VEM's for the numerical solution of PDE's

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From Joint works with L. Beirão da Veiga, L.D. Marini, and A. Russo

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Papers on VEMs from our group

- L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A. Russo: Basic principles of Virtual Element Methods, M3AS **23** (2013) 199-214.
- F. Brezzi, L.D. Marini: Virtual Element Method for plate bending problems, CMAME **253** (2013) 455-462.
- L. Beirão da Veiga, F. Brezzi, L.D. Marini: Virtual Elements for linear elasticity problems, SIAM JNA **51** (2013) 794-812
- B. Ahmad, A. Alsaedi, F. Brezzi, L.D. Marini, A. Russo: Equivalent projectors for virtual element methods, Comput. Math. Appl. **66** (2013) 376-391
- F. Brezzi, R.S. Falk, L.D. Marini: Basic principles of mixed Virtual Element Method, M2AN **48** (2014), 1227-1240
- L. Beirão da Veiga, F. Brezzi, L.D. Marini, A. Russo: The hitchhiker guide to the Virtual Element Method, M3AS **24** (2014) 1541-1573
- L. Beirão da Veiga, F. Brezzi, L.D. Marini, A. Russo: $H(\mathrm{div})$ and $H(\mathbf{curl})$ VEM, (submitted; arXiv preprint arXiv:1407.6822)

Outline

- The practical interest of Scientific Computing is known to (almost) everybody.
- Here I will discuss a (minor) part of the role of Mathematics in Scientific Computation
- Within the **M.S.O.** (Modelization, Simulation, Optimization) paradigm I will focus on the "S" part.
- In particular, I will deal with "basic instruments to compute an approximate solution (as accurate as needed) to a (system of) PDE's".
- I apologize to Numerical Analysts for the first part of this lecture. I hope it will not be too boring.

Basic physical laws

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \wedge \mathbf{E} = 0 \qquad \frac{\partial \mathbf{D}}{\partial t} - \nabla \wedge \mathbf{H} = \mathbf{J}$$

Phenomenological laws (material dependent)

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

Compatibility of the right-hand sides

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Incompressible Navier-Stokes Equations

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla + \nabla^T)\,\mathbf{u} \qquad \boldsymbol{\sigma} = (2\mu\boldsymbol{\varepsilon} + \mathbb{I}_{id}\,p)$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\cdot\boldsymbol{\sigma} = -\mathbf{f}$$

$$\nabla\cdot\mathbf{u} = 0$$

Let us consider the **simplest possible** problem: *Given a polygon Ω and $f \in L^2(\Omega)$:*

$$\text{find } u \in V \text{ such that } -\Delta u = f \text{ in } \Omega,$$

where $V \equiv H_0^1(\Omega) \equiv \{v |\, v \in L^2(\Omega), \, \mathbf{grad} v \in (L^2(\Omega))^2$ such that $v = 0$ on $\partial\Omega\}$. The *variational form* of this problem consists in looking for a function $u \in V$ such that:

$$\int_\Omega \mathbf{grad} u \cdot \mathbf{grad} v \mathrm{d}x = \int_\Omega f \, v \mathrm{d}x \qquad \forall\, v \in V.$$

The Galerkin method consists in choosing a finite dimensional $V_h \subset V$ and looking for $u_h \in V_h$ such that

$$\int_\Omega \mathbf{grad} u_h \cdot \mathbf{grad} v_h \mathrm{d}x = \int_\Omega f\, v_h \mathrm{d}x \qquad \forall\, v_h \in V_h.$$

It is an easy exercise to show that such a u_h exists and is unique in V_h, and satisfies the estimate

$$\int_\Omega |\mathbf{grad}(u - u_h)|^2 \mathrm{d}x \leq \; C \inf_{v_h \in V_h} \int_\Omega |\mathbf{grad}(u - v_h)|^2 \mathrm{d}x$$

bounding the *error* $\|u - u_h\|$ with the best approximation that could be given of u within the subspace V_h.

Sequences of approximations

More generally, the *analysis, from the mathematical point of view,* of these procedures assumes that we are given a *sequence* of subspaces $\{V_h\}_h$ and proves, under suitable assumptions on the subspaces, that the sequence of solutions $\{u_h\}_h$ converges to the exact solution u when h tends to 0.

As far as possible, one also tries to connect the *speed* of this convergence with suitable properties of the sequence $\{V_h\}_h$, and hence to find what are the sequences of subspaces that would provide the best speed, plus possibly other convenient properties (e.g.computability, positivity, conservation of physical quantities, etc.).

In the FEM's one decomposes the domain Ω in small pieces and takes V_h as the space of functions that are piece-wise polynomials. The most classical case is that of decompositions in *triangles*

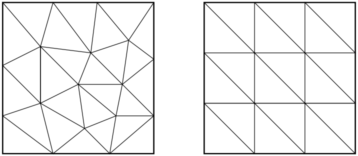


Figure: Triangulations of a square domain: non-uniform or uniform

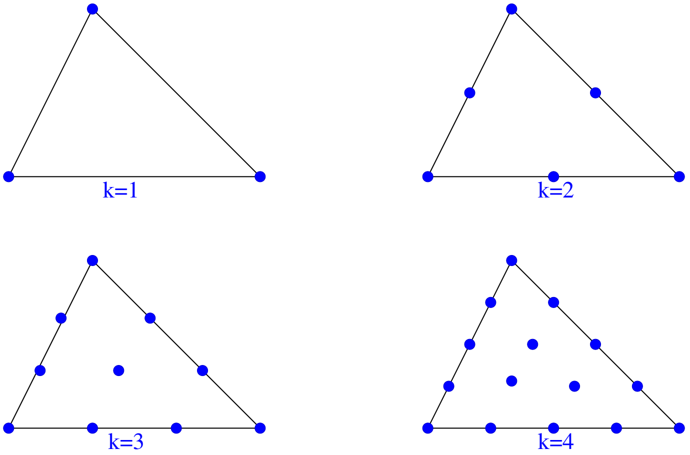
taking then V_h as the space of functions that are polynomials of degree ≤ 1 in each triangle.

Instead of p.w. polynomials of degree ≤ 1 one can take piecewise polynomials of degree $\leq k$ $(k = 2, 3, ...)$.

For the analysis we consider a sequence of decompositions $\{\mathcal{T}_h\}_h$, and piecewise polynomials of degree $\leq k$, and try to express the speed of convergence (of u_h to u) in terms of k, of h = biggest diameter among the elements of $\mathcal{T}_h$, and of some additional geometric property θ (e.g. the *minimum angle* of all triangles of all decompositions):

$$\|\mathbf{grad}(u - u_h)\|_{L^2(\Omega)} \leq C_\theta \, h^k \, \|D^{k+1}u\|_{L^2(\Omega)}.$$

Lagrange FEM's



Triangular elements and their degrees of freedom

Typical Functional Spaces

Here are the functional spaces most commonly used in variational formulations of PDE problems

- $L^2(\Omega)$ (ex. pressures, densities)
- $H(\mathrm{div};\Omega)$ (ex. fluxes, $\mathbf{D}$, $\mathbf{B}$)
- $H(\mathbf{curl};\Omega)$ (ex. vector potentials, $\mathbf{E}$, $\mathbf{H}$)
- $H(\mathbf{grad};\Omega)$ (H^1) (ex. displacements, velocities)
- $H(\mathbb{D}^2;\Omega)$ (H^2) (ex. in K-L plates, Cahn-Hilliard)

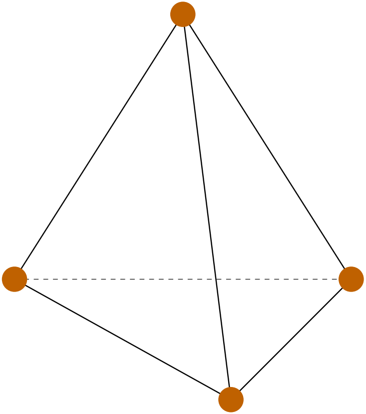
Continuity requirements

For a **piecewise smooth** vector valued function, at the common boundary between two elements,

in order to belong to	**you need to match**
$L^2(\Omega)$	nothing
$H(\mathrm{div};\Omega)$	normal component
$H(\mathbf{curl};\Omega)$	tangential components
$H(\mathbf{grad};\Omega)$	all the components
$H(\mathbb{D}^2;\Omega)$	$w,\ w_x\ w_y$

Note that *the freedom you gain by relaxing the continuity properties can be used to satisfy other properties*

Elegance of FEM spaces

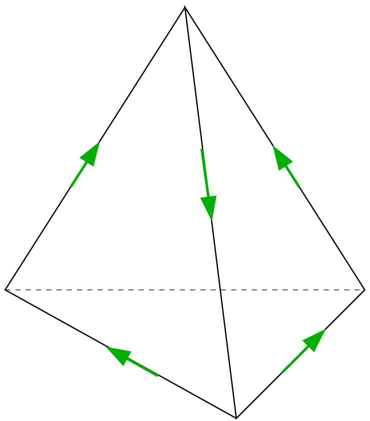


$$\mathbb{P}_1 := \{v = a + \mathbf{c}\cdot\mathbf{x}\} \text{ with } a \in \mathbb{R} \text{ and } \mathbf{c} \in \mathbb{R}^3$$

(1 d.o.f. per **node**)

$$H(\mathbf{grad};\Omega) \sim \{v \in H(\mathbf{grad};\Omega) \text{ s.t. } v_{|T} \in \mathbb{P}_1 \quad \forall T \in \mathcal{T}_h\}.$$

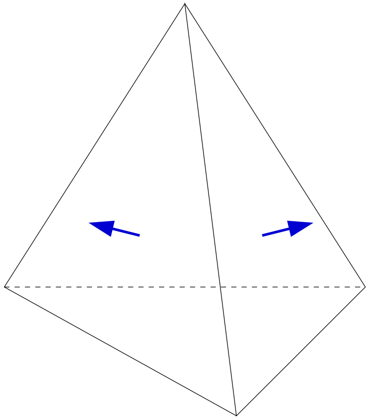
Elegance of FEM spaces



$N_0 := \{\varphi = \mathbf{a} + \mathbf{c} \wedge \mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{c} \in \mathbb{R}^3$

(1 d.o.f. per **edge**)

$H(\mathbf{curl};\Omega) \sim \{\varphi \in H(\mathbf{curl};\Omega)$ s.t. $\varphi_{|T} \in N_0 \quad \forall T \in \mathcal{T}_h\}$.

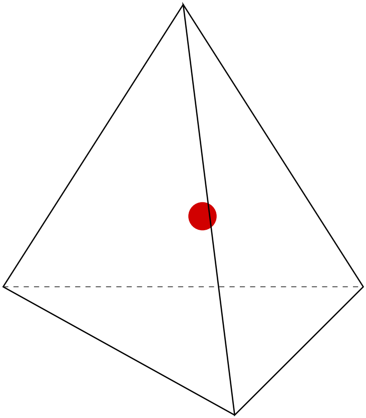


$RT_0 := \{\boldsymbol{\tau} = \mathbf{a} + c\mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

(1 d.o.f. per **face**)

$H(\mathrm{div};\Omega) \sim \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ s.t. $\boldsymbol{\tau}_{|T} \in \quad \forall T \in \mathcal{T}_h\}$.

Elegance of FEM spaces

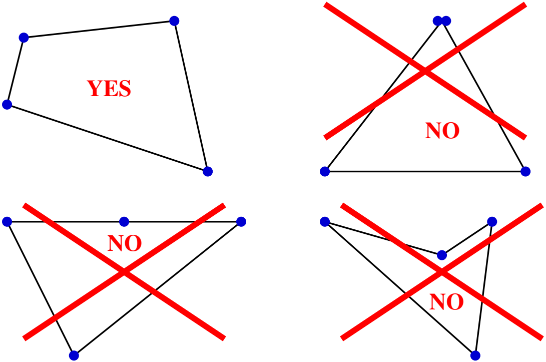


$\mathbb{P}_0 := \{\text{constants}\}$ (1 d.o.f. per **element**)

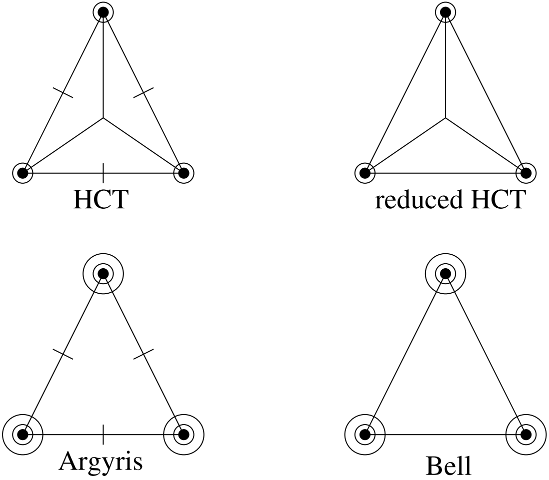
$$L^2(\Omega) \sim \{q \in L^2(\Omega) \text{ such that } q_{|T} \in \mathbb{P}_0 \quad \forall T \in \mathcal{T}_h\}.$$

Difficulties with FEM's: distorted elements

Distorted quads can degenerate in many ways:



There are relatively few C^1 Finite Elements on the market. Here are some:

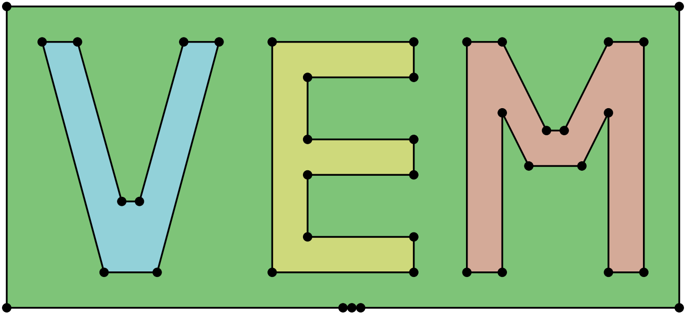


Programming C^1 elements

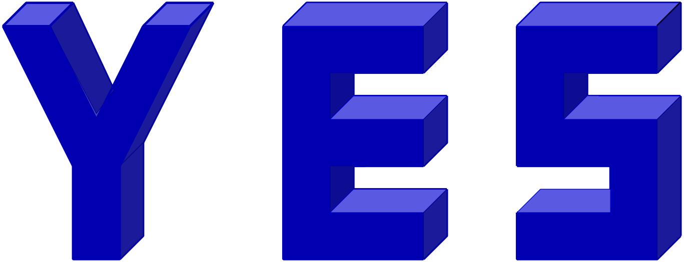
Cod liver oil
(Olio di fegato di merluzzo, Ribje olje)

A flavor of VEM's

For a decomposition in general sub-polygons, FEM's face **considerable** difficulties. With VEM, instead, you can take a decomposition like



having four elements with 8 12 14, and 41 nodes, respectively! **Can we work in 3D as well?**



WE CAN !! These are three possible 3D elements

Polygonal and Polyhedral elements

There is a wide literature on Polygonal and Polyhedral Elements

- Rational Polynomials (Wachspress, 1975, 2010)
- Voronoi tassellations (Sibson, 1980; Hiyoshi-Sugihara, 1999; Sukumar et als, 2001)
- Mean Value Coordinates (Floater, 2003)
- Metric Coordinates (Malsch-Lin-Dasgupta, 2005)
- Maximum Entropy (Arroyo-Ortiz, 2006; Hormann-Sukumar, 2008)
- Harmonic Coordinates (Joshi et als 2007; Martin et als, 2008; Bishop 2013)

Today, there are several similar methods that generalize Finite Element Methods to Polygonal/Polyhedral elements

- HDG Methods (e.g. Bernardo Cockburn, Jay Gopalakhrisnan),
- Weak Galerkin Methods (e.g. Junping Wang, Xiu Ye),
- Discrete Gradient Reconstruction (e.g. Daniele Di Pietro, Alexandre Ern).

There are also a strong connections between these methods and

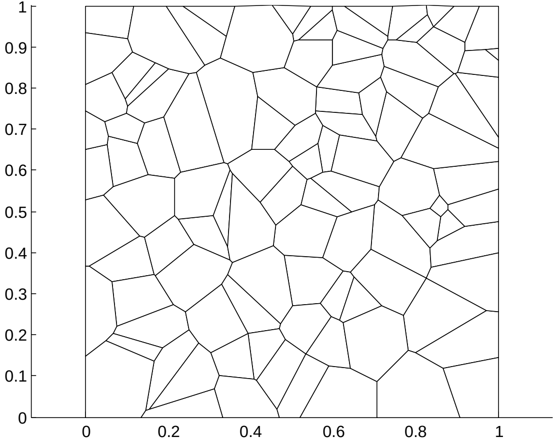
- Mimetic Finite Diffrences (e.g. Mikhail Shashkov, Konstantin Lipnikov),

Why Polygonal/Polyhedral Elements

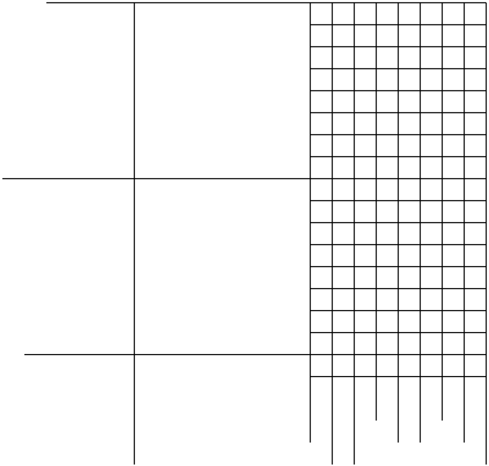
There are several types of problems where Polygonal and Polyhedral elements are used:

- Crack propagation and Fractured materials (e.g. T. Belytschko, N. Sukumar)
- Topology Optimization (e.g. O. Sigmund, G.H. Paulino)
- Computer Graphics (e.g. M.S. Floater)
- Fluid-Structure Interaction (e.g. W.A. Wall)
- Complex Micro structures (e.g. N. Moes)
- Two-phase flows (e.g. J. Chessa)

Voronoi Meshes

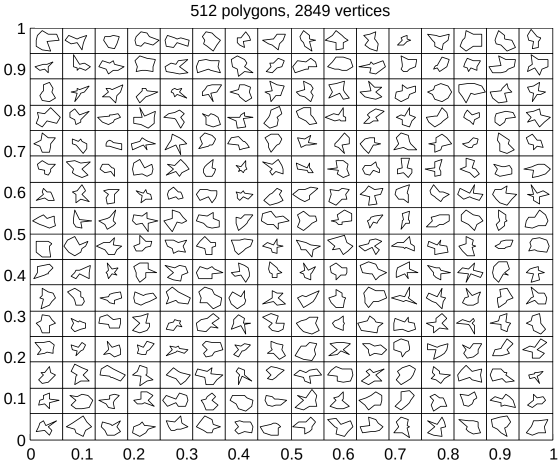


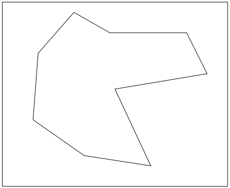
Local refinements and hanging nodes



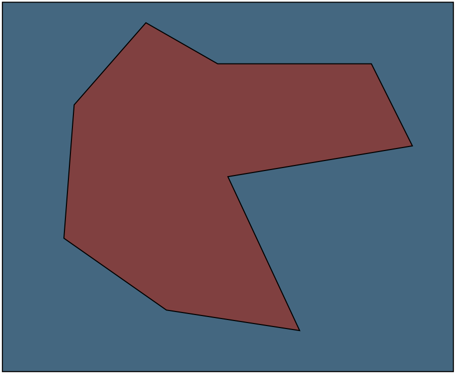
The "interface" big squares are treated as polygons with 11 edges.

Possible Inclusions



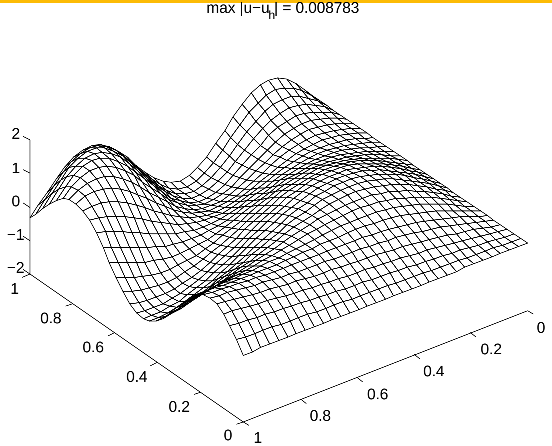


Robustness of VEMs - General elements



Note that the pink element is a polygon with 9 edges, while the blue element is a polygon (not simply connected) with 13 edges. We are exact on linears...

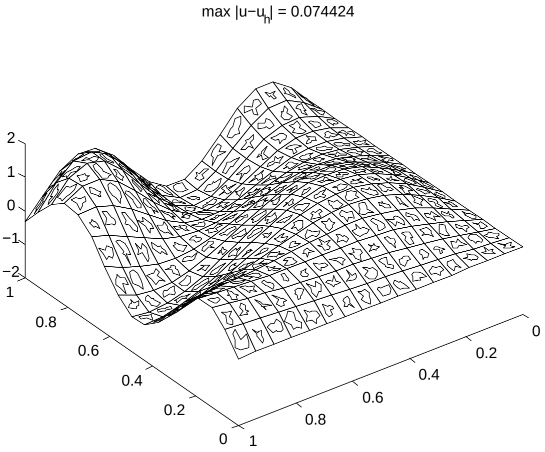
The exact solution of the PDE



For reasons of "glastnost", we take as exact solution

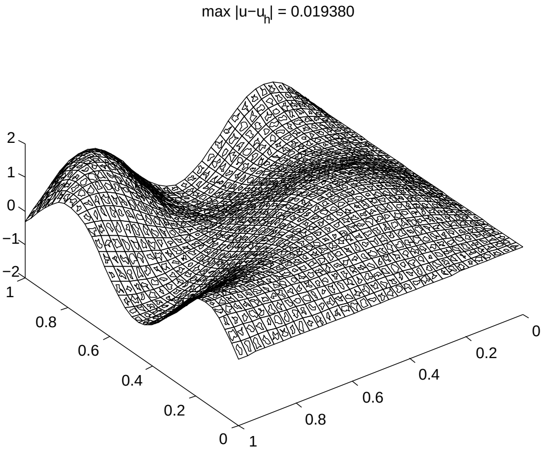
$$w = x(x - 0.3)^3(2 - y)^2 \sin(2\pi x)\sin(2\pi y) + \sin(10xy)$$

Robustness of the method



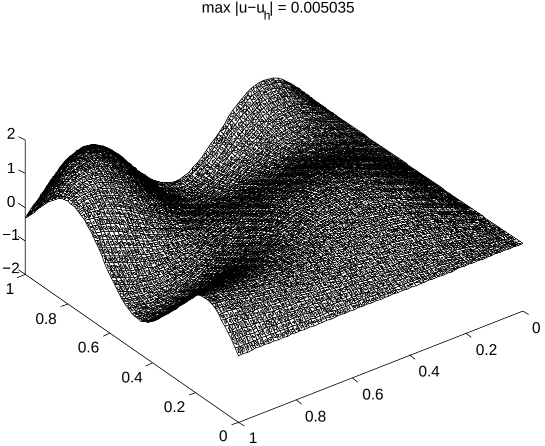
Mesh of 512 ($16 \times 16 \times 2$) elements. Max-Err=0.074

Finer grids



Mesh of 2048 ($32 \times 32 \times 2$) elements. Max-Err$=$0.019

Solution on the finer grid



Mesh of 8192 ($64 \times 64 \times 2$) elements. Max-Err$=$0.005
Note the $O(h^2)$ convergence in L^∞.

The next steps? (by M.C. Escher)

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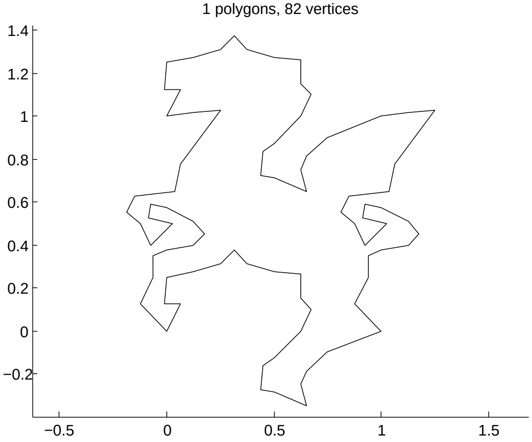
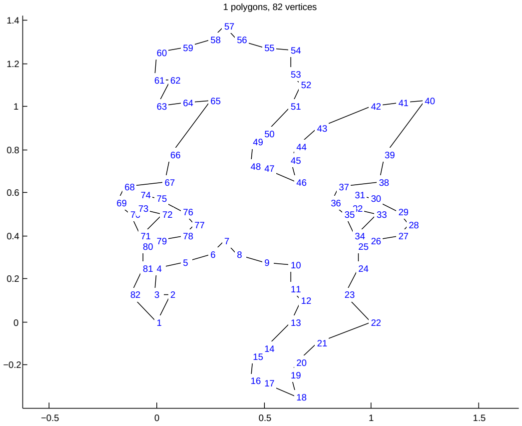
Going berserk



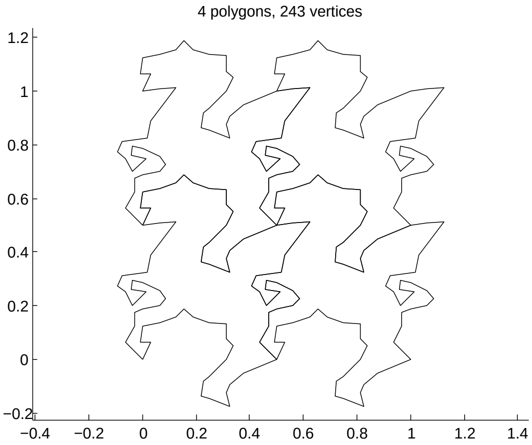
The first step: a pegasus-shaped polygon with **82 edges**.

Going berserk



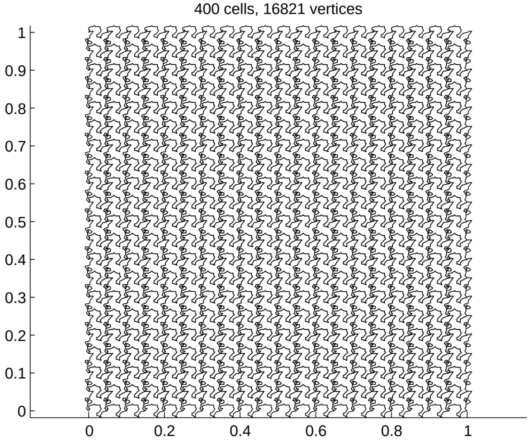
The second step: local numbering of the **82 nodes**.

Going berserk



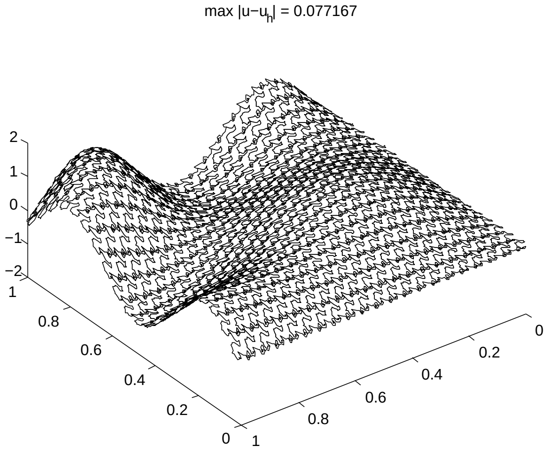
The third step: a mesh of 2×2 pegasus.

Going totally berserk



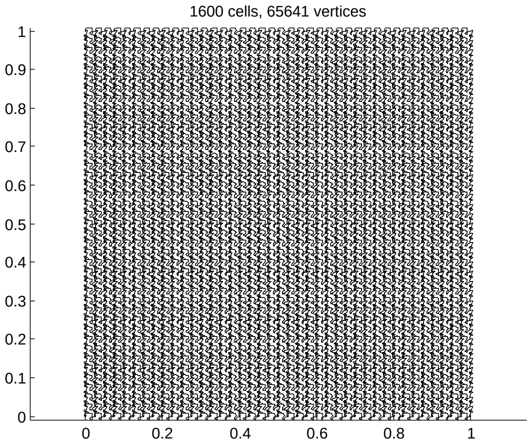
A mesh of 20 $\times$ 20 pegasus.

Going totally berserk



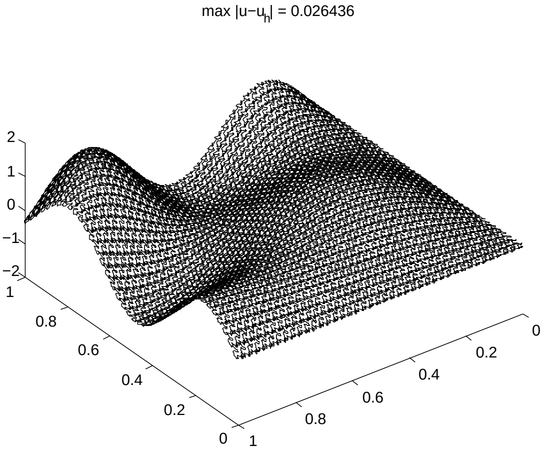
Solution on a 20×20-pegasus mesh. Max-Err=0.077

Going **totally** berserk



A mesh of 40×40 pegasus.

Going **totally** berserk



Solution on a 40 $\times$ 40-pegasus mesh. Max-Err=0.026

As for other methods on polyhedral elements

- the trial and test functions inside each element are rather complicated (e.g. solutions of suitable PDE's or systems of PDE's).

Contrary to other methods on polyhedral elements,

- they **do not** require the approximate evaluation of trial and test functions at the integration points.
- In most cases they satisfy the *patch test* **exactly** (up to the computer accuracy).
- We have now a full family of spaces.

The general philosophy

In every element, to *define* the generic (scalar or vector valued) element v of our VEM space:

- You start from the **boundary** d.o.f. and use a 1D edge-by edge reconstruction
- Then you define v **inside** as the solution of a (system of) PDE's, typically with a polynomial right-hand side.
- The construction is such that **all polynomials** of a certain degree belong to the local space. In general the local space also contains some additional elements.

Let us see some examples.

We take, for every integer $k \geq 1$

$$V_h^E = \{v | \; v_{|e} \in \mathbb{P}_k(e) \, \forall \text{ edge } e \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}$$

It is easy to see that the local space will contain all $\mathbb{P}_k$.
As degrees of freedom we take:

- the values of v at the vertices,
- the moments $\int_e v \, p_{k-2} \mathrm{d}e$ on each edge,
- the moments $\int_E v \, p_{k-2} \mathrm{d}E$ inside.

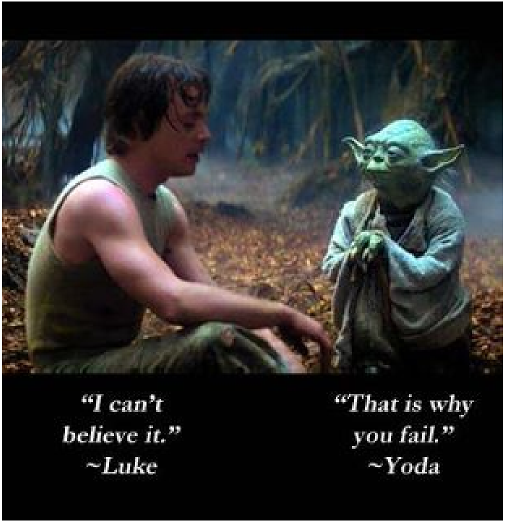
It is easy to see that these d.o.f. are *unisolvent*.

The L^2-projection

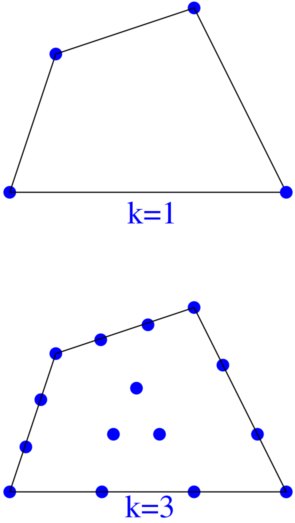
A fantastic trick (sometimes called *The Three Card Monte trick*), often allows the *exact* computation of the moments of order $k-1$ and k of every $v \in V_h^E$.

This is very useful for dealing with the 3D case.

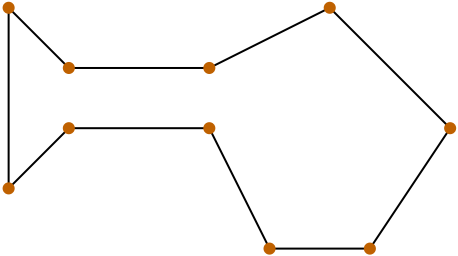
The *Three Card Monte Trick* is hard to believe



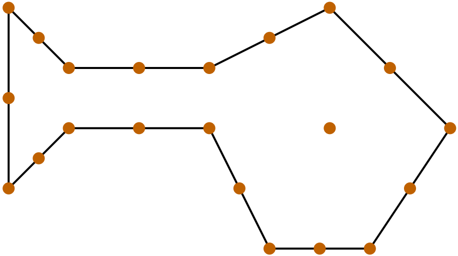
Example: Degrees of freedom of nodal VEM's in 2D



More general geometries $k = 1$



More general geometries $k = 2$



Approximations of $H^1(\Omega)$ in 3D

For a given integer $k \geq 1$, and for every element E, we set

$$V_h^E = \{v \in H^1(E) |\, v_{|e} \in \mathbb{P}_k(e) \,\forall \text{ edge } e,$$
$$v_{|f} \in V_h^f \,\forall \text{ face } f, \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}$$

with the degrees of freedom:

- values of v at the vertices,
- moments $\int_e v\, p_{k-2}\,(e)$ on each edge e,
- moments $\int_f v\, p_{k-2}\,(f)$ on each face f, and
- moments $\int_E v\, p_{k-2}\,(E)$ on E.

Ex: for $k = 3$ the number of degrees of freedom would be: the number of vertices, plus $2\times$ the number of edges, plus $3\times$ the number of faces, plus 4. On a cube this makes $8 + 24 + 18 + 4 = 54$ against 64 for $\mathbb{Q}_3$.

The spaces $\mathcal{G}_k$, $\mathcal{G}_k^\perp$, $\mathcal{R}_k$, and $\mathcal{R}_k^\perp$

In the sequel it will be convenient to introduce the following notation

- $\mathcal{G}_k := \mathbf{grad}(\mathbb{P}_{k+1})$
- $\mathcal{R}_k := \mathbf{rot}(\mathbb{P}_{k+1})$ (in 2 dimensions)
- $\mathcal{R}_k := \mathbf{curl}(\mathbb{P}^3_{k+1})$ (in 3 dimensions)

Moreover, for every vector valued polynomial space $\mathcal{P}_k(E) \subset \mathbb{P}^d_k(E)$ we denote

- $\mathcal{P}_k^\perp(E) := \{\mathbf{q} \in \mathbb{P}^d(E) \text{ s.t. } (\mathbf{q},\mathbf{p})_{0,E} = 0\ \forall \mathbf{p} \in \mathcal{P}_k(E)\}$

VEM approximations of $H(\mathrm{div};\Omega)$ in $2d$ and in $3d$

In each element E, and for each integer k, we define

$$\mathcal{B}_k(\partial E) := \{g \mid g_{|e} \in \mathbb{P}_k \,\forall \text{ edge } e \in \partial E\} \text{ in 2d}$$

$$\mathcal{B}_k(\partial E) := \{g \mid g_{|f} \in \mathbb{P}_k \,\forall \text{ face } f \in \partial E\} \text{ in 3d.}$$

Then we define, in 2 dimensions:

$$V_k(E) = \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_k(\partial E), \nabla \mathrm{div} \boldsymbol{\tau} \in \mathcal{G}_{k-2}, \mathrm{rot} \boldsymbol{\tau} \in \mathbb{P}_{k-1}\}$$

and in 3 dimensions

$$V_k(E) = \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_k(\partial E), \nabla \mathrm{div} \boldsymbol{\tau} \in \mathcal{G}_{k-2}, \mathbf{curl} \boldsymbol{\tau} \in \mathcal{R}_{k-1}\}.$$

Variants of VEMs in $H(\mathrm{div};\Omega)$

For k, r and s integer, we define, in 2 dimensions:

$$V_{(k,r,s)}(E) = \{\boldsymbol{\tau} | \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_k(\partial E), \nabla \mathrm{div} \boldsymbol{\tau} \in \mathcal{G}_{r-1}, \mathrm{rot} \boldsymbol{\tau} \in \mathbb{P}_s\}$$

and in 3 dimensions

$$V_{(k,r,s)}(E) = \{\boldsymbol{\tau} | \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_k(\partial E), \nabla \mathrm{div} \boldsymbol{\tau} \in \mathcal{G}_{r-1}, \mathbf{curl} \boldsymbol{\tau} \in \mathcal{R}_s\}.$$

In general we might say that

$$V_k \equiv V_{(k,k-1,k-1)} \simeq BDM_k \quad \text{and} \quad V_{(k,k,k-1)} \simeq RT_k$$

On a triangle: $V_{(0,0,-1)} = RT_0$. We point out that $\forall k \geq 0$

$$(\mathbb{P}_k)^d \subset V_{(k,k-1,k-1)} \text{ and } \nabla(P_{k+1}) \subset V_{(k,k-1,-1)}$$

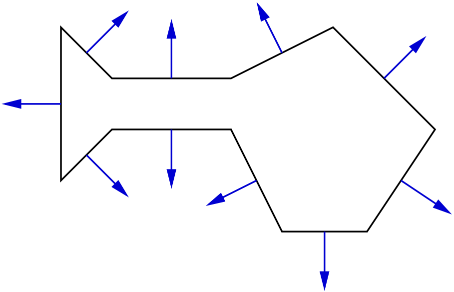
Degrees of freedom in $V_k \equiv V_{(k,k-1,k-1)}(E)$ in $2d$

In $V_k \equiv V_{(k,k-1,k-1)}$ we can take the following d.o.f.

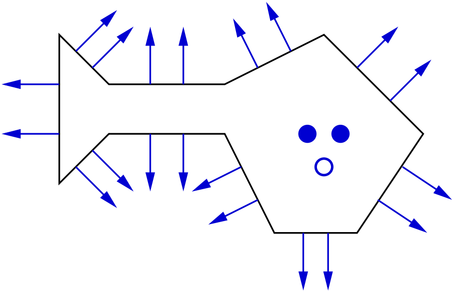
- $\int_e \boldsymbol{\tau} \cdot \mathbf{n}\, q_k \mathrm{d}e \qquad \forall q_k \in \mathbb{P}_k(e) \;\; \forall$ edge e
- $\int_E \boldsymbol{\tau} \cdot \mathbf{grad} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\int_E \boldsymbol{\tau} \cdot \mathbf{g}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{g}_k^{\perp} \in \mathcal{G}_k^{\perp}$

with natural variants for other spaces of the type $V_{(k,s,r)}$.

Degrees of freedom for $V_{(0,0,-1)}$



Degrees of freedom for $V_{(1,1,0)}$



Degrees of freedom in $V_k \equiv V_{(k,k-1,k-1)}(E)$ in $3d$

In $V_k \equiv V_{(k,k-1,k-1)}$ we can take the following d.o.f.

- $\displaystyle\int_f \boldsymbol{\tau} \cdot \mathbf{n}\, q_k \mathrm{d}f \qquad \forall q_k \in \mathbb{P}_k(f) \ \ \forall \text{ face } f$
- $\displaystyle\int_E \boldsymbol{\tau} \cdot \mathbf{grad} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\displaystyle\int_E \boldsymbol{\tau} \cdot \mathbf{g}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{g}_k^{\perp} \in \mathcal{G}_k^{\perp}$

with natural variants for other spaces of the type $V_{(k,s,r)}$.

VEM approximations of $H(\mathrm{rot};\Omega)$ in $2d$

In each element E, and for each integer k, we recall

$$\mathcal{B}_k(\partial E) := \{g|\ g_{|e} \in \mathbb{P}_k \,\forall \text{ edge } e \in \partial E\} \text{ in 2d}$$

Then we set

$$V_k(E) = \{\varphi | \varphi \cdot \mathbf{t} \in \mathcal{B}_k(\partial E), \mathrm{div}\varphi \in \mathbb{P}_{k-1}, \mathbf{rot}\mathrm{rot}\varphi \in \mathcal{R}_{k-2}\}$$

and for integers k, r, and s

$$V_{(k,r,s)}(E) = \{\varphi | \varphi \cdot \mathbf{t} \in \mathcal{B}_k(\partial E), \mathrm{div}\varphi \in \mathbb{P}_r, \mathbf{rot}\mathrm{rot}\varphi \in \mathcal{R}_{s-1}\}$$

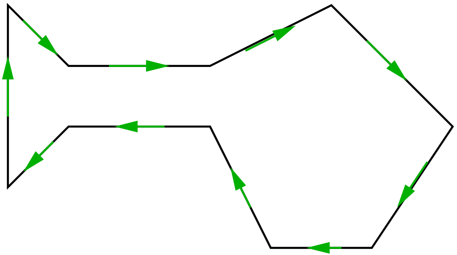
Degrees of freedom in $V_k \equiv V_{(k,k-1,k-1)}(E)$ in $2d$

In $V_k \equiv V_{(k,k-1,k-1)}$ in $2d$ we can take the following d.o.f.

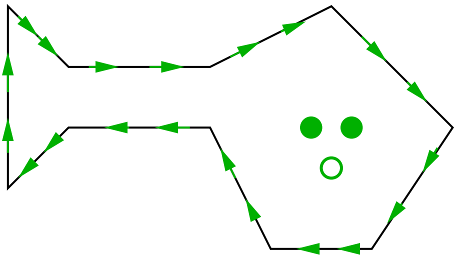
- $\int_e \varphi \cdot \mathbf{t}\, q_k \mathrm{d}e \qquad \forall q_k \in \mathbb{P}_k(e) \;\; \forall$ edge e
- $\int_E \varphi \cdot \mathbf{rot} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}$
- $\int_E \varphi \cdot \mathbf{r}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{r}_k^{\perp} \in \mathcal{R}_k^{\perp}$

with natural variants for other spaces of the type $V_{(k,r,s)}$.

Degrees of freedom for $V_{(0,-1,0)}$



Degrees of freedom for $V_{(1,0,1)}$



VEM approximations of $H(\mathbf{curl};\Omega)$ in $3d$

In each element E, and for each integer k, we set

$$\mathcal{B}_k(\partial E) := \{\boldsymbol{\varphi} |\ \boldsymbol{\varphi}_{|f} \in V_k(f) \,\forall \text{ face } f \in \partial E \text{ and } \boldsymbol{\varphi}\cdot\mathbf{t}_e \text{ is single valued at each edge } e \in \partial E\}$$

Then we set

$$V_k(E) = \{\boldsymbol{\varphi} |\ \text{such that } \boldsymbol{\varphi}\cdot\mathbf{t} \in \mathcal{B}_k(\partial E),\ \mathrm{div}\boldsymbol{\varphi} \in \mathbb{P}_{k-1}, \mathbf{curlcurl}\boldsymbol{\varphi} \in \mathcal{R}_{k-2}\}$$

Degrees of freedom in $V_k(E) \equiv V_{(k,k-1,k-1)}$ in 3d

- for every edge e $\displaystyle\int_e \varphi \cdot \mathbf{t}\, q_k \mathrm{d}e \quad \forall q_k \in \mathbb{P}_k(e)$
- for every face f

$$\int_f \varphi \cdot \mathbf{rot} q_{k-1} \mathrm{d}f \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}(f)$$

$$\int_f \varphi \cdot \mathbf{r}_k^{\perp} \mathrm{d}f \qquad \forall \mathbf{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(f)$$

- and inside E

$$\int_E \varphi \cdot \mathbf{curl} q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in (\mathbb{P}_{k-1}(E))^3$$

$$\int_E \varphi \cdot \mathbf{r}_k^{\perp} \mathrm{d}E \qquad \forall \mathbf{r}_k^{\perp} \in \mathcal{R}_k^{\perp}(E)$$

VEM approximations of $L^2(\Omega)$

In each element E, and for each integer k, we set

$$V_k(E) := \mathbb{P}_k(E),$$

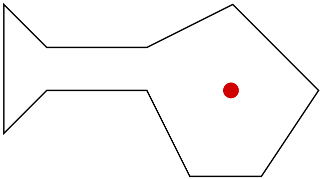
and then obviously

$$V_k(\Omega) = \{v| \text{ such that } v_{|E} \in \mathbb{P}_k(E), \forall\, E \text{ in } \mathcal{T}_h\}$$

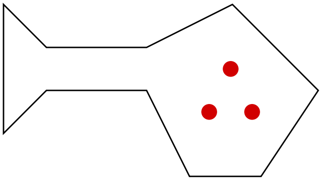
Degrees of freedom in $V_k(E)$

As degrees of freedom in $V_k(E)$ we can obviously choose

$$\int_E v\, q_k \mathrm{d}E \qquad \forall q_k \in (\mathbb{P}_k(E))^3$$



$k = 0$



$k = 1$

VEM approximations of $H^2(\Omega)$

For r, s, k, with

- $r \geq k$,
- $s \geq k - 1$

we set

$$V_h := \{v \in V : \ v \in \mathbb{P}_r(e),\ v_n \in \mathbb{P}_s(e)\ \forall \text{ edge } e, \\ \text{and } \Delta^2 v \in \mathbb{P}_{k-4}(E)\ \forall \text{ element } E\}$$

It is clear that for every element E the restriction V_h^E of V_h to E contains all the polynomials of degree $\leq k$.

Degrees of freedom for K-L plate elements

We had:

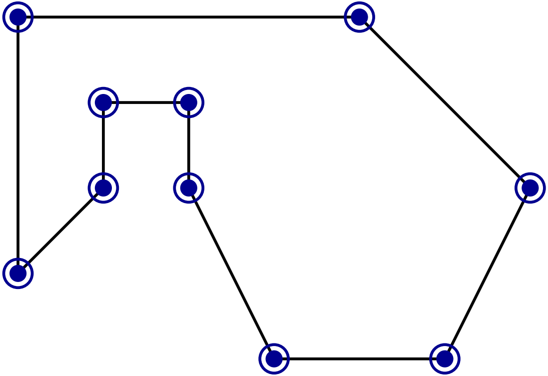
$$V_h := \{v \in V : \; v \in \mathbb{P}_r(e), \; v_n \in \mathbb{P}_s(e) \\ \forall \text{ edge } e \text{ and } \Delta^2 v \in \mathbb{P}_{k-4}(E) \; \forall \text{ element } E\}$$

In each E the functions in V_h^E are identified by

- their value and the value of their derivatives on ∂E,
- (for $k > 3$) the moments up to the order $k-4$ in E

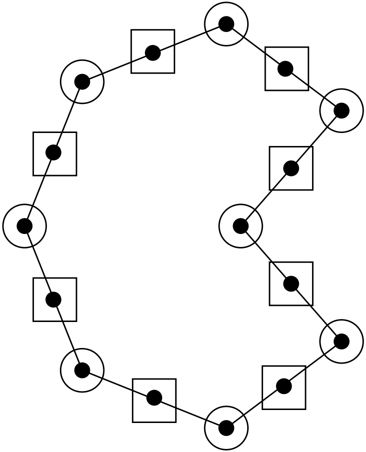
Hence we have to worry only for the boundary degrees of freedom.

Example: $r = 3$, $s = 1$



On each vertex we assign v, v_x, v_y.

Example: $r = 4$, $s = 3$ (Pac-Plate)



On each vertex we assign v, v_x, v_y. On each midpoint we assign v, v_n, v_{nt}.

Manipulating VEM's

When we deal with VEM, we cannot manipulate them as we please. As we don't want to use approximate solutions of the PDE problems in each element, we have to use only the *degrees of freedom* and all the information that you can deduce *exactly* from the degrees of freedom.

In a sense, is like doing *Robotic Surgery*

Example of manipulation

For instance if you know a function v on ∂E and its mean value in E you can compute

$$\int_E \nabla v \cdot \mathbf{q}_1 \mathrm{d}E = \int_{\partial E} v\, \mathbf{q}_1 \cdot \mathbf{n} \mathrm{d}s - \int_E v\, \mathrm{div} \mathbf{q}_1 \mathrm{d}E$$

for every vector valued polynomial $\mathbf{q}_1 \in (\mathbb{P}_1)^2$.

A very useful property

We observe that the classical differential operators *grad*, *curl*, and *div* send these VEM spaces one into the other (up to the obvious adjustments for the polynomial degrees). Indeed:

$$\mathbf{grad}(VEM, nodal) \subseteq VEM, edge$$

$$\mathbf{curl}(VEM, edge) \subseteq VEM, face$$

$$\mathrm{div}(VEM, face) \subseteq VEM, volume$$

$$\mathbb{R} \xrightarrow{i} V_k^{\mathrm{ver}}(\Omega) \xrightarrow{\mathbf{grad}} V_{k-1}^{\mathrm{edg}}(\Omega) \xrightarrow{\mathbf{curl}} V_{k-2}^{\mathrm{fac}}(\Omega) \xrightarrow{\mathrm{div}} V_{k-3}^{\mathrm{vol}}(\Omega) \xrightarrow{o} 0$$

.

The crucial feature common to all these choices is the possibility **to construct** (starting from the degrees of freedom, and without solving approximate problems in the element) **a symmetric bilinear form** $[\mathbf{u},\mathbf{v}]_h$ such that, on each element E, we have

$$[\mathbf{p}_k,\mathbf{v}]_h^E=\int_E \mathbf{p}_k\cdot\mathbf{v}\,\mathrm{d}E\ \forall\mathbf{p}_k\in(\mathbb{P}_k(E))^d,\ \forall\mathbf{v}\text{ in the VEM space}$$

and $\exists\,\alpha^*\geq\alpha_*>0$ independent of h such that

$$\alpha_*\|\mathbf{v}\|^2_{L^2(E)}\leq[\mathbf{v},\mathbf{v}]_h^E\leq\alpha^*\|\mathbf{v}\|^2_{L^2(E)},\quad\forall\mathbf{v}\text{ in the VEM space}$$

The crucial feature - 2

In other words: In each VEM space (nodal, edge, face, volume) we have a corresponding **inner product**

$$[\cdot,\cdot]_{VEM,nodal}, \; [\cdot,\cdot]_{VEM,edge}, \; [\cdot,\cdot]_{VEM,face}, \; [\cdot,\cdot]_{VEM,volume}$$

that **scales properly**, and reproduces **exactly** the L^2 inner product when **at least one of the two entries** is a **polynomial** of degree $\leq k$.

General idea on the construction of Scalar Products

- First note that you can always integrate a polynomial

$$\int_E x^3 \mathrm{d}E = \int_{\partial E} \frac{x^4}{4} n_x \mathrm{d}s.$$

- You construct $\Pi_k : \; V^E \to (\mathbb{P}_k(E))^d$ defined by

$$\int_E (\mathbf{v} - \Pi_k \mathbf{v}) \cdot \mathbf{p}_k \mathrm{d}E = 0 \,\forall\, \mathbf{p}_k,$$

- and then set, for all **u** and **v** in V^E

$$\Big[\mathbf{u}, \mathbf{v}\Big]_E := \int_E (\Pi_k \mathbf{u} \cdot \Pi_k \mathbf{v}) \mathrm{d}E + S(\mathbf{u} - \Pi_k \mathbf{u}, \mathbf{v} - \Pi_k \mathbf{v})$$

where the *stabilizing* bilinear form S is for instance the measure of E times the Euclidean inner product in $\mathbb{R}^N$.

Strong formulation of Darcy's law

- p = pressure
- $\mathbf{u}$ = velocities *(volumetric flow per unit area)*
- f = source
- $\mathbb{K}$ = material-depending (full) tensor

- $\mathbf{u} = -\mathbb{K}\nabla p$ (Constitutive Equation)
- $\operatorname{div} \mathbf{u} = f$ (Conservation Equation)

$$-\operatorname{div}(\mathbb{K}\nabla p) = f \qquad \text{in } \Omega,$$

$$p = 0 \qquad \text{on } \partial\Omega, \qquad \text{for simplicity.}$$

The **variational formulation** of Darcy problem is:
find $p \in H^1_0(\Omega)$ *such that*

$$\int_\Omega \mathbb{K}\nabla p \cdot \nabla q \,\mathrm{d}x = \int_\Omega f\, q \mathrm{d}x \qquad \forall q \in H^1_0(\Omega).$$

and as **VEM approximate problem** we can take:
find $p_h \in$ *VEM,nodal such that:*

$$[\mathbb{K}\nabla p_h, \nabla q_h]_{VEM,edge} = [f,\, q_h]_{VEM,nodal} \; \forall q_h \in \text{VEM,nodal}$$

Variational formulation - Mixed

Darcy problem, in *mixed form*, is instead:
find $p \in L^2(\Omega)$ *and* $\mathbf{u} \in H(\mathrm{div};\Omega)$ *such that:*

$$\int_\Omega \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}V = \int_\Omega p\,\mathrm{div}\mathbf{v}\,\mathrm{d}V \quad \forall \mathbf{v} \in H(\mathrm{div};\Omega)$$

and

$$\int_\Omega \mathrm{div}\mathbf{u}\, q\,\mathrm{d}V = \int_\Omega f\, q\,\mathrm{d}V \qquad \forall q \in L^2(\Omega).$$

The approximate mixed formulation can be written as:

find $p_h \in VEM, volume$ *and* $\mathbf{u}_h \in VEM, face$ *such that*

$$[\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v}_h]_{VEM,face} = [p_h, \mathrm{div}\mathbf{v}_h]_{VEM,volume} \; \forall \mathbf{v}_h \in VEM, face$$

and

$$[\mathrm{div}\mathbf{u}_h, q_h]_{VEM,volume} = [f, \; q_h]_{VEM,volume} \; \forall q_h \in VEM, volume.$$

Strong formulation of Magnetostatic problem

- $\mathbf{j}$ = *divergence free* current density
- μ = magnetic permeability
- $\mathbf{u}$ = vector potential with the gauge $\mathrm{div}\,\mathbf{u} = 0$
- $\mathbf{B} = \mathbf{curl}\,\mathbf{u}$ = magnetic induction
- $\mathbf{H} = \mu^{-1}\mathbf{B} = \mu^{-1}\mathbf{curl}\,\mathbf{u}$ =magnetic field
- $\mathbf{curl}\,\mathbf{H} = \mathbf{j}$

The classical magnetostatic equations become now

$$\mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\mathbf{u} = \mathbf{j} \text{ and } \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega,$$

$$\mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega.$$

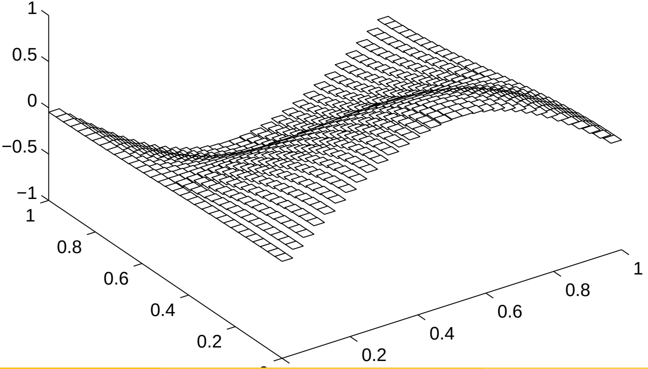
The **variational formulation** of the magnetostatic problem (setting $\mathbf{B} = \mu\mathbf{H} = \mathbf{curl}\,\mathbf{u}$) is :

$$\begin{cases} \textit{Find } \mathbf{u} \in H_0(\mathbf{curl},\Omega) \textit{ and } p \in H_0^1(\Omega) \textit{ such that :} \\ (\mu^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) - (\nabla p, \mathbf{v}) = (\mathbf{j}, \mathbf{v})\ \forall\,\mathbf{v} \in H_0(\mathbf{curl};\Omega) \\ \qquad\qquad (\mathbf{u}, \nabla q) = 0 \qquad \forall\, q \in H_0^1(\Omega), \end{cases}$$

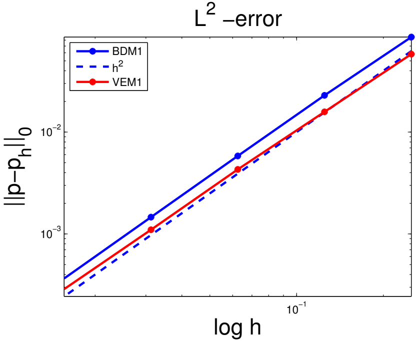
and the **VEM approximation** can be chosen as:

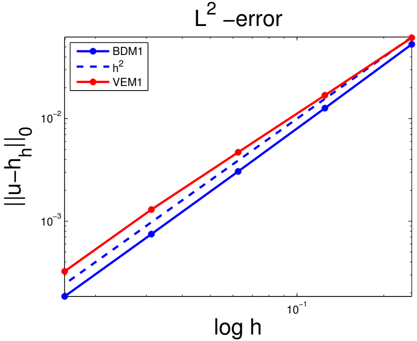
$$\begin{cases} \textit{Find } \mathbf{u}_h \in \textit{VEM,edges and } p_h \in \textit{VEM,nodal such that:} \\ [\mu^{-1}\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h]_{VEM,face} - [\mathbf{grad}\,p_h, \mathbf{v}_h]_{VEM,edge} \\ = [\mathbf{j}, \mathbf{v}_h]_{VEM,edge}\ \forall\,\mathbf{v}_h \in \textit{VEM, edge}, \\ [\mathbf{u}, \mathbf{grad}\,q_h]_{VEM,edge} = 0 \quad \forall\, q_h \in \textit{VEM, nodal}. \end{cases}$$

Mesh of squares 4x4, 8x8, ...,64x64
Exact solution p=sin(2x)cos(3y)

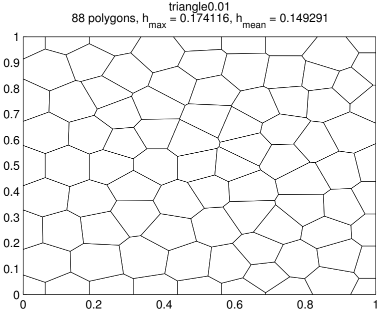


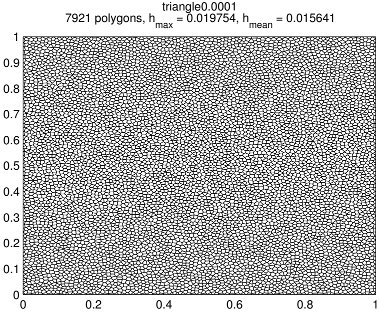
Numerical results-Squares



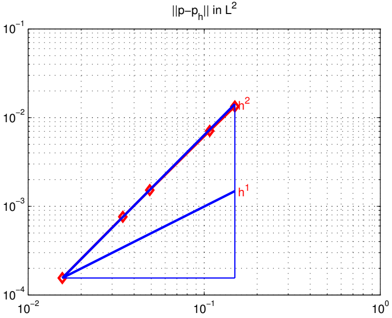


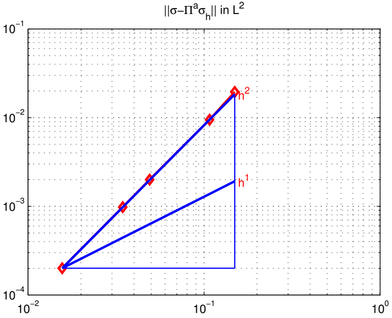
Voronoi polygons 88,...,7921
Exact solution p=sin(2x)cos(3y)





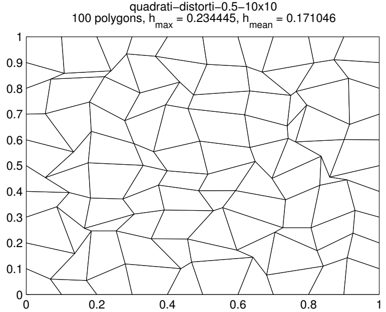
Numerical results-Voronoi

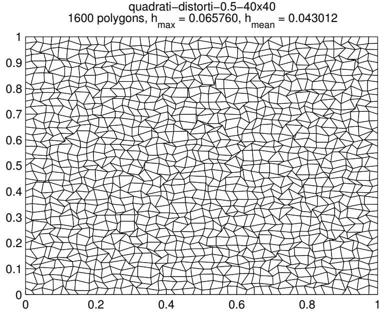




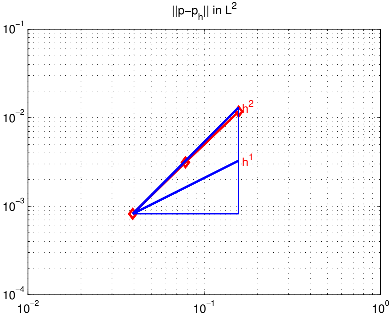
Mesh of distorted quads: 10x10; 20x20; 40x40

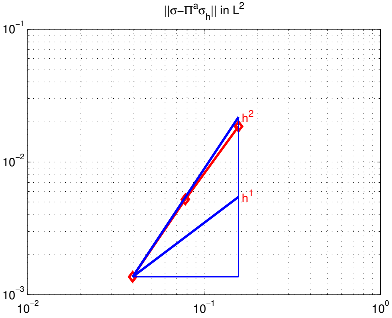
Exact solution: $p = \sin(2x)\cos(3y)$





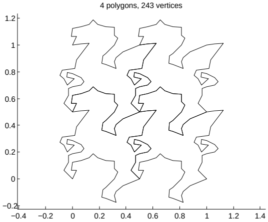
Numerical results–Distorted Quads

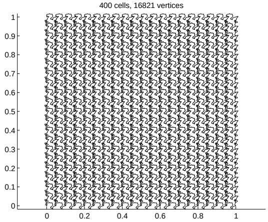




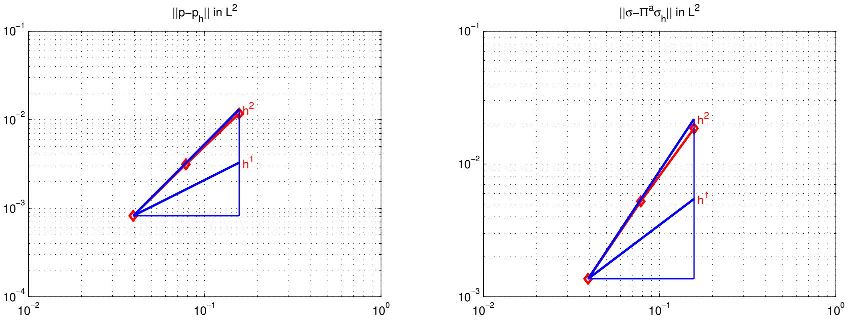
Mesh of horses: 4x4; 8x8; 10x10; 16x16

Exact solution: $p = \sin(2x)\cos(3y)$





Numerical results–Winged horses



Conclusions

- Virtual Elements are a new method, and a lot of work is needed to assess their *pros* and *cons*.
- Their major interest is on polygonal and polyhedral elements, but their use on distorted quads, hexa, and the like, is also quite promising.
- For triangles and tetrahedra the interest seems to be concentrated in higher order continuity (e.g. plates).
- The use of VEM mixed methods seems to be quite interesting, in particular for their connections with other methods for polygonal/polyhedral elements.