S_0T_0S_1T_1 \cdots S_{m-1}T_{m-1}S_0$ is a Hamilton cycle in X_ρ we may, without loss of generality, assume that $l(S_iT_i) = l(T_iS_{i+1}) = 0$ for every $i \in \mathbb{Z}_m$. Let $i \in \mathbb{Z}_m \setminus \{m-2, m-1\}$. Then

$$C_i = S_0T_iS_iT_{i-1}S_{i-1} \cdots S_1T_0S_0 \quad \text{and} \quad C_i' = S_{i+1}T_{m-1}S_{m-1}T_{m-2} \cdots S_{i+2}T_{i+1}S_{i+1}$$

are two disjoint cycles such that $V(X_\rho) = V(C_i) \cup V(C_i')$, $l(C_i) = l(S_0T_i)$ and $l(C_i') = l(S_{i+1}T_{m-1})$. If $l(C_i) = l(S_0T_i) \neq 0$ then, by the assumption made in the preceding paragraph, we have that $l(C_i') = l(S_{i+1}T_{m-1}) = 0$. Next, since

$$C_i'' = S_0T_{i+1}S_{i+2}T_{i+2}S_{i+3} \cdots S_{m-1}T_{m-1}S_{i+1}T_iS_iT_{i-1} \cdots S_1T_0S_0$$

is a Hamilton cycle of X_ρ with $l(C_i'') = l(S_0T_{i+1})$ we have $l(S_0T_{i+1}) = 0$. It follows that

$$D_i = S_0T_{i+1}S_{i+1}T_iS_iT_{i-1} \cdots S_1T_0S_{i+2}T_{i+2}S_{i+3} \cdots T_{m-1}S_0$$

is a Hamilton cycle of X_ρ with $l(D_i) = l(T_0S_{i+2})$, and thus $l(T_0S_{i+2}) = 0$. But then

$$S_0T_iS_iT_{i-1}S_{i-1} \cdots S_1T_0S_{i+2}T_{i+2}S_{i+3} \cdots T_{m-1}S_{i+1}T_{i+1}S_0$$

is a Hamilton cycle of X_ρ with a non-zero label and thus it lifts to a Hamilton cycle of X. It therefore follows that $l(S_0T_i) = 0$ for every $i \in \mathbb{Z}_m \setminus \{m-2\}$. Moreover, by replacing S_0 with an orbit S_j, $j \in \mathbb{Z}_m \setminus \{0\}$, in this argument, one can easily see that we have $l(S_jT_k) = 0$ whenever $|k - j| \neq m-2$. Further, since in the Hamilton cycle

$$C = S_0T_{m-2}S_{m-2}T_{m-3}S_{m-3}T_{m-1}S_{m-1}T_{m-4}S_{m-4}T_{m-5} \cdots S_1T_0S_0$$

of $X_{\mathcal{O}}$ the edge S_0T_{m-2} is the only edge of the form S_iT_{i+m-2}, we have that $l(C) = l(S_0T_{m-2})$, and thus $l(S_0T_{m-2}) = 0$. Since C^{ψ^j} is a Hamilton cycle of $X_{\mathcal{O}}$ and $l(C^{\psi^j}) = l((S_0T_{m-2})^{\psi^j}) = l(S_jT_{j+m-2})$, where

$$\psi = (S_0\ S_1\ \ldots\ S_{m-1})(T_0\ T_1\ \ldots\ T_{m-1}) \in \operatorname{Aut}(X_{\mathcal{O}})$$

and $j \in \mathbb{Z}_m$, we get that all the edges of $X_{\mathcal{O}}$ carry label 0. But then X is disconnected, a contradiction.

Case 2. $X_{\mathcal{O}} \cong K_{m,m} - mK_2$.

We can obtain $X_{\mathcal{O}}$ from the graph in Case 1 in such a way that we delete all the edges of the form $\{S_iT_{i+1} \mid i \in \mathbb{Z}_m\}$. Since none of the edges in the cycles, used in the proof of Case 1, is of the form S_iT_{i+1}, $i \in \mathbb{Z}_m$, we can apply the same argument as in Case 1 to show that X has a Hamilton path. ■

6.2 Genuinely imprimitive graphs of order 10p

Throughout this section let X be a connected genuinely imprimitive graph of order $10p$, $p > 5$ a prime, admitting an imprimitive subgroup G of $\mathrm{Aut}(X)$ with a non-transitive minimal normal subgroup $N \triangleleft G$. Let the set of orbits of N be denoted by $\mathcal{B}$. Then $\mathcal{B}$ is a complete imprimitivity block system of G.

Lemmas 6.2.1, 6.2.3, 6.2.4, 6.2.5, 6.2.6 and 6.2.7, each of which covers a particular size of the blocks in $\mathcal{B}$, combined together imply that every connected genuinely imprimitive graph of order $10p$, $p > 7$ a prime, possesses a Hamilton path.

Lemma 6.2.1 *If the size of blocks in $\mathcal{B}$ is 2 then X has a Hamilton path.*

Proof. Since $X_{\mathcal{B}}$ is a connected vertex-transitive graph of order $5p$, by Proposition 6.1.4, it has a Hamilton cycle C. By Proposition 2.2.5, X has a $(5p, 2)$-semiregular automorphism whose set of orbits equals $\mathcal{B}$. Thus, by Proposition 6.1.6, either C lifts to a Hamilton cycle of X or it lifts to a disjoint union of two cycles of length $5p$. Since X is connected a Hamilton path exists in X. ■

The following proposition about the graphs whose quotient graph with respect to $\mathcal{B}$ is isomorphic to the Petersen graph will be used in the proof of Lemma 6.2.3. The proposition is a direct generalization of [64, Lemma 3.2]. We omit the proof.

Proposition 6.2.2 *If the size of blocks in $\mathcal{B}$ is p and the quotient graph $X_{\mathcal{B}}$ is isomorphic to the Petersen graph then X has a Hamilton path.*

In the proof of the next lemma we will be using the following notation. Let $C_n = (0, 1, \dots, n-1)$ be an n-cycle. A graph C_n^+ is a graph with $V(C_n^+) = V(C_n)$ and $E(C_n^+) = E(C_n) \cup \{\{i, i+n/2\} \mid i \in \mathbb{Z}_n\}$ (clearly C_n^+ is well defined only for even integers n). A graph $C_n(k)$ is a graph with $V(C_n(k)) = V(C_n)$ and $E(C_n(k)) = E(C_n) \cup \{\{i, i+k\} \mid i \in \mathbb{Z}_n\}$. A graph $C_n(k,l)$ is a graph with $V(C_n(k,l)) = V(C_n)$ and $E(C_n(k,l)) = E(C_n) \cup \{\{i, i+k\}, \{i, i+l\} \mid i \in \mathbb{Z}_n\}$. Also, recall that the *direct product* $Y \times Z$ of graphs Y and Z is a graph with $V(Y \times Z) = V(Y) \times V(Z)$ and $E(Y \times Z) = \{\{(a,x), (b,y)\} \mid ab \in E(Y) \text{ and } xy \in E(Z)\}$.

Lemma 6.2.3 *If the size of blocks in $\mathcal{B}$ is p then X has a Hamilton path.*

Proof. The quotient graph $X_{\mathcal{B}}$ is a connected vertex-transitive graph of order 10. By Proposition 6.2.2, we may assume that $X_{\mathcal{B}}$ is not isomorphic to the Petersen graph. By Proposition 2.2.4 the blocks of $\mathcal{B}$ coincide with the orbits of some $(10, p)$-semiregular automorphism $\rho \in G$ of X, which exists by Proposition 2.2.5. Let $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_{10}\}$ denote the set of orbits of ρ.

There exist eighteen connected vertex-transitive graphs of order 10 of which one is isomorphic to the Petersen graph, see [90]. In particular, the quotient graph $X_{\mathcal{S}} = X_{\mathcal{B}}$ is isomorphic to one of the following seventeen graphs:

$$\begin{array}{llllll}
C_{10}, & K_{5,5}, & C_{10}^+, & C_5 \times K_2, & (K_5 \times K_2)^c, & C_{10}(4), \\
C_{10}(2), & C_{10}(2,5), & C_{10}(4,5), & K_5 \times K_2, & GP(5,2)^c, & (C_5 \times K_2)^c, \\
(C_{10}^+)^c, & (2C_5)^c, & C_{10}^c, & (5K_2)^c, & K_{10}. &
\end{array}$$

It is easy to see that in all these cases for any edge e of $X_{\mathcal{S}}$ there exists a Hamilton cycle of $X_{\mathcal{S}}$ containing e. Hence, by Proposition 6.1.6, we may assume that no multiedge exists in X_ρ. Since $X_{\mathcal{S}}$ is hamiltonian we may label the orbits of ρ in such a way that $S_i \sim S_{i+1}$ for every $i \in \mathbb{Z}_{10}$. If there exists a Hamilton cycle of $X_{\mathcal{S}}$ whose lift contains a Hamilton cycle of X, there is nothing to prove. Therefore, we can assume that no such Hamilton cycle of $X_{\mathcal{S}}$ exists. Consequently, we may assume that $l(S_iS_{i+1}) = 0$ for every $i \in \mathbb{Z}_{10}$. Note also that we can assume that $X\langle S_i\rangle = pK_1$ for all $i \in \mathbb{Z}_{10}$. Namely, all the subgraphs $X\langle S_i\rangle$ are of the same valency, and if the subgraphs $X\langle S_i\rangle$ are of valency 2, then a Hamilton cycle of X exists by [3, Theorem 3.9], and if the subgraphs $X\langle S_i\rangle$ are of valency at least 4, then [12, Theorem 4] implies that each of $X\langle S_i\rangle$ is Hamilton-connected (that is, there exists a Hamilton path in $X\langle S_i\rangle$ connecting any two vertices), and so a Hamilton cycle of X clearly exists.

We distinguish seventeen different cases depending on which of the seventeen connected vertex-transitive graphs of order 10 the quotient graph $X_{\mathcal{S}}$ is isomorphic to.

If $X_{\mathcal{S}} \cong C_{10}$ then S_iS_{i+1}, where $i \in \mathbb{Z}_{10}$, are the only edges of $X_{\mathcal{S}}$, and so X is not connected, a contradiction.

If $X_{\mathcal{S}} \cong K_{5,5}$ or $X_{\mathcal{S}} \cong K_{5,5} - 5K_2 \cong (K_5 \times K_2)^c$ then, by Proposition 6.1.8, X has a Hamilton path.

If $X_{\mathcal{S}} \cong C_{10}^+$ then in addition to the edges S_iS_{i+1}, also S_0S_5, S_1S_6, S_2S_7, S_3S_8, $S_4S_9 \in E(X_{\mathcal{S}})$. Let $r_0 = l(S_0S_5)$, $r_1 = l(S_1S_6)$, $r_2 = l(S_2S_7)$, $r_3 = l(S_3S_8)$, and $r_4 = l(S_4S_9)$. Since

$$S_0S_5S_4S_3S_2S_1S_6S_7S_8S_9S_0, \quad S_0S_1S_2S_3S_4S_9S_8S_7S_6S_5S_0,$$
$$S_0S_1S_6S_5S_4S_3S_2S_7S_8S_9S_0, \quad S_0S_1S_2S_3S_8S_7S_6S_5S_4S_9S_0$$

are Hamilton cycles of $X_{\mathcal{S}}$, Proposition 6.1.6 implies that $r_0 + r_1 = 0$, $r_4 - r_0 = 0$, $r_1 + r_2 = 0$ and $r_3 + r_4 = 0$. It follows that $r_1 = r_3, r_0 = r_2 = r_4$, $r_1 = -r_4$. If $r_1 = 0$ then since p is odd it follows that $r_0 = r_1 = r_2 = r_3 = r_4 = 0$ and thus X is disconnected, a contradiction. If, however, $r_1 \neq 0$ then $r_0 \neq 0$ and since $S_0S_5S_6S_1S_2S_7S_8S_3S_4S_9S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$, Proposition 6.1.6 implies that $r_0 - r_1 + r_2 - r_3 + r_4 = 0$, and so $3r_0 = 2r_1$. But then, since $r_1 = -r_0$, it follows that $5r_0 \equiv 0 \pmod p$, implying that $p = 5$, a contradiction.

If $X_{\mathcal{S}} \cong C_5 \times K_2$ then we may assume that in addition to the edges S_iS_{i+1}, also

$$S_1S_8, S_2S_7, S_3S_6, S_4S_0, S_5S_9 \in E(X_{\mathcal{S}}).$$

Let $r_0 = l(S_1S_8)$, $r_1 = l(S_2S_7)$, $r_2 = l(S_3S_6)$, $r_3 = l(S_4S_0)$, and $r_4 = l(S_5S_9)$. Since

$$S_0S_1S_8S_9S_5S_6S_7S_2S_3S_4S_0, \quad S_0S_1S_2S_7S_8S_9S_5S_6S_3S_4S_0,$$
$$S_0S_1S_2S_3S_6S_7S_8S_9S_5S_4S_0, \quad S_0S_9S_5S_6S_7S_8S_1S_2S_3S_4S_0,$$

are Hamilton cycles of $X_{\mathcal{S}}$, Proposition 6.1.6 implies that $r_0 - r_4 - r_1 + r_3 = 0$, $r_1 - r_4 - r_2 + r_3 = 0$, $r_2 - r_4 + r_3 = 0$, and $-r_4 - r_0 + r_3 = 0$. Combing these equations we get that $r_1 = 2r_0 = 2r_2$ and $r_0 + r_2 = 0$. Since p is odd it follows from the first of these two equations that $r_0 = r_2$, and then we get from the second equation that $r_0 = r_2 = 0$. Hence $r_0 = r_1 = r_2 = 0$ and then from the above equations we get that $r_3 = r_4$. In view of the connectedness of X, we have that $r_3 = r_4 \neq 0$. But then $S_0S_1S_2S_3S_4S_0$ and $S_9S_8S_7S_6S_5S_9$ are disjoint 5-cycles in $X_{\mathcal{S}}$ that lift to $5p$-cycles in X. Since the vertex sets of the obtained $5p$-cycles are disjoint and X is connected, it follows that X has a Hamilton path.

The remaining twelve cases are dealt with in a similar manner. We leave the details to the reader. ■

Lemma 6.2.4 *If $p > 7$ and the size of blocks in $\mathcal{B}$ is 5 then X has a Hamilton path.*

Proof. By Proposition 2.2.5 there exists a $(2p, 5)$-semiregular automorphism $\varphi \in G$ whose orbit set coincides with $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_{2p}\}$. Since $p > 5$ the graph $X_{\mathcal{B}}$ is a vertex-transitive graph of order $2p$, not isomorphic to the Petersen graph, and therefore, by Proposition 6.1.3, it contains a Hamilton cycle, say $C = B_0B_1 \cdots B_{2p-1}B_0$. In view of Proposition 6.1.6 we can assume that the lift of C consists of five disjoint $2p$-cycles. So $d(B_i, B_{i+1}) = 1$ for all $i \in \mathbb{Z}_{2p}$. Moreover, we can assume that $X\langle B\rangle = 5K_1$ for all $B \in \mathcal{B}$. Namely, if for some $B \in \mathcal{B}$ we have $X\langle B\rangle \cong Y$, where $Y \in \{C_5, K_5\}$, then since $\mathcal{B}$ is an imprimitivity block system we have $X\langle B'\rangle \cong Y$ for every block $B' \in \mathcal{B}$. Further, since $X_{\mathcal{B}}$ has a Hamilton cycle and since between any two adjacent blocks of $\mathcal{B}$ we have a perfect matching (as $\mathcal{B}$ is the set of orbits of a normal subgroup) one can easily see that X has a Hamilton path.

Let K be the kernel of the action of G on $\mathcal{B}$. Then depending on the (im)primitivity of the action of $\bar{G} = G/K$ on $X_{\mathcal{B}}$ three cases need to be considered. In particular, since $X_{\mathcal{B}}$ is of order $2p$ either $\bar{G}$ acts primitively on $X_{\mathcal{B}}$ or it acts imprimitively with blocks of size 2 or p. Following the notation given in [61] we denote these possible types of action of G by $(2p : 5)$, $(2 : p : 5)$, and $(p : 2 : 5)$, respectively.

Case 1. G is of type $(2p : 5)$.

In this case $\bar{G} = G/K$ acts primitively on $X_{\mathcal{B}}$. Since $p > 5$, Proposition 2.1.10, implies that $\bar{G}$ is doubly transitive on $X_{\mathcal{B}}$, and so $X_{\mathcal{B}}$ is isomorphic to K_{2p}. Since, by Proposition 6.1.2, every edge of K_{2p} is contained in some Hamilton cycle, by Proposition 6.1.6, we may assume that $X_{\mathcal{B}} = X_\varphi \cong K_{2p}$. Clearly, $X_{\mathcal{B}}$ can be viewed as the graph whose vertices are circles in Frucht's notation of X with respect to φ and edges are the edges between the circles. Observe that if there exists a Hamilton cycle C of $X_{\mathcal{B}}$ such that $l(C) \neq 0$, where $l(C)$ is the sum of the labels of the arcs belonging to C, then X has a Hamilton cycle. Therefore, we can assume that no such Hamilton cycle of $X_{\mathcal{B}}$ exists.

Let us relabel the vertices of $X_{\mathcal{B}} = X_\varphi \cong K_{2p}$ in such a way that $V(X_{\mathcal{B}}) = \{v_i \mid i \in \mathbb{Z}_{2p}\}$. Let $C = v_0v_1v_2v_3v_4 \cdots v_{2p-1}v_0$ be a Hamilton cycle of $X_{\mathcal{B}}$. Then $l(C) = 0$, and moreover we may assume that all the edges of $X_{\mathcal{B}}$ contained in C carry label 0. Further, since $v_0v_2v_1v_3v_4 \cdots v_{2p-1}v_0$ is a Hamilton cycle of $X_{\mathcal{B}}$, we have $l(v_0v_2) = t = -l(v_1v_3)$. In addition, observe that for every $i \in \mathbb{Z}_{2p}$

$$v_0v_1v_2 \cdots v_{i+1}v_iv_{i+2} \cdots v_{2p-1}v_0$$

is a Hamilton cycle of $X_{\mathcal{B}}$ and thus we have

$$l(v_i v_{i+2}) = \begin{cases} t & \text{if } i \text{ is even} \\ -t & \text{if } i \text{ is odd} \end{cases}$$

where $t \in \mathbb{Z}_5$.

If $p \cdot t \not\equiv 0 \pmod 5$ then p-cycles $v_0 v_2 \cdots v_{2p-2} v_0$ and $v_1 v_3 \cdots v_{2p-1} v_1$ of $X_{\mathcal{B}}$ lift to two disjoint $5p$-cycles in X. Since X is connected it is clear that X contains a Hamilton path in this case. We may therefore assume that $p \cdot t \equiv 0 \pmod 5$, that is, $t = l(v_i v_{i+2}) = 0$ for every $i \in \mathbb{Z}_{2p}$. Further,

$$C_i \quad = \quad v_0 v_1 \cdots v_{i-1} v_{i+1} v_{i+2} v_i v_{i+3} v_{i+4} \cdots v_{2p-1} v_0, \quad i \in \mathbb{Z}_{2p}$$

is a Hamilton cycle of $X_{\mathcal{B}}$, and since $l(C_i) = l(v_i v_{i+3})$ we have that $l(v_i v_{i+3}) = 0$ for every $i \in \mathbb{Z}_{2p}$. Next,

$$C_i' \quad = \quad v_{i+4} v_i v_{i+3} v_{i+2} v_{i+1} v_{i-1} v_{i-2} \cdots v_{i+5} v_{i+4}, \quad i \in \mathbb{Z}_{2p}$$

is a Hamilton cycle of $X_{\mathcal{B}}$ with $l(C_i') = l(v_i v_{i+4})$ and thus we have that also $l(v_i v_{i+4}) = 0$ for every $i \in \mathbb{Z}_{2p}$. Continuing inductively, we get that all the edges of $X_{\mathcal{B}}$ have label 0. But then X is disconnected, a contradiction.

Case 2. $(2 : p : 5)$.

Then the action of $\bar{G}$ on $X_{\mathcal{B}}$ gives an imprimitivity block system with two blocks, say $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$, of size p. Let $\mathcal{C}$ and $\mathcal{D}$ be the corresponding blocks of size $5p$ of G in X, and let H be the index 2 subgroup of G such that $\bar{H} = \bar{G}_{\mathcal{C}} = \bar{G}_{\mathcal{D}}$ is the corresponding block stabilizer. Therefore, for a block $B \in \mathcal{B}$ and a vertex $v \in B$, we have a sequence of groups $G_v \leq G_B \leq H \leq G$ giving the type $(2 : p : 5)$.

Now let $C = \{x_0, x_1, \ldots, x_4\} \in \mathcal{C}$ and $D = \{y_0, y_1, \ldots, y_4\} \in \mathcal{D}$. Since $p > 7$, Proposition 2.2.3 implies that there exists a $(10, p)$-semiregular automorphism $\pi \in G$ such that $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ are orbits of $\bar{\pi}$. Let $x_j^i = x_j^{\pi^i}$ and $y_j^i = y_j^{\pi^i}$, $i \in \mathbb{Z}_p$. Then we have that $\mathcal{C} = \{C_i \mid i \in \mathbb{Z}_p\}$ and $\mathcal{D} = \{D_i \mid i \in \mathbb{Z}_p\}$, where $C_i = \{x_j^i \mid j \in \mathbb{Z}_5\}$ and $D_i = \{y_j^i \mid j \in \mathbb{Z}_5\}$. Clearly, $\mathcal{B} = \{C_i, D_i \mid i \in \mathbb{Z}_p\}$.

Subcase 2.1. $\bar{H}^{\bar{\mathcal{C}}}$ is unfaithful.

Then $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] = K_{p,p}$, and, by Propositions 6.1.8 and 6.1.6, we may assume that $X[C_i, D_j] \cong 5K_2$ for all $i, j \in \mathbb{Z}_p$, that is, $d(C_i, D_j) = 1$ for every $i, j \in \mathbb{Z}_p$. Moreover, all the edges $C_i D_j$ in $X_{\mathcal{B}}$ carry label 0.

Recall that $X\langle B\rangle \cong 5K_1$ for every $B \in \mathcal{B}$. If $X\langle\mathcal{C}\rangle \cong X\langle\mathcal{D}\rangle \cong 5pK_1$ then the edge set of $X_{\mathcal{B}}$ is equal to the edge set of $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}]$, and thus X is disconnected, a contradiction. If, however, $X\langle\mathcal{C}\rangle \cong X\langle\mathcal{D}\rangle \not\cong 5pK_1$, and thus $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle \not\cong pK_1$, then $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ is a connected p-circulant, that is, a Cayley graph on a cyclic group of order p. By Proposition 6.1.2, $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ is hamiltonian and moreover, every edge of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ belongs to a Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$. Let C_H be a particular Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$. If $l(C_H) \neq 0$ then C_H lifts to a Hamilton cycle of $X\langle\mathcal{C}\rangle$ (to a $5p$-cycle in X). Since $X\langle\mathcal{C}\rangle \cong X\langle\mathcal{D}\rangle$, also $X\langle\mathcal{D}\rangle$ contains a cycle of length $5p$, and the connectivity of X implies that X has a Hamilton path. Thus we may assume that

$l(C_H) = 0$, and consequently that $d(C_i, C_j) = 1$ for any pair of adjacent orbits C_i and C_j in $X\langle\mathcal{C}\rangle$. Moreover, we may assume that every Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ as well as every Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ lifts to a disjoint union of five p-cycles.

Assume first that all arcs in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ have label 0. Then all arcs belonging to C_H have label 0. Since X is connected there exists a Hamilton cycle D_H in $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ such that not all arcs belonging to D_H have label 0. This implies that there exists an arc e on D_H such that $l(D_H - e) \neq 0$. Let $e = uv$, and let $e' = u'v'$ be an arc of C_H. Since $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] = K_{p,p}$ we have that $uu', vv' \in E(X)$, and consequently one can easily see that starting at the vertex u, following the cycle C_H till v, then using the edge vv', following the cycle D_H till u', and finally using the edge uu' gives a Hamilton cycle of $X_{\mathcal{B}}$ with non-zero label and thus X has a Hamilton cycle.

Assume now that not all arcs in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ have label 0. However, we can, without loss of generality, assume that there exists an arc e in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ with $l(e) = 0$. Moreover, without loss of generality we may assume that this arc e belongs to C_H. Then $l(C_H - e) = 0$. If all the arcs in $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ have label 0 then, by applying the argument from the preceding paragraph to D_H, one can see that X has a Hamilton path. Thus we may assume that there exists an arc e' in $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ with a non-zero label. Since every edge of $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ is contained in a Hamilton cycle, there exists a Hamilton cycle of D_H containing e'. Since $l(D_H) = 0$ and $l(e') \neq 0$ it follows that $l(D_H - e') \neq 0$. Now we can construct a Hamilton cycle of X in a similar manner as in the preceding paragraph.

Subcase 2.2. $\bar{H}^{\bar{\mathcal{C}}}$ is faithful.

By Proposition 2.1.9, either $\bar{H}^{\bar{\mathcal{C}}}$ is solvable and contains a normal Sylow p-subgroup P, or $\bar{H}^{\bar{\mathcal{C}}}$ is non-solvable and doubly transitive.

Subsubcase 2.2.1. $\bar{H}^{\bar{\mathcal{C}}}$ is solvable.

Then a Sylow p-subgroup P of $\bar{H}^{\bar{\mathcal{C}}}$ is normal in $\bar{H}^{\bar{\mathcal{C}}}$ and thus $\bar{\pi} \in P$. Since $\bar{H}^{\bar{\mathcal{C}}}$ is faithful and solvable, $\bar{H}^{\bar{\mathcal{C}}} \cong \bar{H} \leq A(1,p)$. Since $\bar{H}$ is primitive and $A(1,p)$ is of order $p(p-1)$, P is of order p, and so $\langle\bar{\pi}\rangle = P$. It follows that $\langle\bar{\pi}\rangle$ is a characteristic subgroup of $\bar{H}$, implying that $\langle\bar{\pi}\rangle$ is normal in $\bar{G}$, and finally that $\langle\pi\rangle$ is normal in G. But then X is a genuinely imprimitive graph with respect to an imprimitivity block system consisting of blocks of size p, and so, by Lemma 6.2.3, X has a Hamilton path.

Subsubcase 2.2.2. $\bar{H}^{\bar{\mathcal{C}}}$ is non-solvable.

Then $\bar{H}^{\bar{\mathcal{C}}}$ is doubly transitive, and thus either $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong K_p$ or $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong pK_1$. Moreover, either $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] \cong K_{p,p}$ or $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] \cong K_{p,p} - pK_2$. Observe that for $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] \ncong K_{p,p} - pK_2$ the existence of a Hamilton path in X can be proved as in Subcase 2.1. Thus let us assume that $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] \cong K_{p,p} - pK_2$. By Propositions 6.1.8 and 6.1.6, we may assume that $X[C_i, D_j] \cong 5K_2$ for all $i, j \in \mathbb{Z}_p$, $i \neq j$, that is, $d(C_i, D_j) = 1$ for every edge C_iD_j in $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}]$. Moreover, following the argument in the proof of Proposition 6.1.8 one can see that we may assume that all the edges C_iD_j in $X_{\mathcal{B}}$ carry label 0. Since X is connected it follows that $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle \cong K_p$. Therefore $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ is a connected p-circulant, and, by applying the same argument as in Subcase 2.1, we get that X has a Hamilton path.

Case 3. $(p : 2 : 5)$.

Then G/K acts on $X_{\mathcal{B}}$ imprimitively with p blocks of size 2, and by Proposition 2.2.1, there exists a transitive subgroup H/K of G/K with blocks of size p. Therefore there exists a transitive subgroup of $\mathrm{Aut}(X)$ such that with respect to this subgroup X is of type $(2 : p : 5)$, and so, by Case 2, X has a Hamilton path. ■

Lemma 6.2.5 *If the size of blocks in $\mathcal{B}$ is 10 then X has a Hamilton path.*

Proof. Note that $X_{\mathcal{B}}$ is a connected p-circulant, and so, by Proposition 6.1.2, $X_{\mathcal{B}}$ is edge-hamiltonian. Proposition 2.1.11 implies that N^B is a simple group of degree 10 for every $B \in \mathcal{B}$. By [132], the only transitive simple groups of degree 10 up to permutation isomorphism are the alternating groups A_{10}, A_6 and A_5. Since in each of these three groups subgroups of index 10 are maximal we can conclude that all these groups are primitive, and thus N^B is a primitive group of degree 10.

Suppose first that there exist two adjacent blocks $B, B' \in \mathcal{B}$ such that $X[B, B']$ is of valency no less than 3. Let C be a p-cycle in $X_{\mathcal{B}}$ that contains the edge BB' (such a cycle exists since $X_{\mathcal{B}}$ is edge-hamiltonian). Since the valency of $X[B, B']$ is no less than 3, there exist at least two edges with non-zero voltage, denote them by i and j, $i, j \in \mathbb{Z}_{10}$. If $(i, 10) = 1$, then the lift of C is clearly a cycle of length $10p$, and thus a Hamilton cycle of X. If $(i, 10) = 2$, then C lifts to two $5p$-cycles, and the connectivity of X implies that X has a Hamilton path. If, however, $(i, 10) = 5$, then $(j, 10) \neq 5$. Namely, if also $(j, 10) = 5$ then X has multiedges, which is not possible since X is a simple graph. Thus, either $(j, 10) = 1$ or $(j, 10) = 2$. In both cases X clearly contains a Hamilton path.

We may now assume that the valency between any two adjacent blocks is less than 3. If there exist two adjacent blocks $B, B' \in \mathcal{B}$, such that $X[B, B']$ has valency 2. Then, since $X[B, B']$ is vertex-transitive, it follows that $X[B, B']$ is isomorphic to one of the following graphs: C_{20}, $2C_{10}$ and $5C_4$. However, since N^B is primitive only the first case can occur, in particular $X[B, B'] \cong C_{20}$. Since $X[B, B']$ is of valency 2, there must be one edge with non-zero voltage, denote this voltage by $i \in \mathbb{Z}_{10}$. Since $X[B, B'] = C_{20}$ we have that $(i, 10) = 1$. Since $X_{\mathcal{B}}$ is edge-hamiltonian, there exists a Hamilton cycle in $X_{\mathcal{B}}$ containing the edge BB', and thus one can easily see that X has a Hamilton cycle in this case.

We may therefore assume that for any two adjacent blocks $B, B' \in \mathcal{B}$ the bipartite graph $X[B, B']$ is of valency 1, in particular $X[B, B'] \cong 10K_2$. If $X\langle B\rangle$ is a connected graph, then we can easily see that there is a Hamilton path in X. If $X\langle B\rangle$ is disconnected then $X\langle B\rangle \in \{2C_5, 2K_5, 5K_2, 10K_1\}$. However, since N^B is primitive we must have $X\langle B\rangle \cong 10K_1$. Since N^B is isomorphic to A_5, A_6 or A_{10} there exists a nontrivial automorphism $\alpha \in N$ such that α fixes a vertex in B. But then, since $X[B, B'] \cong 10K_2$ and $\mathcal{B}$ is an imprimitivity block system of G arising from orbits of a normal subgroup N of G, the connectivity of X implies that α fixes all the vertices of X, a contradiction. ■

Lemma 6.2.6 *If $p > 7$ and the size of blocks in $\mathcal{B}$ is $2p$ then X has a Hamilton path.*

Proof. Note that either $X_{\mathcal{B}} \cong K_5$ or $X_{\mathcal{B}} \cong C_5$. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_5\}$. Since $p > 7$, by Proposition 2.2.3, there exists a $(10, p)$-semiregular automorphism $\rho \in G$. Let $\mathcal{S} = \{S_i, S'_i \mid i \in \mathbb{Z}_5\}$ be the set of its orbits. By Proposition 2.2.4, each block in $\mathcal{B}$ is a union of two orbits of ρ. With no loss of generality we can assume that $B_0 = S_0 \cup S'_0$, $B_1 = S_1 \cup S'_1$, $B_2 = S_2 \cup S'_2$, $B_3 = S_3 \cup S'_3$ and $B_4 = S_4 \cup S'_4$.

Consider the subgraph $\bar{X}_{\mathcal{S}}$ of $X_{\mathcal{S}}$, which is obtained from $X_{\mathcal{S}}$ by deleting the edges $S_iS'_i$, $i \in \mathbb{Z}_5$ (if they exist). Observe that for any two adjacent blocks $B, B' \in \mathcal{B}$ we have that either $X_{\mathcal{S}}[B, B'] \cong K_4$ or $X_{\mathcal{S}}[B, B'] \cong 2K_2$.

Suppose that there exist $B, B' \in \mathcal{B}$ such that $X_{\mathcal{S}}[B, B'] \cong K_4$. Suppose that there also exists a pair of adjacent blocks $D, D' \in \mathcal{B}$ such that $X_{\mathcal{S}}[D, D'] \cong 2K_2$. Then, since for any two edges in $X_{\mathcal{B}} \in \{K_5, C_5\}$ there exists a Hamilton cycle of $X_{\mathcal{B}}$ containing both of these two edges, there exists a Hamilton cycle of $X_{\mathcal{B}}$ containing both edges BB' and DD', and thus this cycle gives rise to a Hamilton cycle of $X_{\mathcal{S}}$. Moreover, in view of regularity of X and regularity of the subgraphs $X\langle B\rangle$, $B \in \mathcal{B}$, this cycle contains a multiedge and so, by Proposition 6.1.6, X has a Hamilton cycle. We may therefore assume that the bipartite graphs $X_{\mathcal{S}}[B, B']$, $B, B' \in \mathcal{B}$, are pairwise isomorphic, in particular, either for any two adjacent blocks $B, B' \in \mathcal{B}$ we have $X_{\mathcal{S}}[B, B'] \cong K_4$ or for any two adjacent blocks $B, B' \in \mathcal{B}$ we have $X_{\mathcal{S}}[B, B'] \cong 2K_2$.

Below it will be convenient to have the following notation. For two adjacent blocks $B_i, B_j \in \mathcal{B}$ we will say that the bipartite subgraph $X_{\mathcal{S}}[B_i, B_j]$ is of *type* 0, of *type* 1 and of *type* 2 if, respectively, $E(X_{\mathcal{S}}[B_i, B_j]) = \{S_iS_j, S'_iS'_j\}$, $E(X_{\mathcal{S}}[B_i, B_j]) = \{S_iS'_j, S'_iS_j\}$ and $E(X_{\mathcal{S}}[B_i, B_j]) = \{S_iS_j, S_iS'_j, S'_iS_j, S'_iS'_j\}$. We will say that an edge in $X_{\mathcal{B}}$ is of type k if the corresponding bipartite subgraph in $X_{\mathcal{S}}$ is of type k. Note that, by the above paragraph, either all edges of $X_{\mathcal{B}}$ are of type 2 or there are all of type different from type 2. Moreover, any 5-cycle $C = u_0u_1u_2u_3u_4u_0$, $\{u_i \mid i \in \{0, 1, 2, 3, 4\}\} \subseteq V(X_{\mathcal{B}})$, in $X_{\mathcal{B}}$ can be represented by a vector $[i_0, i_1, i_2, i_3, i_4]$, where i_j is the type of the edge u_ju_{j+1}. In addition, we will say that the cycle C is of *type* $[i_0, i_1, i_2, i_3, i_4]$.

Now suppose that $X_{\mathcal{B}}$ does not contain edges of type 2. Also, suppose that $X_{\mathcal{B}} \cong K_5$. Then the edge set of $X_{\mathcal{B}}$ can be viewed as the set of two disjoint 5-cycles, say $\mathcal{C}$ and $\mathcal{C}'$. Without loss of generality we may assume that one of these 5-cycles, say $\mathcal{C}$, is of type $[\tau, 0, 0, 0, 0]$. If $\tau = 0$ then (for symmetry reasons) we may assume that $\mathcal{C}'$ is of one of the following types: $[1, 0, 0, 0, 0]$, $[1, 1, 0, 0, 0]$, $[1, 0, 1, 0, 0]$, $[1, 1, 1, 0, 0]$, $[1, 1, 0, 1, 0]$, $[1, 1, 1, 1, 0]$, $[1, 1, 1, 1, 1]$, or $[0, 0, 0, 0, 0]$. These give eight possibilities for the graph $\bar{X}_{\mathcal{S}}$. If, however, $\tau = 1$ then, by a detail consideration of all possible types for $\mathcal{C}'$, one can see that we get eight more possibilities for the graph $\bar{X}_{\mathcal{S}}$. In Figure 6.2 we show all these possibilities in the graph $X_{\mathcal{B}}$, whereas in Figure 6.3 we show all possible graphs $\bar{X}_{\mathcal{S}}$. In particular, for $X_{\mathcal{B}} \cong K_5$, $\bar{X}_{\mathcal{S}}$ is isomorphic to one of the graphs Y_i, $i \in \{0, 1, 2, \ldots, 15\}$. In addition, if $X_{\mathcal{B}} \cong C_5$ then we can clearly assume that $\bar{X}_{\mathcal{S}}$ is isomorphic to one of the graphs Y_{16} and Y_{17} in Figure 6.3 (see also Figure 6.2). If, however, $X_{\mathcal{B}}$ contains an edge of type 2 then all edges in $X_{\mathcal{B}}$ are of this type, and thus only two more possibilities occur. Let us denote the graph arising from this case by Y_{18} if $X_{\mathcal{B}} \cong K_5$ and by Y_{19} if $X_{\mathcal{B}} \cong C_5$.

Observe that $Y_0 \cong Y_{13} \cong Y_{14}$, $Y_1 \cong Y_{11} \cong Y_{15}$, $Y_2 \cong Y_5 \cong Y_9 \cong Y_{10}$, and $Y_3 \cong Y_6$. We may therefore assume that $\bar{X}_{\mathcal{S}}$ is isomorphic to one of the following twelve graphs: Y_0, Y_1, Y_2, Y_3, Y_4, Y_7, Y_8, Y_{12}, Y_{16}, Y_{17}, Y_{18} or Y_{19}.

If $X\langle B_0\rangle$ is a connected graph, then for each of its vertices there exists a Hamilton path of $X\langle B_0\rangle$ starting at that vertex, so X clearly has a Hamilton path in this case. We can thus assume that $X\langle B_0\rangle$ is not connected. As it is a vertex-transitive graph, it is isomorphic to $2pK_1$, to pK_2 or it is a disjoint union of two isomorphic connected p-circulants. We consider each of the three cases separately.

Case 1. $X\langle B_0\rangle \cong 2pK_1$.

Since X is connected, the quotient graph $X_{\mathcal{S}} = \bar{X}_{\mathcal{S}}$ is isomorphic to one of the graphs Y_i, $i \in \{0,1,2,3,4,8,12,17,18,19\}$. Then any edge of $X_{\mathcal{S}}$ lies on some Hamilton cycle of $X_{\mathcal{S}}$ and thus Proposition 6.1.6 implies that we can assume that no multiedge exists in X_ρ. By considering all Hamilton cycles in $X_{\mathcal{S}}$ one can easily see that the connectedness of X forces some Hamilton cycle of $X_{\mathcal{S}}$, whose lift contains a Hamilton cycle of X, to exist. The details are left to the reader.

Case 2. $X\langle B_0\rangle \cong pK_2$.

It is clear that $X[S_0, S_1] \cong pK_2$. Suppose first that

$$\bar{X}_{\mathcal{S}} \cong Y_i, \text{ where } i \in \{0,1,2,3,4,8,12,17,18,19\}.$$

Then, by Case 1, we may assume that no multiedge exists in $\bar{X}_\rho$, and moreover that all the edges in $\bar{X}_{\mathcal{S}}$ carry label 0. Observe also that in all cases there exists a 10-cycle C in $\bar{X}_{\mathcal{S}}$ such that the endvertices of the edges S_iS_i', $i \in \mathbb{Z}_5$, are antipodal vertices on the cycle C in $X_{\mathcal{S}}$. Note also that in all cases for any edge S_iS_i', $i \in \mathbb{Z}_5$, there exists a Hamilton cycle of $X_{\mathcal{S}}$ containing this edge, and therefore, by Proposition 6.1.6, we may assume that there is no multiedge in X_ρ. Also, if there exists a Hamilton cycle C of $X_{\mathcal{S}}$ such that $l(C) \neq 0$ then X has a Hamilton cycle. Therefore, we can assume that no such Hamilton cycle of $X_{\mathcal{S}}$ exists.

Let us relabel the vertices of $X_{\mathcal{S}}$ in such a way that $C = u_0u_1u_2u_3u_4u_5u_6u_7u_8u_9u_0$ and let the label of the arc u_iu_{i+5}, $i \in \mathbb{Z}_5$, be denoted by a_i. Since

$$u_0u_5u_4u_3u_2u_1u_6u_7u_8u_9u_0$$
$$u_0u_1u_6u_5u_4u_3u_2u_7u_8u_9u_0$$
$$u_0u_1u_2u_7u_6u_5u_4u_3u_8u_9u_0$$
$$u_0u_1u_2u_3u_8u_7u_6u_5u_4u_9u_0$$
$$u_0u_1u_2u_3u_4u_9u_8u_7u_6u_5u_0$$

are Hamilton cycles in $X_{\mathcal{S}}$ with labels $a_0 + a_1$, $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_4$, $a_4 + a_0$, respectively, we have that $a_0 = a_1 = a_2 = a_3 = a_4 \neq 0$. But, however, $u_0u_5u_6u_1u_2u_7u_8u_3u_4u_9u_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ whose label is equal to a_0, and so $a_0 = 0$, a contradiction.

Suppose now that $\bar{X}_{\mathcal{S}} \cong Y_i$, $i \in \{7,16\}$. Then every edge of $X_{\mathcal{S}}$ is contained on some Hamilton cycle of $X_{\mathcal{S}}$, and so Proposition 6.1.6 implies that we can assume that no multiedge exists in X_ρ. By considering all Hamilton cycles in $X_{\mathcal{S}}$ one can easily see that the connectedness of X forces some Hamilton cycle of $X_{\mathcal{S}}$, whose lift contains a Hamilton cycle of X, to exist. The details are left to the reader.

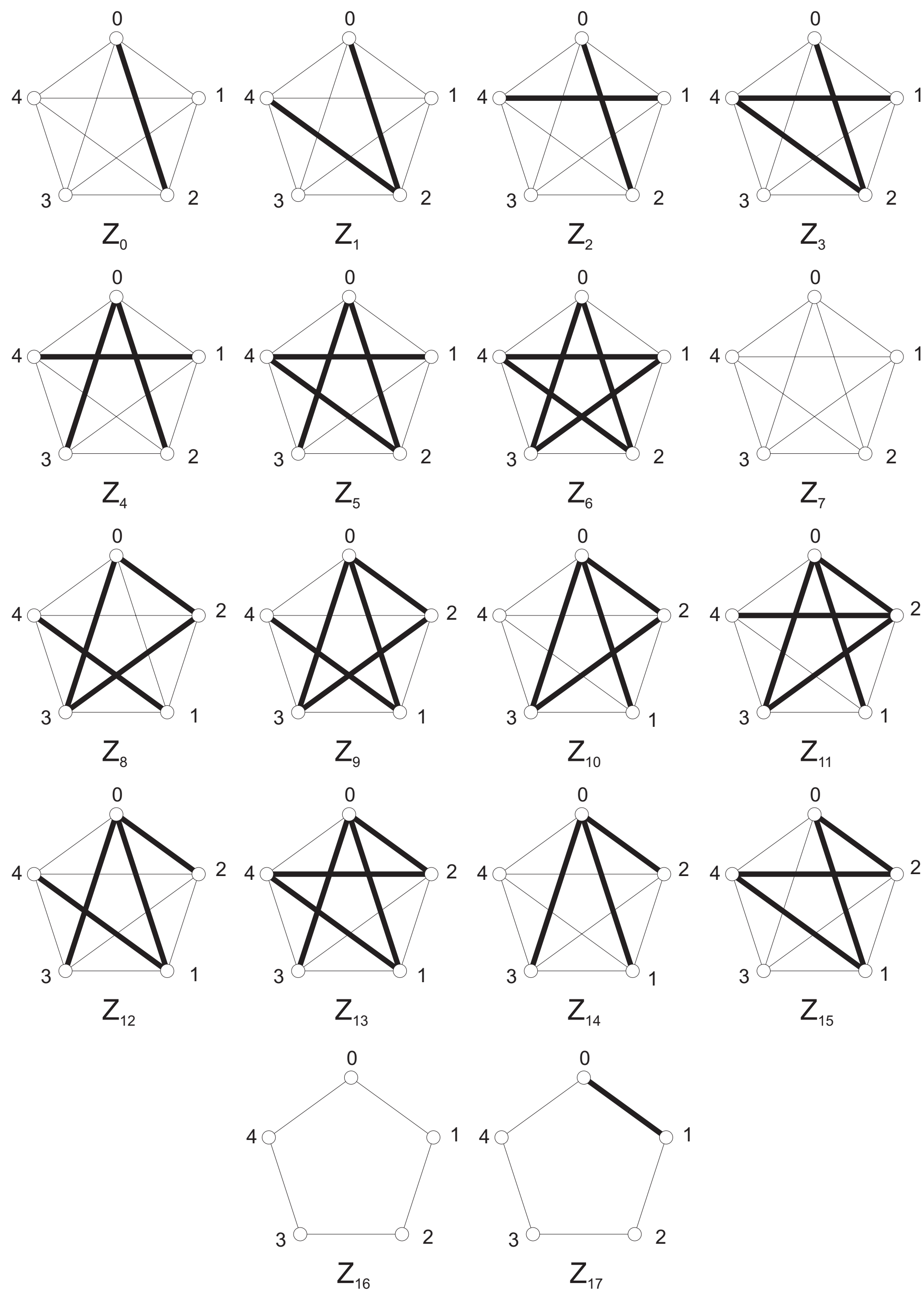

Figure 6.2: All possible structures of $\bar{X}_{\mathcal{S}}$ shown in $X_{\mathcal{B}}$ if $\bar{X}_{\mathcal{S}}$ does not contain edges of type 2. Bold edges are edges of type 1.

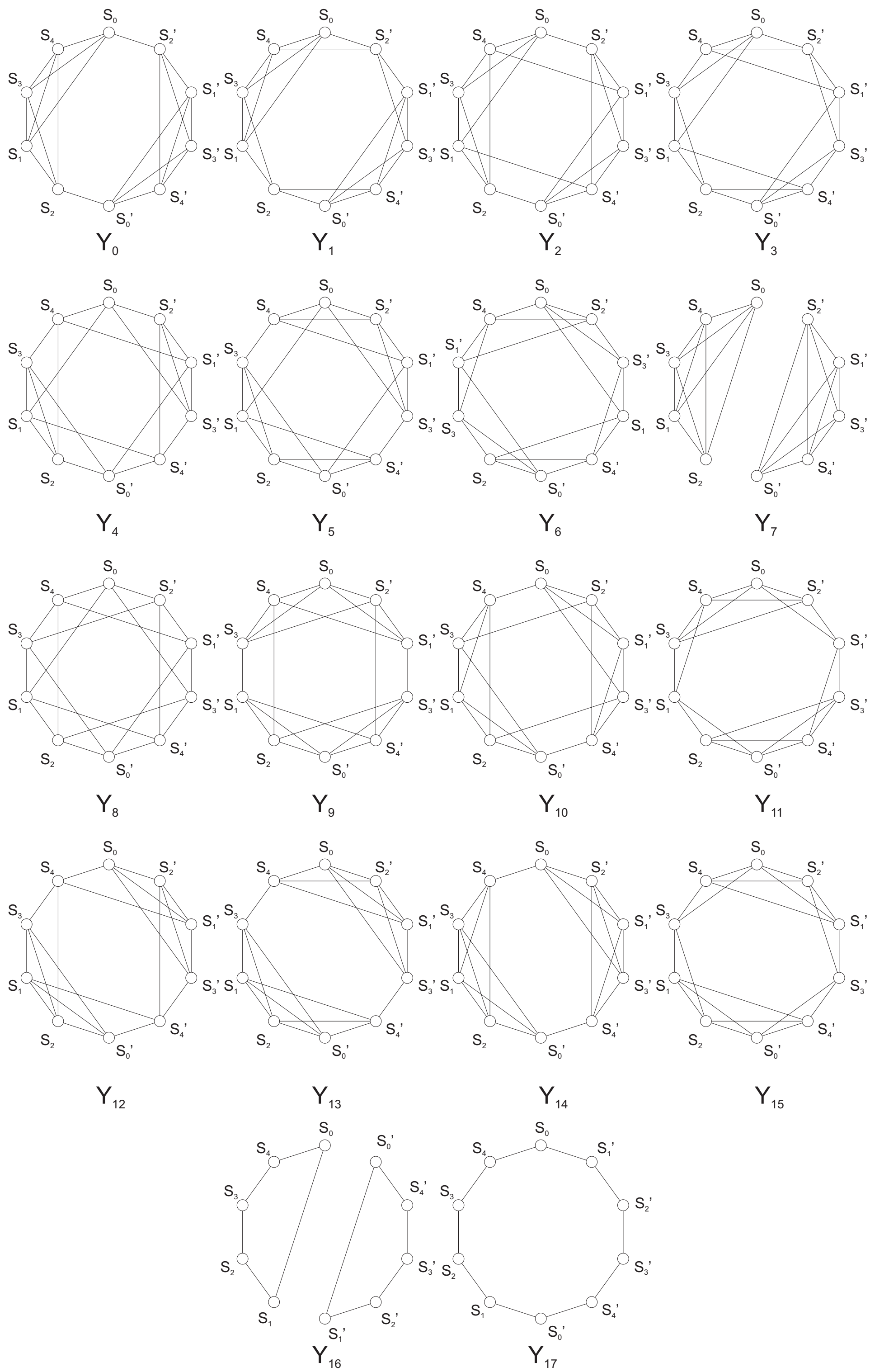

Figure 6.3: All possible graphs for $\bar{X}_{\mathcal{S}}$ where the graph Y_i corresponds to the graph Z_i in Figure 6.2.

CASE 3. $X\langle B_0\rangle$ is isomorphic to a disjoint union of two isomorphic connected p-circulants.

In view of connectedness of X the quotient graph $X_{\mathcal{S}} = \bar{X}_{\mathcal{S}}$ is isomorphic to

$$Y_i,\ i \in \{0, 1, 2, 3, 4, 8, 12, 17, 18, 19\}.$$

As the p-circulants are precisely the graphs $X\langle S_i\rangle$, where $i \in \mathbb{Z}_{10}$, a Hamilton path exists in X. This completes the proof. ∎

Lemma 6.2.7 *If $p > 7$ and the size of blocks in $\mathcal{B}$ is $5p$ then X has a Hamilton path.*

PROOF. Note that $|\mathcal{B}| = 2$ and $X_{\mathcal{B}} \cong K_2$. Let us denote the two blocks of $\mathcal{B}$ by B and B'. By Proposition 2.2.3 there exists a $(10, p)$-semiregular automorphism $\rho \in G$ of X. Let $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_{10}\}$ be the set of its orbits. By Proposition 2.2.4 each block in $\mathcal{B}$ is a union of five orbits of ρ. With no loss of generality we can assume that $B = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$ and $B' = S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9$.

By Proposition 2.1.11, for every $B \in \mathcal{B}$ the group N^B is simple. In addition, by Proposition 2.1.3, either N^B is primitive, or $N^B \in \{\mathrm{PSL}(2, 11), \mathrm{PSL}(m, q)\}$, where m is a prime and q is a prime power. The lemma will follow from the five claims given below. Throughout the proof we will frequently use the following fact about the number of edges between orbits of ρ in the subgraph $\bar{X}_{\mathcal{S}}$ of $X_{\mathcal{S}}$, which is obtained from $X_{\mathcal{S}}$ by deleting the edges between the orbits inside the blocks B and B' (if they exist):

$$\sum_{j \in \{5,\dots,9\}} d(S_i, S_j) = \sum_{j \in \{0,\dots,4\}} d(S_j, S_k) \tag{6.1}$$

for every $i \in \{0, 1, 2, 3, 4\}$ and $k \in \{5, 6, 7, 8, 9\}$.

CLAIM 1. If $X\langle B\rangle \cong X\langle B'\rangle$ is connected then X contains a Hamilton path.

Since $X\langle B\rangle \cong X\langle B'\rangle$ is a connected vertex-transitive graph of order $5p$, by Proposition 6.1.4, it has a Hamilton cycle, and thus, since X is connected, we can conclude that X contains a Hamilton path.

CLAIM 2. If $X\langle B\rangle \cong X\langle B'\rangle = 5pK_1$ then X contains a Hamilton path.

Since $X\langle B\rangle \cong X\langle B'\rangle \cong 5pK_1$ the graph $\bar{X}_{\mathcal{S}} = X_{\mathcal{S}}$ is a connected bipartite graph of order 10 with bipartition sets of size 5. Moreover, (6.1) implies that its minimal valency is not less than 2. From the list of all bipartite graphs of order 10, given in [90], we get, with the help of the program package MAGMA [7], that there exist 600 (of which five are regular) nonisomorphic bipartite graphs of order 10 with bipartition sets of size 5 and minimal valency no less than 2. We consider two cases depending on whether $X_{\mathcal{S}}$ is irregular or regular.

CASE 2.1. X_S is irregular.

SUBCASE 2.1.1. $X_{\mathcal{S}}$ possesses a Hamilton cycle.

In every such graph one can, with the help of MAGMA, find such a Hamilton cycle that in the corresponding multigraph this Hamilton cycle contains a multiedge, and thus, by Proposition 6.1.6, X contains a Hamilton cycle.

SUBCASE 2.1.2. $X_{\mathcal{S}}$ does not possess a Hamilton cycle.

With the exception of the two graphs shown in Figure 6.4 every graph belonging to this subfamily contains a disjoint union of a 4-cycle and a 6-cycle such that in the corresponding multigraph these two cycles both contain a multiedge. Therefore these two cycles lift to a disjoint union of a $4p$-cycle and $6p$-cycle in X, implying that X contains a Hamilton path.

We may now assume that $X_{\mathcal{S}}$ is one of the two graphs shown in Figure 6.4. We will show that both cases lead to a contradiction.

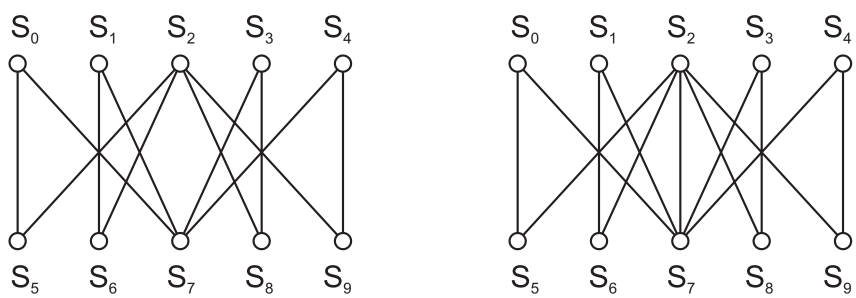

Figure 6.4: The two bipartite graphs of order 10 and minimal valency 2 not possessing a disjoint union of two cycles such that the union of their vertices covers the vertex set of the graph.

Let the vertices of X be labeled in such a way that $S_i = \{s_i^j \mid j \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_{10}$. Suppose first that $X_{\mathcal{S}}$ is the graph shown in the left-hand side picture of Figure 6.4. Then $X_{\mathcal{S}}$ has eight vertices of valency 2 and two vertices of valency 4. Since X is regular the edges S_iS_{i+5}, $i \in \mathbb{Z}_{10} \setminus \{2,7\}$, are multiedges in X_ρ. Hence the 6-cycles

$$S_2S_5S_0S_7S_1S_6S_2 \text{ and } S_2S_9S_4S_7S_3S_8S_2$$

in X_ρ both lift to a $6p$-cycle in X. Consequently, each of the vertices s_i^j, where $i \in \mathbb{Z}_{10} \setminus \{2,7\}$ and $j \in \mathbb{Z}_p$, is contained on at least one $6p$-cycle. On the other hand, since the above 6-cycles in X_ρ both contain S_2 and S_7, it follows that the vertices s_2^j and s_7^j, $j \in \mathbb{Z}_p$, are contained on at least two different $6p$-cycles in X. Now vertex-transitivity of X implies that also the vertices s_i^j, where $i \in \mathbb{Z}_{10} \setminus \{2,7\}$ and $j \in \mathbb{Z}_p$, are contained on at least two different $6p$-cycles. But since any $6p$-cycle in X containing a vertex s_i^j, $i \in \mathbb{Z}_{10} \setminus \{2,7\}$ and $j \in \mathbb{Z}_p$, and not arising from the above mentioned 6-cycles in X_ρ, must contain at least one vertex from S_7 (respectively, S_2), vertices from the orbits S_i, $i \in \mathbb{Z}_{10} \setminus \{2,7\}$ lie on less $6p$-cycles than those from the orbits S_2 and S_7. But this is clearly impossible in view of vertex-transitivity of X. That the other case (when $X_{\mathcal{S}}$ is isomorphic to the graph shown in the right-hand side picture of Figure 6.4) is not possible can be proved with a similar argument. The details are left to the reader.

CASE 2.2. $X_{\mathcal{S}}$ is a regular graph.

There are five regular bipartite graphs of order 10 with the two bipartition sets of size 5: C_{10}, $K_{5,5}$, $K_{5,5}-5K_2$, C_{10}^{+}, and $K_{5,5}-(C_6 \cup C_4)$. Observe that the first four of these graphs are vertex-transitive graphs, and thus the same argument as in the proof of Lemma 6.2.3 applies. We may, therefore, assume that $X_{\mathcal{S}} = K_{5,5}-(C_6 \cup C_4)$. Let the vertices of $X_{\mathcal{S}}$ be labeled in such a way as shown in Figure 6.5. Since

$$S_0S_9S_1S_8S_2S_5S_3S_7S_4S_6S_0$$

is a Hamilton cycle of $X_{\mathcal{S}}$, by Proposition 6.1.6, we can assume that all the edges on this cycle are single edges in X_ρ. Moreover, we can assume that all the edges on this cycle carry label 0. Further, since every edge of $X_{\mathcal{S}}$ lies on some Hamilton cycle of $X_{\mathcal{S}}$ we can assume that $X_{\mathcal{S}} = X_\rho$, that is, no multiedge exists in X_ρ. Next, since

$$\begin{gathered} S_0S_5S_3S_8S_2S_7S_4S_6S_1S_9S_0, \\ S_0S_5S_2S_7S_3S_8S_1S_6S_4S_9S_0, \\ S_0S_5S_3S_7S_2S_8S_1S_6S_4S_9S_0, \\ S_0S_6S_1S_8S_2S_5S_3S_7S_4S_9S_0, \\ S_0S_6S_4S_7S_2S_5S_3S_8S_1S_9S_0, \end{gathered}$$

are Hamilton cycles in $X_{\mathcal{S}}$, by Proposition 6.1.6, we can assume that for the labels of the arcs of $X_{\mathcal{S}}$ the following equations hold:

$$\begin{aligned} l(S_0S_5)+l(S_3S_8)+l(S_2S_7)+l(S_6S_1) &= 0, \\ l(S_0S_5)+l(S_2S_7)+l(S_3S_8)+l(S_1S_6)+l(S_4S_9) &= 0, \\ l(S_0S_5)+l(S_7S_2)+l(S_1S_6)+l(S_4S_9) &= 0, \\ l(S_6S_1)+l(S_4S_9) &= 0, \\ l(S_7S_2)+l(S_3S_8) &= 0. \end{aligned}$$

Combining together these equations one can easily get that $l(S_0S_5) = l(S_1S_6) = l(S_2S_7) = l(S_3S_8) = l(S_4S_9) = 0$ (using the fact that for $k \in \mathbb{Z}_p$, $p > 7$ a prime, we have $3k \equiv 0 \pmod p$ if and only if $k = 0$), and thus X is disconnected, a contradiction.

CLAIM 3. If $X\langle B\rangle \cong X\langle B'\rangle \cong pC_5$ then X contains a Hamilton path.

Observe that 5-cycles in the blocks B and B' form an imprimitivity block system $\mathcal{C}$ of G. Hence Proposition 2.2.4 implies that either $S_i \cap C = \emptyset$ or $|S_i \cap C| = 1$ for every $i \in \mathbb{Z}_p$ and every $C \in \mathcal{C}$. Since $p > 7$ it follows that $X\langle B\rangle_{\mathcal{S}} \cong X\langle B'\rangle_{\mathcal{S}} \cong C_5$.

The graph $\bar{X}_{\mathcal{S}}$ obtained from $X_{\mathcal{S}}$ by deleting the edges inside the blocks B and B' is clearly a bipartite graph of order 10 with each bipartition set of size 5. (Note that $\bar{X}_{\mathcal{S}}$ can be disconnected.) Checking the list of all bipartite graphs of order 10 given in [90], and using (6.1), one can see that either $\bar{X}_{\mathcal{S}}$ is isomorphic to the graph shown in the left-hand side picture of Figure 6.4 or $\bar{X}_{\mathcal{S}}$ contains $5K_2$.

CASE 3.1. $\bar{X}_{\mathcal{S}}$ is isomorphic to the graph shown in the left-hand side picture of Figure 6.4.

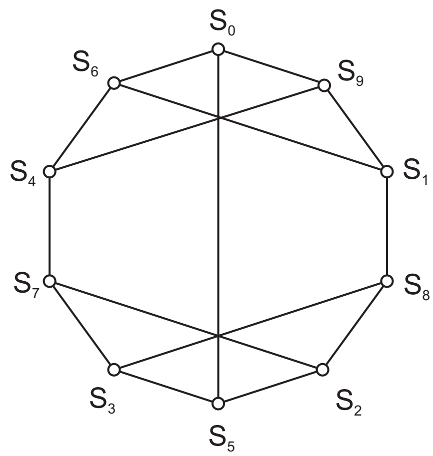

Figure 6.5: The graph $K_{5,5} - (C_6 \cup C_4)$.

Then we can, without loss of generality, assume that the graph $\tilde{X}_{\mathcal{S}}$ obtained from $X_{\mathcal{S}}$ by deleting the edges in B' is the graph shown in Figure 6.6. Also, the regularity of X and $X\langle B\rangle \cong X\langle B'\rangle$ combined together imply that $d(S_0, S_5) > 1$, and consequently any Hamilton cycle of $X_{\mathcal{S}}$ containing this edge, by Proposition 6.1.6, gives rise to a Hamilton cycle of X.

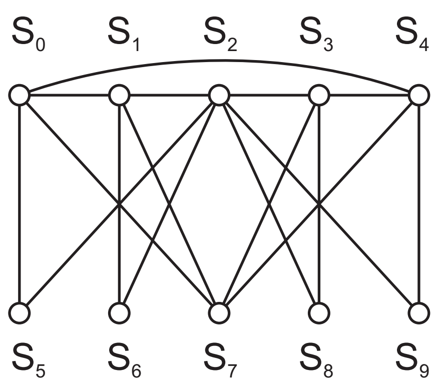

Figure 6.6: The graph $\tilde{X}_{\mathcal{S}}$ in case $\bar{X}_{\mathcal{S}}$ is isomorphic to the graph shown on the left-hand side picture of Figure 6.4.

Since $X\langle B'\rangle \cong C_5$, the vertex S_7 is adjacent to two of the vertices from the set $\{S_5, S_6, S_8, S_9\}$. In particular, we can assume (for symmetry reasons) that one of the following occurs in $X_{\mathcal{S}}$:

(i) $S_7S_8, S_7S_6 \in E(X_{\mathcal{S}})$;

(ii) $S_7S_8, S_7S_9 \in E(X_{\mathcal{S}})$;

(iii) $S_7S_8, S_7S_5 \in E(X_{\mathcal{S}})$;

(iv) $S_7S_9, S_7S_5 \in E(X_{\mathcal{S}})$.

If (i) occurs then $S_0S_5S_2S_9S_4S_3S_8S_7S_6S_1S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Next, if (ii) occurs then since $X\langle B'\rangle_{\mathcal{S}} \cong C_5$, we have that S_8 is either adjacent to S_5 or it is adjacent to S_6. For the first case,

$$S_0S_5S_8S_7S_9S_4S_3S_2S_6S_1S_0$$

is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. For the latter case, $S_0S_1S_6S_8S_7S_9S_4S_3S_2S_5S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Further, if (iii) occurs then $S_4S_9S_2S_6S_1S_0S_5S_7S_8S_3S_4$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Finally, if (iv) occurs then $S_1S_6S_2S_8S_3S_4S_9S_7S_5S_0S_1$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Therefore in all these cases Proposition 6.1.6 applies.

Case 3.2. $\bar{X}_{\mathcal{S}}$ contains $5K_2$.

Then $X_{\mathcal{S}}$ contains one of the four graphs Y_i, $i \in \{1,2,3,4\}$ shown in Figure 6.7, and thus four cases need to be considered. However, recall that all the edges in $X\langle B\rangle_{\mathcal{S}}$ and $X\langle B'\rangle_{\mathcal{S}}$ are single edges in X_ρ, and moreover, each edge in these two subgraphs carries label 0.

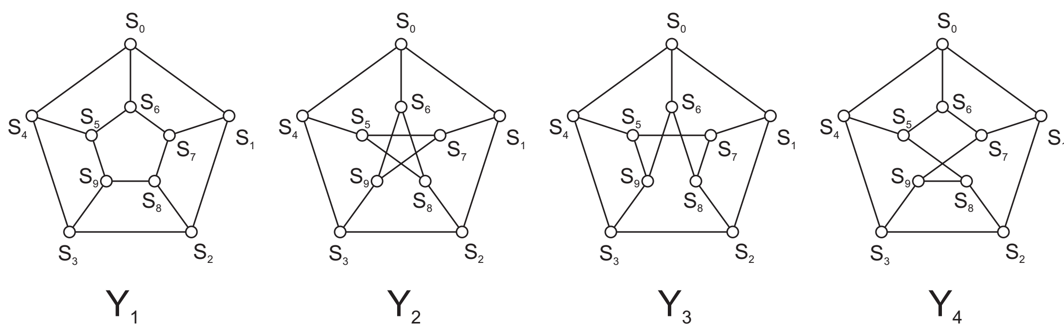

Figure 6.7: Possibilities for a subgraph of $X_{\mathcal{S}}$ in case $\bar{X}_{\mathcal{S}}$ contains $5K_2$.

Subcase 3.2.1. $X_{\mathcal{S}}$ contains Y_1.

Observe that every edge of Y_1 is contained in a Hamilton cycle and thus we may assume that all the edges of Y_1 are single edges in X_ρ. Further, since

$$C = S_0S_6S_5S_9S_8S_7S_1S_2S_3S_4S_0$$

is a Hamilton cycle of Y_1, we can assume that it carries label 0. Now, observe that with permuting the indices of the orbits S_i in C with the permutation

$$(0\ 1\ 2\ 3\ 4)(6\ 7\ 8\ 9\ 5)$$

we get four more Hamilton cycles in Y_1. Consequently, we can assume that all the edges of Y_1 carry label 0. Since X is connected, it follows that Y_1 is a proper subgraph of X_S, implying that there must exist an arc e in X_S carrying a non-zero label, without loss of generality we can assume that S_0 is an endvertex of this arc.

Since edges of Y_1 are single edges in X_ρ and Y_1 is symmetric we can assume that either $e = S_0S_5$ or $e = S_0S_8$. In the first case $S_0S_5S_6S_7S_1S_2S_8S_9S_3S_4S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ carrying a non-zero label, and in the second case $S_0S_8S_9S_5S_6S_7S_1S_2S_3S_4S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ carrying a non-zero label. Thus we can conclude that in both cases X has a Hamilton cycle.

Subcase 3.2.2. $X_{\mathcal{S}}$ contains Y_2 (the Petersen graph).

Recall that all the edges in $X\langle B\rangle_{\mathcal{S}}$ and $X\langle B'\rangle_{\mathcal{S}}$ are single edges in X_ρ, and moreover, each edge in these two subgraphs carries label 0. Since for any pair of edges from the set $A = \{S_0S_6, S_1S_7, S_2S_8, S_3S_9, S_4S_5\}$ there exists a disjoint union of two 5-cycles we can assume that at most one edge from A is a multiedge in X_ρ. (Namely, if two such edges exist in Y_2 then we have two disjoint 5-cycles in Y_2, each containing one of these edges, and thus they both give rise to a $5p$-cycle in X, implying that X has a Hamilton path.) If, however, exactly one of the edges from A is a multiedge in X_ρ, say that this edge is the edge S_0S_6, then the regularity of X implies that S_1 is an endvertex of an edge of $X_{\mathcal{S}}$ which is not contained in Y_2 and is not incident to neither of the vertices S_0 and S_6. This shows that S_1S_8 or S_1S_9 or S_1S_5 is an edge of $X_{\mathcal{S}}$. In each of these cases one can find a Hamilton cycle of X_ρ containing the multiedge S_0S_6, and thus Proposition 6.1.6 implies that X contains a Hamilton cycle. In particular

(a) if $S_1S_8 \in E(X_{\mathcal{S}})$ then $S_0S_6S_8S_1S_2S_3S_9S_7S_5S_4S_0$ is a Hamilton cycle of X_ρ containing the multiedge S_0S_6;

(b) if $S_1S_9 \in E(X_{\mathcal{S}})$ then $S_0S_6S_8S_5S_7S_9S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_ρ containing the multiedge S_0S_6;

(c) if $S_1S_5 \in E(X_{\mathcal{S}})$ then $S_0S_6S_8S_2S_1S_5S_7S_9S_3S_4S_0$ is a Hamilton cycle of X_ρ containing the multiedge S_0S_6;

We may therefore assume that no edge in A is a multiedge in X_ρ. Let the labels of the arcs S_0S_6, S_1S_7, S_2S_8, S_3S_9, S_4S_5 be denoted, respectively, by a, b, c, d and e. Observe that if there exist two disjoint 5-cycles in Y_2, whose lifts both contain a $5p$-cycle, then the connectedness of X implies that X has a Hamilton path. We can thus assume that no two such 5-cycles exist in $X_{\mathcal{B}}$. Considering all possible disjoint 5-cycles in Y_2 we have

$$a = c \text{ or } d = e, \quad b = d \text{ or } a = e, \quad c = e \text{ or } a = b, \quad a = d \text{ or } b = c, \quad b = e \text{ or } c = d.$$

Assume first that we have $a = b = c = d = e$. Then, since X is connected we get that Y_2 is a proper subgraph of $X_{\mathcal{S}}$. In particular, there exists a vertex which is an endvertex of an edge of $X_{\mathcal{S}}$, such that it is not contained in Y_2 and that the arc with the tail in this vertex carries a label $t \notin \{0, a\}$. Since Y_2 is symmetric we can assume that such a vertex is the vertex S_1, and thus for the edge having S_1 for one of its endvertices we again have possibilities (a)-(e) listed above. However, in this case also S_1S_6 can be such an edge. But since all the edges in Y_2 are single edges in X_ρ, we have (in view of the symmetry of Y_2) that it suffices to consider possibilities (a), (b) and (c). But in all these cases one can easily see that since $t \notin \{0, a\}$ the listed Hamilton cycles all give rise to a Hamilton cycle of X also in this case.

Assume now that not all labels a, b, c, d and e are equal. With no loss of generality assume that $a \neq b$, and so $c = e$. Suppose first that $a = d$. Then $b \neq d$, and so $d = a = e = c$. The reader may check that then the vertices of S_1 are contained on precisely two 5-cycles arising from Y_2, whereas the vertices of S_0 are contained on precisely four 5-cycles arising from Y_2, which in view of vertex-transitivity of X implies that Y_2 is a proper subgraph of $X_{\mathcal{S}}$. In particular, since edges in Y_2 are single edges it follows that each vertex of $X_{\mathcal{S}}$ lies on an edge that is not contained in Y_2. Consider all possibilities for such an edge with endvertex S_1. Let $t \in \mathbb{Z}_p$ be the label of the corresponding arc with the tail in S_1. For symmetry reasons (since $d = a = e = c$) it suffices to assume that either $S_1S_8 \in E(X_{\mathcal{S}})$ or $S_1S_9 \in E(X_{\mathcal{S}})$.

First, suppose that $S_1S_8 \in E(X_{\mathcal{S}})$. Then whenever $t \neq a$ and $a \neq 0$ the Hamilton cycle given in (a) lifts to a Hamilton cycle of X. Thus we may assume that $t = a$ (in addition, S_1S_8 is not a multiedge in X_ρ). But then the Hamilton cycle $S_0S_6S_9S_3S_2S_8S_1S_7S_5S_4S_0$ of $X_{\mathcal{S}}$ has a non-zero label $-t+b \neq 0$ (since $t = a \neq b$), and so it gives rise to a Hamilton cycle of X.

And second, suppose that $S_1S_9 \in E(X_{\mathcal{S}})$. Then whenever $t \neq a$ the Hamilton cycle given in (b) lifts to a Hamilton cycle of X. Thus we may assume that $t = a$. But then the Hamilton cycle $S_0S_6S_8S_2S_3S_9S_1S_7S_5S_4S_0$ of $X_{\mathcal{S}}$ has a non-zero label $b - a$, and so it gives rise to a Hamilton cycle of X.

If, however $a \neq d$ then $b = c$ and thus also $d = e = c = b$. As in the previous case in view of vertex-transitivity of X we get that Y_2 is a proper subgraph of $X_{\mathcal{S}}$. Also, if we consider all possibilities for edges of $X_{\mathcal{S}}$ lying outside the subgraph Y_2 and containing S_0 (in such a way as for S_1 in the previous case) we get that X has a Hamilton cycle also in this case.

Subcase 3.2.3. $X_{\mathcal{S}}$ contains Y_3.

Observe that every edge of Y_3 is contained in a Hamilton cycle and thus we may assume that all the edges of Y_3 are single edges in X_ρ. Further, since

$$\begin{array}{ll} S_0S_6S_9S_5S_4S_3S_2S_8S_7S_1S_0 & S_0S_6S_8S_7S_1S_2S_3S_9S_5S_4S_0 \\ S_0S_1S_2S_3S_9S_6S_8S_7S_5S_4S_0 & S_0S_1S_7S_5S_9S_6S_8S_2S_3S_4S_0 \end{array}$$

are Hamilton cycles in Y_3, we can assume that they all carry label 0. Combining together the corresponding equations for the labels of arcs in Y_3 imply that all the edges in Y_3 carry label 0. It follows that Y_3 is a proper subgraph of X_S, implying that there must exist an arc e in X_S carrying a non-zero label. From symmetry reasons we may assume that either S_0, or S_1, or S_2 is an endvertex of this arc. In particular, the following cases need to be considered: $e = S_0S_5$, $e = S_0S_8$, $e = S_1S_8$, $e = S_1S_9$, $e = S_1S_5$, $e = S_1S_6$, $e = S_2S_7$, $e = S_2S_9$, $e = S_2S_5$, and $e = S_2S_6$. However, since we have the following:

- if $e = S_0S_5$ then $S_0S_5S_7S_1S_2S_8S_6S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_0S_8$ then $S_0S_8S_6S_9S_5S_7S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;

- if $e = S_1S_8$ then $S_0S_6S_9S_5S_7S_8S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_9$ then $S_0S_6S_8S_7S_5S_9S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_5$ then $S_0S_6S_9S_3S_2S_8S_7S_1S_5S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_6$ then $S_0S_1S_6S_9S_5S_7S_8S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_2S_9$ then $S_0S_6S_8S_7S_5S_4S_3S_9S_2S_1S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_2S_5$ then $S_0S_1S_7S_8S_6S_9S_5S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;

we can assume that $e = S_2S_7$. Since X is regular and edges in Y_3 are single edges there exists an edge f with endvertex S_1 that is not contained in Y_3. In particular, either $f = S_1S_5$, or $f = S_1S_6$, or $f = S_1S_8$, or $f = S_1S_9$. Assume first that $f \neq S_1S_6$. Then, in view of the first part of this paragraph, we can assume that the edge f carries label 0, and consequently the following hold:

- if $e = S_1S_5$ then $S_0S_6S_8S_7S_2S_1S_5S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_8$ then $S_0S_6S_8S_1S_2S_7S_5S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_9$ then $S_0S_6S_8S_2S_7S_5S_4S_3S_2S_1S_0$ is a Hamilton cycle of X_S carrying a non-zero label.

It follows that either X has a Hamilton path or $f = S_1S_6$. If, however, $f = S_1S_6$ then since $S_0S_1S_6S_9S_5S_7S_8S_2S_3S_4S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ we can assume that it has label 0, and consequently that f carries label 0. But then $S_0S_1S_6S_8S_2S_7S_5S_9S_3S_4S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ carrying a non-zero label, and thus we can conclude that X possesses a Hamilton path also in this case.

Subcase 3.2.4. $X_{\mathcal{S}}$ contains Y_4.

Observe that every edge of Y_4 is contained in a Hamilton cycle and thus we may assume that all the edges of Y_4 are single edges in X_ρ. Further, since

$$\begin{array}{ll} S_0S_1S_2S_8S_5S_6S_7S_9S_3S_4S_0 & S_0S_6S_5S_8S_9S_7S_1S_2S_3S_4S_0 \\ S_0S_1S_2S_3S_4S_5S_8S_9S_7S_6S_0 & S_0S_6S_7S_1S_2S_3S_9S_8S_5S_4S_0 \\ S_0S_6S_5S_4S_3S_2S_8S_9S_7S_1S_0 & \end{array}$$

are Hamilton cycles in Y_4, we can assume that they all carry label 0. Combining together the corresponding equations for the labels of arcs in Y_4 imply that either X has a Hamilton cycle or all the edges in Y_4 carry label 0. In particular, we may assume that Y_4 is a proper subgraph of X_S, implying that there must exist an arc e in X_S with a non-zero label. From symmetry reasons we may assume that either

S_0, or S_1, or S_2 is an endvertex of this arc. In particular, the following cases need to be considered: $e = S_0S_5$, $e = S_0S_8$, $e = S_1S_8$, $e = S_1S_9$, $e = S_1S_5$, $e = S_1S_6$, $e = S_2S_7$, $e = S_2S_9$, $e = S_2S_5$, and $e = S_2S_6$. However, since the following hold:

- if $e = S_0S_5$ then $S_0S_5S_4S_3S_9S_8S_2S_1S_7S_6S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_0S_8$ then $S_0S_8S_9S_7S_1S_2S_3S_4S_5S_6S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_8$ then $S_0S_1S_8S_2S_3S_9S_7S_6S_5S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_9$ then $S_0S_1S_9S_7S_6S_5S_8S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_5$ then $S_0S_1S_5S_6S_7S_9S_8S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_6$ then $S_0S_6S_1S_7S_9S_3S_2S_8S_5S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_2S_7$ then $S_0S_1S_2S_7S_6S_5S_8S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_2S_9$ then $S_0S_1S_7S_6S_5S_8S_9S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_2S_5$ then $S_0S_1S_7S_6S_5S_2S_8S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_2S_6$ then $S_0S_1S_7S_9S_8S_5S_6S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label.

we can conclude that X has a Hamilton cycle in this case.

Claim 4. If N^B is primitive on B then X contains a Hamilton path.

Since N^B is primitive either $X\langle B\rangle$ is a connected graph or it is totally disconnected. In the former case Claim 1 applies whereas in the latter case Claim 2 applies.

Claim 5. If N^B is imprimitive on B then X contains a Hamilton path.

Let $T = N^B$. Since T is a non-abelian simple group, it is quasiprimitive on B. Let Δ be the corresponding imprimitivity block system of T on B. Since T is simple the kernel of the action of T on $X\langle B\rangle_\Delta$ is trivial, and so, by Proposition 2.1.2, T is a transitive group of degree $|\Delta|$. It follows that T is isomorphic to a subgroup of $S_{|\Delta|}$. Observe that Δ cannot consist of blocks of size p. Namely, if this is the case then $|\Delta| = 5$ and consequently $T \leq S_5$. But this is clearly impossible as $p > 7$ divides $|T|$ (since T is a group of degree $5p$). We therefore have that $\Delta = \{\Delta_i \mid i \in \mathbb{Z}_p\}$ consists of p blocks of size 5. Then $X\langle \Delta_i\rangle$, $i \in \mathbb{Z}_p$, is a vertex-transitive graph of order 5, and thus it is isomorphic to $5K_1$, C_5 or K_5. Observe also that the corresponding quotient action on $X\langle B\rangle_\Delta$ is primitive, implying that either $X\langle B\rangle_\Delta \cong K_p$ or $X\langle B\rangle_\Delta \cong pK_1$.

Suppose first that $X\langle \Delta_i\rangle \cong 5K_1$. If $X\langle B\rangle_\Delta \cong pK_1$ then $X\langle B\rangle \cong 5pK_1$, and, by Claim 2, X has a Hamilton path. We may therefore assume that $X\langle B\rangle_\Delta \cong K_p$. If

$X\langle B\rangle$ is a connected graph then, by Claim 1, X has a Hamilton path. If, however, $X\langle B\rangle$ is a disconnected graph (but clearly not totally disconnected) then, since $p > 5$ and since, by assumption the graphs induced on the blocks Δ_i, $i \in \mathbb{Z}_p$, are isomorphic to $5K_1$, the connected components of $X\langle B\rangle$ are of size p. However these connected components form an imprimitivity block system $\mathcal{D}$ of T on B consisting of blocks of size p, which in view of the argument given in the first paragraph of the proof of this claim is impossible.

Next, suppose that $X\langle \Delta_i\rangle \cong C_5$. If $X\langle B\rangle_\Delta \cong K_p$ then $X\langle B\rangle$ is a connected graph, and, by Claim 1, X has a Hamilton path. If, however, $X\langle B\rangle_\Delta \cong pK_1$ then $X\langle B\rangle$ is disconnected and $X\langle B\rangle \cong X\langle B'\rangle \cong pC_5$, and thus, by Claim 3, X has a Hamilton path.

Finally, suppose that $X\langle \Delta_i\rangle \cong K_5$. If $X\langle B\rangle$ is a connected graph then, by Claim 1, X has a Hamilton path. If, however, $X\langle B\rangle$ is disconnected then $X\langle B\rangle \cong pK_5$, and clearly also $X\langle B'\rangle \cong pK_5$. The imprimitivity block system Δ on B gives rise to an imprimitivity block system of G on X, and in addition, the quotient graph with respect to this imprimitivity block system is a bipartite connected vertex-transitive graph of order $2p$. Let $V(X) = \{u_i^j \mid i \in \mathbb{Z}_{10}, j \in \mathbb{Z}_p\}$ such that the sets $\{u_i^j \mid j \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_{10}$ are orbits of ρ. Then, without loss of generality, we may assume that $B = \{u_i^j \mid i \in \{0,1,2,3,4\}, j \in \mathbb{Z}_p\}$, $B' = \{u_i^j \mid i \in \{5,6,7,8,9\}, j \in \mathbb{Z}_p\}$, and that $F_j = \{u_i^j \mid i \in \{0,1,2,3,4\}\}$ and $T_j = \{u_i^j \mid i \in \{5,6,7,8,9\}\}$, $j \in \mathbb{Z}_p$, are the connected components of $X\langle B\rangle$ and $X\langle B'\rangle$, respectively. Then $\mathcal{C} = \{F_j, T_j \mid j \in \mathbb{Z}_p\}$ is an imprimitivity block system of G on X with blocks of size 5. Since X is connected there must exist two vertices in F_0 that have neighbors in two different blocks of $\mathcal{C}$ lying in B'. Since the graph induced on F_0 is isomorphic to K_5, we may, without loss of generality, assume that u_0^0 is adjacent to u_5^0 and that u_1^0 is adjacent to u_k^j, where $j \neq 0$ and $k \in \{5,6,7,8,9\}$. Now one can, with the use of a $(2,p)$-semiregular automorphism of $X_{\mathcal{C}}$ (arising from the $(10,p)$-semiregular automorphism ρ of X), see that each of these two edges gives a perfect matching in $X_{\mathcal{C}}$. Moreover, since $j \neq 0$, we have that the union of these two perfect matchings is a Hamilton cycle of $X_{\mathcal{C}}$. Since $X\langle F_j\rangle \cong K_5$ and $X\langle T_j\rangle \cong K_5$ are Hamilton-connected we can clearly conclude that X has a Hamilton path. ■

6.3 Quasiprimitive graphs of order 10p

Throughout this section let X denote a connected quasiprimitive graph of order $10p$, $p \geq 7$ a prime. In [89] a complete characterization of quasiprimitive graphs of order pqr, where p, q and r are distinct primes, was given via the well known generalized orbital graph construction relative to certain simple groups having an imprimitive permutation representation of degree pqr. All the possible group actions are given in Tables A and B in [89, pp. 298-299]. For our purposes (we require that $pqr = 10p'$) only a handful of group actions needs to be considered. They are given in Table 6.1. Note that only row 16 of Table 6.1 corresponds to an infinite family of actions giving rise to quasiprimitive graphs of order $10p$. As for the other rows of Table 6.1, each case is investigated separately. More precisely, we consider all the possible generalized orbital graphs and study their structural properties (using program package MAGMA [7]) which allows us to easily find a Hamilton cycle in these graphs.

row	p	Action
1	7	A_7 on cosets of $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$
2	11	$\mathrm{PSL}(2,11)$ on cosets of D_6
3	11	$\mathrm{PSL}(2,11)$ on cosets of $\mathbb{Z}_6$
4	31	$\mathrm{PSL}(3,5)$ on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_4.D_{12})$
5	31	$\mathrm{PSL}(3,5)$ on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_4.\mathrm{A}_4)$
6	11	M_{11} on cosets of M_9
7	31	$\mathrm{PSL}(3,5)$ on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_{24}.\mathbb{Z}_2)$
8	7	A_7 on cosets of $\mathrm{A}_4 \times \mathbb{Z}_3$
9	7	$\mathrm{PSL}(4,2)$ on cosets of $\mathbb{Z}_2^4 \rtimes (\mathrm{A}_3 \times S_3)$
10	7	$\mathrm{PSL}(4,2)$ on cosets of $\mathbb{Z}_2^4 \rtimes (\mathrm{A}_3 \rtimes S_3)$
11	31	$\mathrm{PSL}(5,2)$ on cosets of $\mathbb{Z}_2^6 \rtimes (\mathrm{A}_3 \times \mathrm{PSL}(3,2))$
12	13	$\mathrm{PSL}(2,25)$ on cosets of $\mathrm{PSL}(2,5)$
13	11	M_{11} on cosets of $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_8$
14	11	M_{11} on cosets of $\mathbb{Z}_3^2 \rtimes Q_8$
15	11	A_{11} on cosets of A_9
16	$\frac{k+1}{2}$	$\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$ where $5 \mid \frac{k-1}{2}$ and $k = s^m$

Table 6.1: Actions giving rise to quasiprimitive graphs of order $10p$.

Let G be a group acting on the cosets of its subgroup H in a natural way. Following the terminology of [64] we say that the set $\mathcal{O}(G,H)$ of generalized orbital graphs (in short GOGs) of this action is a *minimal connected orbital graph set* for this action if each connected GOG corresponding to this action contains some graph of $\mathcal{O}(G,H)$ as a spanning subgraph. As we are only interested in whether a given GOG contains a Hamilton path (or a Hamilton cycle) Proposition 6.1.1 implies that we can disregard the graphs from $\mathcal{O}(G,H)$ whose valencies are at least $[G:H]/3$. We let the remaining set of GOGs be the set $\mathcal{R}(G,H)$ of *relevant graphs* for this action. It is now clear that in order to show that each GOG corresponding to the above mentioned action of G contains a Hamilton path (Hamilton cycle) we only need to show that each GOG of $\mathcal{R}(G,H)$ has this property.

A graph X admitting an (m,n)-semiregular automorphism is completely determined by the so-called *symbol*. However, we define it here only for graphs admitting a $(10,p)$-semiregular automorphism. Let ρ be a $(10,p)$-semiregular automorphism and let S_i, $i \in \mathbb{Z}_{10}$, be its orbits. Choose $s_i \in S_i$ and define the following subsets of $\mathbb{Z}_p$. For $i,j \in \mathbb{Z}_{10}$, we let $R_{i,j} = \{r \in \mathbb{Z}_p \mid s_i \sim s_j^{\rho^r}\}$. Note that $R_{j,i} = -R_{i,j}$. It is clear that the collection of all $R_{i,j}$ completely determines X. The 10×10-matrix $\mathrm{M}_\rho(X) = (R_{i,j})_{i,j}$, whose (i,j)-th entry is the set $R_{i,j}$, is the *symbol* of X relative to $(\rho,\ s_0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ s_8,\ s_9)$. The symbols will be used in Sections 6.3

and 6.4 to give relevant quasiprimitive and primitive graphs of order $10p$, p a prime.

We now describe the method of obtaining $\mathcal{R}(G,H)$ for the action of row 2 of Table 6.1 in full detail. The other actions are dealt with in a similar way, so we only give the relevant graphs and leave the details to the reader. Each relevant graph X will be represented in a structural way given by some semiregular automorphism φ of X from which the existence of a Hamilton cycle will be clear. In the case when φ is $(10,p)$-semiregular its symbol (for the definition see Section 2.2.5) will be given.

Graphs corresponding to row 2 of Table 6.1: Note that these graphs are of order 110. In the action of $\mathrm{PSL}(2,11)$ on the cosets of D_6, we get that D_6 has 21 nontrivial suborbits, 9 of which are self-paired. Of the nine self-paired suborbits, one is of length 1 and two are of length 3, the others are of length 6. Of the twelve non-self-paired suborbits, 2 are of length 3, the others are of length 6. Denote these 21 nontrivial suborbits by U_i, $i \in \{1,2,\ldots,21\}$, where U_1 is of length 1, U_2 and U_3 are of length 3, the others are of length 6, $U_1, U_2, \ldots, U_9$ are the self-paired suborbits, and U_{2i} is paired with U_{2i+1} for $i \in \{5,6,\cdots,10\}$.

The unions $U_{2i} \cup U_{2i+1}$, where $i \in \{5,6,\ldots,10\}$, give rise to five nonisomorphic graphs, all of them are connected. Of these five graphs, three graphs admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, and thus, by Proposition 6.1.5, have a Hamilton cycle. The other two graphs are isomorphic to X_2 and X_3 of Table 6.2, respectively. Using an argument similar to the one used in the proof of Proposition 6.1.6, one can see that these two graphs both contain a Hamilton cycle.

For $i \in \{1,2,3,\ldots,9\}$ the graphs arising from the suborbits U_i, $i \in \{6,7,8,9\}$, are all connected. Moreover, the graphs arising from the suborbits U_7 and U_8 admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, and thus Proposition 6.1.5 implies that these graphs contain a Hamilton cycle. The graph arising from the suborbit U_6 is isomorphic to the graph arising from the suborbit U_9, and is isomorphic to the graph X_1 in Table 6.2. Proposition 6.1.6 implies that X_1 contains a Hamilton cycle.

The graphs arising from the suborbits U_i, $i \in \{1,2,3,4,5\}$, are disconnected, whereas the graphs arising from $U_1 \cup U_i$, $i \in \{2,3,4,5\}$, are connected and give rise to two nonisomorphic graphs X_4 and X_5 in Table 6.2. Proposition 6.1.6 implies that both graphs contain a Hamilton cycle. The graph X_4 is also given in Figure 6.1.

Finally, the unions $U_i \cup U_j$, where $i,j \in \{2,3,4,5\}$, give rise to three nonisomorphic connected graphs. Two of which admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, and thus, by Proposition 6.1.5, have a Hamilton cycle. The third graph is isomorphic to the graph X_6 in Table 6.2. Proposition 6.1.6 implies that this graph has a Hamilton cycle.

We have now clearly considered all the relevant graphs $\mathcal{R}(\mathrm{PSL}(2,11), D_6)$, and we can conclude that each connected GOG arising from the action of $\mathrm{PSL}(2,11)$ on the cosets of D_6, contains a Hamilton cycle.

Graphs corresponding to row 1 of Table 6.1: The relevant graphs are given in Table 6.3, and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 3 of Table 6.1: The relevant graphs are given in Table 6.4, and, by Proposition 6.1.6, each of them contains a Hamilton cycle.

Graphs corresponding to row 4 of Table 6.1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 5 of Table 6.1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 6 of Table 6.1: There are four connected relevant graphs. They all admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order. Thus Proposition 6.1.5 implies that these graphs have a Hamilton cycle.

Graphs corresponding to row 7 of Table 6.1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 8 of Table 6.1: The relevant graphs are given in Table 6.5. By Proposition 6.1.6, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 9 of Table 6.1: The relevant graphs are given in Table 6.6. By Proposition 6.1.6, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 10 of Table 6.1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 11 of Table 6.1: There is only one connected relevant graph. It admits a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order. By Proposition 6.1.5, this graph thus has a Hamilton cycle.

Graphs corresponding to row 12 of Table 6.1: The relevant graphs are given in Table 6.9. By Proposition 6.1.6, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 13 of Table 6.1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 14 of Table 6.1: The relevant graphs are given in Table 6.7. By Proposition 6.1.6, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 15 of Table 6.1: There are two connected relevant graphs. They both admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order. By Proposition 6.1.5, these graphs thus have a Hamilton cycle.

In view of the fact that the connected quasiprimitive graphs of orders $4p$, $2p^2$, and $6p$ (except for the truncation of the Petersen graph) contain a Hamilton cycle (see [61, 62, 79]), the results of this section imply that the following proposition holds.

Proposition 6.3.1 *Let X be a connected quasiprimitive graph of order $10p$, p a prime, which is not isomorphic to a quasiprimitive graph arising from the action of* $\mathrm{PSL}(2, k)$ *on cosets of* $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$. *Then X is the truncation of the Petersen graph or X is hamiltonian.*

	X_1	X_2	X_3	X_4	X_5	X_6
p	11	11	11	11	11	11
$\lvert V(X_i)\rvert$	110	110	110	110	110	110
val	6	12	10	4	7	9
$R_{0,0}$	$\emptyset$	$\pm4,\pm5$	±4	$\emptyset$	$\emptyset$	$\emptyset$
$R_{1,1}$	$\emptyset$	$\pm2,\pm5$	±4	$\emptyset$	$\emptyset$	$\emptyset$
$R_{2,2}$	±3	$\pm1,\pm3$	±1	$\emptyset$	$\emptyset$	$\emptyset$
$R_{3,3}$	$\emptyset$	$\emptyset$	±1	$\emptyset$	$\emptyset$	$\emptyset$
$R_{4,4}$	$\emptyset$	$\emptyset$	±5	$\emptyset$	$\emptyset$	$\emptyset$
$R_{5,5}$	±2	$\emptyset$	±5	$\emptyset$	$\emptyset$	$\emptyset$
$R_{6,6}$	±1	$\emptyset$	±3	$\emptyset$	$\emptyset$	$\emptyset$
$R_{7,7}$	±5	$\pm1,\pm4$	±2	$\emptyset$	$\emptyset$	$\emptyset$
$R_{8,8}$	±4	$\pm2,\pm3$	±3	$\emptyset$	$\emptyset$	$\emptyset$
$R_{9,9}$	$\emptyset$	$\pm4,\pm5$	±2	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,1}$	0	0, 7	0, 8	0	0	0
$R_{0,2}$	0, 3	0, 9	0, 10	0	0, 10	0, 8
$R_{0,3}$	$\emptyset$	0	0, 10	0	0	0, 3
$R_{0,4}$	0	0, 6	0, 7	0	0	0
$R_{0,5}$	0	0, 7	0, 4	$\emptyset$	0	0, 6
$R_{0,6}$	0	0	$\emptyset$	$\emptyset$	0	0
$R_{0,7}$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,8}$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,9}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{1,2}$	$\emptyset$	1	6, 7	$\emptyset$	0	8
$R_{1,3}$	$\emptyset$	10	6, 7	7	0	$\emptyset$
$R_{1,4}$	$\emptyset$	$\emptyset$	3, 7	$\emptyset$	0	0, 8
$R_{1,5}$	$\emptyset$	6	0, 7	0	$\emptyset$	0, 5
$R_{1,6}$	0	6, 10	$\emptyset$	0	$\emptyset$	$\emptyset$
$R_{1,7}$	0	6	$\emptyset$	$\emptyset$	0	0, 9
$R_{1,8}$	0, 4	$\emptyset$	$\emptyset$	$\emptyset$	0, 3	0
$R_{1,9}$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{2,3}$	0	3, 5	1, 10	$\emptyset$	0	6
$R_{2,4}$	$\emptyset$	3	$\emptyset$	1	$\emptyset$	$\emptyset$
$R_{2,5}$	$\emptyset$	3	$\emptyset$	$\emptyset$	0	$\emptyset$
$R_{2,6}$	$\emptyset$	$\emptyset$	0, 8	1	$\emptyset$	3
$R_{2,7}$	$\emptyset$	$\emptyset$	$\emptyset$	0	0	3, 10
$R_{2,8}$	$\emptyset$	$\emptyset$	0, 8	$\emptyset$	$\emptyset$	3, 7
$R_{2,9}$	7	0	$\emptyset$	$\emptyset$	0	$\emptyset$
$R_{3,4}$	7	1, 3	$\emptyset$	$\emptyset$	$\emptyset$	8, 9
$R_{3,5}$	$\emptyset$	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{3,6}$	2, 3	5, 8	0, 8	$\emptyset$	0, 6	4, 8
$R_{3,7}$	$\emptyset$	8	$\emptyset$	10	$\emptyset$	$\emptyset$
$R_{3,8}$	5	$\emptyset$	0, 8	0	0	8
$R_{3,9}$	$\emptyset$	0, 10	$\emptyset$	$\emptyset$	0	0
$R_{4,5}$	0	$\emptyset$	9, 10	3	0, 9	0
$R_{4,6}$	$\emptyset$	6	$\emptyset$	$\emptyset$	0	$\emptyset$
$R_{4,7}$	2, 8	6, 9	0, 6	$\emptyset$	0	$\emptyset$
$R_{4,8}$	10	2, 7	$\emptyset$	$\emptyset$	0	0, 9
$R_{4,9}$	9	5	0, 6	0	$\emptyset$	3
$R_{5,6}$	$\emptyset$	9, 10	$\emptyset$	$\emptyset$	0	0
$R_{5,7}$	$\emptyset$	$\emptyset$	2, 7	10	0	0
$R_{5,8}$	$\emptyset$	2, 7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{5,9}$	5	0, 2	2, 7	0	0	2, 3
$R_{6,7}$	$\emptyset$	2, 10	2, 4	$\emptyset$	$\emptyset$	0, 6
$R_{6,8}$	$\emptyset$	5	3, 8	0	0	$\emptyset$
$R_{6,9}$	$\emptyset$	9	2, 4	2	0	3, 4
$R_{7,8}$	$\emptyset$	2	7, 9	7	0	0
$R_{7,9}$	0	$\emptyset$	2, 9	$\emptyset$	0, 7	3
$R_{8,9}$	$\emptyset$	0	2, 4	0	0	1, 3

Table 6.2: Relevant graphs corresponding to the action of row 2 of Table 6.1.

	X_1	X_2
p	7	7
$\|V(X_i)\|$	70	70
val	6	18
$R_{0,0}$	$\emptyset$	$\emptyset$
$R_{1,1}$	$\emptyset$	±3
$R_{2,2}$	$\emptyset$	±2
$R_{3,3}$	$\emptyset$	±1
$R_{4,4}$	±3	±1
$R_{5,5}$	$\emptyset$	±1
$R_{6,6}$	$\emptyset$	±3
$R_{7,7}$	$\emptyset$	±2
$R_{8,8}$	±1	±2
$R_{9,9}$	±2	±3
$R_{0,1}$	0	0
$R_{0,2}$	0	0
$R_{0,3}$	0	0
$R_{0,4}$	$0,3$	$0,3$
$R_{0,5}$	0	0
$R_{0,6}$	$\emptyset$	0
$R_{0,7}$	$\emptyset$	0
$R_{0,8}$	$\emptyset$	$0,1,4,5$
$R_{0,9}$	$\emptyset$	0
$R_{1,2}$	2	$\emptyset$
$R_{1,3}$	0	±1
$R_{1,4}$	$\emptyset$	$1,3,5$
$R_{1,5}$	0	$\emptyset$
$R_{1,6}$	0	$3,5$
$R_{1,7}$	0	$4,5,6$
$R_{1,8}$	$\emptyset$	$2,3$
$R_{1,9}$	$\emptyset$	$1,2,3$
$R_{2,3}$	$\emptyset$	$1,4$
$R_{2,4}$	$0,3$	$2,3,6$
$R_{2,5}$	$\emptyset$	$\emptyset$
$R_{2,6}$	5	$1,2$
$R_{2,7}$	5	$1,4,5$
$R_{2,8}$	$\emptyset$	$3,4$
$R_{2,9}$	$\emptyset$	$4,5,6$
$R_{3,4}$	$\emptyset$	$1,5$
$R_{3,5}$	0	$0,4$
$R_{3,6}$	$\emptyset$	$\emptyset$
$R_{3,7}$	3	$1,4$
$R_{3,8}$	$0,1$	$\emptyset$
$R_{3,9}$	$\emptyset$	$4,6$
$R_{4,5}$	$\emptyset$	$1,3,5$
$R_{4,6}$	$\emptyset$	$1,3$
$R_{4,7}$	$\emptyset$	$\emptyset$
$R_{4,8}$	$\emptyset$	$3,6$
$R_{4,9}$	$\emptyset$	$\emptyset$
$R_{5,6}$	6	$1,3$
$R_{5,7}$	$\emptyset$	$2,3,6$
$R_{5,8}$	$\emptyset$	$2,6$
$R_{5,9}$	$0,2$	$1,3,5$
$R_{6,7}$	0	$4,5$
$R_{6,8}$	$\emptyset$	$\emptyset$
$R_{6,9}$	$1,3$	$1,6$
$R_{7,8}$	$4,5$	$1,2$
$R_{7,9}$	$\emptyset$	$\emptyset$
$R_{8,9}$	$\emptyset$	$4,5$

Table 6.3: Relevant graphs corresponding to the action of row 1 of Table 6.1.

	X_1	X_2
p	11	11
$\|V(X_i)\|$	110	110
val	12	6
$R_{0,0}$	±3	±1
$R_{1,1}$	±1	$\emptyset$
$R_{2,2}$	±2	$\emptyset$
$R_{3,3}$	±2	±2
$R_{4,4}$	±1	$\emptyset$
$R_{5,5}$	±3	±5
$R_{6,6}$	±4	$\emptyset$
$R_{7,7}$	±4	±4
$R_{8,8}$	±5	$\emptyset$
$R_{9,9}$	±4	±3
$R_{0,1}$	$0,8$	$0,1$
$R_{0,2}$	$0,2$	$0,10$
$R_{0,3}$	$0,2$	$\emptyset$
$R_{0,4}$	$0,8$	$\emptyset$
$R_{0,5}$	$0,5$	$\emptyset$
$R_{0,6}$	$\emptyset$	$\emptyset$
$R_{0,7}$	$\emptyset$	$\emptyset$
$R_{0,8}$	$\emptyset$	$\emptyset$
$R_{0,9}$	$\emptyset$	$\emptyset$
$R_{1,2}$	$\emptyset$	10
$R_{1,3}$	$\emptyset$	$0,2$
$R_{1,4}$	±1	0
$R_{1,5}$	$0,8$	$\emptyset$
$R_{1,6}$	$0,10$	$\emptyset$
$R_{1,7}$	$0,10$	$\emptyset$
$R_{1,8}$	$\emptyset$	$\emptyset$
$R_{1,9}$	$\emptyset$	$\emptyset$
$R_{2,3}$	±2	$\emptyset$
$R_{2,4}$	$\emptyset$	$\emptyset$
$R_{2,5}$	$6,8$	$0,6$
$R_{2,6}$	$\emptyset$	0
$R_{2,7}$	$\emptyset$	$\emptyset$
$R_{2,8}$	$0,5$	$\emptyset$
$R_{2,9}$	$0,6$	$\emptyset$
$R_{3,4}$	$\emptyset$	$0,9$
$R_{3,5}$	$6,8$	$\emptyset$
$R_{3,6}$	$\emptyset$	$\emptyset$
$R_{3,7}$	$\emptyset$	$\emptyset$
$R_{3,8}$	$0,5$	$\emptyset$
$R_{3,9}$	$0,6$	$\emptyset$
$R_{4,5}$	$0,8$	$\emptyset$
$R_{4,6}$	$0,10$	$\emptyset$
$R_{4,7}$	$0,10$	$0,4$
$R_{4,8}$	$\emptyset$	0
$R_{4,9}$	$\emptyset$	$\emptyset$
$R_{5,6}$	$\emptyset$	$0,5$
$R_{5,7}$	$\emptyset$	$\emptyset$
$R_{5,8}$	$\emptyset$	$\emptyset$
$R_{5,9}$	$0,2$	$\emptyset$
$R_{6,7}$	$4,7$	$\emptyset$
$R_{6,8}$	±2	1
$R_{6,9}$	$4,8$	$0,8$
$R_{7,8}$	±2	$0,7$
$R_{7,9}$	$4,8$	$\emptyset$
$R_{8,9}$	$0,1$	$7,10$

Table 6.4: Relevant graphs corresponding to the action of row 3 of Table 6.1.

	X_1	X_2	X_3
p	7	7	7
$\lvert V(X_i)\rvert$	70	70	70
val	8	12	12
$R_{0,0}$	±3	±1	$\emptyset$
$R_{1,1}$	±3	±3	$\emptyset$
$R_{2,2}$	$\emptyset$	±2	$\emptyset$
$R_{3,3}$	±2	±1	$\emptyset$
$R_{4,4}$	$\emptyset$	±1	$\emptyset$
$R_{5,5}$	$\emptyset$	±1	$\emptyset$
$R_{6,6}$	±2	±3	$\emptyset$
$R_{7,7}$	±2	±2	$\emptyset$
$R_{8,8}$	±3	±2	$\emptyset$
$R_{9,9}$	±1	±3	$\emptyset$
$R_{0,1}$	0	0, 6	0, 1
$R_{0,2}$	0	0, 6	0
$R_{0,3}$	0, 6	0, 2	0, 2
$R_{0,4}$	0	0, 1	0, 6
$R_{0,5}$	0	0	0, 6
$R_{0,6}$	$\emptyset$	0	0
$R_{0,7}$	$\emptyset$	$\emptyset$	0, 5
$R_{0,8}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,9}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{1,2}$	5	0	5
$R_{1,3}$	3	$\emptyset$	2, 6
$R_{1,4}$	$\emptyset$	1	0, 3
$R_{1,5}$	5	2, 5	2, 5
$R_{1,6}$	0	3, 6	1
$R_{1,7}$	0	$\emptyset$	$\emptyset$
$R_{1,8}$	0	$\emptyset$	0, 6
$R_{1,9}$	0	$\emptyset$	$\emptyset$
$R_{2,3}$	3	4	5
$R_{2,4}$	2	1, 5, 6	1, 5, 6
$R_{2,5}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{2,6}$	1	$\emptyset$	$\emptyset$
$R_{2,7}$	1	1, 6	0, 1
$R_{2,8}$	5	0, 4	2, 6
$R_{2,9}$	5	$\emptyset$	0, 2
$R_{3,4}$	4	2	3, 5
$R_{3,5}$	4	0, 5	1, 6
$R_{3,6}$	$\emptyset$	0, 5	3
$R_{3,7}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{3,8}$	$\emptyset$	0, 3	$\emptyset$
$R_{3,9}$	$\emptyset$	$\emptyset$	2, 5
$R_{4,5}$	5	$\emptyset$	$\emptyset$
$R_{4,6}$	0	$\emptyset$	$\emptyset$
$R_{4,7}$	0	1, 3	1
$R_{4,8}$	0	1, 5	4
$R_{4,9}$	0	0	6
$R_{5,6}$	1	0, 2, 6	0, 2, 6
$R_{5,7}$	1	5	5
$R_{5,8}$	5	4	4
$R_{5,9}$	5	0, 1	4
$R_{6,7}$	±2	4	4, 5
$R_{6,8}$	$\emptyset$	1	1, 5
$R_{6,9}$	$\emptyset$	4, 5	0, 2
$R_{7,8}$	$\emptyset$	$\emptyset$	4, 5
$R_{7,9}$	$\emptyset$	4, 6	1, 3
$R_{8,9}$	1, 6	3, 4	3, 6

Table 6.5: Relevant graphs corresponding to the action of row 8 of Table 6.1.

	X_1	X_2
p	7	7
$\lvert V(X_i)\rvert$	70	70
val	16	16
$R_{0,0}$	$\emptyset$	$\emptyset$
$R_{1,1}$	±2	±2
$R_{2,2}$	±1	±3
$R_{3,3}$	±1	$\emptyset$
$R_{4,4}$	±3	$\emptyset$
$R_{5,5}$	$\emptyset$	±2
$R_{6,6}$	±2	$\emptyset$
$R_{7,7}$	$\emptyset$	±1
$R_{8,8}$	±3	±3
$R_{9,9}$	$\emptyset$	$\emptyset$
$R_{0,1}$	0	0
$R_{0,2}$	0, 1, 6	0, 1, 6
$R_{0,3}$	0	0, 2, 5
$R_{0,4}$	0, 1, 4	0, 2, 6
$R_{0,5}$	0, 4, 6	0, 3, 6
$R_{0,6}$	0, 2, 4	0
$R_{0,7}$	0	0
$R_{0,8}$	0	0
$R_{0,9}$	$\emptyset$	$\emptyset$
$R_{1,2}$	$\emptyset$	2, 3
$R_{1,3}$	0, 5	3, 6
$R_{1,4}$	$\emptyset$	1, 4, 5
$R_{1,5}$	4	4, 5
$R_{1,6}$	2, 4	1
$R_{1,7}$	0, 2, 4	$\emptyset$
$R_{1,8}$	0, 4	$\emptyset$
$R_{1,9}$	0, 2, 4	1, 2, 4
$R_{2,3}$	0, 6	$\emptyset$
$R_{2,4}$	1, 2	4
$R_{2,5}$	4, 5, 6	$\emptyset$
$R_{2,6}$	1, 6	2, 3, 4
$R_{2,7}$	1	2, 4
$R_{2,8}$	$\emptyset$	3, 5
$R_{2,9}$	4	3
$R_{3,4}$	$\emptyset$	3
$R_{3,5}$	6	$\emptyset$
$R_{3,6}$	$\emptyset$	1, 4, 6
$R_{3,7}$	0, 1, 2	3, 6
$R_{3,8}$	0, 6	1, 3
$R_{3,9}$	4, 5, 6	0
$R_{4,5}$	2, 3, 6	2
$R_{4,6}$	0, 4	$\emptyset$
$R_{4,7}$	0	2, 4, 6
$R_{4,8}$	3, 6	2, 3, 4
$R_{4,9}$	2	1
$R_{5,6}$	0, 2, 5	2, 5, 6
$R_{5,7}$	$\emptyset$	3, 6
$R_{5,8}$	0	5, 6
$R_{5,9}$	0	5
$R_{6,7}$	0	3
$R_{6,8}$	$\emptyset$	5
$R_{6,9}$	5	2, 4, 5
$R_{7,8}$	0, 3, 6	$\emptyset$
$R_{7,9}$	2, 3, 5	0, 3, 5
$R_{8,9}$	0, 3, 6	1, 2, 3

Table 6.6: Relevant graphs corresponding to the action of row 9 of Table 6.1.

	X_1
p	11
$\|V(X_i)\|$	110
val	18
$R_{0,0}$	± 4
$R_{1,1}$	± 5
$R_{2,2}$	± 3
$R_{3,3}$	± 2
$R_{4,4}$	± 1
$R_{5,5}$	± 2
$R_{6,6}$	± 1
$R_{7,7}$	± 5
$R_{8,8}$	± 4
$R_{9,9}$	± 2
$R_{0,1}$	0
$R_{0,2}$	0
$R_{0,3}$	02, 3, 5
$R_{0,4}$	0, 34, 10
$R_{0,5}$	0, 4
$R_{0,6}$	0
$R_{0,7}$	0
$R_{0,8}$	0
$R_{0,9}$	$\emptyset$
$R_{1,2}$	2, 3, 7, 9
$R_{1,3}$	7
$R_{1,4}$	3, 8
$R_{1,5}$	1
$R_{1,6}$	1, 2, 9, 10
$R_{1,7}$	1
$R_{1,8}$	0
$R_{1,9}$	0
$R_{2,3}$	10
$R_{2,4}$	5
$R_{2,5}$	5
$R_{2,6}$	3
$R_{2,7}$	0, 8
$R_{2,8}$	9
$R_{2,9}$	7
$R_{3,4}$	0
$R_{3,5}$	1
$R_{3,6}$	0, 2, 6, 8
$R_{3,7}$	5
$R_{3,8}$	3, 5
$R_{3,9}$	8
$R_{4,5}$	2, 5, 8, 9
$R_{4,6}$	0
$R_{4,7}$	6
$R_{4,8}$	0, 5, 9, 10
$R_{4,9}$	3
$R_{5,6}$	9
$R_{5,7}$	7
$R_{5,8}$	2
$R_{5,9}$	0, 1, 2, 9
$R_{6,7}$	3
$R_{6,8}$	8
$R_{6,9}$	9, 10
$R_{7,8}$	1, 3, 7, 10
$R_{7,9}$	3, 4, 6, 9
$R_{8,9}$	2

Table 6.7: Relevant graphs corresponding to the action of row 14 of Table 6.1.

	X_1
p	19
$\|V(X_i)\|$	190
val	36
$R_{0,0}$	± 5
$R_{1,1}$	± 9
$R_{2,2}$	± 1
$R_{3,3}$	± 3
$R_{4,4}$	± 8
$R_{5,5}$	± 7
$R_{6,6}$	$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9$
$R_{7,7}$	± 2
$R_{8,8}$	± 6
$R_{9,9}$	± 4
$R_{0,1}$	7, 12, 13, 18
$R_{0,2}$	0, 1, 5, 6
$R_{0,3}$	0, 11, 14, 16
$R_{0,4}$	0, 3, 8, 14
$R_{0,5}$	0, 5, 12, 17
$R_{0,6}$	0, 14
$R_{0,7}$	0, 3, 5, 17
$R_{0,8}$	0, 5, 6, 11
$R_{0,9}$	0, 4, 14, 18
$R_{1,2}$	5, 6, 14, 15
$R_{1,3}$	0, 6, 9, 16
$R_{1,4}$	0, 8, 9, 17
$R_{1,5}$	5, 7, 14, 17
$R_{1,6}$	0, 9
$R_{1,7}$	3, 5, 12, 14
$R_{1,8}$	1, 5, 11, 14
$R_{1,9}$	0, 4, 9, 13
$R_{2,3}$	10, 11, 13, 14
$R_{2,4}$	2, 3, 13, 14
$R_{2,5}$	50, 11, 12, 18
$R_{2,6}$	13, 14
$R_{2,7}$	0, 16, 17, 18
$R_{2,8}$	0, 5, 6, 18
$R_{2,9}$	13, 14, 17, 18
$R_{3,4}$	0, 3, 8, 11
$R_{3,5}$	1, 5, 8, 17
$R_{3,6}$	0, 3
$R_{3,7}$	3, 5, 6, 8
$R_{3,8}$	5, 8, 11, 14
$R_{3,9}$	0, 3, 4, 7
$R_{4,5}$	5, 9, 16, 17
$R_{4,6}$	0, 11
$R_{4,7}$	3, 5, 14, 16
$R_{4,8}$	3, 5, 11, 16
$R_{4,9}$	0, 4, 11, 15
$R_{5,6}$	2, 14
$R_{5,7}$	0, 5, 7, 17
$R_{5,8}$	0, 6, 7, 13
$R_{5,9}$	2, 6, 14, 18
$R_{6,7}$	3, 5
$R_{6,8}$	5, 11
$R_{6,9}$	0, 4
$R_{7,8}$	0, 2, 6, 8
$R_{7,9}$	1, 14, 16, 18
$R_{8,9}$	8, 12, 14, 18

Table 6.8: Relevant graphs corresponding to the action of row 2 of Table 6.10.

	X_1	X_2	X_3	X_4	X_5
p	13	13	13	13	13
$\|V(X_i)\|$	130	130	130	130	130
val	12	20	30	30	24
$R_{0,0}$	$\emptyset$	±1	$\pm1, \pm5$	±4	$\emptyset$
$R_{1,1}$	$\pm2, \pm6$	$\pm2, \pm4$	±4	±5	$\emptyset$
$R_{2,2}$	$\pm3, \pm4$	±5	$\pm1, \pm5$	±6	$\emptyset$
$R_{3,3}$	$\emptyset$	±1	±3	±1	$\emptyset$
$R_{4,4}$	±5	$\pm3, \pm6$	±2	±1	$\emptyset$
$R_{5,5}$	±1	$\pm2, \pm4$	±2	±6	$\emptyset$
$R_{6,6}$	$\emptyset$	±1	±6	±5	$\pm3, \pm5, \pm6$
$R_{7,7}$	$\emptyset$	±5	±3	±4	$\pm1, \pm2, \pm4$
$R_{8,8}$	$\emptyset$	$\pm3, \pm6$	$\emptyset$	$\pm2, \pm3$	$\pm3, \pm5, \pm6$
$R_{9,9}$	$\emptyset$	$\emptyset$	±4	$\pm2, \pm3$	$\pm1, \pm2, \pm4$
$R_{0,1}$	0	0, 2	0, 2, 6, 8, 9, 12	0, 4, 5, 8	0, 3
$R_{0,2}$	0	0, 3, 8	0, 8, 10, 11	0, 4, 7, 10	0, 1, 8, 9
$R_{0,3}$	0, 9, 11	0, 1, 2	0, 10	0, 7, 9	0, 12
$R_{0,4}$	0, 11	0, 3	0, 11	0, 4, 6	0, 2, 10, 12
$R_{0,5}$	0	0, 11	0, 11	0, 2, 4, 11	0, 6, 7, 12
$R_{0,6}$	0	0, 11, 12	0, 3, 4, 7, 9, 11	$\emptyset$	0
$R_{0,7}$	0, 9	0, 5, 10	0, 3	0, 2	0, 2, 6
$R_{0,8}$	0	0, 10	0	0, 12	0
$R_{0,9}$	$\emptyset$	$\emptyset$	0	0, 1	0, 2, 9
$R_{1,2}$	$\emptyset$	3, 7, 11	0	2, 6, 12	2, 4
$R_{1,3}$	2	0	6, 7, 12	0, 3, 7, 8, 9	3, 6, 10, 12
$R_{1,4}$	$\emptyset$	$\emptyset$	1, 6, 9, 12	$\emptyset$	2, 7
$R_{1,5}$	3, 11	0, 2, 7, 9	2, 4, 7, 10	3, 7, 10	1, 2
$R_{1,6}$	4	11	$\emptyset$	8, 12	0, 6, 8
$R_{1,7}$	$\emptyset$	0, 4, 8	6, 11, 12	2, 7, 11, 12	2, 11, 12
$R_{1,8}$	6	$\emptyset$	6, 7, 9, 11, 12	2, 4, 5, 7, 8, 10	2, 4, 10
$R_{1,9}$	0, 1	0, 11	6, 12	$\emptyset$	0, 1, 10
$R_{2,3}$	8	6	1, 4	0, 1, 2, 7	4, 12
$R_{2,4}$	7, 8	0, 6	8, 10	2, 7, 8, 9	1, 2, 6, 10
$R_{2,5}$	$\emptyset$	0, 4, 8	6, 8	4, 7	1, 9, 11, 12
$R_{2,6}$	9	4	6	2, 6, 12	1, 4, 10
$R_{2,7}$	4, 12	2, 5	1, 4	0, 2, 9, 11	7
$R_{2,8}$	7	10	2, 3, 6, 8, 10, 12	3, 8	0, 3, 7
$R_{2,9}$	$\emptyset$	0, 5, 10	0, 1, 4, 5, 7, 11	$\emptyset$	12
$R_{3,4}$	7, 9	0, 1, 7	3, 4, 8, 12	0, 6	1, 12
$R_{3,5}$	12	11	0, 5, 6, 12	0, 6, 7, 8	2, 5
$R_{3,6}$	8	10, 12	8, 9, 10, 12	$\emptyset$	4, 9, 11
$R_{3,7}$	8, 12	4	7, 9	0, 4, 11	6, 9, 10
$R_{3,8}$	8	4, 10, 11	6, 8, 9, 10	1, 3, 4, 5, 6, 8	3, 5, 10
$R_{3,9}$	$\emptyset$	0, 11, 12	5, 6, 11	5	6, 7, 10
$R_{4,5}$	$\emptyset$	$\emptyset$	4, 7	2, 7, 8, 9	0, 7, 8, 12
$R_{4,6}$	4	$\emptyset$	7, 8, 12	0, 4, 7, 11, 12	0, 4, 10
$R_{4,7}$	$\emptyset$	10	1, 2, 7, 8	2, 4, 11	11
$R_{4,8}$	10	0, 3, 4, 7	3, 4, 12	8	1, 4, 10
$R_{4,9}$	1, 4	0, 10	1, 4, 6, 11	5, 7, 8, 9, 10	5
$R_{5,6}$	2, 12	0	1, 2, 10	5, 9, 12	2
$R_{5,7}$	$\emptyset$	2, 6, 10	1, 5, 6, 10	2, 5, 8, 11	0, 4, 6
$R_{5,8}$	7, 10	$\emptyset$	3, 11, 12	0, 8	5
$R_{5,9}$	$\emptyset$	0, 2	1, 4, 9, 11	3, 11	3, 5, 9
$R_{6,7}$	$\emptyset$	6	1, 2, 3, 5	0, 4, 9, 12	6, 11
$R_{6,8}$	1, 4, 7	0, 6, 12	1, 5	9	$\emptyset$
$R_{6,9}$	2, 8	0, 1, 2	0, 1, 3, 5, 6	1, 2, 4, 5, 7, 12	1, 2
$R_{7,8}$	$\emptyset$	0	3, 4, 5, 7	5, 6	0, 12
$R_{7,9}$	0, 4, 5, 12	0, 3, 8	2, 7, 8	5, 6	$\emptyset$
$R_{8,9}$	2, 9	0, 3	$\emptyset$	0, 1, 3, 11	4, 9

Table 6.9: Relevant graphs corresponding to the action of row 12 of Table 6.1.

6.4 Primitive graphs of order 10p

Throughout this section let X denote a primitive graph of order $10p$, p a prime. In [43] the complete characterization of primitive graphs of order $2pq$, where p and q are distinct odd primes, was given. Extracting the information about graphs of order $10p$ we find that the only primitive graphs of order $10p$, p a prime, are the ones arising from the actions given in Table 6.10. Below we show that each of the corresponding graphs has a Hamilton cycle. We let the GOGs and the relevant graphs corresponding to some action be defined as in Section 6.3.

row	p	Action of $\mathrm{Aut}(X)$
1	13	$\mathrm{PSL}(4,3)$ on cosets of P_2
2	19	S_{20} on pairs

Table 6.10: Primes p for which there exists a graph X on $10p$ vertices such that $\mathrm{Aut}(X)$ and all vertex-transitive subgroups of $\mathrm{Aut}(X)$ act primitively on X.

Graphs corresponding to row 1 of Table 6.10: It turns out that $\mathcal{R}(G,H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 2 of Table 6.10: The relevant graphs are given in Table 6.8, and so it is clear that each GOG arising from this action contains a Hamilton cycle.

The results of this section imply that the following proposition holds.

Proposition 6.4.1 *A primitive graph of order* $10p$*,* p *a prime, contains a Hamilton cycle.*

6.5 The proof of the main theorem

Proof of Theorem 6.0.1: If X is not genuinely imprimitive, then either Proposition 6.3.1 or Proposition 6.4.1 applies. If, however, X is genuinely imprimitive, then in view of the fact that the connected vertex-transitive graphs of orders $4p$, $6p$ and $2p^2$ contain a Hamilton path (see [61, 62, 79]), we may assume that $p > 7$. Now apply one of Lemma 6.2.1, Lemma 6.2.3, Lemma 6.2.4, Lemma 6.2.5, Lemma 6.2.6 and Lemma 6.2.7, depending on the size of the corresponding blocks.

6.6 Observations and conclusions

In this section it is proved that with the exception of quasiprimitive graphs arising from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, every connected vertex-transitive graph of order $10p$, $p \neq 7$ a prime, has a Hamilton path. This result gives a contribution to Lovász question about existence of Hamilton paths in connected vertex-transitive graphs. The proof of this result is based on lifting Hamilton cycles approach and analyzing (im)primitivity of actions of automorphism groups.

Chapter 7

Conclusion

A number of research problems concerning vertex-transitive graphs are solved. A complete classification of cubic non-Cayley vertex-transitive graphs of order $4p^2$, $p > 7$ a prime, and a characterization of cubic non-Cayley vertex-transitive graphs of order $2p^k$, $p > 7$ a prime and $k \leq p$, are given. It is also proved that cubic non-symmetric Cayley graphs on a non-abelian simple group

$$G \in \{\mathrm{M}_{11}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{J}_2, \mathrm{Suz}, \mathrm{PSL}(2,11), \mathrm{A}_n \mid n \in \{5, 11, 23, 47\}\}$$

are normal. Further, a complete classification of tetravalent one-regular graphs of order $4p^2$, p a prime, is given. And finally, it is proved that connected vertex-transitive graphs of order $10p$, p a prime, different from quasiprimitive graphs arising from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, possess a Hamilton path.

The first two results give partial solutions to Problems 3.0.1 and 3.0.2. In particular, it is shown that every cubic non-Cayley vertex-transitive graph of order $4p^2$, $p > 7$ a prime, is a generalized Petersen graph (see Theorem 3.1.4), and that every cubic non-Cayley vertex-transitive graph of order $2p^k$, $p > 7$ a prime and $k \leq p$, is a 2-Cayley graph on a cyclic group (see Theorem 3.2.3).

Next, Theorem 4.0.4 partially solves an open problem about normality of Cayley graphs on non-abelian simple groups posed in [34]. However, for a complete solution of this problem the normality of cubic Cayley graphs on the following non-abelian simple groups still need to be consider: A_n, $n = 2^m - 1, m \geq 3$, $\mathrm{PSL}(2, 2^e)$, $\mathrm{PSL}(3, 2^e)$, $\mathrm{U}_3(2^e)$, $\mathrm{PSp}_4(2^e)$, $\mathrm{E}_8(2^e)$, $\mathrm{F}_4(2^e)$, ${}^2\mathrm{F}_4(2^e)'$, $\mathrm{G}_2(2^e)$, and $\mathrm{Sz}(2^e)$.

Tetravalent one-regular graphs of orders p, pq and $2pq$, p and q primes, are classified in [16, 100, 102, 113, 120, 122, 129]. And, in the PhD Theiss the next step needed to be taken if one is to obtain a complete classification of all tetravalent one-regular graphs, is done. In particular, Theorem 5.4.1, gives a complete classification of tetravalent one-regular graphs of order $4p^2$, p a prime.

And finally, the PhD Thesis also gives a partial solution to the hamiltonicity problem of connected vertex-transitive graphs. In particular, by Theorem 6.0.1 every connected vertex-transitive graph of order $10p$, $p \neq 7$ a prime, which is not isomorphic to a quasiprimitive graph arising from the action of $\mathrm{PSL}(2,k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, contains a Hamilton path.

To wrap up, the results of this PhD Thesis represent a contribution to a number of long standing open problems in algebraic graph theory.

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Index

Povzetek v slovenskem jeziku

Disertacija obravnava štiri teme s podorčja algebraične teorije grafov:

- ne-Cayleyjevi točkovno tranzitivni grafi,
- grupe avtomorfizmov točkovno tranzitivnih grafov,
- ena-regularni grafi in
- hamiltonske poti v točkovno tranzitivnih grafih.

Vse te teme govorijo o *točkovno tranzitivnih grafih*, t.j. grafih, katerih grupe avtomorfizmov delujejo tranzitivno na množici točk grafa, kar pojasnuje že naslov disertacije. V literaturi so točkovno tranzitivni grafi včasih poimenovani *točkovno simetrični grafi*. Motivacija za obravnavane teme izhaja iz štirih odprtih problemov, ki jih prestavljamo v nadaljevanju.

Ne-Cayleyjevi točkovno tranzitivni grafi

Če grupa avtomorfizmov grafa premore podgrupo, ki deluje regularno na množici točk grafa, grafu pravimo *Cayleyjev graf*. Vsak Cayleyjev graf je seveda točkovno tranzitiven, vendar obrat ne velja. Obstajajo točkovno tranzitivni grafi, ki niso Cayleyjevi. Najmanjši primer takega točkovno tranzitivnega grafa je Petersenov graf. Red ne-Cayleyjevega grafa imenujemo *ne-Cayleyjevo število*.

Leta 1983 je Marušič [77] vprašal, za katera pozitivna števila n obstaja ne-Cayleyjev točkovno tranzitiven graf na n točkah. V literaturi obstajajo številna znanstvena dela, ki delno rešijo ta problem (glej [4, 5, 20, 55, 69, 73, 78, 86, 87, 88, 89, 91, 92, 100, 102, 108, 109, 111, 131]), vendar je v splošnem še vedno odprt. Na primer, v [78] je dokazano, da je vsak točkovno tranzitiven graf reda p^k, kjer je p liho praštevilo in $k \leq 3$, Cayleyjev graf. Družino ne-Cayleyjevih točkovno tranzitivnih grafov reda p^k, kjer je $p \geq 5$ praštevilo in $k \geq 4$, sta konstruirala McKay in Praeger v [91]. Leta 1971 so Frucht, Graver in Watkins [42] konstruirali družino ne-Cayleyjevih točkovno tranzitivnih grafov reda $2p$, kjer je $p \equiv 1 \pmod 4$ praštevilo. Nato sta leta 1979 Alspach in Sutcliffe [5] dokazala, da je za praštevilo p število $2p$ ne-Cayleyjevo natanko tedaj, ko je $p \equiv 1 \pmod 4$. Rezultati iz [4], [86] in [102] dajo klasifikacijo ne-Cayleyjevih točkovno tranzitivnih grafov, katerih red je produkt dveh praštevil. Karakterizacija ne-Cayleyjevih števil oblike $2pq$, p in q praštevili, sledi iz [57] in [94]. Leta 1996 sta McKay in Praeger [92] dokazala, da je vsako pozitivno število n, ki je deljivo s kvadratom praštevila, z izjemo $n \in \{12, p^2, p^3\}$, ne-Cayleyjevo. Ne-Cayelyjeva števila, ki so produkt treh različnih praštevil, je obravnaval Seress v [108].

Nedavno pa sta Li in Seress [69] določila tista pozitivna števila n prostih kvadratov, za katera obstaja ne-Cayleyjev točkovno tranzitiven graf reda n s primitivno grupo avtomorfizmov.

Graf X je *m-Cayleyjev graf* grupe G, če premore polregularno grupo avtomorfizmov G, ki ima natanko m orbit, vse enake dožine n. Glede na domnevo [75, 83] o obstoju polregularnih avtomorfizmov v vsakem točkovno tranzitivnem grafu je smiselno postaviti naslednji problem.

Problem 1: *Za dani ne-Cayleyjev točkovno tranzitiven graf X določi najmanjše celo število m, tako da je X m-Cayleyjev graf ciklične grupe.*

Na osnovi zgoraj omenjenega raziskovalnega dela je Feng [40] vprašal po določitvi najmanjše valence $\vartheta(n)$ med valencami ne-Cayleyjevih točkovno tranzitivnih grafov reda n. Ta problem je tudi rešil za grafe reda p^n, kjer je p liho praštevilo. Očitno je ne-Cayleyjevo število n, za katerega je $\vartheta(n) = 3$, sodo število. Znano je tudi, da je posplošeni Petersenov graf $\mathrm{GP}(n, t)$ ne-Cayleyjev točkovno tranzitiven graf natanko tedaj, ko je $t^2 \equiv -1 \pmod n$ ali $(n, t) = (10, 2)$ (glej [42, 72, 96]). Zato za vsako tako pozitivno število n, da 4 deli $\phi(n)$, velja $\vartheta(2n) = 3$. Glede na ta dejstva je smiselno raziskovanje ne-Cayleyjevih števil usmeriti v reševanje spodaj navedenih problemov.

Problem 2: *Klasificiraj ne-Cayleyjeva števila n, za katera je $\vartheta(n) = 3$.*

Problem 3: *Za ne-Cayleyjevo število n z $\vartheta(n) = 3$ klasificiraj vse povezane kubične ne-Cayleyjeve točkovno tranzitivne grafe reda n. Za katera števila so posplošeni Petersenovi grafi edini taki grafi?*

Problem 3 je rešen za števila $2p$, $4p$, $2p^2$ in $2pq$, kjer sta p in q lihi praštevili (glej, [5, 42, 78, 127, 128, 131]). V disertaciji je ta problem rešen za števila oblike $4p^2$, kjer je $p > 7$ praštevilo. Z uporabo rezultatov s področja teorije grup in kombinatoričnih tehnik je dokazano, da so posplošeni Petersenovi grafi edini primeri povezanih kubičnih ne-Cayleyjevih točkovno tranzitivnih grafov reda $4p^2$ (glej Theorem 3.1.4). Z drugimi besedami, dokazano je, da je vsak kubični ne-Cayleyjev točkovno tranzitiven graf takega reda 2-Cayleyjev graf ciklične grupe. Torej disertacija reši problem 1 za ta poseben razred ne-Cayleyjevih točkovno tranzitivnih grafov. Poleg tega je v disertaciji dokazano, da je vsak kubični ne-Cayleyjev točkovno tranzitiven graf reda $2p^n$, kjer je $p > 7$ praštevilo in $n \leq p$, 2-Cayleyjev graf p-grupe P, ki je generirana z dvema elementoma a in b istega reda in premore tak avtomorfizem $\phi \in \mathrm{Aut}(P)$ reda 4, da je $a^\phi = b$ in $b^\phi = a^{-1}$ (glej Theorem 3.2.3).

Grupe avtomorfizmov posebnih točkovno tranzitivnih grafov

Cayleyjevemu grafu $X = \mathrm{Cay}(G, S)$ pravimo *normalen Cayleyjev graf*, če je grupa G edinka v $\mathrm{Aut}(X)$, torej, če je $N_{\mathrm{Aut}(X)}(G) = \mathrm{Aut}(X)$. Li [66] je dokazal, da je vsak povezan kubični simetrični Cayleyjev graf ne-abelove enostavne grupe, ki je različna od

$$\mathrm{A}_5, \mathrm{PSL}(2, 11), \mathrm{M}_{11}, \mathrm{A}_{11}, \mathrm{M}_{23}, \mathrm{A}_{23} \text{ in } \mathrm{A}_{47},$$

normalen. Nato sta Xu in Xu [124] dokazala, da so tudi povezani kubični simetrični Cayleyjevi grafi alternirajoče grupe A_5 normalni. Z uporabo rezultatov iz [33] je kasneje Xu [123] dokazal tudi, da kubični simetrični Cayleyjevi grafi grupe A_{47}

niso normalni, medtem ko so kubični simetrični Cayleyjevi grafi vseh drugih zgoraj navedenih enostavnih grup normalni. Za povezane kubične ne-simetrične Cayleyjeve grafe ne-abelovih enostavnih grup je Fang [34] dokazal, da so skorajda vsi normalni. Natančneje, dokazal je, da so za vse ne-abelove enostavne grupe z izjemo M_{11}, M_{22}, M_{23}, J_2, Suz, A_n, kjer $n \in \{5, 11, 23, 47\} \cup \{2^m - 1 \mid m \geq 3\}$, $\mathrm{PSL}(2, 11)$, $\mathrm{PSL}(2, 2^e)$, $\mathrm{PSL}(3, 2^e)$, $\mathrm{U}_3(2^e)$, $\mathrm{PSp}_4(2^e)$, $\mathrm{E}_8(2^e)$, $\mathrm{F}_4(2^e)$, ${}^2\mathrm{F}_4(2^e)'$, $\mathrm{G}_2(2^e)$ in $\mathrm{Sz}(2^e)$ pripadajoči povezani kubični ne-simetrični Cayleyjevi grafi normalni. Za navedene izjeme pa Fang ni uspel določiti, ali so pripadajoči povezani kubični ne-simetrični ne-Cayleyjevi grafi normalni ali ne. V disertaciji je izboljšan Fangov rezultat. Dokazano je, da so povezani kubični ne-simetrični Cayleyjevi grafi normalni tudi za grupe M_{11}, M_{22}, M_{23}, J_2, Suz, A_n, kjer $n \in \{5, 11, 23, 47\}$, in $\mathrm{PSL}(2, 11)$.

Ena-regularni grafi

Če grupa avtomorfizmov grafa deluje regularno na množici lokov grafa, pravimo, da je graf *ena-regularen*. Ena-regularni graf brez izoliranih točk je povezan in je valence 2 natanko tedaj, ko je cikel. Prvi kubični ena-regularni graf je konstruiral Frucht [40]. V zadnjih desetletjih so bili ena-regularni grafi predmet številnih raziskav (glej [19, 25, 35, 36, 37, 38, 95]). V disertaciji je narejena popolna klasifikacija štirivalentnih ena-regularnih grafov reda $4p^2$, kjer je p praštevilo, glej Theorem 5.4.1.

Hamiltonskost točkovno tranzitivnih grafov

Leta 1969 je Lovász [71] postavil vprašanje ali ima vsak povezan točkovno tranzitiven graf hamiltonsko pot. Vsi znani povezani točkovno tranzitivni grafi imajo hamiltonsko pot. Še več, z izjemo štirih grafov (to so: Petersenov graf, Coxeterjev graf in grafa dobljena iz prvih dveh grafov, tako da vsako točko nadomestimo s trikotnikom) imajo vsi tudi hamiltonski cikel.

Znano je, da imajo povezani točkovno tranzitivni grafi reda p, $2p$ (z izjemo Petersenovega grafa), $3p$, $4p$ (z izjemo Coxeterjevega grafa), $5p$, p^2, p^3, p^4 in $2p^2$, kjer je p praštevilo, hamiltonski cikel (glej [1, 15, 29, 61, 64, 78, 79, 80, 84, 85, 110]). Za povezane točkovno tranzitivne grafe reda $6p$ pa je znano le, da imajo hamiltonsko pot (glej [64]). Med najpomembnejše rezultate pa zagotovo sodi rezultat, da povezani točkovno tranzitivni grafi, katerih grupa avtomorfizmov vsebuje tranzitivno podgrupo s ciklično komutatorsko podgrupo moči p^k, kjer je p praštevilo, z izjemo Petersenovega grafa $\mathrm{GP}(5, 2)$, premorejo hamiltonski cikel, glej [27].

V disertaciji je Lovászov problem, ki sodi med najpomembnejše odprte probleme na področju algebraične teorije grafov, obravnavan za povezane točkovno tranzitivne grafe reda $10p$, kjer je p praštevilo. Dokazano je, da vsak povezan točkovno tranzitiven graf reda $10p$, $p \neq 7$ praštevilo, ki ni izomorfen kvaziprimitivnemu grafu glede na delovanje grupe $\mathrm{PSL}(2, k)$ na odsekih po $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, premore hamiltonsko pot, glej Theorem 6.0.1.

Navedeno dokazuje, da rezultati disertacije predstavljajo pomemben prispevek k številnim odprtim problemom v algebraični teoriji grafov.

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- Y.-Q. Feng, K. Kutnar, D. Marušič and C. Zhang, Tetravalent one-regular graphs of order $4p^2$, poslano v objavo.
- K. Kutnar, D. Marušič and C. Zhang, Hamilton paths and cycles in vertex-transitive graphs of order $10p$, poslano v objavo.

Declaration

I declare that this PhD Thesis does not contain any materials previously published or written by another person except where due reference is made in the text.

Cui Zhang